Sheet 1

Source: https://personalpages.manchester.ac.uk/staff/Marcus.Tressl/teaching/MathematicalLogic/index.php

Question 1. First read definition 1.1.1 and examples 1.1.2.

(i) Give an example of a poset having two elements, which is not a chain.
(ii) Give an example of a partial order $\leq$ on $\{1, 2, 3\}$ such that $2 \leq 1$.
(iii) Give an example of a partial order $\leq$ on $\mathbb{N}$ such that $1 \leq 3$ and $3 \leq 2$.
(iv) Verify that the powerset of the set $\mathbb{N}$ together with the inclusion of subsets is a poset by checking the conditions in the definition of a partially ordered set.
(v) In the poset from (iv), give an example of incomparable elements of this poset and an example of an infinite chain in this poset.
(vi) Let $(X, \leq)$ be a poset and define a new binary relation $\leq_{\text{inv}}$ on X by

$$x \leq_{\text{inv}} y \iff y \leq x.$$

Show that $\leq_{\text{inv}}$ is again a partial order (which is called the **inverse** of $\leq$).
(vii) Let $X = (X, \leq)$ be a poset and let $x < y$ be the strict partial order as defined in 1.1.1. Show that $<$ satisfies the following properties:
 (a) For all $x \in X$ we have not $x < x$.
 (b) For all x, y, z, if $x < y$ and $y < z$, then $x < z$.
(viii) Now assume $\lhd$ is a binary relation on X satisfying properties (a) and (b) of (vii), so you need to read (a) and (b) with $\lhd$ instead of $<$. Show that the relation defined by $x \unlhd y \iff (x \lhd y \text{ or } x = y)$ is a partial order on X whose strict order relation is the given $\lhd$.

Question 2. First read the constructions in 1.1.3.

(i) Prove that the product of two posets is again a poset.
Now consider the product of $(\mathbb{R}, \leq)$ with itself, where $\leq$ denotes the natural order of real numbers. We write the product poset as $(\mathbb{R} \times \mathbb{R}, \sqsubseteq)$.
 (a) Consider the two elements $a = (1, 1)$ and $b = (3, 2)$ of $\mathbb{R} \times \mathbb{R}$. Draw a picture of the **interval** $[a, b]$ of the poset $(\mathbb{R} \times \mathbb{R}, \sqsubseteq)$, hence draw a picture of the set of all elements $c \in \mathbb{R} \times \mathbb{R}$ with $a \sqsubseteq c \sqsubseteq b$.
 (b) Name an infinite subset of $\mathbb{R} \times \mathbb{R}$ such that all distinct elements in that set are incomparable for $\sqsubseteq$.
(ii) Consider the chains $X = \{1, 2, 3, 4\}$ and $Y = \{1, 2, 3\}$ in their natural order. Draw a grid of width 4 and height 3 and label the intersection points with 1,2,...,12 according to the lexicographic order $X \times_{lex} Y$.
(iii) Consider the following seven words, each consisting of 5 letters: "lemon, apple, grain, melon, mango, guava, grape". When we order them alphabetical, this looks as follows: apple, grain, grape, guava, lemon, mango, melon.
Explain, in terms of lexicographic orders, how this is achieved. Hence explain how words with five (lower case) letters are brought into a total order. A more challenging (not examinable) question is: Explain in a mathematical rigorous way how words of arbitrary (finite) length are brought into alphabetic order.

(iv) Prove that the lexicographic product of two chains is again a chain.

Question 3. First read definition 1.1.4 and examples 1.1.5.

(i) Let $(X \leq)$ be a poset. Show that the following conditions are equivalent:
  (a) $(X, \leq)$ is a chain.
  (b) For all $S \subseteq X$, every maximal element of S is the largest element of S.
  (c) Every $S \subseteq X$ has at most one maximal element.

(ii) In example 1.1.5(i), consider the subset $S = \emptyset$ of $X = \{a, b, c\}$. Does S have a supremum in X? Does S have an infimum in X?

(iii) Let $(X, \leq_1)$ be the poset from example 1.1.5(i) again and let $(Y, \leq_2)$ be a chain with $X \cap Y = \emptyset$. Let $(Z, \leq)$ be the ordered sum of $(X, \leq_1)$ and $(Y, \leq_2)$ as defined in 1.1.2(iv). Show that X has a supremum in Z for $\leq$ if and only if Y has a smallest element.

(iv) Let $(X, \leq_1)$ be the poset from example 1.1.5(i) again. Find a poset $(Y, \leq_2)$ that has $(X, \leq_1)$ as a subposet such that b and c have no infimum in $(Y, \leq_2)$.

(v) Consider the binary relation $|$ on $\mathbb{N}$ defined by $k|n \iff k$ divides n. Show that
  (a) $(\mathbb{N}, |)$ is a poset.
  (b) For $k, n \in \mathbb{N}$, show that the supremum of $\{k, n\}$ in this poset is the least common multiple of k and n.
  (c) For $k, n \in \mathbb{N}$, show that the infimum of $\{k, n\}$ in this poset exists.
  (d) Is $(\mathbb{N}, |)$ a subposet of $(\mathbb{N}, \leq)$, where $\leq$ is the natural order of natural numbers?

Question 4. First read the definition of "initial segment".

(i) Let X be a partially ordered set. Let I be an arbitrary index set and let $Y_i \Subset X$ for each $i \in I$. Show that the union $\bigcup_{i \in I} Y_i$ and the intersection $\bigcap_{i \in I} Y_i$ are again initial segments of X.

(ii) Let X, Y, Z be posets and let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be maps with $f(X) \Subset Y$ and $g(Y) \Subset Z$. Assume that g is a poset embedding. Show that $g(f(X)) \Subset Z$.

(iii) For a chain X, and $A, B \subseteq X$ we also write $A \Subset B$ if A is an initial segment of the subposet of X induced on B.
  (a) Show that $(\mathcal{P}(X), \Subset)$ is a partially ordered set.
  (b) Is $(\mathcal{P}(X), \Subset)$ a subposet of $(\mathcal{P}(X), \subseteq)$?
  (c) Show that the set $\{A \subseteq X \mid A \Subset X\}$ is a chain in $(\mathcal{P}(X), \Subset)$.

(iv) Consider the following properties of a chain X.
  (1) If $a \in X$, then a is the supremum of $X_{<a}$.
  (2) If I is any index set and $a_i \in X$ for all $i \in I$, then there is some $b \in X$ such that $\bigcup_{i \in I} X_{<a_i} = X_{<b}$.

  Give an example of
  (a) an infinite chain such that (1) holds.
  (b) an infinite chain such that (1) fails.
  (c) an infinite chain such that (2) holds.
  (d) an infinite chain such that (2) fails.

Question 5. First read 1.1.8 and 1.1.9, then answer the following questions:

(i) let $(X, \leq_1)$ be the poset from example 1.1.5(ii) and let $(Y, \leq_2)$ be the poset from example 1.1.5(i). Write down a surjective monotone map $f : X \longrightarrow Y$.

(ii) Let $(X, \leq_1), (Y, \leq_2)$ be posets with $Y \neq \emptyset$. Show that there is a monotone map $f : X \longrightarrow Y$.

(iii) Give an example of a bijective monotone map f between posets of your choice such that f is **not** an isomorphism.

(iv) Suppose $f : X \longrightarrow Y$ is a monotone map between posets $(X, \leq_1), (Y, \leq_2)$ and let $x, x' \in X$ such that the infimum of x, x' exists in $(X, \leq_1)$.

  (a) Show that in general the infimum of $f(x), f(x')$ does not need to exist in $(Y, \leq_2)$.

  (b) Suppose now that the infimum of $f(x), f(x')$ does exist in $(Y, \leq_2)$. Is it necessarily true that $f(\inf\{x, x'\}) = \inf\{f(x), f(x')\}$?

(v) Show that every chain of size n is poset-isomorphic to $\{1, \dots, n\}$ with its natural order. Hint: Do an induction on n.

(vi) Let $(X, \leq_1), (Y, \leq_2)$ be posets and let $f : X \longrightarrow Y$ be a monotone map. Suppose $(X, \leq_1)$ is a chain. Show the following.

  (a) If f is surjective, then also $(Y, \leq_2)$ is a chain.

  (b) If f is injective, then f is an embedding. (Note that this implication fails in general without the assumption that $(X, \leq_1)$ is a chain; see (iii))

(vii) Show that in general the projection onto the second component of a lexicographic product is **not** a monotone map.