Idempotent tropical matrices:
graphs, groups and metric spaces

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(joint work with Zur Izhakian and Mark Kambites)
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Let $\mathbb{F}_T$ denote the tropical semifield $\mathbb{F}_T = (\mathbb{R}, \oplus, \otimes)$, where

$$a \oplus b := \max(a, b), \quad a \otimes b := a + b.$$ 

and let $M_n(\mathbb{F}_T)$ denote the set of all $n \times n$ matrices over $\mathbb{F}_T$, with multiplication $\otimes$ defined in the obvious way.

It is easy to see that $(M_n(\mathbb{F}_T), \otimes)$ is a semigroup.

We are interested in the algebraic structure of this semigroup, much of which can be neatly described using some geometric ideas.
Let $\mathbb{FT}^n$ denote the set of all real $n$-tuples $v = (v_1, \ldots, v_n)$ with obvious operations of addition and scalar multiplication:

$$(v \oplus w)_i = v_i \oplus w_i, \quad (\lambda \otimes v)_i = \lambda \otimes v_i.$$ 

Given a finite subset $X = \{x_1, \ldots, x_r\} \subset \mathbb{FT}^n$, the tropical polytope generated by $X$ is the $\mathbb{FT}$-linear span of $X$:

$$\{\lambda_1 \otimes x_1 \oplus \cdots \oplus \lambda_r \otimes x_r : \lambda_i \in \mathbb{FT}\}.$$
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Let $A \in M_n(\mathbb{FT})$. We define the row space $R(A) \subseteq \mathbb{FT}^n$ to be the tropical polytope generated by the rows of $A$.

Similarly, we define the column space $C(A) \subseteq \mathbb{FT}^n$ to be the tropical polytope generated by the columns of $A$. 
Some tropical polytopes in $\mathbb{FT}^3$
On the structure of semigroups

**Green’s relations**: Equivalence relations that can be defined upon any semigroup $S$ and encapsulate the **ideal and subgroup structure** of $S$.

For $A, B \in S$...

- $A \mathrel{L} B$ if $\exists X, Y \in S^1$ such that $A = XB$ and $B = YA$.
- $A \mathrel{R} B$ if $\exists X, Y \in S^1$ such that $A = BX$ and $B = AY$.
- $A \mathrel{H} B$ if $A \mathrel{L} B$ and $A \mathrel{R} B$.  

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- $A \mathcal{R} B$ if $\exists X, Y \in S^1$ such that $A = BX$ and $B = AY$.
- $A \mathcal{H} B$ if $A \mathcal{L} B$ and $A \mathcal{R} B$.

In $M_n(\mathbb{FT})$:

$A \mathcal{L} B$ if and only if $R(A) = R(B)$.

$A \mathcal{R} B$ if and only if $C(A) = C(B)$.

$A \mathcal{H} B$ if and only if $R(A) = R(B)$ AND $C(A) = C(B)$.
Let $S$ be a semigroup.
The **idempotent elements** $(E \in S, E^2 = E)$ play a special role in the study of the subgroup structure of $S$.

Around every idempotent element there is a unique **maximal subgroup** $H_E$. This is the $\mathcal{H}$-equivalence class of $E$.

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- What are the maximal subgroups of $M_n(\mathbb{F}T)$? (i.e. What are the $\mathcal{H}$-equivalence classes of idempotents?)
- What *kinds* of group arise? (i.e. What are these groups up to isomorphism?)
Maximal subgroups of $M_n(\mathbb{F}_2)$

Given an idempotent $E \in M_n(\mathbb{F}_2)$ it is clear from the previous definitions that

$$H_E = \{ A \in M_n(\mathbb{F}_2) : R(A) = R(E) \text{ and } C(A) = C(E) \}$$
Maximal subgroups of $M_n(\mathbb{F}_T)$

Given an idempotent $E \in M_n(\mathbb{F}_T)$ it is clear from the previous definitions that

$$H_E = \{ A \in M_n(\mathbb{F}_T) : R(A) = R(E) \text{ and } C(A) = C(E) \}$$

**Theorem** Let $E$ be an idempotent in $M_n(\mathbb{F}_T)$. Then

- $H_E$ is isomorphic to the group of $\mathbb{F}_T$-linear automorphisms of the column space $C(E)$
- $H_E$ is isomorphic to the group of $\mathbb{F}_T$-linear automorphisms of the row space $R(E)$.
Three notions of dimension

Let $V \subseteq \mathbb{FT}^n$ be a tropical polytope.

- The tropical dimension of $V$ is the maximum topological dimension of $V$ regarded as a subset of $\mathbb{R}^n$. We say that the tropical dimension is pure if the open (within $V$) subsets of $V$ all have the same topological dimension.

- The generator dimension of $V$ is the minimum cardinality of a generating set for $V$.

- The dual dimension of $V$ is the minimum $k$ such that $V$ embeds linearly into $\mathbb{FT}^k$.

In general, these dimensions can differ.
### Dimensions of tropical polytopes

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<th>dual dim</th>
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![Diagram of tropical polytopes]
Theorem  Let $V \subseteq \mathbb{FT}^n$ be a tropical polytope.

There is a positive integer $k$ such that $V$ has pure tropical dimension $k$, generator dimension $k$ and dual dimension $k$

if and only if

$V$ is the column space of an idempotent

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If $E$ is an idempotent in $M_n(\mathbb{FT})$, we say that $E$ has rank $k$ if the dimension (in any sense) of $C(E)$ is $k$. (Note: $1 \leq \text{rank}(E) \leq n$)
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- Idempotents of full rank $n$ have a particularly nice structure; their row and column spaces are convex in the ordinary sense.
Let $\mathbb{T} = \mathbb{FT} \cup \{-\infty\}$.
The **units** in $M_n(\mathbb{T})$ are the tropical monomial matrices.
Maximal subgroups for idempotents of full rank

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**Theorem**
Let $E$ be an idempotent of rank $n$ in $M_n(\mathbb{FT})$ and define $G_E = \{G : G \text{ is a unit in } M_n(\mathbb{T}) \text{ and } GE = EG\}$. Then $H_E \cong G_E$. 
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Then $H_E \cong G_E$.

**Corollary**

Every $\mathbb{FT}$-module automorphism of $C(E)$

(i) extends to an automorphism of $\mathbb{FT}^n$ and

(ii) is a (classical) affine linear map.
Maximal subgroups for idempotents of full rank

Let $E$ be an idempotent of rank $n$ in $M_n(\mathbb{F} \mathbb{T})$, so that $H_E \cong G_E$.

**Theorem**

Let $R = \{ \lambda \otimes I_n \}$ and $\Sigma = \{ G \in G_E : G$ has eigenvalue 0$\}$. Then $G_E = R \times \Sigma$. 
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Let $R = \{\lambda \otimes I_n\}$ and $\Sigma = \{G \in G_E : G$ has eigenvalue 0\}. Then $G_E = R \times \Sigma$.

It is clear that $R \cong \mathbb{R}$ and not hard to show that the map $\Sigma \to S_n$ sending each unit $G$ to its associated permutation is injective, giving:

**Theorem**

Let $E$ be an idempotent of rank $n$ in $M_n(\mathbb{F}T)$. Then $H_E \cong \mathbb{R} \times \Sigma$, for some $\Sigma \leq S_n$. 
Maximal subgroups of $M_n(\mathbb{F}T)$

So, for an idempotent $E$ of full rank $n$, the corresponding maximal subgroup is isomorphic to a direct product of $\mathbb{R}$ with a finite group $\Sigma \leq S_n$. What about when $E$ has rank $< n$?
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Let $E$ be an idempotent of rank $k$ in $M_n(\mathbb{F}T)$. Then there is an idempotent $F \in M_k(\mathbb{F}T)$ such that $F$ has rank $k$ and $H_E \cong H_F$. 
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**Corollary**
Let $H$ be a maximal subgroup of $M_n(\mathbb{F}T)$ containing a rank $k$ idempotent. Then $H \cong \mathbb{R} \times \Sigma$, for some $\Sigma \leq S_k$. 
Let \([n] = \{1, \ldots, n\}\) and let \(d: [n] \times [n] \to \mathbb{R}\) be a metric. Consider the \(n \times n\) matrix \(E\) with \(E_{i,j} = -d(i, j)\).

Then

- \(E \otimes E = E\);
- \(E\) has full rank \(n\).
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**Theorem [JK]**
The columns of $E$ with respect to $d_H$ form a metric space isometric to $([n], d)$.
Idempotents, groups and finite metrics

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$H_E \cong \mathbb{R} \times I$, where $I$ is the isometry group of the finite metric space $([n], d)$.

**Corollary [JK]**

Let $G$ be a finite group. Then $\mathbb{R} \times G$ is a maximal subgroup of $M_n(\mathbb{F}_T)$, for $n$ sufficiently large.