Course outline for Mathematical Logic (MATH33021) 2024-25

Unit code: MATH33021

Credit Rating: 20

Unit level: Level 3

Teaching period(s): Semester 1

Offered by Department of Mathematics

Available as a free choice unit: No

Aims

To provide a concise base of Mathematical Logic, including Set Theory, Predicate Logic and Model Theory.

Overview

The course captures the beginning of first order logic and leads up to applications of Mathematical Logic in Algebra and Analysis.

In Set Theory we will first give a non-axiomatic approach to infinite numbers and how to do basic calculations with them. Historically this is how the subject began, when G. Cantor realised that ordinary arithmetic can be extended to the infinite. We will focus on ordinal and cardinal numbers and start with a brief introduction to ordered sets.

In Predicate Logic we will set up so called first order languages in which mathematics can be formalized and mathematical methods can be applied. Hence the informal notion of a 'formula' will become a mathematical object, amenable to tools and methods from the subject. For example one can ask if there is a computer, which in principle is able to find all true statements about mathematics (this was a driving force at the beginning of the discipline). It turned out that such a computer cannot exist. The crux here is that this statement has a rigorous mathematical proof, which necessitates the translation of clauses like "true statements about mathematics" and "can be proved" into mathematical statements itself. This part of the course will give a thorough exposition of this translation together with the fundamental theorems saying that the translation is correct (Soundness Theorem) and optimal (Completeness Theorem).

General mathematical structures (like groups, vector spaces or ordered sets) will be used to exemplify formulas of first order logic. A formula can be thought of a generalisation of an equation, but now we are also allowing quantifiers. The tools developed in the course will be used to analyse solution sets of such formulas (called 'definable sets'). Furthermore the methods allow a classification of mathematical structures according to the properties of their definable sets. This for example connects a priori different looking structures (think of a group and an ordered set) in surprising ways. The course will make first steps in this direction with illustrations in the complex and the real field.

After having established the fundamentals of Predicate Logic we will revisit Set Theory from an axiomatic point of view. We state and discuss Zermelo Fraenkel Set Theory as well as immediate applications of the Completeness Theorem to Set Theory.

Prerequisites

Familiarity with rigorous treatment of the basic mathematical language (sets, functions and relations) is indispensable. Simple properties of groups (as for example taught in Groups & Geometry) will be assumed, and will be used mainly in examples. The definition of fields and vector spaces over fields will be helpful in examples, but is not strictly assumed.

The course has a continuations at level 4 in the *Model Theory* module and provides valuable preparation for the modules *Category Theory* and *Computation and Complexity*.

Intended Learning Outcomes

(1) State the fundamental definitions and theorems of various classes of partially ordered sets (totally ordered, well-ordered, product orders and sums) and answer simple combinatorial questions testing knowledge of the definitions.
(2) Define an ordinal and perform simple operations using the main theorems about ordinals and well-ordered sets.
(3) Define a cardinal beyond the finite case and compute cardinalities of infinite sets in simple examples by using the main theorems about cardinal arithmetic.
(4) Formalize mathematical statements in first order logic and conversely translate the meaning of first-order sentences by constructing structures satisfying the sentences.
(5) Explain formal proofs in first order logic and formulate the Soundness Theorem and the Completeness Theorem.
(6) State and prove the compactness theorem.
(7) Prove the existence of structures with given specific properties and compare structures using completeness and compactness.
(8) Explain definability in structures and confirm definability of sets in a given structure in simple cases.
(9) State the axioms of set theory and establish consequences for set theory using the completeness and the compactness theorem

Assessment methods

There will be a 3 hours final exam in January, worth 100% in the module. In weeks 5 and 10 there will be formative tests to provide feedback on progress and to prepare for the final exam.

Syllabus

- Set Theory (4 weeks, 10 lectures)
  Ordered and partially ordered sets [3 lectures]. Well ordered sets and the well ordering principle, Zorn's Lemma [2 lectures]. Ordinal numbers [2 lectures]. Cardinal numbers [2 lectures]. The requirement of formal languages in set theory. [1 lecture]
- Predicate Logic (5 weeks, 15 lectures)
  Syntax and semantics of Propositional Logic [2 lectures]. Proof system and completeness of Propositional Logic [2 lectures]. First order languages [2 lectures]. First order structures [2 lectures]. Examples: Groups and partially ordered sets [3 lectures]. Formal proofs [2 lectures]. Soundness, Completeness and Compactness of Predicate Logic[2 lectures].
- First steps in axiomatice set theory (2 weeks, 6 lectures)

Axioms of set theory [2 lectures] and naïve set theory in this setup. Basic applications of Predicate Logic to models of set theory [2 lecture]. Outlook: Independence results [2 lecture].

Recommended reading

A full set of lecture notes will be provided. Further reading may be found in the references of these notes. The following two books are not text books for the course, but will give interested students a good impression what the subject is about.

(1) Goldrei, Derek; Propositional and Predicate Calculus: A Model of Argument; Springer London, 2005. ISBN : 9781846282294 https://manchester.primo.exlibrisgroup.com/permalink/44MAN_INST/bofker/alma992976946311601631

(2) Kunen, Kenneth; The foundations of mathematics; Studies in Logic (London). College Publications, London, 2009. Mathematical Logic and Foundations; volume 19 https://manchester.primo.exlibrisgroup.com/permalink/44MAN_INST/bofker/alma992976007810801631

(3) Cori, René, Lascar, Daniel; Mathematical logic. A course with exercises. Part I. Propositional Calculus, Boolean algebras, predicate calculus. Oxford University Press, Oxford, 2000. xx+338 pp. ISBN: 0-19-850049-1; 0-19-850048-3 http://man-fe.hosted.exlibrisgroup.com/MU_VU1:44MAN_ALMA_DS21154144760001631

Learning and Teaching Processes

There are 3 lectures and 2 tutorials each week.

Feedback methods

Feedback tutorials will provide an opportunity for students' work to be discussed and provide feedback on their understanding. Formative in-class tests provide an opportunity for students to receive feedback. Students can also get feedback on their understanding directly from the lecturer, for example during the lecturer's office hour. There will be a discussion board on Piazza, accessible through BlackBoard©,™.

Study hours

Lectures - 33 hours
Tutorials - 22 hours
Independent study hours (including assessments and revision) - 145 hours

Teaching staff

Gareth O Jones, Marcus Tressl