Products of random Max-plus matrices

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Products of random i.i.d. Max-plus matrices

Let $[A(k)]_{k=1}^{\infty}$ be an i.i.d. sequence of $\mathbb{R}_{Max}^{N\times N}$-valued random variables. Then

$$P(n) = A(n) \otimes A(n-1) \otimes \dots \otimes A(1) = \bigotimes_{k=1}^{n} A(k)$$

Example

$$A(k) = \begin{pmatrix} -\infty & t_{1,2}(k) & t_{1,3}(k) \\ t_{2,1}(k) & -\infty & t_{2,3}(k) \\ t_{2,3}(k) & -\infty & -\infty \end{pmatrix}$$

where the $t_{i,j}(k)$ are all i.i.d. uniform-$[0,1]$. Simulation

$$P(2) = \begin{pmatrix} -\infty & 0.8147 & 0.9058 \\ 0.1270 & -\infty & 0.9134 \\ 0.6324 & -\infty & -\infty \end{pmatrix} \otimes \begin{pmatrix} -\infty & 0.0975 & 0.7922 \\ 0.2785 & -\infty & 0.5469 \\ 0.9575 & -\infty & -\infty \end{pmatrix}$$

$$= \begin{pmatrix} 1.4246 & -\infty & 1.0109 \\ 1.1793 & 1.0932 & 1.1843 \\ -\infty & 1.7722 & 1.8633 \end{pmatrix}$$

Max paths

We can think of everything in terms of the paths through an associated weighted graph G with vertices $\{1, 2, ..., N\}$ and an edge (i, j) whenever $A_{i,j} \neq -\infty$.

$$P(n)_{i,j} = \max_{\sigma : \sigma j \mapsto i : |\sigma| = n} W(\sigma)$$

Where $W(\sigma) = \sum_{k=1}^{n} A_{\sigma(k+1),\sigma(k)}(k)$. Example

$$A(k) = \begin{pmatrix} -\infty & t_{1,2}(k) & t_{1,3}(k) \\ t_{2,1}(k) & -\infty & t_{2,3}(k) \\ t_{2,3}(k) & -\infty & -\infty \end{pmatrix}$$

A matrix distribution whose individual edges are either always finite or always infinite is said to have fixed support.

Max-plus multiplicative ergodic theorem

If $[A(k)]_{k=1}^{\infty}$ is a sequence of i.i.d. Max-plus matrices with fixed irreducible support then

$$\lim_{n\to\infty} \frac{1}{n} \bigotimes_{k=1}^{n} A(k) = \lambda$$

exists and is a Max-plus matrix whose components are all equal to the same constant which we denote λ.

λ is the step averaged weight of the maximally weighted path of length n as $n \to \infty$.

So that for a Max-plus linear system $X(n+1) = A(n) \otimes X(n)$ we have

$$\lim_{n\to\infty} \frac{X(n)}{n} = \lambda$$

which we can think of as the reciprocal of some service rate or throughput.

Componentwise homogeneous Max-plus matrices

Given a directed graph G on $\{1, 2, ..., N\}$ which may contain multiple (parallel) edges between the same to vertices as well as an array of i.i.d. random variables $[t_e(k)]_{k=1,e\in E}^{\infty}$ we define an associated sequence of i.i.d. componentwise homogeneous Max-plus matrices by

$$A(k)_{i,j} = \max_{e:j\mapsto i} t_e(k)$$

A componentwise homogeneous Max-plus matrix is any Max-plus matrix which can be defined in this way.

We further define an associated adjacency matrix by $\mathcal{A}_{i,j} =$ number of edges from j to i in G. Perron frobenius theory tells us that provided G is irreducible $\mathcal{A}$ will have a maximal positive eigenvalue $\mathcal{X}$ with unique eigenvector supported positively on every node.

Bounds on exponent

For a componentwise homogeneous Max-plus matrix with irreducible support we have

$$\sum_v \pi_v E(d_v) \leq \lambda \leq \lim_{n\to\infty} \frac{1}{n}\widehat{E}(n, \mathcal{X}^n)$$

where d_v is the out degree of v in G,

$$E(d) = \mathbb{E} \max_{i=1}^{d} t_i \quad \widehat{E}(n,m) = \mathbb{E} \max_{i=1}^{m} \sum_{j=1}^{n} t_{i,j}$$

π is the invariant distribution of the uniform walk on G and $\mathcal{X}$ is the greatest classical eigenvalue of the support graphs adjacency matrix (Perron root).

Importance of $\mathcal{X}$

For all our examples we will provide an explicit link between $\mathcal{X}$ and λ. What is it?

Let $D_i(n)$ be the number of paths of length n that end at i

$$D(n+1) = \mathcal{A}D(n)$$

So that

$$\lim_{n\to\infty} \frac{1}{n} \log ||D(n)|| = \log \mathcal{X}$$

is the exponential growth rate for the number of paths of length n.

Idea behind lower bound

λ is the step averaged weight of the maximally weighted path of length n as $n \to \infty$.

We can bound λ below by the step averaged weight of any other path - lets try to choose one with a high weight.

The greedy strategy constructs a path σ by choosing $\sigma(k) \mapsto \sigma(k+1)$ such that

$$A(k)_{\sigma(k+1),\sigma(k)} = \max_j A(k)_{j,\sigma(k)}$$

Greedy strategy analysis

The greedy strategy gives rise to a Markov chain on G with transition probabilities

$$Q_{i,j} = \frac{\mathcal{A}_{i,j}}{d_j = \sum_k \mathcal{A}_{k,j}}$$

and invariant distribution π.

We have

$$\lambda \geq \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} A_{\sigma(k+1),\sigma(k)}(k)$$

The law of large numbers gives

$$\lambda \geq \sum_i \pi_i \underbrace{\mathbb{E}[\max_{j=1}^{d_i} t_j]}$$

where the underbraced term is the expected weight of the chosen edge leaving i.

Example distribution - Exponential

For a componentwise mean-1 exponential matrix we have

$$\lambda \geq \sum_i \pi_i \mathbb{E}[\max_{j=1}^{d_i} t_j]$$

$$= \sum_i \pi_i \sum_{k=1}^{d_i} \frac{1}{k} \geq \sum_i \pi_i \log d_i = \log \mathcal{X}$$

Red Lemma is valid for any graph G including directed case where π has no tractable general form.

Idea behind upper bound

A sequence of random variables $x_1, ..., x_N$ is associated if

$$cov[f(x_1, ..., x_N), g(x_1, ..., x_N)] \geq 0$$

for all f, g non-decreasing in every component.

If $(x_1, ..., x_N)$ and $(y_1, ..., y_N)$ are identically distributed but the x_i are associated while the y_i are independent then

$$\mathbb{E} \max_{i=1}^{N} x_i \leq \mathbb{E} \max_{i=1}^{N} y_i$$

Path weights are associated

...So

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \mathbb{E} \max_{\sigma|_n} W(\sigma) \leq \lim_{n\to\infty} \frac{1}{n} \mathbb{E} \max_{\sigma|_n} \widehat{W}(\sigma)$$

and

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{E} \max_{\sigma|_n} \widehat{W}(\sigma) = \lim_{n\to\infty} \frac{1}{n} \mathbb{E} \max_{i=1}^{\mathcal{X}^n} \sum_{k=1}^{n} t_i(k)$$

Example distribution - Uniform

For componentwise $[0, 1]$-uniform Max-plus matrices define the sequence of generalized medians $(\mu_n)_{n=1}^{\infty}$ by

$$\mathbb{P}\{\max_{i=1}^{\mathcal{X}^n} \sum_{k=1}^{n} t_i(k) < \mu_n\} = 1 - \frac{1}{n}$$

So that

$$\lambda \leq \lim_{n \to \infty} \frac{\mu_n}{n}$$

Now

$$\mathbb{P}\{\sum_{k=1}^{n} t_i(k) < n - \mu_n\} = 1 - (1 - \frac{1}{n})^{\frac{1}{\mathcal{X}^n}}$$

and we can bound this probability above as follows

$$\mathbb{P}[\sum_{k=1}^{n} t_i(k) < n - \mu_n] = \int_{\Delta^*(n-\mu_n)} d\underline{u} \leq \int_{\Delta(n-\mu_n)} d\underline{u} = \frac{(n-\mu_n)^n}{n!}$$

where

$$\Delta^*(n-\mu_n) = \{\underline{u} \in \mathbb{R}^n : \sum_{i=1}^{n} u_i \leq n - \mu_n : 0 \leq u_i \leq 1\}$$

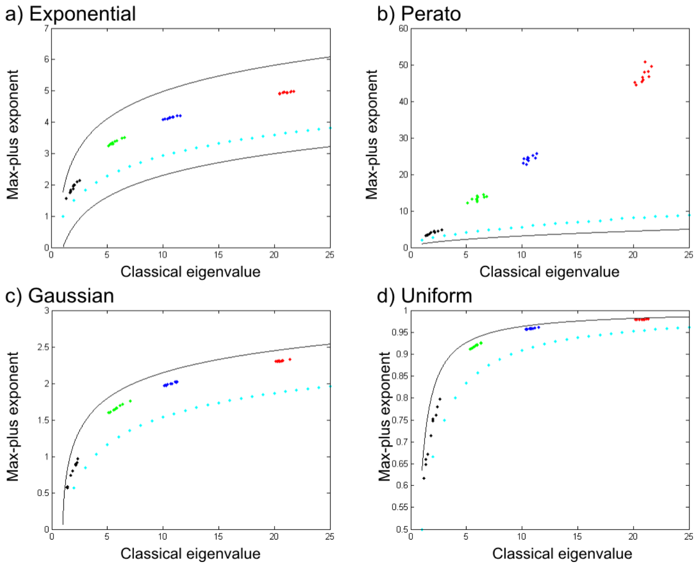
and

$$\Delta(n-\mu_n) = \{\underline{u} \in \mathbb{R}^n : \sum_{i=1}^{n} u_i \leq n - \mu_n : 0 \leq u_i\}$$

Therefore

$$\lambda \leq \lim_{n\to\infty} \frac{\mu_n}{n} = 1 - \frac{1}{\mathbf{e}\mathcal{X}}$$

Results and numerical examples



Conclusions/ongoing work

- Explicit link between Max-plus exponent and graph structure for componentwise homogeneous matrices.
- More connections $\Rightarrow$ greater $\mathcal{X}$ $\Rightarrow$ greater λ
- Upper bounds can be used to give a guaranteed service rate.
- Q: Can we extend these results to an arbitrary fixed matrix plus some homogeneous noise? A: It looks like it ...hopefully!

Thank you for listening!!!