180 minutes

THE UNIVERSITY OF MANCHESTER

MATHEMATICAL LOGIC

XX January 2025

XX.XX – XX.XX

Answer ALL NINE questions (100 marks in total).

The use of calculators is NOT permitted in this exam.

Overall feedback and common mistakes: There were three main causes for loss of marks:

(i) Many marks were lost because questions (or parts of questions) were not read properly and therefore the answers could possibly not gain marks.

(ii) Answers have not been formulated in an understandable way. Possibly, the material has not been revised correctly, e.g., answers to example questions have not been written out by hand and in full, when revising the material. When this is done the first time during the exam, it is understandably a challenge to express possibly correct thoughts in a coherent way.

The issue here points more to a weakness in the ability of reading and writing mathematics, rather than a deficit in factual knowledge of Mathematical Logic.

1. Let X be a partially ordered set.

(i) What does it mean to say that a subset $S \subseteq X$ is an *initial segment* of X? [2 marks]

(ii) (a) Suppose that Y and Z are also partially ordered sets, and that $f : X \to Y$ and $g : Y \to Z$ are functions such that $f(X)$ is an initial segment of Y and $g(Y)$ is an initial segment of Z. Suppose that g is an embedding of partially ordered sets (that is, a poset embedding). Show that $g(f(X))$ is an initial segment of Z. [5 marks]

(b) Is the conclusion in (a) also true if g is only assumed to be monotone (that is, order preserving), rather than an embedding of partially ordered sets? Justify your answer. [3 marks]

[End of Question 1; 10 marks total]

Feedback on common mistakes and guide to the solution:
Guide to the solution: Item (i) is in the lecture notes, (ii)(a) is on the example sheets and was done in a tutorial. (ii)(b) is unseen, though the question was mentioned in tutorial. Drawing a picture in (ii)(b) is fine as long as the picture is explained.
Common mistakes: In the proof of (ii)(a) the communication of the (possibly correct idea) used a poor setup of notations which implied errors. For example, solutions read as follows. Take $x, y \in Z$ with $y \in g(f(X))$ and $x \leq y$. Then there is some $z \in X$ with $g(f(z)) = y$. Since $g(Y)$ is an initial segment of Z there is some $z' \in Y$ with $g(z') = y$. Since g is an embedding we get $z' \leq f(y)$. Since $f(X)$ is an initial segment of Y we see that x is in $g(f(X))$. Is this a correct solution? Compare with the solution on the example sheet.

In (ii)(b) it was said that the proof in (ii)(a) makes use of the assumption that g is an embedding and from this it was concluded that the conclusion in (a) fails. But this is not sound, because there might be another proof of (a) that does not need the assumption that g is an embedding. The correct answer requires an explicit counterexample.

2.

(i) What does it mean to say that a partially ordered set $(X, \leq)$ is a *well-ordered set*? [3 marks]

(ii) Suppose that α and β are ordinals. For the following sets, say whether they are necessarily ordinals. Justify your answer.

(a) $\alpha \cup \beta$. [4 marks]

(b) $\alpha \setminus \beta$. [4 marks]

(iii) Let α be an ordinal. Suppose that $f : \alpha \to \alpha$ is a function such that for all $\beta \in \alpha$ we have $f(\beta) \leq \beta$. Suppose also that there is a $\gamma \in \alpha$ such that $f(\gamma) < \gamma$. Show that f is *not* injective.

[4 marks]

[End of Question 2; 15 marks total]

Feedback on common mistakes and guide to the solution:
Guide to the solution: Item (i) is in the lecture notes, (ii)(a) has been seen in more generality (the union of a set of ordinals is an ordinal) but also follows from the frequently used property that ordinals are comparable for the subset relation. (ii)(b) is unseen, a solution is obtained by testing the statement with the ordinals 1 and 2. The full solutions has to make this explicit. The argument for the proof of (iii) was seen often in proofs of the lectures and example sheets: Take the smallest γ with $f(\gamma) < \gamma$ and then compare f on $f(\gamma)$ and γ.
Common mistakes: In (ii)(a) a direct proof was attempted, but then some parts of the definition of "ordinal" were not checked. For example that ε is a transitive relation on $\alpha \cup \beta$. In (iii) examples where shown with a finite ordinal; but this is not a proof. Some solutions misread the assumption as "For all β: $f(\beta) \leq \beta$ and/or for all γ: $f(\gamma) < \gamma$".

3. Show that there is no set X such that $\text{card}(\mathcal{P}(X)) = \omega$. [Hint: suppose that X is such a set and apply results from the lecture notes to obtain a contradiction.] [8 marks]

[End of Question 3; 8 marks total]

Feedback on common mistakes and guide to the solution:
Guide to the solution: The question is testing if the theorem on the size of the powerset is present. Together with the hint in the question, this leads to the solution quickly.
Common mistakes: This questions went well, up to standard problems on how to communicate mathematics like, 'what is assumed?', 'what do letters stand for that pop up on the page?', 'how are the various statements of the solution logically connected?'.

4.

(i) Show from the axioms of ZF that $V = \{x : x \text{ is a set }\}$ is not a set. [3 marks]

(ii) (a) State the *Axiom of Foundation*. [2 marks]

(b) Show that there do not exists sets a, b such that $a \in b$ and $b \in a$. [5 marks]

[End of Question 4; 10 marks total]

Feedback on common mistakes and guide to the solution:
Guide to the solution: (i) is on the example sheet, (ii)(a) is in the lecture notes and the solution to (ii)(b) is following a very similar line of argument as the question "show that there is no set a with $a \in a$" from the example sheets.
Common mistakes: In (i), Russell's paradox was quoted as a reason; this is the correct idea but is not a proof why ZF actually implies that V is not a set. In (i) it was not made explicit which axiom(s) of set theory would be violated by the assumption that V is a set.

In (ii)(b), the assumptions $a \in b$, $b \in a$ where confused with the assumptions $a \subseteq b$, $b \subseteq a$.

5. Let $\mathscr{L}$ be the language $\{f, P\}$ where f is a unary function symbol and P is a unary relation symbol. Consider the following sentences of $\mathscr{L}$:

φ_1 which is $\exists x \exists y \exists z \forall v \, (v \doteq x \vee v \doteq y \vee v \doteq z)$;
φ_2 which is $\forall x \, (P(x) \to P(f(x)))$;
φ_3 which is $\exists x \, P(x)$;
φ_4 which is $\forall x \forall y \, (f(x) \doteq f(y) \to x \doteq y)$;
φ_5 which is $\exists x \, (\neg P(x) \wedge x \neq f(x))$.

Is there an $\mathscr{L}$-structure $\mathscr{M}$ that is a model of $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5\}$? Justify your answer with an explanation of why there is no such structure, or by giving an example of such a structure and explaining why it is an example. [7 marks]

[End of Question 5; 7 marks total]

Feedback on common mistakes and guide to the solution:
Guide to the solution: This follows a similar pattern to a question on the example sheets. First describe what the various sentences express in a model, then take these descriptions as instructions on how to find a common model - which exists!
Common mistakes: This question went well. Some solutions misinterpreted what some sentences say about a model and concluded that there is no model of all of the sentences. Some solutions forgot to say (or did not say clearly) how f or P are interpreted in their suggested model.

6. Let $\mathscr{L}$ be a language and let ψ be an $\mathscr{L}$-sentence.

(i) Given $\mathscr{L}$-formulas $\varphi_1, \dots, \varphi_n$ what does it mean to say that $(\varphi_1, \dots, \varphi_n)$ *is a proof from* $\{\psi\}$.

[3 marks]

(ii) Suppose now that ψ is a tautology of Predicate Logic. Is there an $\mathscr{L}$-sentence φ such that $\psi \nvdash \varphi$ and $\psi \nvdash \neg\varphi$? Justify your answer.

[5 marks]

[**End of Question 6; 8 marks total**]

Feedback on common mistakes and guide to the solution:

Guide to the solution: (i) is a major definition in the course. For (ii) one can take any sentence that can be true in some $\mathscr{L}$-structure and false in some other $\mathscr{L}$-structure. The full solutions has to make this explicit and give references to a theorem in the notes.

Common mistakes: In (i) a sizeable number of answers did not state the correct definition of what a formal proof is. In (ii), by far the biggest problem was that $\psi \nvdash \varphi$ was understood as $\psi \vdash \neg\varphi$. The argument said "By the completeness theorem $\psi \nvdash \varphi$ is equivalent to $\psi \not\models \varphi$ [which is correct] and $\psi \not\models \varphi$ is equivalent to $\psi \models \neg\varphi$ by Tarski's truth definition" However, Tarski's truth definition says that in an $\mathscr{L}$-structure $\mathscr{M}$ we have $\mathscr{M} \not\models \varphi$ if and only if $\mathscr{M} \models \neg\varphi$, whereas $\psi \not\models \varphi$ is **not** equivalent to $\psi \models \neg\varphi$ in general.

7.

(i) State the *Completeness Theorem* of Predicate Logic and use it to show that every consistent set of sentences in any language has a model.
[5 marks]

(ii) State the *Compactness Theorem* of Predicate Logic and use (i) to prove the Compactness Theorem.
[6 marks]

[End of Question 7; 11 marks total]

Feedback on common mistakes and guide to the solution:
Guide to the solution: They can all be found in the lecture notes.
Common mistakes: This went well, but a number of solutions were already not stating the theorems correctly. The status of these theorems in the course are made clear in the notes and these theorems were talked about a lot in the lectures. A guess here is that students revised by mainly using past papers, where this question was also asked.

8. Let $\mathscr{L}$ be a language.

(i) Let $\mathscr{M}$ be an $\mathscr{L}$-structure and let $n \in \mathbb{N}$. What does it mean for a set $X \subseteq |\mathscr{M}|^n$ to be *definable in* $\mathscr{M}$?
[4 marks]

(ii) Show that for every definable set X as in (i) and each $a \in |\mathscr{M}|$, the set $X \times \{a\} \subseteq |\mathscr{M}|^{n+1}$ is definable in $\mathscr{M}$ as well.
[4 marks]

(iii) Now let $\mathscr{L} = \{\varepsilon\}$ be the language of set theory and suppose we are given a model $\mathscr{M}$ of ZF. Write down an $\mathscr{L}$-formula φ in one free variable such that the set defined by φ in $\mathscr{M}$ is the set of ordinals of $\mathscr{M}$.
[6 marks]

[End of Question 8; 14 marks total]

Feedback on common mistakes and guide to the solution:
Guide to the solution: (i) is in the lecture notes, (iii) is on the example sheets and (ii) was done in the tutorials (and similar questions were done on the example sheets).
Common mistakes: In (ii) it was not stated which formula defines $X \times \{a\}$. In (iii) some parts of the definition of an ordinal was forgotten, most of the time this was *transitivity of the* ε*-relation in a set* (not to be confused with the notion of a *transitive set*).

9.

(i) State both Skolem-Löwenheim theorems for a given finite language $\mathscr{L}$. [4 marks]

(ii) Let $\mathscr{L} = \mathscr{L}_\emptyset$ be the empty language. Show that all infinite $\mathscr{L}$-structures are elementarily equivalent. [7 marks]

(iii) Here $\mathscr{L}$ is the language $\{\varepsilon\}$ of set theory. Assuming that the $\mathscr{L}$-theory ZF is consistent, show that ZF has a countable model. [6 marks]

[End of Question 9; 17 marks total]

Feedback on common mistakes and guide to the solution:
Guide to the solution: (i) is in the lecture notes. (ii) is also in the lecture notes and was done in detail in the lectures. (iii) Is an application of Skolem-Löwenheim downwards, but first one needs to argue why ZF has an infinite model under the assumptions of the question.
Common mistakes: In (ii) it was argued that because the language is empty, there are essentially no sentences and therefore the statement holds. However there are many sentences in the empty language (see the sections on $\mathscr{L}_\emptyset$ in the lecture notes and the related example sheets) and it is not straightforward to check directly for such a sentence whether it is true in all infinite structures or false in all infinite structures. A frequent oversight was the lack of references to the implication "there is a bijection $|\mathscr{M}| \longrightarrow |\mathscr{N}|$" and "there is an isomorphism $\mathscr{M} \longrightarrow \mathscr{N}$"; notice that this implication is far from true in general. In (iii) it was said that ω is a countable model of ZF. It was not explained what this statement meant, but – assuming it meant ω in some model together with the interpretation of the ε-relation – this is not correct, because the axiom of infinity fails. This cannot be remedies by replacing ω by a countable ordinal. In (iii) incomplete explanations why ZF has an infinite model were given, or the explanation was not present at all.

END OF EXAMINATION PAPER