Lie representations of $GL(V)$

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Outline

▶ Tensor representations of $GL(V)$
▶ Lie representations of $GL(V)$
▶ Klyachko’s Theorem
▶ A combinatorial proof
Tensor representations of $GL(V)$

- $V$ a finite dimensional vector space over a field of characteristic zero
Tensor representations of $GL(V)$

- $V$ a finite dimensional vector space over a field of characteristic zero
- $T$ the tensor algebra on $V$

\[
T = \bigoplus_{n \geq 0} T_n \quad T_n = V^\otimes n
\]
Tensor representations of $GL(V)$

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$$T = \bigoplus_{n \geq 0} T_n \quad T_n = V \otimes^n$$

- $V$ is the natural module for $GL(V)$
- $T$ is a $GL(V)$-module
- Each $T_n$ is a $GL(V)$-submodule of $T$ called the $n$th tensor representation.
Tensor representations of $GL(V)$

Schur (1901, 1923): The $T_n$ are semisimple $GL(V)$-modules and the irreducible components are parameterised by partitions of $n$

$$T_n \cong \bigoplus_{\lambda \vdash n} t_\lambda[\lambda]$$
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$[\lambda]$ - irreducible $GL(V)$-module corresponding to $\lambda$  
([\lambda] = 0 if \lambda has more than dim V parts).  
$t_\lambda$ - multiplicity
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Recall that a partition of $n$ is a sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $\lambda_1 \geq \cdots \geq \lambda_k$ and $\lambda_1 + \cdots + \lambda_k = n$
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e.g. $\lambda = (4, 2, 2, 1, 1) = (4, 2^2, 1^2) \vdash 10$
Tensor representations of $GL(V)$

- A **Young diagram** of shape $\lambda$ is a collection of $n$ boxes arranged with $\lambda_i$ boxes in the $i$th row.

```
1 2 5 6
3 7
4 8
9
10
```

It turns out that $t_\lambda = \text{number of standard tableaux of shape } \lambda$. 
A Young diagram of shape $\lambda$ is a collection of $n$ boxes arranged with $\lambda_i$ boxes in the $i$th row.

Example:
$\lambda = (4, 2^2, 1^2)$

\[
\begin{array}{cccc}
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{array}
\]
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<tr>
<td>3</td>
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Tensor representations of $\text{GL}(V)$

Example:

$T_4 \cong [4] \oplus 3 [3, 1] \oplus 2 [2^2] \oplus 3 [2, 1^2] \oplus [1^4]$
Lie representations of $Gl(V)$

- Turn $T$ into a Lie algebra by setting $[u, v] = u \otimes v - v \otimes u$
Lie representations of $GL(V)$

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- $L$ the Lie subalgebra generated by $V$ in $T$

$$L = \bigoplus_{n \geq 1} L_n \quad L_n = L \cap T_n$$
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- $L_n$ is a $GL(V)$-submodule of $T_n$ called the \textit{$n$th Lie representation}.
- Hence

\[ L_n \cong \bigoplus_{\lambda \vdash n} l_\lambda [\lambda] \quad 0 \leq l_\lambda \leq t_\lambda \]
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$$L_n \cong \bigoplus_{\lambda \vdash n} l_\lambda[\lambda] \quad 0 \leq l_\lambda \leq t_\lambda$$

- What is $l_\lambda$?
Lie representations of $GL(V)$

<table>
<thead>
<tr>
<th>Decomposition into irreducibles</th>
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<tbody>
<tr>
<td>$L_1 \cong [1]$</td>
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<tr>
<td>$L_2 \cong [1^2]$</td>
<td>[2]</td>
</tr>
<tr>
<td>$L_3 \cong [2, 1]$</td>
<td>[3], [1^3]</td>
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<tr>
<td>$L_4 \cong [3, 1] \oplus [2, 1^2]$</td>
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<tr>
<td>$L_5 \cong [4, 1] \oplus [3, 2] \oplus [3, 1^2] \oplus [2^2, 1] \oplus [2, 1^3]$</td>
<td>[5], [1^5]</td>
</tr>
<tr>
<td>$L_6 \cong [5, 1] \oplus [4, 2] \oplus 2[4, 1^2] \oplus [3^2] \oplus 3[3, 2, 1]$ $\oplus [3, 1^3] \oplus 2[2^2, 1^2] \oplus [2, 1^4]$</td>
<td>[6], [2^3], [1^6]</td>
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Lie representations of $GL(V)$

Wever (1949):

$$l_{\lambda} = \frac{1}{n} \sum_{d \mid n} \mu(d)\chi_{\lambda}(\tau^{n/d})$$
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$\mu$ - the Möbius function
$\chi_\lambda$ - the character of the irreducible $S_n$-module corresponding to $\lambda$
$\tau$ - a cycle of length $n$ in $S_n$
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It is difficult to see in general which modules actually occur in the decomposition of $L_n$, that is, for which $\lambda$ we have $l_\lambda > 0$. 
Klyachko’s Theorem (1974)

Let \( n \geq 3 \) and let \( \lambda \vdash n \) with no more than \( \dim(V) \) parts. Then

\[ l_\lambda > 0 \iff \lambda \neq (1^n), (n), (2^2), (2^3). \]
Let $n \geq 3$ and let $\lambda \vdash n$ with no more than $\dim(V)$ parts. Then

$$l_{\lambda} > 0 \iff \lambda \neq (1^n), (n), (2^2), (2^3).$$

In other words, almost every irreducible $GL(V)$ module occurs in the Lie representation.
It turns out that $l_\lambda$ also has a nice combinatorial description in terms of standard tableaux.
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Let $T$ be a standard tableau. An entry $i$ is a descent in $T$ if $i + 1$ occurs in any row below the row containing $i$. 

We shall write $D(T)$ for the set of all descents in $T$. We define the major index of $T$ to be the sum of all descents in $T$: 

$$\text{maj}(T) = \sum_{i \in D(T)} i$$
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Standard tableaux, descents and major index

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\[
\begin{array}{cccccc}
\hline
5 & 3 & 2 & 1 & \text{} & \text{}\\
\hline
\end{array}
\]
Standard tableaux, descents and major index

Example: \( \lambda = (5, 3, 2, 1) \vdash 11 \)

\[
\begin{array}{cccc}
1 & 2 & 4 & 8 \ 9 \\
3 & 5 & 11 \\
6 & 10 \\
7 \\
\end{array}
\]

\( \text{maj}(T) = 2 + 4 + 5 + 6 + 9 = 26 \)

Remarks:

\( D(T) \subseteq \{1, \ldots, n-1\} \)

\( k - 1 \leq |D(T)| \leq n - \lambda_1 \)
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Kraśkiewicz-Weyman Theorem (1987)

Let $i$ and $n$ be coprime.

$$l_{\lambda} = \text{number of standard tableaux } T \text{ of shape } \lambda \text{ with } \text{maj}(T) \equiv i \mod n.$$
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- Note that $i$ can be any fixed number which is coprime to $n$.
- It is natural to try to prove Klyachko’s Theorem using the Kraśkiewicz-Weyman Theorem.
Theorem

Let $n \geq 3$, $\lambda \vdash n$.

$\exists$ a standard tableau of shape $\lambda$ with major index coprime to $n$

$\iff \lambda \neq (1^n), (n), (2^2), (2^3)$
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Main Idea

- We look at standard tableaux with “small” descent sets.
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- Let $\lambda \vdash n$ into $k$ parts.
  We can construct a standard tableau of shape $\lambda$ with at most $k$ descents which has major index coprime to $n$. 
We look at standard tableaux with “small” descent sets.

Let \( \lambda \vdash n \) into \( k \) parts. We can construct a standard tableau of shape \( \lambda \) with at most \( k \) descents which has major index coprime to \( n \).

Strategy:

- Two part partitions.
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Strategy:
- Two part partitions.
- Rectangles.
- Non-rectangular partitions into more than two parts.
Two part partitions

\[ n = 2m + 1, \; \lambda = (n - s, s): \]

<table>
<thead>
<tr>
<th>1</th>
<th>...</th>
<th>s</th>
<th>...</th>
<th>m</th>
<th>m+s+1</th>
<th>...</th>
<th>2m+1</th>
</tr>
</thead>
<tbody>
<tr>
<td>m+1</td>
<td>...</td>
<td>m+s</td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

\[ n = 2m, \; \lambda = (n - s, s), \; 1 < s < m: \]

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>...</th>
<th>s</th>
<th>...</th>
<th>m-1</th>
<th>m+1</th>
<th>m+2</th>
<th>m+s+2</th>
<th>...</th>
<th>2m</th>
</tr>
</thead>
<tbody>
<tr>
<td>m</td>
<td>m+3</td>
<td>...</td>
<td>m+s+1</td>
<td></td>
<td></td>
<td></td>
<td></td>
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</table>

\[
1 \quad 3 \quad ... \quad 2m
group\[1, 2\]
\]

\[
1 \quad 2 \quad 3 \quad ... \quad m-1 \quad m+2
group\[\{1, 2, 3\}\]
\]

\[
1 \quad 2 \quad 3 \quad ... \quad m-1 \quad m+2
\]

\[
1 \quad 2 \quad 3 \quad ... \quad m-1 \quad m+2
\]

\[ m \quad m+1 \quad m+3 \quad ... \quad 2m-1 \quad 2m \]
Rectangles

Let \( n = mk \), \( \lambda = (m^k) \vdash n \) \( 0 \leq i \leq k - 2 \) \( 1 \leq s \leq m - 1 \).

\[
T = \begin{array}{cccc}
1 & \cdots & \cdots & m \\
\vdots & & & \vdots \\
(i-1)m+1 & \cdots & \cdots & im \\
im+1 & im+2 & \cdots & im+s \phantom{|} \phantom{|} im+s+2 \phantom{|} \cdots \phantom{|} (i+1)m+1 \\
im+s+1 & (i+1)m+2 & \cdots & (i+2)m \\
& \vdots & & \vdots \\
(k-2)m+1 & \cdots & \cdots & (k-1)m \\
(k-1)m+1 & \cdots & \cdots & km
\end{array}
\]

\[ \text{maj}(T) = \frac{mk(k-1)}{2} + im + s + 1 \]

Show that one of these is coprime to \( n \) (technical)
Let $\lambda$ be a non-rectangular partition of $n$ into $k > 2$ parts.
The rest

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- Write $n = mk + r$ where $0 \leq r < k < n$
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- Write \( n = mk + r \) where \( 0 \leq r < k < n \)
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- Set \( \lambda^{(k)} = \lambda \)
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The lower rim
The rest

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- Set $\lambda^{(k)} = \lambda$

Can remove $m_i$ boxes from the lower rim of $\lambda^{(i)}$ to obtain a Young diagram $\lambda^{(i-1)}$ which has $i - 1$ rows.
The rest

- For every choice $m_1, \ldots, m_k$ we can construct a standard tableau $T$ of shape $\lambda$ with descent set

$$D(T) = \{m_1, m_1 + m_2, \ldots, m_1 + m_2 + \cdots + m_{k-1}\}$$
The rest

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Put the entries

$$m_1 + \cdots + m_{i-1} + 1, \ldots, m_1 + \cdots + m_{i-1} + m_i$$

from left to right in $\lambda^{(i)} \setminus \lambda^{(i-1)}$
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It can be shown that one of these descent sets gives major index which is coprime to $n$. 