Dedekind cuts in polynomially bounded, o-minimal expansions of real closed fields

Dissertation

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Introduction

We are concerned with o-minimal theories. The main result (the Box Theorem 17.3) is about polynomially bounded, o-minimal expansions T of real closed fields with archimedean prime model. In this Introduction, T always denotes such a theory - the reader may think of T as the theory of real closed fields.

A Dedekind cut p ('cut' for short) of a real closed field M is a pair (p^L, p^R) of subsets p^L, p^R (the Left, Right options of p) of M such that $p^L < p^R$ and $M = p^L \cup p^R$. We examine the structure (M, D), where D is a symbol for the set p^L. Sometimes p cuts out an algebraic substructure of M. For example $A := \{a \in M \mid |a| \in p^L\}$ can be a convex subgroup or a convex valuation ring of M. In the latter case, (M, D) is the same as the real valued field (M, A). This case is treated in real valuation theory which is therefore a main tool here. In fact we are far under results like the examination of the real valuation spectra in general (we have no analysis of the type space $S_n(M, D)$). Rather our first goals are model completeness results for all structures (M, D) in small, definable expansions of (M, D) (§16).
There are six sorts of such expansions according to certain properties of the cut p. I describe these properties and explain the word "sort":

If a is an element of a model M of T, then the cut $a^+ := ((-\infty, a], (a, +\infty))$ has definable options (for real closed fields, definable sets/maps are semi algebraic sets/maps), that is (M, D) is a definable expansion of M. If s is an M-definable map, then s moves a^+ to another cut with M-definable options. This means, that $\lim\limits_{x \searrow a} s(x)$ exists in $M \cup \{\pm\infty\}$.

In general if p and q are cuts over M, we say that p and q are equivalent, if there is an M-definable map, which moves p to q (this is an equivalence relation).
If p is a cut of M we write $W_0(p)$ for the convex subgroup $\{a \in M \mid a + p = p\}$ and A_p for the convex valuation ring $\{b \in M \mid b \cdot W_0(p) \subseteq W_0(p)\}$ of M. Let $\hat{p}$ be the cut of M with $\hat{p}^R = \{a \in M \mid a > W_0(p)\}$ (thus $\hat{p}$ is 'the upper boundary of $W_0(p)$') and let p^* be the cut of M with right options $\{a \in M \mid a > A_p\}$. Observe that $\hat{p}$ has M-definable options if p has M-definable options. Now I define the sort of p as follows:

Sort 1: p has M-definable options (we say "p is definable").
Sort 2: p is not definable and $\hat{p}$ is definable, thus $\hat{p} = 0^+$.
Sort 3: $\hat{p}$ is not definable, $\hat{p}$ is equivalent to p and to p^*.
Sort 4: p is equivalent to $\hat{p}$ and $\hat{p}$ is not equivalent to p^*.
Sort 5: $\hat{p}$ is not definable, not equivalent to p, but equivalent to p^*.
Sort 6: p is not equivalent to $\hat{p}$ and $\hat{p}$ is not equivalent to p^*.

We'll show that each cut p belongs to exactly one of these sorts. I want to mention that the sort is defined by the equivalence relations which holds between the cuts p, $\hat{p}$ and p^* (a closer description of the sorts is not necessary in the moment).

The Box Theorem says:

if q is a non definable cut of M, such that the options of q are definable in the structure (M, p^L) and X is a set of representatives for the equivalence classes in $\{p, \hat{p}, p^*\}$ (thus $n := \operatorname{card} X \in \{1, 2, 3\}$), then there is a unique n-type r of M such that the projections of r are in X and there is an element $r \in X$ such that $q \sim r$.

The name "Box" comes from the fact that the type r has a basis of neighbourhoods of open boxes in $S_n(M)$.

Consequences are:

(1) If q is another cut of M, equivalent to p, then p and q have the same sort (18.1).

(2) If q is a non definable cut of M, such that q^L is definable in the structure (M, p^L), then $A_p = A_q$ (18.2).

(3) If $M \prec N$ are models of T, with $\dim N/M = n < \infty$ (dim is dimension relative T), then N induces, in a canonical way, at most n convex valuation rings on M (the extension $M \subseteq N$ becomes more easier if the number of these valuation rings grows; cf §11).

(4) If $M \prec N$ are models of T with $\dim N/M = 1$, B is a convex valuation ring of N and $A := B \cap M$, then either the value groups or the residue fields of A, B respectively are equal (18.12).

(5) Suppose T has quantifier elimination and an universal system of axioms in the language $\mathscr{L}$. Let G be a convex subgroup of $(M, +)$ and let $A(G) = \{a \in M \mid a \cdot G \subseteq G\}$ be the convex valuation ring as defined above. Suppose G is **not** of the form $b \cdot A(G)$ or $b \cdot \mathfrak{m}(G)$, where $b \in M$ and $\mathfrak{m}(G)$ is the maximal ideal of $A(G)$. Then the theory of $(M, A(G), G)$ has quantifier elimination in the language $\mathscr{L}$ augmented by two predicates $\mathscr{O}$ and $\mathcal{G}$ for $A(G)$ and G (19.2).

Let me bring out the instrument which is central, and everywhere in the work. It is the notion of an heir of a cut p. Suppose T is model complete and q is a cut of an elementary extension N of M which lies over p. Then q is an heir of p if the extension $(M, p^L) \subseteq (N, q^L)$ is existential. This is not the general definition, but in the moment it is the right way to see heirs. In particular if one thinks of the Cherlin-Dickmann theorem, which says now: if B is a convex valuation ring extending A, then the upper

boundary of B is an heir of the upper boundary of A. For polynomially bounded theories, this was proved by L. van den Dries and A. Lewenberg in [vdD-Lew]. This is the crucial ingredient we need, in order to use our methods for polynomially bounded, o-minimal theories.
We can prove the Box Theorem and most of the consequences, because we have a sufficiently strong description of heirs. In particular for the case that $(M, p^L) \subseteq (N, q^L)$ is existential but not elementary. Unfortunately I have only ad hoc algebraic descriptions of heirs in the case of real closed fields.

I have to say something about the organization of the work. There are four chapters, divided in what kind of o-minimal theories we are working in. We go from the general to the special:

Chapter I is about o-minimal expansions of dense linear ordered sets without endpoints. We give the general results about heirs of cuts (mainly Theorem 2.5). Furthermore, I want to emphasize §3 in this first chapter, where a rank is defined, which measures the complexity of the basis of neighbourhoods of an n-type.

Chapter II is about o-minimal expansions of divisible ordered, abelian groups. We use the results later on for o-minimal expansions of real closed fields M, as well as for the structure which is imposed by M on the multiplicative group $(M^{>0}, \cdot)$.

Chapter III is about o-minimal expansions of real closed fields. The main result there, is the rigidity of convex valuation rings (§11), which is a weakening of consequence (2) above, but which holds for arbitrary o-minimal expansions of real closed fields.

Chapter IV is about polynomially bounded, o-minimal expansions of real closed fields with archimedean prime model. We prove the Box Theorem in §17 and the consequences in the last two paragraphs.

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Contents

§0 Preliminaries and notations

We want to describe some tools which are well known in the model theory of o-minimal structures, without having a standard notation. The non trivial part of this paragraph can be found in [PS] and [KPS]. Furthermore we always work in a fixed monster $\mathfrak{M}$ and use standard notations for types, type spaces, languages, formulas and parameters, which we do not repeat.

o-minimal structures

An o-minimal theory T is always assumed to be an expansion of the theory DLO of dense linear ordered sets without endpoints. Furthermore all o-minimal theories are assumed to be complete.

We write $DOAG$ for the theory of divisible, ordered, abelian groups in the language $\{+,-,<,0\}$ and RCF for the theory of real closed fields in the language $\{+,-,\cdot,\ ^{-1},<,0,1\}$ (note that this language has the symbol $\ ^{-1}$).

Cuts

If M is an o-minimal structure in the language $\mathscr{L}$ then the 1-types of M are uniquely determined by the cuts they generate over M. Thus we may identify a cut $p = (p^L, p^R)$ of M with a 1-type. By definition we have $p^L < p^R$ and $p^L \cup p^R = M$. Therefore, a cut is always a non realized 1-type.
An extension of a cut p on an elementary extension N of M is a 1-type over N, which extends p. Therefore this extension need not be a cut itself.

Definable maps

Because definable maps are in the focus of o-minmal structures we want to repeat here the tools, which we use:

Let $\bar{x}$ be an n-tuple and $\bar{y}$ be a k-tupel of variables. Let M be an arbitrary $\mathscr{L}$-structure and let $F : M^n \longrightarrow M^k$ be an M-definable map. Then F induces a map $\tilde{F} : \mathrm{Fml}\,\mathscr{L}_k(M) \longrightarrow \mathrm{Fml}\,\mathscr{L}_n(M)$ by $\tilde{F}(\varphi(\bar{y})) := \varphi(F(\bar{x}))$. More precisely: if the graph of F is defined by $\vartheta(\bar{x}, \bar{y})$, then $\tilde{F}(\varphi(\bar{y})) = \exists \bar{y}\ \vartheta(\bar{x}, \bar{y}) \wedge \varphi(\bar{y})$.
Futhermore $\tilde{F}$ respects $Th(M, M)$-equivalence of $\mathscr{L}(M)$-formulas:
If $X \subseteq M^k$ is defined by $\varphi(\bar{y}) \in \mathrm{Fml}\,\mathscr{L}(M)$, then $F^{-1}(X)$ is defined by $\tilde{F}(\varphi(\bar{y}))$. Hence $\tilde{F}$ is an homomorphism of boolean algebras of M-definable sets and induces a continuous map $S_n(M) \longrightarrow S_k(M)$. We write F for this map again. Explicitly we have

$$F(p) = \{\varphi(\bar{y}) \in \mathrm{Fml}\,\mathscr{L}_k(M) \mid \varphi(F(\bar{x})) \in p\}$$

for $p \in S_n(M)$.
If N is an elementary extension of M, then F has an extension to a map $N^n \longrightarrow N^k$ and we can apply the above construction to this map. If $\bar{a} \in N^n$ we have

$$F(t(\bar{a}/M)) = t(F(\bar{a})/M)$$

If $p \in S_n(M)$, $q = F(p) \in S_k(M)$, $\bar{\alpha}$ is a realization of p and $\bar{\beta}$ is a realization of q, then the map $\beta_i \mapsto F_i(\bar{\alpha})$ induces an elementary M-embedding of the algebraic closures $\mathrm{acl}(M\bar{\beta}) \longrightarrow \mathrm{acl}(M\bar{\alpha})$.

We usually write $F, G, ...$ for definable maps. The expression "definable" for maps will only appear if it is not necessary to know about what set definablity holds.

Isomorphisms

Let $f : A \longrightarrow B$ be an isomorphism of subsets of an $\mathscr{L}$-structure M (this means that f is an automorphism of M (or better: of the monster $\mathfrak{M}$) with $f(A) = B$). We write f again for all the maps $f \times ... \times f : A^n \longrightarrow B^n$. The isomorphism f induces a map

$$\hat{f} : \mathrm{Fml}\,\mathscr{L}_n(B) \longrightarrow \mathrm{Fml}\,\mathscr{L}_n(A)$$

by $\hat{f}(\varphi(\bar{x}, \bar{b})) = \varphi(\bar{x}, f^{-1}(\bar{a}))$. If $X \subseteq M^n$ is defined by $\varphi(\bar{x}, \bar{b}) \in \mathrm{Fml}\,\mathscr{L}_n(B)$, then $f^{-1}(X)$ is defined by $\hat{f}(\varphi(\bar{x}, \bar{b}))$. $\hat{f}$ induces a continuous map $S_n(A) \longrightarrow S_n(B)$. We write f for this map again. Explicitely

$$f(p) = \{\varphi(\bar{x}, \bar{b}) \mid \varphi(\bar{x}, f^{-1}\bar{b}) \in p\} = \{\varphi(\bar{x}, f\bar{a}) \mid \varphi(\bar{x}, \bar{a}) \in p\}$$

If f is defined for $\bar{c}$, then $f(t(\bar{c}/A)) = t(f\bar{c}/B)$.

Definable closures

From now on we assume M to be a dense ordered o-minmal $\mathscr{L}$-structure without endpoints

We write $\mathrm{cl}\,A$ for the algebraic closure of A, which is the definable closure of A at the same time. If $A \prec M$ and $B \subseteq M$ we write $A\langle B\rangle$ instead of $\mathrm{cl}(A \cup B)$.

We frequently use the following point of view for definable closures, definable maps and isomorphisms:

Let M be o-minimal and $p \in S_n(M)$. Let $\mathfrak{X}_0$ be the set of M-definable maps $F : M^n \longrightarrow M^n$ with $F(p) = p$. We call two maps $F, G \in \mathfrak{X}_0$ equivalent if $F(\bar{x}) = G(\bar{x})$

lies in p, that is: there is an M-definable subset X of M^n such that $p \in X$ and such that $F = G$ on X. Let $\mathfrak{X}$ be the set of equivalence classes of $\mathfrak{X}_0$ with respect to this equivalence relation. If $\bar{\alpha}$ is a realization of p, then

$$\begin{aligned} \mathfrak{X} &\longrightarrow \mathrm{Aut}(M\langle\bar{\alpha}\rangle/M) \\ [F] &\longmapsto (\bar{\alpha} \mapsto F(\bar{\alpha})) \end{aligned}$$

is bijective. The map is well defined, because the relation $b \in \mathrm{cl}\, B$ is a dependence relation between elements b and subsets B of M as formulated from van der Waerden [vdW].

$\dim B/A$ always means the dimension of B over A with respect to the theory of M.

If M is o-minimal and $A \subseteq M$ is definable closed (this means $A = \mathrm{cl}\, A$) and if $\varphi(x) \in \mathrm{Fml}\, \mathscr{L}_1(A)$, then there are intervals $I_1, ..., I_k \subseteq M$ with boundary points in A, such that $Th(M, A) \vdash \forall x\ (\varphi(x) \leftrightarrow x \in I_1 \vee ... \vee x \in I_k)$. In other words:

If A is definable closed, then each formula in one variable, with parameters from A is equivalent modulo $Th(M, A)$ to a quantifier free formula in the language $\{<\}$ with parameters from A.

Some consequences are:

(1) If $A \subseteq M$ is definably closed without endpoints then A is dense ordered if and only if A is an $\mathscr{L}$-structure and an elementary restriction of M. This follows with the above remark from the Tarski-Vaught test, since the isolated points of $S_1(A)$ are dense in $S_1(A)$ ([PS].

(2) If $A \subseteq M$ is definably closed and $p, q \in S_1(A)$ are different, then there is some $a \in A$ with $p \leq a \leq q$ or $q \leq a \leq p$. If $A_0 \subseteq M$ is an arbitrary set and $p_0 \in S_1(A_0)$ then p has a unique extension to $\mathrm{cl}\, A_0$. Therefore it makes sense to write $p_0 < q_0$ for 1-types living over the same set.

(3) Let $M \subseteq B \subseteq N$, let B be definably closed, let N be a prime model over B and let $p \in S_1(M)$. If p is omitted in B, then p is omitted in N.

(4) Let T be an o-minimal expansion of the theory $DOAG$ of divisible ordered abelian groups in the langauage $\{+, -, <, 0\}$ and let M be a model of T. Then each definably closed subset of M which contains an element different from 0 is an elementary restriction of M. If $A \subseteq M$ is a divisible, ordered abelian subgroup of $(M, +)$, then A is dense (with respect to the ordering) in the topological closure $\overline{A}$ of A in M. If A is not archimedean in M, then $A = \overline{A}$.

Cell decomposition and definable maps

We use the notation of [PS]: if $F, G : M^n \longrightarrow M$ are M-definable maps and $X \subseteq M^n$ is M-definable, we write

$$(F, G)_X := \{(\bar{a}, b) \in M^{n+1} \mid F(\bar{a}) < b < G(\bar{a})\}$$

The following facts are implied by cell decomposition for a subset A of M:

(1) Let $\bar{b}$ be an n-tuple and $\bar{c}$ be a k-tuple with $\operatorname{cl} A\bar{b} = \operatorname{cl} A\bar{c}$.
Then there is an A-definable map $F : M^n \longrightarrow M^k$ with $F(\bar{b}) = \bar{c}$. If F is such a map and if $U \subseteq M^n$, $V \subseteq M^k$ are A-definable with $\bar{b} \in U$, $\bar{c} \in V$, then there are A-definable, definably connected subsets $X \subseteq U$ and $Y \subseteq V$ with $\bar{b} \in X$, $\bar{c} \in Y$, such that $F|_X : X \longrightarrow Y$ is an homeomorphism. If in addition $\dim(\bar{b}/A) = n$, we can choose X to be an open cell. If in addition $\dim(\bar{b}/A) = n = k$, we can choose Y to be open. ("open" with respect to the order topology on M^n)

(2) Let $\bar{b}$ be an n-tuple and $\bar{c}$ be an k-tuple with $\bar{b} \in \operatorname{cl} A\bar{b}$.
Then there is an A-definable map $F : M^n \longrightarrow M^k$ with $F(\bar{b}) = \bar{c}$. If F is such a map and if $U \subseteq M^n$, $V \subseteq M^k$ are A-definable with $\bar{b} \in U$, $\bar{c} \in V$, then there are A-definable, definable connected subsets $X \subseteq U$ and $Y \subseteq V$ with $\bar{b} \in X$, $\bar{c} \in Y$, such that $F|_X : X \longrightarrow Y$ is a continuous surjective map. If in addition $\dim(\bar{b}/A) = n$, we can choose X to be an open cell. If in addition $\dim(\bar{b}/A) = k$, we can choose Y to be open.

(3) If $F : M^n \longrightarrow M^k$ is an A-definable map and $p \in S_n(A)$, then there are A-definable subsets $X \subseteq M^n$, $Y \subseteq M^k$ with $p \in X$, $F(p) \in Y$, with the following properties:
there is a projection $pr : M^n \longrightarrow M^l$ and A-definable subsets $\hat{X} \subseteq M^n$, $Z \subseteq M^l$ together with A-definable homeomorpisms $\hat{F} : X \longrightarrow \hat{X}$, $G : Z \longrightarrow Y$ such that $F|_X = G \circ pr \circ \hat{F}$.

Hence each A-definable map $M^n \longrightarrow M^k$ is locally (in the sense of the topology of $S_n(M)$) a composition of A-definable homeomorphisms and projections. Note that for a real point $p \in M^n$ the latter sentence means nothing at all.

Chapter I Heirs of Cuts

§1 Heirs, strong heirs and bases of formulas

This paragraph contains general model theoretic material adapted to our purposes. Some statements are known, or they are more or less clear. We give the detailed proofs only for facts, which are less clear.

The language $\mathscr{L}^{\mathrm{def}\,C}$, the structure (M, d^p) and inheritances

Preliminary Remark

Let $\mathscr{L}$ and $\mathscr{L}'$ be signatures. For each formula $\varphi \in \operatorname{Fml}\mathscr{L}$ of the form $t_1 = t_2$ or $\mathcal{R}(t_1, ..., t_k)$ ($\mathcal{R}$ an $\mathscr{L}$-relation, $t_1, ..., t_k$ $\mathscr{L}$-terms) let $\varphi' \in \operatorname{Fml}\mathscr{L}'$ with $\operatorname{Fr}\varphi = \operatorname{Fr}\varphi'$. We extend this assignment inductively for all $\mathscr{L}$-formulas φ as follows:

$(\neg\varphi)' = \neg\varphi'$, $(\varphi_1 \wedge \varphi_2)' = \varphi_1' \wedge \varphi_2'$, as well as $(\forall u\ \varphi)' = \forall u\ \varphi'$.

We call this map the canonical extension of $'$. It is only used in this paragraph. Obviously $\operatorname{Fr}\varphi = \operatorname{Fr}\varphi'$ if φ is any $\mathscr{L}$-formula.

Let M be an $\mathscr{L}$-structure and let M' be an $\mathscr{L}'$-structure with same universe $\operatorname{dom} M = \operatorname{dom} M'$. Further suppose:

> If $\varphi \in \operatorname{Fml}\mathscr{L}$ is a formula of the easy form as described above and $\bar{m} \subseteq \operatorname{dom} M = \operatorname{dom} M'$, then $M \models \varphi(\bar{m}) \Leftrightarrow M' \models \varphi'(\bar{m})$.

Then we have for all formulas $\varphi \in \operatorname{Fml}\mathscr{L}$ and $\bar{m} \subseteq \operatorname{dom} M = \operatorname{dom} M'$:

$$M \models \varphi(\bar{m}) \Leftrightarrow M' \models \varphi'(\bar{m}).$$

The proof is obvious. //

Suppose $\mathscr{L}'$ contains all functions and constant symbols of $\mathscr{L}$. For each $\mathscr{L}$-Relation R with arity k and mutually distinct variables $w_1, ..., w_k$ let $R(w_1, ..., w_k)'$ be an $\mathscr{L}'$-formula with $\operatorname{Fr} R(w_1, ..., w_k)' = \{w_1, ..., w_k\}$.

We extend this assignment to all $\mathscr{L}$-formulas of the form $t_1 = t_2$ and $R(t_1, ..., t_k)$ as follows: $(t_1 = t_2)' := (t_1 = t_2)$ and $R(t_1, ..., t_k)' = R(w_1, ..., w_k)'(t_1, ..., t_k)$, if $w_1, ..., w_k$ are mutually distinct variables. By assumption on the variables of $R(w_1, ..., w_k)'$ this makes sense. Let M, M' be $\mathscr{L}$, $\mathscr{L}'$-structures respectively with the same universe and the same interpretation of the constants and the function symbols. Assume $M \models R(\bar{m}) \Leftrightarrow M' \models R'(\bar{m})$ ($\bar{m} \subseteq M$) for each $\mathscr{L}$-relation symbol R. Then we have again

$$M \models \varphi(\bar{m}) \Leftrightarrow M' \models \varphi'(\bar{m})$$

for all $\bar{m} \subseteq M$ and each $\mathscr{L}$-formula φ.

Let $\mathscr{L}$ be a language and let C be a set of new constants. We define the following expansion $\mathscr{L}^{\operatorname{def} C}$ of $\mathscr{L}$:

$\mathscr{L}^{\operatorname{def} C}$ contains $\mathscr{L}$ and for each $\mathscr{L}(C)$-formula φ a new predicate d_φ of arity $\operatorname{card} \operatorname{Fr} \varphi$. Observe, that if we replace the variables in φ we take a different predicate for the new formula. This is not important for our purposes but leads to some syntactical complications. In any case it is the reason why this section looks a little bit inflated.

If M is an $\mathscr{L}$-structure and p is a C-type over M, then we define the following $\mathscr{L}^{\operatorname{def} C}$-expansion (M, d^p) of M:

If φ is an $\mathscr{L}(C)$-formula with free variables $\operatorname{Fr} \varphi = \{z_1, ..., z_k\}$:

$$d_\varphi^{(M,d^p)} := d_\varphi^p := \{(a_1, ..., a_k) \in M^k \mid \varphi(a_1, ..., a_k) \in p\}$$

If N is an $\mathscr{L}$-structure and q is a C-type over N, then (M, d^p) is a substructure of (N, d^q) iff N is an elementary extension of M and q is an extension of p.

Definition 1.1. q is an heir of p iff the extension $(M, d^p) \subseteq (N, d^q)$ is existential.

q is a strong heir of p iff the extension $(M, d^p) \subseteq (N, d^q)$ is elementary.

The construction (M, d^p) is from Bruno Poizat's book ([Poi]). The next Proposition is well known, I include the proof:

Proposition 1.2.
Let $M \prec N$ be $\mathscr{L}$-structures, $A \subseteq M$, let $p \in S_C(M)$ and let q be an extension of p on N. Then q is a (strong) heir of p, if and only if q is a (strong) heir of p in the sense of the language $\mathscr{L}(A)$ and the $\mathscr{L}(A)$-structures (M, A), (N, A) respectively.

PROOF. Let φ be an $\mathscr{L}(A)(C)$-formula. Only for this proof we write δ_φ for the relation symbol, which corresponds to φ with respect to the language $\mathscr{L}(A)^{defC}$. We have $\mathscr{L}^{defC}(A) \subseteq \mathscr{L}(A)^{defC}$ and $\mathscr{L}^{defC}(\emptyset) = \mathscr{L}(\emptyset)^{defC}$. To each $\mathscr{L}(A)^{defC}$-formula ϑ we assign an $\mathscr{L}^{defC}(A)$-formula ϑ^* in the following way:

if ϑ is an $\mathscr{L}(A)$-formula, we take $\vartheta^* = \vartheta$. Furthermore let $\varphi(\bar{y}, \bar{z})$ be an $\mathscr{L}(C)$-formula, let $\bar{t}$ be a $\bar{y}$-tuple of $\mathscr{L}(A)$-terms, let $\bar{a}$ be a $\bar{z}$-tuple from A ($\varphi(\bar{y}, \bar{a})$ is now an $\mathscr{L}(A)(C)$-formula) and $\vartheta = \delta_{\varphi(\bar{y},\bar{a})}(\bar{t})$. Then we take

$$\vartheta^* = d_{\varphi(\bar{y},\bar{z})}(\bar{t}, \bar{a})$$

Finally we get ϑ^* by the canonical extension described in the preliminary note. We have for a model (M', A) of the theory $Th(M, A) = Th(N, A)$, a type $p' \in S_C(M')$, a formula $\vartheta(\bar{y}) \in \mathrm{Fml}\,\mathscr{L}(A)^{defC}$ and all $\bar{m}' \in M'^{\bar{y}}$:

$$(M', A, \delta^{p'}) \models \vartheta(\bar{m}') \;\Leftrightarrow\; (M', d^{p'}) \models \vartheta^*(\bar{m}').$$

//

Existence of heirs

In Poizat's book we find ([Poi] 11.02) :

Proposition 1.3. *Let $M \prec N$ be $\mathscr{L}$-structures, $M \subseteq B \subseteq N$ and let Σ be a set of $\mathscr{L}$-formulas with n free variables and parameters from N, such that $Th(N, N) \cup \Sigma$ is consistent and such that Σ contains the n-type $p \in S_n(M)$. Then the following conditions are equivalent:*

(i) if $\varphi(\bar{x}, \bar{u}, \bar{v})$ is an $\mathscr{L}$-formula, $\bar{a} \in M^{\bar{u}}$ and $\bar{b} \in N^{\bar{v}}$ such that $\Sigma \vdash \varphi(\bar{x}, \bar{a}, \bar{b})$, then there is $\bar{b}' \in M^{\bar{v}}$ such that $\varphi(\bar{x}, \bar{a}, \bar{b}') \in p$.
(ii) there is an heir q of p on N with $\Sigma \subseteq q$.

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We need more detailed information about "heirs of partial types".

Definition 1.4. Let $\mathscr{L}$ be a signature. We call a non empty set Δ of $\mathscr{L}$-formulas a **base of formulas** if the following conditions hold:

(a) if $\delta(x_1, ..., x_n) \in \Delta$, with free variables $x_1, ..., x_n$ $(x_i \neq x_j,\ i \neq j)$ and if $y_1, ..., y_n$ are variables such that x_i is free for y_i in δ, then $\delta(y_1, ..., y_n) \in \Delta$.
(b) if $\delta, \delta' \in \Delta$, then $\delta \wedge \delta'$ and $\delta \vee \delta' \in \Delta$.

If $n \in \omega$, we define $\Delta_n := \Delta \cap \mathrm{Fml}\,\mathscr{L}_n$, the set of formulas from Δ with at most n free (fixed) variables.

Definition 1.5. Let Δ be a base of $\mathscr{L}$-formulas and let A be a subset of an $\mathscr{L}$-structure M. We define

$$\Delta(A) := \{\delta(x_1, ..., x_n, a_1, ..., a_k) \mid k, n \in \mathbb{N},\ \delta(x_1, ..., x_n, y_1, ..., y_k) \in \Delta,\ a_i \in A\}$$

$\Delta(A)$ is a base of formulas in the language $\mathscr{L}(A)$. Note that $\Delta(A)$ possibly is not $\wedge$-stable if we require only condition (b) in the definition of "base of formula".

If $n \in \mathbb{N}$ we define

$$\Delta_n(A) := (\Delta(A))_n$$

and

$$S_n^\Delta(M,A) := \{p \cap \Delta_n(M,A) \mid p \in S_n(M,A)\}$$

where we write $S_n(A)$ and $S_n^\Delta(A)$ instead of $S_n(M,A)$ and $S_n^\Delta(M,A)$ as usual.

If $p \in S_n(A)$ we write p_Δ for $p \cap \Delta_n(A)$.

Definition 1.6. Let M be an $\mathscr{L}$-structure, $B \supseteq M$ a set from an elementary extension of M, $\mathfrak{p} \in S_n^\Delta(M)$ and $\mathfrak{q} \in S_n^\Delta(B)$. We call $\mathfrak{q}$ an **extension** of $\mathfrak{p}$ iff $\mathfrak{q} \cap \Delta_n(M) = \mathfrak{p}$. We call $\mathfrak{q}$ an **heir** of $\mathfrak{p}$ on B if the following condition holds:

If $\delta_1(\bar{x},\bar{y}), \delta_2(\bar{x},\bar{y}) \in \Delta(M)$, $\psi(\bar{y}) \in \mathrm{Fml}\,\mathscr{L}(M)$ and if $\bar{b}$ is a $\bar{y}$-tuple from B with

$$\delta_1(\bar{x},\bar{b}) \in \mathfrak{q},\ \ \delta_2(\bar{x},\bar{b}) \notin \mathfrak{q} \text{ and } \models \psi(\bar{b})$$

then there is some $\bar{y}$-tuple $\bar{a}$ from M such that

$$\delta_1(\bar{x},\bar{a}) \in \mathfrak{p},\ \ \delta_2(\bar{x},\bar{a}) \notin \mathfrak{p} \text{ and } \models \psi(\bar{a})$$

Proposition 1.7. *Let $M \prec N$ be $\mathscr{L}$-structures, $M \subseteq B \subseteq N$, $\mathfrak{p} \in S_n^\Delta(M)$ and let $\mathfrak{q}$ be an extension of $\mathfrak{p}$ on B. Then the following conditions are equivalent:*

(i) *$\mathfrak{q}$ is an heir of $\mathfrak{p}$.*
(ii) *there is a type $q \in S_n(B)$ such that $q_\Delta = \mathfrak{q}$ and q is an heir of $q \restriction M$.*
(iii) *if $p \in S_n(M)$ such that $p_\Delta = \mathfrak{p}$, then there is an heir q of p on N such that $q_\Delta \restriction B = (q \restriction B)_\Delta = \mathfrak{q}$.*

Proof. Obviously we have (iii)⇒(ii)⇒(i).

(i)⇒(iii) Let Θ be the set

$$\Theta := \{\varphi(\bar{x},\bar{v}) \in \mathrm{Fml}\,\mathscr{L}(M) \mid \text{there is some } \bar{a} \in M^{\bar{v}} \text{ with } \varphi(\bar{x},\bar{a}) \in p\}$$

Let T_1 be the deductive closure of

$$Th(M,B) \cup \mathfrak{q} \cup \{\neg\delta \mid \delta \in \Delta_n(B) \setminus \mathfrak{q}\} \cup \{\neg\varphi(\bar{x},\bar{d}) \mid \varphi(\bar{x},\bar{v}) \in \mathrm{Fml}\,\mathscr{L}(M) \setminus \Theta, \bar{d} \in N^{\bar{v}}\}$$

with respect to the language $\mathscr{L}_n(N)$. $T_1 \restriction \mathscr{L}_n(M) \cup p$ is consistent:
Let $\varphi(\bar{x}) \in T_1 \restriction \mathscr{L}_n(M)$. We have to show that $\varphi(\bar{x}) \in p$: There are $\psi(\bar{v}) \in \mathrm{Fml}\,\mathscr{L}(M)$, $\delta_1(\bar{x},\bar{v}), \delta_2(\bar{x},\bar{v}) \in \Delta(M)$, $\bar{c} \in B^{\bar{v}}$ with $\delta_1(\bar{x},\bar{c}) \in \mathfrak{q}$, $\delta_2(\bar{x},\bar{c}) \notin \mathfrak{q}$, $\models \psi(\bar{c})$, $\varphi_1(\bar{x},\bar{v}_1), ..., \varphi_k(\bar{x},\bar{v}_k) \in \mathscr{L}(M) \setminus \Theta$ and $\bar{v}_i$-tuple $\bar{d}_i \in N$ such that

$$\vdash \Big[\delta_1(\bar{x},\bar{c}) \wedge \neg\delta_2(\bar{x},\bar{c}) \wedge \psi(\bar{c}) \wedge \neg\varphi_1(\bar{x},\bar{d}_1) \wedge ... \wedge \neg\varphi_k(\bar{x},\bar{d}_k)\Big] \to \varphi(\bar{x})$$

Let

$$\gamma(\bar{w},\bar{v},\bar{v}_1,...,\bar{v}_k) := \Big[\psi(\bar{v}) \wedge \delta_1(\bar{w},\bar{v}) \wedge \neg\delta_2(\bar{w},\bar{v}) \wedge \neg\varphi_1(\bar{w},\bar{v}_1) \wedge ... \wedge \neg\varphi_k(\bar{w},\bar{v}_k)\Big]$$

The formula

$$\exists \bar{v}_1,...,\bar{v}_k \forall \bar{w}\ \gamma(\bar{w},\bar{v},\bar{v}_1,...,\bar{v}_k) \rightarrow \varphi(\bar{w})$$

is realized by $\bar{c}$. Because $\mathfrak{q}$ is an heir of $\mathfrak{p}$, there is some $\bar{a} \in M^{\bar{v}}$ such that

$$\delta_1(\bar{x},\bar{a}) \in \mathfrak{p},\ \ \delta_2(\bar{x},\bar{a}) \notin \mathfrak{p} \text{ and } \models \exists \bar{v}_1,...,\bar{v}_k \forall \bar{w}\ \gamma(\bar{w},\bar{a},\bar{v}_1,...,\bar{v}_k) \rightarrow \varphi(\bar{w})$$

Thus there are $\bar{v}_i$-tuple $\bar{b}_i \in M$ such that

$$\models \forall \bar{w}\ \psi(\bar{a}) \wedge \delta_1(\bar{w},\bar{a}) \wedge \neg\delta_2(\bar{w},\bar{a}) \wedge \neg\varphi_1(\bar{w},\bar{b}_1) \wedge ... \wedge \neg\varphi_k(\bar{w},\bar{b}_k) \rightarrow \varphi(\bar{w})$$

Because of $p_\Delta = \mathfrak{p}$ we have $\delta_1(\bar{x},\bar{a}) \wedge \neg\delta_2(\bar{x},\bar{a}) \in p$. From the choice of the φ_i we get $\neg\varphi_i(\bar{x},\bar{b}_i) \in p$ hence $\varphi(\bar{x}) \in p$.

If $q \in S_n(N)$ such that $T_1 \cup p \subseteq q$, then q is the type we are looking for. //

Theorem 1.8. *Let M be an $\mathscr{L}$-structure, $B \supset M$, $\mathfrak{p} \in S_n^\Delta(M)$ and let $\mathfrak{q}$ be an heir of $\mathfrak{p}$ on B. Let $k \geq 0$. Then for every $\mathfrak{p}' \in S_{n+k}^\Delta(M)$ with $\mathfrak{p} \subseteq \mathfrak{p}'$ there is some $\mathfrak{q}' \in S_{n+k}^\Delta(B)$ which is an heir of $\mathfrak{p}'$ such that $\mathfrak{q} \subseteq \mathfrak{q}'$.*

Proof. By Proposition 1.7 we may assume that $N = B$ is an elementary extension of M. Let $q \in S_n^\Delta(N)$ such that $q_\Delta = \mathfrak{q}$ and such that q is an heir of $p := q \restriction M$. Let $p' \in S_{n+k}(N)$ such that $p'_\Delta = \mathfrak{p}'$. Let Θ be the set

$$\Theta := \{\varphi(\bar{x},\bar{y},\bar{v}) \in \mathrm{Fml}\,\mathscr{L}(M) \mid \text{there is some } \bar{a} \in M^{\bar{v}} \text{ with } \varphi(\bar{x},\bar{y},\bar{a}) \in p'\}$$

Let T_1 be the deductive closure of

$$Th(N,N) \cup \mathfrak{q} \cup \{\neg\varphi(\bar{x},\bar{y},\bar{d}) \mid \varphi(\bar{x},\bar{y},\bar{v}) \in \mathrm{Fml}\,\mathscr{L}(M) \setminus \Theta, \bar{d} \in N^{\bar{v}}\}$$

in the language $\mathscr{L}_{n+k}(N)$.

$p' \cup T_1 \restriction \mathscr{L}_{n+k}(M)$ is consistent:
For each $\varphi(\bar{x},\bar{y}) \in T_1 \restriction \mathscr{L}_{n+k}(M)$ we have to show that $\varphi(\bar{x},\bar{y}) \in p'$:

there are $\delta(\bar{x},\bar{z}) \in \Delta(M)$, $\psi(\bar{z}) \in \mathrm{Fml}\,\mathscr{L}(M)$, $\bar{c} \in N^{\bar{z}}$, $\varphi_1(\bar{x},\bar{y},\bar{v}_1),...,\varphi_k(\bar{x},\bar{y},\bar{v}_k) \in \mathrm{Fml}\,\mathscr{L}(M) \setminus \Theta$ and $\bar{v}_i$-tuple $\bar{d}_i \in N$ such that

$$\delta(\bar{x},\bar{c}) \in \mathfrak{q} \text{ and } \vdash \Big[\delta(\bar{x},\bar{c}) \wedge \psi(\bar{c}) \wedge \neg\varphi_1(\bar{x},\bar{y},\bar{d}_1) \wedge ... \wedge \neg\varphi_k(\bar{x},\bar{y},\bar{d}_k)\Big] \rightarrow \varphi(\bar{x},\bar{y})$$

Therefore we have

$$\delta(\bar{x},\bar{c}) \wedge \forall \bar{v},\bar{w}\ \Big[\psi(\bar{c}) \wedge \delta(\bar{v},\bar{c}) \wedge \neg\varphi_1(\bar{v},\bar{w},\bar{d}_1) \wedge ... \wedge \neg\varphi_k(\bar{v},\bar{w},\bar{d}_k) \rightarrow \varphi(\bar{v},\bar{w})\Big]\ \in q$$

Because q is an heir of p, there are $\bar{a} \in M^{\bar{z}}$ and $\bar{b}_1 \in M^{\bar{v}_1}, ..., \bar{b}_k \in M^{\bar{v}_k}$ such that

$$\delta(\bar{x},\bar{a}) \wedge \forall \bar{v},\bar{w}\ \Big[\psi(\bar{a}) \wedge \delta(\bar{v},\bar{a}) \wedge \neg\varphi_1(\bar{v},\bar{w},\bar{b}_1) \wedge ... \wedge \neg\varphi_k(\bar{v},\bar{w},\bar{b}_k) \rightarrow \varphi(\bar{v},\bar{w})\Big]\ \in p$$

Therefore $\delta(\bar{x},\bar{a}) \in \mathfrak{p} \subseteq p'$ and because of $\psi(\bar{a}) \wedge \neg\varphi_1(\bar{x},\bar{y},\bar{b}_1) \wedge ... \wedge \neg\varphi_k(\bar{x},\bar{y},\bar{b}_k) \in p'$ we get $\varphi(\bar{x},\bar{y}) \in p'$.

Especially T_1 itself is consistent. If $q' \in S_{n+k}(N)$ with $T_1 \cup p' \subseteq q'$, we can take $\mathfrak{q}' := q'_\Delta$ as the element we looked for. //

If we equip the set $S_n^\Delta(A)$ with the topology, which has the sets

$$D(\delta) := \{\mathfrak{p} \mid \delta \notin \mathfrak{p}\}\ (\delta \in \Delta_n(A))$$

as a basis of open sets, then $S_n^\Delta(A)$ is a spectral space. For example if we work in the theory of RCF formalized in the language of ordered rings with the symbol $\leq$ (not with the symbol $<$!), and we take Δ to be the set of positive, quantifier free formulas, then $S_n^\Delta(A)$ becomes the space $\operatorname{Sper} K[x_1, ..., x_n]$ (K is the real closure of A) with the Harrison topology.

The theorem for $k = 0$ implies some going up and going down information for the spectral map $S_n^\Delta(B) \longrightarrow S_n^\Delta(M)$, concerning heirs:

Corollary 1.9. *Let M be a model of T, $B \supset M$, $\mathfrak{p} \in S_n^\Delta(M)$ and let $\mathfrak{q}$ be an heir of $\mathfrak{p}$ on B.*

(i) for each $\mathfrak{p}' \in S_n^\Delta(M)$ with $\mathfrak{p} \subseteq \mathfrak{p}'$ there is some $\mathfrak{q}' \in S_n^\Delta(B)$, which is an heir of $\mathfrak{p}'$, such that $\mathfrak{q} \subseteq \mathfrak{q}'$.

(ii) for each $\mathfrak{p}_0 \in S_n^\Delta(M)$ with $\mathfrak{p}_0 \subseteq \mathfrak{p}$ there is some $\mathfrak{q}_0 \in S_n^\Delta(B)$, which is an heir of $\mathfrak{p}_0$, such that $\mathfrak{q}_0 \subseteq \mathfrak{q}$.

Proof. (i) is exactly the theorem for $k = 0$
(ii) is (i) applied to the base of formulas generated by the set

$$\{\neg\delta \mid \delta \in \Delta\}$$

//

Corollary 1.10. *Let M be a model of T, $B \supset M$, $\mathfrak{p} \in S_n^\Delta(M)$, $\mathfrak{p}' \in S_m^\Delta(M)$ and let $\mathfrak{q}$ be an heir of $\mathfrak{p}$ on B. Then there is an element $\mathfrak{r} \in S_{n+m}^\Delta(B)$ such that $\mathfrak{q}(x_1, ..., x_n) = \mathfrak{r} \cap \Delta_n(B)$, $\mathfrak{p}'(y_1, ..., y_m) = \mathfrak{r} \cap \Delta_m(M)$ and such that $\mathfrak{r}$ is an heir of $\mathfrak{r} \cap \Delta_{n+m}(M)$. Here x_i and y_j are distinct variables. We want to concatenate the elements $\mathfrak{p}$ and $\mathfrak{p}'$.*

Proof. Let $p \in S_n(M)$ and $p' \in S_m(M)$ be types with $p_\Delta = \mathfrak{p}$ and $p'_\Delta = \mathfrak{p}'_\Delta$. Choose a type $p'' \in S_{n+m}(M)$ with $p(\bar{x}) \cup p'(\bar{y}) \subseteq p''$ and apply the theorem to $\mathfrak{p}$, $\mathfrak{q}$ and p''_Δ. //

Inheritances are preserved under definable maps

We work again with complete types. Let $C = \{c_1, ..., c_n\}$ be finite and let $D = \{d_1, ..., d_k\}$ be another finite set of new constants. Let $F_1(\bar{v}, v'), ..., F_k(\bar{v}, v')$ be $\mathscr{L}$-formulas with arity $n+1$, where v' shall occur in each F_i.
We write $F = (F_1, ..., F_k)$. If $(u_1, ..., u_k)$ is a k-tuple of variables and $\varphi(u_1, ..., u_k, \bar{w})$ is an $\mathscr{L}$-formula we write $\varphi(F(C), \bar{w})$ instead of

$$\exists u_1, ..., u_k F_1(\bar{c}, u_1) \wedge ... \wedge F_k(\bar{c}, u_k) \wedge \varphi(u_1, ..., u_k, \bar{w})$$

We define a map

$$^F : \mathrm{Fml}\, \mathscr{L}^{defD} \longrightarrow \mathrm{Fml}\, \mathscr{L}^{defC}$$

as follows: if $\varphi(\bar{u}_0, \bar{u})$ is an $\mathscr{L}$-formula (with free variables $\bar{u}_0 \cup \bar{u}$) and $\bar{d} \in D^{\bar{u}_0}$, we take

$$(d_{\varphi(\bar{d}, \bar{u})}(\bar{w}))^F := d_{\varphi(F(C), \bar{u})}(\bar{w})$$

where $\bar{w}$ is an $\bar{u}$-tuple of variables. Again we take φ^F to be the canonical extension of this assignment; note that $\mathscr{L}^{defC}$ and $\mathscr{L}^{defD}$ are built by new relation symbols from $\mathscr{L}$ only. By the preliminary remark it is enough to give a translation for the new predicates of $\mathscr{L}^{defD}$ into formulas of $\mathscr{L}^{defC}$. We get at once:

Lemma 1.11. *Let T be an $\mathscr{L}$-theory $F_1, ..., F_k$ be T-definable maps of arity n. We write $F = (F_1, ..., F_k)$. Then for each model M of T, each type $p \in S_C(M)$, each formula ϑ of $\mathscr{L}^{defD}$ and all $\bar{m} \subseteq M$ we have*

$$(M, d^{F(p)}) \models \vartheta(\bar{m}) \;\Leftrightarrow\; (M, d^p) \models \vartheta^F(\bar{m})$$

//

The effort was made to prove

Proposition 1.12. *Let $p \in S_C(M)$, q be an extension of p on N and let $F : M^C \longrightarrow M^D$ be M-definable. Then*

(i) if q is an heir of p, then $F(q)$ is an heir of $F(p)$.
(ii) if q is a strong heir of p, then $F(q)$ is a strong heir of $F(p)$.

Proof. (i) if $F(q) \neq q$ we have some $\varphi(\bar{x}, \bar{y}) \in \mathrm{Fml}\,\mathscr{L}(M)$ and $\bar{b} \in N^{\bar{y}}$ with $\varphi(\bar{x}, \bar{b}) \in q$ and $\varphi(F(\bar{x}), \bar{b}) \notin q$. Since q is an heir of p, there is some $\bar{a} \in M^{\bar{y}}$ such that $\varphi(\bar{x}, \bar{a}) \in p$ and $\varphi(F(\bar{x}), \bar{a}) \notin p$, hence $F(p) \neq p$.

(ii) by Proposition 1.2 we can assume, that F is $\emptyset$-definable. Now apply Lemma 1.11 to $T = Th(M) = Th(N)$. //

Coheirs

We repeat the definition and easy facts about coheirs.

If $M \prec N$ are $\mathscr{L}$-structures and $M \subseteq B \subseteq N$ is a set, then a type $q \in S_n(B)$ is called a coheir over $q \restriction M$ (or over M) if q is in the closure of $\{t(\bar{a}/B) \mid \bar{a} \in M^n\}$.

If $\bar{b} \in N^n$, then $t(\bar{b}/B)$ is a coheir of $t(\bar{b}/M)$ if and only if $t(B/M\bar{b})$ is an heir of $t(B/M)$.

Let $M \prec N$ be $\mathscr{L}$-structures, $M \subseteq B \subseteq N$ and $n \in \mathbb{N}$. If $f : S_n(M) \longrightarrow S_n(B)$ is a continuous map with $f(t(\bar{a}/M)) = t(\bar{a}/B)$ for all $\bar{a} \in M^n$, then $f(p)$ is the unique coheir of p on B. Consequently, there is at most one such map. If $\varphi(\bar{x}) \in \mathrm{Fml}\,\mathscr{L}_n(B)$ and $\psi(\bar{x}) \in \mathrm{Fml}\,\mathscr{L}_n(M)$ with $f^{-1}(\langle\varphi(\bar{x})\rangle) = \langle\psi(\bar{x})\rangle$, then $\psi[M^n] = \varphi[N^n] \cap M^n$ (where $\langle\varphi(\bar{x})\rangle$ denotes the set of all types, which contains $\varphi(\bar{x})$ and $\varphi[N^n]$ are the realizations of φ in N^n).

Let $M \prec N$ be $\mathscr{L}$-structures, and let $\bar{a}, \bar{b}$ be tuples from an elementary extension $\mathfrak{M}$ of N. Assume that $\mathrm{cl}\,A$ is an elementary restriction of $\mathfrak{M}$ for all sets $A \supseteq M$. Then

(i) If $t(\bar{\alpha}\hat{\ }\bar{\beta}/N)$ is an heir of $t(\bar{\alpha}\hat{\ }\bar{\beta}/M)$, then $t(\bar{\alpha}/N)$ is an heir of $t(\bar{\alpha}/M)$ and $t(\bar{\beta}/N\langle\bar{a}\rangle)$ is an heir of $t(\bar{\beta}/M\langle\bar{\alpha}\rangle)$.

(ii) If $t(\bar{\beta}/M\langle\bar{\alpha}\rangle)$ is definable, then we have equivalence in (i).

(iii) If $t(\bar{\alpha}/N)$ is a coheir of $t(\bar{\alpha}/M)$ and $t(\bar{\beta}/N\langle\bar{\alpha}\rangle)$ is a coheir of $t(\bar{\beta}/M\langle\bar{\alpha}\rangle)$, then $t(\bar{\alpha}\hat{\ }\bar{\beta}/N)$ is a coheir of $t(\bar{\alpha}\hat{\ }\bar{\beta}/M)$.

Strong heirs for cuts in o-minimal structures

Let $(X,<)$ be a totally ordered set. The notion 'initial segment' of X can be described in two ways: on the one hand one can say, that an initial segment is a subset U of X with $a<b\in U\Rightarrow a\in U$. On the other hand one can define: a subset I of X is an initial segment of X, if there is some totally ordered set $Y\supseteq X$ and some $c\in Y$, such that I consists of the elements $x\in X$ with $x<c$. Thus, statements about initial segments can be expressed in different expansions of the language $\{<\}$. This is done now:

Let $\mathscr{L}$ be a signature which contains the symbol $<$ and let U be a new unary predicate. We think of $<$ as a total ordering and of U as an initial segment of this ordering. The following manipulation though occurs at a syntactical level only.

Let c be a new constant with respect to $\mathscr{L}$. We define maps

$$^{def} : \mathrm{Fml}\,\mathscr{L}(U) \longrightarrow \mathrm{Fml}\,\mathscr{L}^{\mathrm{def}\,c}$$

$$^{U} : \mathrm{Fml}\,\mathscr{L}^{def\,c} \longrightarrow \mathrm{Fml}\,\mathscr{L}(U)$$

as follows: we fix a variable w_0 and put $U(w)^{def}=d_{w_0<c}(w)$ as well as

$$d_{\varphi(c,\bar v)}(\bar w)^U = \exists w_1 w_2\ U(w_1)\wedge\neg U(w_2)\wedge(\forall z\ w_1<z<w_2\rightarrow\varphi(z,\bar w)),$$

if $\varphi(v_0,\bar v)$ is some $\mathscr{L}$-formula with free variables $v_0\hat{\ }\bar v$ and $\bar w$ is a $\bar v$-tuple of variables. The maps we are looking for, are the canonical extensions of these assignments: the languages $\mathscr{L}(U)$ and $\mathscr{L}^{def\,c}$ have the same function and the same constant symbols.

Lemma 1.13. *If T is an o-minimal $\mathscr{L}$-theory, then for $\vartheta\in\mathrm{Fml}\,\mathscr{L}^{def\,c}$ and $\psi\in\mathrm{Fml}\,\mathscr{L}(U)$ we have: if M is a model of T, $p\in S_c(M)$ $(=S_1(M))$ is not realized, $U_p=\{a\in M\mid a<p\}$ and $\bar m\subseteq M$, then:*

$$(M,d^p)\models\vartheta(\bar m)\ \Leftrightarrow\ (M,U_p)\models\vartheta^U(\bar m)$$

$$(M,U_p)\models\psi(\bar m)\ \Leftrightarrow\ (M,d^p)\models\psi^{def}(\bar m)$$

Proof. The lemma follows immediately from the definition by applying the preliminary remark of this paragraph. //

Therefore if q is a cut of N extending p, then q is a strong heir of p iff $(M,p^L)\subseteq(N,q^L)$ is elementary.

§2 Heirs of cuts

In this section again, 'o-minimal' means 'o-minimal expansion of a dense ordered set without endpoints'.

Let Δ be the set of quantifier free formulas in the language $\{<\}$. Obviously Δ is a base of $\mathscr{L}$-formulas.

If M is o-minimal, then the map

$$pr = (pr_1, ..., pr_n) : S_n^{\Delta}(M) \longrightarrow S_1(M)^n$$

is onto, where pr_i denotes the i-th projection. If $p_1, ..., p_n$ are cuts of M, then $pr^{-1}(p_1, ..., p_n)$ consists of exactly one element if and only if $p_i \neq p_j$ for all $i \neq j$. We are interested mainly in this case and look at $(p_1, ..., p_n)$ as an element of $S_n^{\Delta}(M)$ if $p_i \neq p_j$ $(i \neq j)$.

Proposition 2.1. *Let $M \prec N$ be o-minimal $\mathscr{L}$-structures, let $p_1, ...p_n$ be mutually different cuts of M and let $q_1, ..., q_n$ be cuts of N extending $p_1, ..., p_n$ respectively. Then the following conditions are equivalent:*

(i) $(q_1, ..., q_n)$ *is an heir of* $(p_1, ..., p_n)$.

(ii) $(M, p_1^L, ..., p_n^L) \subseteq (N, q_1^L, ..., q_n^L)$ *is existential over* $\mathscr{L}$, *that is:*

If $\varphi(\bar{x})$ is an $\mathscr{L}(M)$-formula, $\psi(\bar{x})$ is a quantifier free formula in the language $\mathscr{L}(M, \mathcal{D}_1, ..., \mathcal{D}_n)$ and if

$$(N, q_1^L, ..., q_n^L) \models \exists \bar{x}\ \varphi(\bar{x}) \wedge \psi(\bar{x})$$

then

$$(M, p_1^L, ..., p_n^L) \models \exists \bar{x}\ \varphi(\bar{x}) \wedge \psi(\bar{x})$$

(iii) if $\varphi(\bar{u}, \bar{v}) \in \mathrm{Fml}\,\mathscr{L}_{2n}(M)$ and $\bar{\alpha}, \bar{\beta} \in N^n$ such that

$$\alpha_1 < x < \beta_1 \in q_1, ..., \alpha_n < x < \beta_n \in q_n \text{ and } N \models \varphi(\bar{\alpha}, \bar{\beta})$$

then there are $\bar{a}, \bar{b} \in M^n$ such that

$$a_1 < x < b_1 \in p_1, ..., a_n < x < b_n \in p_n \text{ and } M \models \varphi(\bar{a}, \bar{b})$$

Proof. (i)⇒(ii): By Proposition 1.7 there is a type $p \in S_n(M)$ and an heir q of p on N, which contains $q_1(x_1) \cup ... \cup q_n(x_n)$. Hence $(M, d^p) \subseteq (N, d^q)$ is existential. Now (ii) follows easily.

(ii)⇒(iii) is obvious and (iii)⇒(i) is easy since $p_i \neq p_j$ and no q_i is realized. //

Corollary 2.2. *Let $M \prec N$ be o-minimal, $M \subseteq B \subseteq N$, let $q \in S_1(B)$ be not realized in* $\mathrm{cl}\, B$ *and $p = q \restriction M$. Then the following conditions are equivalent:*

(i) *q is an heir of p.*
(ii) *if $\varphi(u, v) \in \mathrm{Fml}\,\mathscr{L}_2(M)$ and $\alpha, \beta \in \mathrm{cl}\, B$ with $q \vdash \alpha < x < \beta$ and $N \models \varphi(\alpha, \beta)$ then there are elements $a, b \in M$ such that $a < x < b \in p$ and $M \models \varphi(a, b)$.*

//

Lemma 2.3. *Let $M \prec N$ be o-minimal and let p be a non definable cut of M. If $\varphi(x) \in \mathrm{Fml}\,\mathscr{L}_1(M)$, $\beta \in N$ with $N \models \varphi(\beta)$ and if $p \cup \{\beta < x\}$, $p \cup \{\beta > x\}$ are consistent relative N respectively, then there is some $a \in M$ such that $M \models \varphi(a)$ and $a < x \in p$, $a > x \in p$ respectively.*

Proof. Suppose $p \cup \{\beta < x\}$ is consistent relative N and $N \models \varphi(\beta)$. We have $Z := \{a \in M \mid M \models \varphi(a)\} \neq \emptyset$. Suppose $a > x \in p$ for every $a \in Z$. If $a_0 \in M$ with $a_0 > x \in p$ then $N \models \exists y\; y < a_0 \wedge \varphi(y)$. The same is true for M instead of N and $\{a_0 \in M \mid a_0 > x \in p\}$ is defined by $\exists y\; y < z \wedge \varphi(y)$; but p is assumed to be non definable. //

Lemma 2.4. *Let $M \prec N$ be o-minimal, $M \subseteq B \subseteq N$ and let p be a non definable cut of M. The least and the largest extension of p on B are heirs of p. Neither the least extension q' nor the largest extension q of p on B is isolated in B. More precisely:*
if $\beta \in B$ such that $\beta < x \in q'$, $\beta > x \in q$ respectively, then there is some $m \in M$ such that $\beta < m < x \in q'$, $\beta > m > x \in q$ respectively. If N is a prime model over B, then q and q' only have one extension to N and these extensions are again heirs of p.

Proof. We prove the proposition for the largest extension q of p by checking condition (iii) of Proposition 2.1 . Let $\alpha, \beta \in \mathrm{cl}\, B$ and $\varphi(u, v) \in \mathrm{Fml}\,\mathscr{L}_2(M)$ with $\alpha < x < \beta \in q$ and $N \models \varphi(\alpha, \beta)$. Because q is the largest extension of p there is some $b_0 \in M$ with $x < b_0 \in p$ and $b_0 \leq \beta$ (therefore q is not isolated in $S_1(B)$; this gives the second statement). We have $\alpha < x \in q$ and $N \models \psi(\alpha)$ with $\psi(u) := \exists y\; b_0 \leq y \wedge \varphi(u, y) \in \mathrm{Fml}\,\mathscr{L}_1(M)$. By Lemma 2.3 there is some $a \in M$ such that $a < x \in p$ and $M \models \psi(a)$: there is an element $b \in M$ with $b_0 \leq b$ and $M \models \varphi(a, b)$.

The proof shows that q is not isolated in $S_1(B)$, hence q is not realized in a B-prime model. //

Let $p \in S_n(M)$ and let $X \subseteq M^n$ be defined by $\varphi(\bar{x}) \in \mathrm{Fml}\,\mathscr{L}_n(M)$. We write $p \in X$ to express $p \in \langle\varphi(\bar{x})\rangle$.

If M is o-minimal, p is a cut of M and $F : M \longrightarrow M$ is M-definable with $F(p) = p$ then there is an open interval I in M such that $p \in I$ and F is strictly increasing on I. This is clear if p is definable. If p is not definable I is an open interval of M containing p and N is an elementary extension of N, where p is realized in N, then the least and the largest extension of p on N are different. By Lemma 2.4 the map F fixes these extensions, hence F is strictly increasing.

Note that if F is any M-definable map, not constant in every neighbourhood of $p \in S_1(M)$ and N is an elementary extension of M, then F maps the extensions of p on N strictly monotonic onto the extensions of $F(p)$ on N.

Theorem 2.5. *Let $M \prec N$ be o-minimal, $M \subseteq B \subseteq N$, let p be a non definable cut of M and let q be an extension of p on B not realized in* $\mathrm{cl}\,B$. *Then the following conditions are equivalent:*

(i) q is an heir of p.
(ii) If $F : M \longrightarrow M$ is M-definable, $F(p) = p$ and $F(x) > x \in p$, then $F(q) = q$.

Proof. (ii)⇒(i) is Proposition 1.12 (i).
Suppose (ii) holds. We check condition (ii) in Proposition 2.1. Let $\alpha, \beta \in \mathrm{cl}\,B$, $\varphi(u, v) \in \mathrm{Fml}\,\mathscr{L}_2(M)$ such that $\alpha < x < \beta \in q$ and $N \models \varphi(\alpha, \beta)$. Suppose that for all $a, b \in M$ with $a < b$ and $M \models \varphi(a, b)$ we have $a < b < x \in p$ or $x < a < b \in p$. By Lemma 2.4 (here we need that p is not definable) we can assume that α and β are realizations of p. Let $c \in M$ with $x < c \in p$. For $\beta < c$ we can suppose $M \models \forall u, v\ \varphi(u, v) \to u < v < c$. We define $F : M \longrightarrow M$ by

$$F(a) := \begin{cases} \sup\{b \in M \mid M \models \varphi(a, b)\} & \text{if } M \models \exists y \varphi(a, y) \\ a & \text{otherwise} \end{cases}$$

F is well defined and M-definable. We have $F(p) = p$: if $a \in M$ with $a < x \in p$, then $F(a) < x \in p$ by assumption and because of $p \neq d^-$ ($d \in M$). Let I be an open interval in M, $p \in I$, such that F is monotonic on I. From $a \leq F(a) < x \in p$ ($a \in I, a < x \in p$) we know that F must be strictly increasing on I. From

$$\{d \in M \mid d < x \in F(p)\} = \{d \in M \mid \text{there is some } a \in I \text{ with } a < x \in p \text{ and } d \leq F(a)\}$$

we get $F(p) = p$.
But $F(\alpha) \geq \beta$, hence $F(q) \neq q$. By shrinking I we get $F(x) > x \in p$. //

Corollary 2.6. *If p is a non definable cut of M, then the following conditions are equivalent:*

(i) *p has a definable heir.*
(ii) *All one dimensional extensions of p are heirs of p.*
(iii) *If α is a realization of p and N is a prime model of $M\alpha$, then α is the unique realization of p in N.*
(iv) *If α is a realization of p, then α is the unique realization of p in $M\langle\alpha\rangle$.*
(v) *If $F : M \longrightarrow M$ is M-definable and $F(p) = p$, then $F(x) = x \in p$, that is $F(x) = x$ in some open interval containing p.*
(vi) *If α is a realization of p, then M is dense in $M\langle\alpha\rangle$: if $\beta < \gamma$ are elements of $M\langle\alpha\rangle$, then $(\beta, \gamma) \cap M \neq \emptyset$.*

If M is an expansion of a divisible, ordered, abelian group, then (i)-(vi) are equivalent to

(vii) *For all $a \in M$: if $a + p = p$ then $a = 0$.*

If M is an expansion of a real closed field, then (i)-(vii) are equivalent to

(viii) *For all $a \in M$: if $a \cdot p = p$ then $a = 1$.*

Proof. (v)⇒(ii) follows immediately from the Theorem and (ii)⇒(i) is obvious.

(i)⇒(v) Let $N \succ M$, $\beta \in N$ and suppose β^+ is an heir of p. Let F be strictly monotonic on the M-interval I and $p \in I$. For p is not definable, we have $\beta \in I$ and $\beta \models p$. From $F(p) = p$ we get $\beta^+ = F(\beta^+) = F(\beta)^+$, hence $F(\beta) = \beta$ and $F(x) = x \in p$.

(v)⇒(iv) If $\beta \in M\langle\alpha\rangle$ is a realization of p different from α and $F : M \longrightarrow M$ is M-definable with $F(\alpha) = \beta$ then $F(x) = x \notin p$.

(iv)⇒(iii) Suppose p has different realization α, β in N. Let $\varphi(x) \in \mathrm{Fml}\,\mathscr{L}_1(M\alpha)$ be a formula which isolates $t(\beta/M\langle\alpha\rangle)$. $\varphi(x)$ defines in N an open interval with endpoints in $M\langle\alpha\rangle$ or a single point in $M\langle\alpha\rangle$. If $\varphi(x)$ defines a point, then this point is $\beta \neq \alpha$. If $\varphi(x)$ is of the form $\beta_1 < x < \beta_2$, then β_1 and β_2 are realizations of p.

(iii)⇒(i) if α is the unique realization of p in N, then α^+ and α^- are the only non isolated extensions of p on N. One of them must be an heir of p.

We know that (i)-(v) are equivalent. Clearly (vi)⇒(iv) holds.

(i)⇒(vi) Let $\gamma < \beta$ be elements of $M\langle\alpha\rangle$ and suppose $\gamma \notin M$. There is $F : M \longrightarrow M$ definable in M with $F(\alpha) = \gamma$. The cut $t(\gamma/M)$ is not definable and fulfills (i) again. (iv) says, that γ is the unique realization of $t(\gamma/M)$ in $M\langle\gamma\rangle = M\langle\alpha\rangle$, hence we can find an element $m \in M$ with $\gamma < m < \beta$.

Suppose now M expands $DOAG$ or RCF. Clearly (v)⇒(vii) and (v)⇒(viii) holds.

(vii)⇒(v). Suppose $F(p) = p$ and $F(x) = x \notin p$ say $F(x) > x \in p$. Let I be an interval with $p \in I$ such that F is strictly increasing on I. There is a closed interval $I' \subseteq I$ with $p \in I'$ such that $F(x) - x$ is monotonic or constant. In any case there is some $a \in M$ such that $F(x) - x > a > 0$ for all $x \in I'$. Certainly we have $a + p = p$ in this case.

(viii)⇒(v) is similar to (vii)⇒(v). //

If one of these conditions is true for a non definable cut of M, then $M\langle\alpha\rangle$ is an elementary extension of M, because (vi) tells us that $M\langle\alpha\rangle$ is dense ordered.

Definition 2.7. A cut p of M is called dense if one of the equivalent conditions of the above Corollary holds for p.

Corollary 2.8. *Let p be a non definable cut of M and let β be some element from an elementary extension N of M, such that N is the prime model of $M\beta$. Then there are at most two heirs of p on N: the least and the largest extension of p on N.*

Proof. Let q denote the least and q' denote the largest extension of p on N. We can assume that β is a realization of p. We already know that q and q' are heirs of p. Suppose there is an heir r of p on N such that $q < r < q'$ and $q \restriction M\langle\beta\rangle = r \restriction M\langle\beta\rangle$. then $q \restriction M\langle\beta\rangle$ is isolated and not realized in $M\langle\beta\rangle$. Hence $q \restriction M\langle\beta\rangle$ is isolated by some formula $\beta_1 < x < \beta_2$ with $\beta_1, \beta_2 \in M\langle\beta\rangle$. Because q is the least extension of p we must have that $q = \beta_1^+$ is definable. But this contradicts Corollary 2.6 (iii): p has three nonisolated extensions on N. The same argument for q' gives: $q \restriction M\langle\beta\rangle < r \restriction M\langle\beta\rangle < q' \restriction M\langle\beta\rangle$. If $\alpha, \gamma \in M\langle\beta\rangle$ with $q < \alpha < r < \gamma < q'$, then α and γ are realizations of p and we can map α to γ via some M-definable map: this is not possible because r is an heir of p. //

Proposition 2.9. *If $M \prec N$ is o-minimal, $M \subseteq B \subseteq N$, p is a cut of M and q is an extension of p on B, then q is an heir of p if and only if $q \restriction M\beta$ is an heir of p for all $\beta \in \operatorname{cl} B$.*

Proof. This is clear if p is definable. Suppose p is not definable and $F : M \longrightarrow M$ is M-definable with $F(p) = p$ and $F(q) \neq q$. Take an open interval $I \ni p$ such that F is monotonic on I. For some $\beta \in \operatorname{cl} B$ we have $\beta \in I$, $\beta > x \in q$ and $\beta < F(x) \in q$, that is $q \restriction M\beta$ is not an heir of p //

Corollary 2.10. *If p is not definable and q is an extension of p on $B \subseteq N$, which is not an heir of p, then there are $\alpha, \beta \in \operatorname{cl} B$ such that $\alpha < q < \beta$ and such that the interval (α, β) contains no heir of p on B.* //

Corollary 2.11.
If p is not definable and q is an extension of p on $B \subseteq N$, then the following conditions are equivalent:

(i) *q is an heir of p*
(ii) *for all realizations $\alpha, \beta \in \operatorname{cl} B$ of p with $\alpha \in \operatorname{cl} M\beta$ we have* $x < \alpha \in q \Leftrightarrow x < \beta \in q$
(iii) *for all realizations $\alpha, \beta \in N$ of p with $\alpha < q < \beta$ the type* $t(\alpha/M\beta)$ *is an heir of p.*

//

If q is a strong heir we have a weak converse of Theorem 2.5:

Lemma 2.12. *Let $p \in S_1(M)$, $M \prec N$ and let q be a strong heir of p on N. Let $F : M^{k+1} \longrightarrow M$ be an M-definable map.*

If there is some $\bar{\alpha} \in N^k$ such that $F(q, \bar{\alpha}) = q$, then there is some $\bar{a} \in M^k$ such that $F(p, \bar{a}) = p$

Proof. The condition "there is some $\bar{v}$ in N such that $F(q, \bar{v}) = q$" can be written in the language $\mathscr{L}(M)^{defc}$. This gives the Lemma. //

Coheirs

Coheirs in dimension 1 are easy to describe:

Proposition 2.13. *Let M be o-minimal and $M \subseteq B$. If p is a cut of M then*

(i) *If $p^L \neq \emptyset$ and p^L does not have a maximum, then the least extension of p on B is a coheir of p.*
(ii) *If $p^R \neq \emptyset$ and p^L does not have a minimum, then the largest extension of p on B is a coheir of p.*
(iii) *If $q \in S_1(B)$ is a coheir of p, then q is the least or the largest extension of p on B.*

Especially: for 1-types the notion "coheir" does only depend on the structure $(M, <)$.

The proof is obvious. //

Corollary 2.14. *If $t(\alpha/M)$ is not definable and N is a prime model over $M\beta$, then the following conditions are equivalent:*
(i) $t(\alpha/M\beta)$ *is an heir of* $t(\alpha/M)$.
(ii) $t(\alpha/M\beta)$ *is a coheir of* $t(\alpha/M)$
(iii) $t(\alpha/N)$ *is an heir of* $t(\alpha/M)$
(iv) $t(\alpha/N)$ *is a coheir of* $t(\alpha/M)$

Proof. From Corollary 2.8 and the above Proposition we get (i)⇔(ii) and (iii)⇔(iv). (iii)⇒(i) is obvious. (i)⇒(iii) is from Lemma 2.4. //

If $t(\alpha/M)$ is arbitrary and $t(\beta/M)$ is not definable, then we only have (i)⇔(ii) in general.

Remark: We'll see later, that if $p \in S_1(M)$ is not realized in $N \succ M$, then the unique extension of p on N need not be a strong heir.
Suppose M is an o-minimal expansion of a real closed field. Then, in general, strong heirs are very different from coheirs: we 'll see, that a non definable cut p which has a proper coheir that is also a strong heir, has options very similar to a convex valuation ring.

Morley sequences

A special extension q of a type $p \in S_n(M)$ is the restriction of a global extension of p, which is setwise fixed under every M-automorphism of the monster. In [Poi] we find: 'every coheir is special'. The next Proposition tells us, that the notion 'special extension' standardize 'least/largest extension' for definable and non definable cuts.

Proposition 2.15. *The special extensions of a cut p on $N \succ M$ are exactly the least and the largest extensions of p on N.*

Proof. Assume, there are elements $\alpha, \beta \in N$ which realizes p such that $\alpha < q < \beta$. Then there is an M-automorphism σ of the monster, such that $\sigma\alpha = \beta$. Therefore σ does not fix any global extension of q, thus q can not be special. As mentioned above, every coheir is special. If p is definable, then the unique heir of p on N is special, since the unique heir of p on the monster is fixed by every M-automorphism of the monster. //

Definition 2.16. Let $M \prec N$ be an o-minimal expansions of $DOAG$, and let λ be an element of ω or $\lambda = \omega$. Let p be a cut of M and $\alpha_i \in N$ $(i < \lambda)$. $(\alpha_i)_{i<\lambda}$ is an **extreme sequence**, if α_0 is a realization of p and if for all $i,\ i+1 < \lambda$:

$$t(\alpha_{i+1}/M\langle\alpha_0, ..., \alpha_i\rangle) \text{ is a special extension of } t(\alpha_i/M\langle\alpha_0, ..., \alpha_{i-1}\rangle)$$

We call $(\alpha_i)_{i<\lambda}$ a **left (right) Morley sequence** of p if for all i

$$t(\alpha_{i+1}/M\langle\alpha_0, ..., \alpha_i\rangle) \text{ is the least (largest) extension of } t(\alpha_i/M\langle\alpha_0, ..., \alpha_{i-1}\rangle)$$

Lemma 2.17. *Every left (right) Morley sequence is indiscernible.*

Proof.
Let $(\alpha_i)_{i<\lambda}$ be a left Morley sequence of p and let $N \succ M$ be $|M|^+$-saturated. Let q be the least extension of p on N. Then q is special over M. Let $(\beta_i)_{i<\lambda}$ be a left Morley sequence of q. Then (β_i) is a special sequence of (q, M) over M as described in [Poi], thus indiscernible. Certainly (β_i) is a left Morley sequence of p, too. therefore it is enough to show that $t((\alpha_i)_{i<\lambda}/M) = t((\beta_i)_{i<\lambda}/M)$. By an obvious induction on $n < \omega$ we can show that $t((\alpha_i)_{i<n}/M) = t((\beta_i)_{i<n}/M)$, hence the assertion follows.//

Observe that an extreme sequence is not indiscernible if it is not a left or a right Morley sequence. In order to avoid confusion with the notion 'Morley sequence' from stability theory, we always add the side left/right if we speak about indiscernible, extreme sequences.

§3 Rank of realization

Again, 'o-minimal' means 'o-minimal expansion of a dense ordered set without end-points'.

If p and q are cuts of M we write $p \sim q$ if $F(p) = q$ for an M-definable map $F : M \longrightarrow M$. David Marker proved in [Ma], Lemma 3.1, that $\sim$ is an equivalence relation. This is the germ of an exchange lemma, which leads to the realization rank.

We begin with the abstract independence properties of this rank:

Weak Dependence

We call a relation between elements a and subsets B of a fixed set $\mathfrak{M}$ **weak dependence** and write $a \lhd B$ if the following conditions are fulfilled:

(1) $a \lhd \{a\}$.
(2) if $a \lhd B$ and $B \subseteq C$ then $a \lhd C$.
(3) if $a \lhd B$ then there is a finite subset B_0 of B, such that $a \lhd B_0$.
(4) (Exchange lemma)
if B is finite, $a \lhd B \cup \{d\}$ and $a \not\lhd B$, then $d \lhd B \cup \{a\}$.
(5) (weak transitivity)
if B is finite, $a \lhd B \cup C$, and $b \lhd C \cup D$ for every $b \in B$ and if

$$(*) \quad \text{for all } b \in B : b \not\lhd C \cup (B \setminus \{b\})$$

then $a \lhd C \cup D$.

The only difference of a weak dependence to a dependence relation in the style of van der Waerden (see [vdW]) is the additional condition $(*)$ in (5). Although this difference is crucial, a weak dependence leads to the notion of basis and dimension.

If (1)-(5) are fulfilled then (4) and (5) also holds if we drop the assumption 'B finite'.

Definition 3.1. A set B is called **weak independent over a set** C, if no $b \in B$ is weak dependent of $C \cup (B \setminus \{b\})$. Otherwise B is called weak dependent over C. From (1)-(4) we get at once:

(i) if B is weak independent over C and $d \not\lhd C \cup B$, then $B \cup \{d\}$ is weak independent over C.

(ii) if $B' \subseteq B$ is weak independent over C then there is some maximal $B_0 \subseteq B$ containing B', such that B_0 is weak independent over C. We call each maximal subset which is weak independent over C a **weak basis** of B over C.

Proposition 3.2. *Two weak bases of B over C have the same cardinality.*

Proof. Suppose first that B is finite, $\{b_1, ..., b_n\}$ is a weak basis of B over C and $\{d_1, ..., d_k, a\} \subseteq B$ is weak independent over C, such that $a \notin \{d_1, ..., d_k\}$. Then there is some $i \in \{1, ..., n\}$, such that $b_i \not\lhd \{d_1, ..., d_k\} \cup C$:
Suppose $b_i \lhd C \cup \{d_1, ..., d_k\}$ for all i. We have $a \lhd C \cup \{b_1, ..., b_n\}$ and $\{b_1, ..., b_n\}$ is weak independent over C. The weak transitivity says $a \lhd C \cup \{d_1, ..., d_k\}$: contradiction.

Using this argument repeated we get the Proposition for finite B.

Obviously this fact gives the Proposition for arbitrary B if there is a finite basis of B over C.

Finally we get the full Proposition as in the case of an independence relation. //

We write $\mathrm{rk}(B/C)$ for the cardinality of a weak basis of B over C.

We want to define a special weak dependence in *o*-minimal structures. Some properties of this relation are already true for the abstract notion:

Proposition 3.3. *Suppose $B_0 \subseteq B$ is a weak basis of B over C and $C \subseteq D$. If $B_1 \subseteq B_0$ is a weak basis of B_0 over D, then B_1 is also a weak basis of B over D.*

Proof. Let $B_2 = B_0 \setminus B_1$. Every element of B_2 is weak dependent over $D \cup B_1$. Let $b \in B$. Since $b \lhd B_2 \cup (C \cup B_1)$ and B_2 is weak independent over $C \cup B_1$ we get from weak transitivity: $b \lhd D \cup B_1$. But this means that B_1 is a weak basis of B over D. //

Corollary 3.4. *Let $C \subseteq D$ and B be a set.*

(i) If B_0 is a weak basis of B over C, then there is a weak basis of B over D inside B_0.

(ii) If $\mathrm{rk}(B/D)$ is finite, then $\mathrm{rk}(B/C) = \mathrm{rk}(B/D)$ if and only if every subset of B, which is weak independent over C is weak independent over D.

Proof. (i) is immediate from the Proposition.
Suppose $\mathrm{rk}(B/C) = \mathrm{rk}(B/D)$ is finite and $B_0 \subseteq B$ is weak independent over C. We can suppose, that B_0 is a weak basis of B over C. Since B_0 is finite and $\mathrm{card}\, B_0 = \mathrm{rk}(B/D)$ we get from (ii) that B_0 is a weak basis of B over D. //

Remark
If $C \subseteq B \subseteq D$ and $d \lhd B$ for all $d \in D$ it does not follow $\mathrm{rk}(D/C) = \mathrm{rk}(B/C)$ in general.
Suppose $\mathrm{rk}(B/C), \mathrm{rk}(D/C)$ is finite. Then the symmetry

$$\mathrm{rk}(B/C \cup D) = \mathrm{rk}(B/C) \;\Rightarrow\; \mathrm{rk}(D/C \cup B) = \mathrm{rk}(D/C)$$

does not hold in general. We see a counter example now:

Rank of realization

We return to the *o*-minimal situation. We fix an *o*-minimal monster $\mathfrak{M}$ and a small subset A of $\mathfrak{M}$. A is always assumed to be definable closed.

Lemma 3.5. *If $p \in S_1(A)$ and $A \subseteq B \subseteq \mathfrak{M}$, then the following conditions are equivalent.*

(i) p has a unique extension on B.

(ii) If p is realized in $\mathrm{cl}\, B$ *then p is realized in A.*

Proof. The set A is definably closed. Therefore each formula with parameters in A with one free variable is equivalent to a quantifier free formula of the language $\{<\}$ with parameters in A. Now the lemma follows easily. //

Definition 3.6. If B is a subset of $\mathfrak{M}$ and if c is an element from $\mathfrak{M}$, we say that c is dominated by B over A (or A-dominated by B), if $t(c/A)$ is realized in $\mathrm{cl}\, A \cup B$; otherwise c is called **A-indominated** by B.

We show that A-dominance is a weak dependence relation between elements and subsets of $\mathfrak{M}$.

A-dominance is not a dependence relation in the sense of van der Waerden: for example if $A = R_0$ is the real closure of $\mathbb{Q}$ in $\mathbb{R}$, a is some realization of 0^+ over $\mathbb{R}$ and $c = a + \pi$ we have:
(a) $a \in R_0(c, \pi)$, thus a is R_0-dominated by $\{c, \pi\}$
(b) c is R_0-dominated by $\{\pi\}$.
(c) a is R_0-indominated by $\{\pi\}$.

In the sequel we suppress the index A and write dominated or indominated only. The set A is always fixed and, as mentioned in the beginning, definably closed.

Certainly we have:

(1) c is dominated by $\{c\}$.
(2) c dominated by B, $B \subseteq C \Rightarrow c$ dominated by C.
(3) if c is dominated by B, then there is a finite subset B_0 of B, such that c is dominated by B_0.

From Lemma 3.5 we know for any element $c \notin A$ the equivalence of

(i) c is indominated by B.
(ii) $t(c/A \cup B)$ is the unique extension of $t(c/A)$ on $A \cup B$.
(iii) If $c' \in \mathfrak{M}$ such that $t(c/A) = t(c'/A)$, then $c' \notin \mathrm{cl}\, A \cup B$.

Exchange Lemma for A-dominance
If c is indominated by B and dominated by Bd, then d is dominated by Bc.

Proof. We search a realization of $t(d/A)$ in $\mathrm{cl}\, ABc$. Since c is dominated by Bd there is some realization $c' \in \mathrm{cl}\, ABd$ of $t(c/A)$. Because c is indominated by B we have $c' \notin \mathrm{cl}\, AB$. From the exchange lemma for the algebraic relation in $\mathfrak{M}$ we get

$d \in \operatorname{cl} ABc'$. Since c is indominated by B it follows $t(c/AB) = t(c'/AB)$. Let σ be an $(A \cup B)$-automorphism of $\mathfrak{M}$ such that $\sigma(c') = c$. Then $\sigma(d) \in \operatorname{cl} ABc$ is the promised realization of $t(d/A)$. //

We are going on in checking the weak transitivity for A-dominance. In fact a stronger property will be useful later on:

Proposition 3.7. *Let I be an index set, let $\{b_i \mid i \in I\}$ (as a set) be indominated by C and suppose that b_i is dominated by $C \cup D$ for every $i \in I$. Then $t((b_i)_{i\in I}/A \cup C)$ is realized in $\operatorname{cl} A \cup C \cup D$. More precisely: If b'_i is a realization of $t(b_i/A)$ in $\operatorname{cl} A \cup C \cup D$, then $(b'_i)_{i\in I}$ is a realization of $t((b_i)_{i\in I}/A \cup C)$.*

Proof. We can assume that $b_i \neq b_j$ if $i \neq j$ and it is enough to prove the Proposition for finite I. We make an induction on $n = \operatorname{card} I$:
$n = 1$: suppose b is dominated by $C \cup D$, indominated by C and b' realizes $t(b/A)$ in $\operatorname{cl} A \cup C \cup D$. Since b is indominated by C the type $t(b/A \cup C)$ is realized by b' too.

$n \to n+1$. Suppose $\{b_1, ..., b_{n+1}\}$ is indominated by C and b_i is dominated by $C \cup D$. Let $b'_1, ..., b'_{n+1} \in \operatorname{cl} A \cup C \cup D$ be realizations of $t(b_1/A), ..., t(b_{n+1}/A)$ respectively. By the induction hypothesis we have $t(b_1, ..., b_n/A \cup C) = t(b'_1, ..., b'_n/A \cup C)$. Let σ be an $A \cup C$-automorphism of $\mathfrak{M}$ such that $\sigma(b_i) = b'_i$ $(1 \leq i \leq n)$. Since b_{n+1} is indominated by $C \cup \{b_1, ..., b_n\}$, we see that $\sigma(b_{n+1})$ is indominated by $C \cup \{b'_1, ..., b'_n\}$, that is $t(f(b_{n+1})/A \cup C \cup \{b'_1, ..., b'_n\}) = t(b'_{n+1}/A \cup C \cup \{b'_1, ..., b'_n\})$. Hence $(b'_1, ..., b'_{n+1})$ is a realization of $t(b_1, ..., b_{n+1}/A \cup C)$. //

We come to the goal of this section:

Theorem 3.8. *A-dominance is a weak dependence relation. If B and C are two sets we write $\operatorname{rk}_A(B/C)$ for the rank of B over C for this relation. We call $\operatorname{rk}_A(B/C)$ the A-rank of realization of B over C. $\operatorname{rk}_A(B/C)$ is the cardinality of each A-dominance basis of B over C. All such A-dominance bases are transcendental over $A \cup C$. We have*

$$\operatorname{rk}_A(B/C) = \min\{\operatorname{card} B_0 \mid B_0 \subseteq B \textit{ and } b \textit{ is } A\textit{-dominated over } C \cup B_0 \textit{ for all } b \in B\}$$

Proof. We already proved conditions (1)-(4) for a weak dependence. A-dominance is weakly transitive (we suppress the letter A again):
Let $\{b_1, ..., b_n\}$ be indominated by C and let b_i be dominated by $C \cup D$. Let a be dominated by $C \cup \{b_1, ..., b_n\}$. We look for a realization of $t(a/A)$ in $\operatorname{cl} A \cup C \cup D$.

Let F be an $A \cup C$-definable map, such that $F(b_1, ..., b_n)$ is a realization of $t(a/A)$. From Proposition 3.7 we know that $t(b_1, ..., b_n/A \cup C)$ is realized in $\operatorname{cl} A \cup C \cup D$ by some n-tuple $(b'_1, ..., b'_n)$. If σ is an $A \cup C$-automorphism of $\mathfrak{M}$ such that $\sigma(b_i) = b'_i$ then $\sigma(F(b_1, ..., b_n)) = F(b'_1, ..., b'_n)$ is a realization of $t(a/A)$ in $\operatorname{cl} A \cup C \cup D$.

Thus we know that A-dominance is a weak dependence relation.

Obviously every A-dominance basis of B over C is transcendental over $A \cup C$.

We prove the equation: certainly $\geq$ holds.
Let B' be a dominance basis of B over C, $B_0 \subseteq B$, such that each $b \in B$ is dominated by $B_0 \cup C$ and suppose $\operatorname{card} B' > \operatorname{card} B_0$.
Then there is some finite $B_1 \subseteq B_0$ and $b_1, ..., b_n \in B'$ with $n > \operatorname{card} B_1$, such that each b_i is dominated by $B_1 \cup C$. By Proposition 3.7 the type $t(b_1, ..., b_n/A \cup C)$ is realized in $\operatorname{cl} A \cup C \cup B_1$. Since $\dim(b_1, ..., b_n/A \cup C) = n$ we get $\dim(B_1/A \cup C) \geq n$; but this contradicts $\operatorname{card} B_1 < n$. //

A set $B_0 \subseteq B$, which is minimal with the property

$$b \in B \ \Rightarrow\ b \text{ is dominated by } C \cup B_0$$

need not be indominated over C:

Example
Let R_0 be the real closure of $\mathbb{Q}$ in $\mathbb{R}$ and b some realization of π^+ over $\mathbb{R}$. Then we have $\operatorname{rk}_{R_0}(R_0(b,\pi)/R_0) = 2$. But the set $\{b, \pi\}$ is not an R_0-dominance basis of $R_0(b,\pi)$ over R_0.

Furthermore we have for $d = b - \pi$:

$$\operatorname{rk}_{R_0}(\{b,\pi\}/R_0 d) = 1 = \operatorname{rk}_{R_0}(\{b,\pi\}/R_0)$$

but

$$\operatorname{rk}_{R_0}(d/R_0 \pi b) = 0 \neq 1 = \operatorname{rk}_{R_0}(d/R_0)$$

That is: the symmetry

$$\operatorname{rk}_A(B/A \cup D) = \operatorname{rk}_A(B/A) \ \Rightarrow\ \operatorname{rk}_A(D/A \cup B) = \operatorname{rk}_A(D/A)$$

does not hold in general.

If A is a subset of $\mathbb{R}$, and B is an arbitrary set, then $\operatorname{rk}_A(B/\mathbb{R}) \leq 1$.
Hence, if $p = t(d, \pi/R_0)$ and (α, β) is another realization of p, we have

$$\operatorname{rk}_A(\{\alpha,\beta\}/\mathbb{R}) < \operatorname{rk}_{R_0}(d,\pi/R_0)$$

We give a geometrical interpretation of the realization rank.

Definition 3.9. Let M be o-minimal and let p be an n-type over M. We say that p is a **box type** if the open boxes, which contains p build a basis of neighbourhoods for p in $S_n(M)$.

Note that if $\bar{a} \in M^n$, then $\{t(\bar{a}/M)\}$ is a neighbourhood of $t(\bar{a}/M)$, which does not contain an open box.

Proposition 3.10. *If $p \in S_n(M)$ then the following conditions are equivalent:*

(i) *For some (hence for each) realization $\bar{\alpha}$ of p we have $\mathrm{rk}_M(\bar{\alpha}/M) = n$.*
(ii) *p is a box type.*
(iii) *p is the unique n-type over M, which extends all the projections $pr_i\, p$ $(1 \le i \le n)$ and none of these projections is realized.*

Therefore, if $p_1, ..., p_n$ are cuts of M we say that $(p_1, ..., p_n)$ is a box type if for all realizations α_i of p_i the type $t(\alpha_1, ..., \alpha_n/M)$ is a box type. Furthermore if $F : M^n \longrightarrow M$ is an M-definable map and $(p_1, ..., p_n) \in S_1(M)^n$ is a box type we may write $F(p_1, ..., p_n)$.

Proof. Note that (i) and (ii) imply $\dim p = n$ at once.

(i)⇒(ii). By induction on n. If $n = 1$, then p is not isolated, thus (ii) holds. Assume $\bar{\alpha}$ to be an $n-1$-tuple and β is an element, such that p is realized by $\bar{\alpha}\hat{\ }\beta$ with $\mathrm{rk}_M(\bar{\alpha}\beta/M) = n$. By the induction hypothesis, $t(\bar{\alpha}/M)$ is a box type. Let X be an M-definable set which contains p. We can suppose that X is an open cell $(F, G)_Y$. We have $F(\bar{\alpha}) < \beta$. For $t(\beta/M)$ is omitted in $M\langle\bar{\alpha}\rangle$, there is some $a_1 \in M$ with $F(\bar{\alpha}) \le a_1 < \beta$. Similar we can find some $a_2 \in M$ with $\beta < a_2 \le G(\bar{\alpha})$. For $t(\bar{\alpha}/M)$ is a box type, there is an open box $Y_0 \subseteq \{\bar{b} \in Y \mid F(\bar{b}) \le a_1,\ a_2 \le G(\bar{b})\}$ with $t(\bar{\alpha}/M) \in Y_0$. Finally $Y \times (a_1, a_2)$ is an open box, which contains p and is contained in X.

(ii)⇒(i). We do again by induction on n. If $n = 1$, then p is not realized in M, thus $\mathrm{rk}_M(\alpha/M) = 1$ for all realizations α of p. Assume $p \in S_n(M)$ is a box type and $\bar{\alpha}\hat{\ }\beta$ is a realization of p. Then $t(\bar{\alpha}/M)$ is a box type too and by the induction hypothesis $\mathrm{rk}_M(\bar{\alpha}/M) = n - 1$. We have to show that $t(\beta/M)$ is omitted in $M\langle\bar{\alpha}\rangle$: Let F be an M-definable map, say $F(\bar{\alpha}) < \beta$. Let $Y \subseteq M^{n-1}$ be an open box and $a_1 < a_2 \in M$ with $p \in Y \times (a_1, a_2)$ and $Y \times (a_1, a_2) \subseteq \{(\bar{b}, b') \in M^n \mid F(\bar{b}) < b'\}$. That is $F(\bar{b}) \le a_1$ for all $\bar{b} \in Y$, hence $F(\bar{\alpha}) \le a_1 < \beta$.

(ii)⇔(iii). If $p_1, ..., p_n$ are the projections of p and each p_i is a cut over M, then the intersection of all open boxes containing p is the set of all n-types q with projections $p_1, ..., p_n$. //

Remark
If $\mathrm{rk}_M(M\langle\bar{\alpha}\rangle/M) = n$, then p need not be a box type; but there is an M-definable map $F : M^n \longrightarrow M^n$, which is an homeomorphism on some open cell containing p, such that $F(p)$ is a box type.

We want to give a more general point of view for looking at the notion 'box type':

Let $\mathscr{L}$ be an arbitrary signature and M an $\mathscr{L}$-structure. We say that a type $p \in S_n(M)$ **is defined by some formula** $\varphi(\bar{x},\bar{u}) \in \mathrm{Fml}\,\mathscr{L}$ if

$$\{\varphi(\bar{x},\bar{a}) \mid \bar{a} \in M^{\bar{u}},\ p \vdash \varphi(\bar{x},\bar{a})\} \vdash p$$

If there there are formulas $\varphi_1(\bar{x},\bar{u}), ..., \varphi_k(\bar{x},\bar{u}) \in \mathrm{Fml}\,\mathscr{L}(M)$ such that

$$(\ \bigcup_{i=1}^{k}\{\varphi_i(\bar{x},\bar{a}) \mid \bar{a} \in M^{\bar{u}},\ p \vdash \varphi_i(\bar{x},\bar{a})\}\ ) \vdash p$$

and if $\varphi_i(\bar{x},\bar{u}) = \varphi_i'(\bar{x},\bar{u},\bar{m})$ for an $\mathscr{L}$-Formula $\varphi_i'(\bar{x},\bar{u},\bar{v})$, then p is defined by $\bigwedge_{i=1}^{k} z_i = z \rightarrow \varphi_i'(\bar{x},\bar{u},\bar{v})$, if $z, z_1, ..., z_k$ are suitable variables.
In this case we say that p is defined by $\varphi_1(\bar{x},\bar{u}), ..., \varphi_k(\bar{x},\bar{u})$.

Some examples:

1. If M is o-minimal, then - by definition - a box type $p \in S_n(M)$ is a type defined by the formulas $x_i < u_i$ and $x_i > u_i$ $(1 \leq i \leq n)$.

2. If M is o-minimal and $p \in S_n(M)$ is defined by $\varphi(\bar{x},\bar{u})$ and if the graph of $F : M^n \longrightarrow M^k$ is defined by $\gamma(\bar{x},\bar{y})$ then $F(p)$ is defined by a formula too:
From cell decomposition we know that F is a composition of projections and homeomorphisms near p. If F is a homeomorphism near p, say on X, then $F(p)$ is defined by

$$\exists\bar{x}\ (\bar{x} \in X \wedge \varphi(\bar{x},\bar{u}) \wedge \gamma(\bar{x},\bar{y}))$$

If F is a projection on the first k coordinates, then $F(p)$ is defined by

$$\exists x_1, ..., x_k\ \varphi(\bar{x},\bar{u})$$

We conclude the paragraph with technical material:

Proposition 3.11. *Let $M \prec N$ be o-minimal, let $\bar{\alpha}$ be an n-tuple and β be an element.*

(i) If $M \subseteq B \subseteq N$, B is definably closed and if $t(\bar{\alpha}\beta/M\gamma)$ is an heir of $t(\bar{\alpha}\beta/M)$ for all $\gamma \in B$, then

$$t(\beta/M) \text{ realized in } M\langle\bar{\alpha}\rangle \;\Rightarrow\; t(\beta/B) \text{ realized in } \operatorname{cl} B\bar{\alpha}$$

(ii) If $t(\bar{\alpha}\beta/N)$ is a strong heir of $t(\bar{\alpha}\beta/M)$, then

$$t(\beta/M) \text{ realized in } M\langle\bar{\alpha}\rangle \;\Leftrightarrow\; t(\beta/N) \text{ realized in } N\langle\bar{\alpha}\rangle$$

PROOF. (i) Suppose $t(\beta/B)$ is realized in $\operatorname{cl} B\bar{\alpha}$ and $F : M^n \longrightarrow M$ is M-definable. Because $t(F(\bar{\alpha})/B) \neq t(\beta/B)$, say $F(\bar{\alpha}) < \beta$, there is some $\gamma \in B$ with $F(\bar{\alpha}) \leq c \leq \beta$. Because $t(\bar{\alpha}\beta/M\gamma)$ is an heir over M, we find some $m \in M$ with $F(\bar{\alpha}) \leq m \leq \beta$. It follows that $t(\beta/M)$ is not realized in $M\langle\bar{\alpha}\rangle$.

(ii) Suppose $t(\beta/M)$ is not realized $M\langle\bar{\alpha}\rangle$. Let F be a 0-definable map $M^{n+k} \longrightarrow M$. We have to show the following:

for all $\bar{n} \in N^k$: $F(\bar{\alpha}, \bar{n}) < \beta \;\Rightarrow$ there is some $n' \in N$ with $F(\bar{\alpha}, \bar{n}) \leq n' \leq \beta$.

Let $\mathcal{P}$ be a new k-ary and let $\mathcal{P}'$ be a new $(k+1)$-ary relation symbol. We expand M to an $\mathscr{L}(\mathcal{P}, \mathcal{P}')$-structure (M, P, P') such that for all $\bar{m} \in M^k$ and all $m' \in M$ we have:

$$\begin{aligned} F(\bar{\alpha}, \bar{m}) < \beta \;&\Leftrightarrow\; (M, P, P') \models \mathcal{P}(\bar{m}) \\ F(\bar{\alpha}, \bar{m}) \leq m' \leq \beta \;&\Leftrightarrow\; (M, P, P') \models \mathcal{P}'(\bar{m}, m'). \end{aligned}$$

Furthermore we expand N to an $\mathscr{L}(\mathcal{P}, \mathcal{P}')$-structure (N, Q, Q') such that for all $\bar{n} \in N^k$ and all $n' \in N$ we have:

$$\begin{aligned} F(\bar{\alpha}, \bar{n}) < \beta \;&\Leftrightarrow\; (N, Q, Q') \models \mathcal{P}(\bar{m}) \\ F(\bar{\alpha}, \bar{n}) \leq n' \leq \beta \;&\Leftrightarrow\; (N, Q, Q') \models \mathcal{P}'(\bar{n}, n'). \end{aligned}$$

Because $t(\bar{\alpha}\beta/N)$ is a strong heir of $t(\bar{\alpha}\beta/M)$ we have $(M, P, P') \prec (N, Q, Q')$. Because $t(\beta/M)$ is not realized in $M\langle\bar{\alpha}\rangle$ we have

$$(M, P, P') \models \forall \bar{z}\; \mathcal{P}(\bar{z}) \to \exists u\; \mathcal{P}'(\bar{z}, u)$$

Thus

$$(N, Q, Q') \models \forall \bar{z}\; \mathcal{P}(\bar{z}) \to \exists u\; \mathcal{P}'(\bar{z}, u)$$

which gives (ii). //

Corollary 3.12.

(i) If B is definably closed, $B \supseteq M$ and $t(A/B)$ is an heir of $t(A/M)$, then

$$\mathrm{rk}_B(A/B) \leq \mathrm{rk}_M(A/M)$$

(ii) If $M \prec N$ and $t(A/N)$ is a strong heir of $t(A/M)$, then

$$\mathrm{rk}_N(A/N) = \mathrm{rk}_M(A/M)$$

//

Lemma 3.13. *Let A be a definably closed set. Then we have:*

(i) Let B, B' be disjoint, such that $B \cup B'$ is A-indominated over C. Then B' is $\mathrm{cl}\, AB$-indominated over C.

(ii) Let $B_0 \subseteq B$ be a basis for A-dominance of B over C. Then for all A-automorphisms f of $\mathfrak{M}$ with $f(B) \subseteq B$ we have that $f(B_0)$ is a basis of A-dominance of B over $f(C)$.

Proof. (i) is immediate.

(ii) Suppose not. Then we find some an element $\beta \in B \setminus f(B_0)$, which is A-indominated by $C \cup f(B)$. $t(\beta/A)$ is realized in $\mathrm{cl}\, AB_0C$ by some element δ. Hence $t(\beta/A)$ is realized by $f(\delta) \in \mathrm{cl}(A \cup f(B_0) \cup f(C))$, a contradiction. //

§4 Dense Types

We want to classify non definable cuts. If p is a dense cut, then p has definable heirs. This fact always disturbs our analysis of non definable cuts. For this reason we put together the information about dense types in this paragraph.

Again in this paragraph, 'o-minimal' means 'o-minimal expansion of a dense ordered set without endpoints'.

Let $A \subseteq B$ be totally ordered sets. We say that A is dense in B, if for $b, b' \in B$ with $b < b'$ there is some $a \in A$ with $b < a < b'$. In particular both sets are dense ordered sets.

Proposition 4.1. *Let $M \prec N$ be o-minimal, and let $M \subseteq B \subseteq N$ be definably closed. The following conditions are equivalent:*

(i) M is dense in B (in this case we have $B \prec N$).

(ii) *If* $q \in S_1(B)$ *is not realized in* B*, then* q *is a coheir over* M*.*
(iii) *If* $q \in S_n(B)$ *with* $\dim q = \dim q \restriction M$*, then* q *is a coheir over* M*.*

PROOF. (iii)⇒(ii). If $q \in S_1(B)$ is not realized in B, then $\dim q = 1 = \dim q \restriction M$.

(ii)⇒(i). If $b < b'$ in B, then there is some $q \in S_1(B)$ with $\dim q = 1$ and $b < x < b' \in q$ (for example $q = b^+/B$). Since q is a coheir over M there is some $m \in M$ with $b < m < b'$.

(i)⇒(iii). We already know that B is an elementary restriction of N. Let $p = q \restriction M$, $\dim p = \dim q = k$, $\bar\alpha$ some k-tuple and $\bar\beta$ some $(n-k)$-tuple with $q = t(\bar\alpha\bar\beta/B)$, $\dim(\bar\alpha/B) = k$ and $\bar\beta \in M\langle\bar\alpha\rangle$. Let $F : M^k \longrightarrow M^{n-k}$ be M-definable with $F(\bar\alpha) = \bar\beta$. If $\varphi(\bar x, \bar y) \in q$, then $\varphi(\bar x, F(\bar x)) \in t(\bar\alpha/B) \in S_k(B)$. The subset of B^k defined by $\varphi(\bar x, F(\bar x))$ has nonempty interior, since $\dim(\bar\alpha/B) = k$. For M is dense in B there is some $\bar m \in M^k$ such that $B \models \varphi(\bar m, F(\bar m))$, which gives (iii). //

Proposition 4.2. *Let* M *be o-minimal and* $p \in S_n(M)$*. Then the following conditions are equivalent:*

(i) *If* $M \subseteq B$*, then all extensions* q *of* p *on* B *with* $\dim q = \dim p$ *are heirs of* p*.*
(ii) *If* p *is realized by* $\bar\alpha$*, then* M *is dense in* $M\langle\bar\alpha\rangle$*.*

PROOF. (i)⇒(ii). We show condition (ii) of Proposition 4.1: if $q \in S_1(M\langle\bar\alpha\rangle)$ and β is a realization of q with $\dim(\beta/M\bar\alpha) = \dim(\beta/M) = 1$, then $\dim(\bar\alpha/M) = \dim(\bar\alpha/M\beta)$. By (i) $t(\bar\alpha/M\beta)$ is an heir of $t(\bar\alpha/M)$, hence q is a coheir over M.

(ii)⇒(i). Let $\bar\beta$ be a finite tuple and let $q \in S_n(M\bar\beta)$ be an extension of p on $M\bar\beta$ with $\dim q = \dim p$. Let $\bar\alpha$ be a realization of q. Then $\dim(\bar\beta/M\bar\alpha) = \dim(\bar\beta/M)$ and by Proposition 4.1 (iii) the type $t(\bar\beta/M\bar\alpha)$ is a coheir of $t(\bar\beta/M)$, thus q is an heir of p. //

Let n be a natural number and $C = \{c_1, ..., c_n\}$ be a fixed set of new constants. We look at the following set of $\mathscr{L}^{defC}$-sentences $\Theta^{dense,n}$:

If $F(x_1, ..., x_n, y_1, ..., y_k)$ and $G(x_1, ..., x_n, y_1, ..., y_k)$ are $\mathscr{L}$-definable maps relative T, take

$$\forall \bar y \ \left[d_{F(\bar c,\bar y)<G(\bar c,\bar y)}(\bar y) \to \exists z \ d_{F(\bar c,\bar y)<z<G(\bar c,\bar y)}(z, \bar y)\right]$$

A consequence of Proposition 4.2 is:

Proposition 4.3. *If* M *is a model of* T *and* $p \in S_n(M)$*, then* p *is dense if and only if* $(M, d^p) \models \Theta^{dense,n}$*. The* $\mathscr{L}^{defC}$*-theory* $Th(M, d^p)$ *is model complete relative* $\mathscr{L}$ *(see Appendix B for the precise description of this notion). If* q *is an extension of* p *on* $N \succ M$*, then* q *is a strong heir of* p *if and only if* q *is dense.*

Proof. The axioms says that a model (M, d^p) is a model of $\Theta^{dense,n}$ if and only if p is dense. By Proposition 4.2 and Appendix B the theory of (M, d^p) is model complete relative $\mathscr{L}$. The description of strong heirs of dense types is obvious now.//

Quantifier elimination for dense cuts

Let T be an o-minimal expansion of the theory of dense ordered sets without end-points, let $\mathcal{D}$ be a new unary predicate and let T^{dense} be the $\mathscr{L}(\mathcal{D})$-theory T together with

$$\mathcal{D} \text{ is the set of left options of a dense cut}$$

Proposition 4.4. *T^{dense} is model complete relative $\mathscr{L}$. Moreover let (M, D) and (N, E) be models of T^{dense} and let (L, U) be a common substructure such that $L \prec M$ and $L \prec N$. Let $\varphi(x)$ be a quantifier free formula with one free variable x in the language $\mathscr{L}(\mathcal{D})$ with parameters from L. If $(M, D) \models \exists x\ \varphi(x)$ then $(N, E) \models \exists x\ \varphi(x)$.*

Proof. T^{dense} is model complete relative $\mathscr{L}$: if $M \prec N$ are models of T and $(M, D) \subseteq (N, E)$ are models of T^{dense}, then this extension is existential, since E^+ is an heir of D^+.

We prove the supplement. It is enough to prove for L-definable maps $F_1, ..., F_k$ and $\alpha \in M$: there is some $\beta \in N$ such that $p := t(\alpha/L) = t(\beta/L)$ and $F_i(\alpha) \in D \Leftrightarrow F_i(\beta) \in E$. If $F_i(\alpha) \leq a \in U$, then $F(x) \leq a \in p$ and we can cross out F_i from the list. Similar if $F_i(\alpha) \geq a \in L \setminus U$. Furthermore we can suppose that α realizes $U^+ \in S_1(L)$, that is $p = U^+$. Now we have $F_i(p) = p$ for all i. In a small neighbourhood of p the maps F_i are comparable and this is written in the type p. Thus it remains to show:

If $\alpha \models p = U^+$ and $F, G : L \longrightarrow L$ are L-definable maps such that $F(p) = G(p) = p$, $F(\alpha) \in D$ and $G(\alpha) \notin D$, then there is some $\beta \in N$ such that $p = t(\beta/L)$, $F(\beta) \in E$ and $G(\beta) \notin E$.

Because E^+ is dense, $G \circ F^{-1}(x) > x \in p$ and $G \circ F^{-1}(p) = p$ we get some $\gamma \in N$ such that $\gamma \models p$, $\gamma \in E$ and $G \circ F^{-1}(\gamma) \notin E$. We choose $\beta = F^{-1}(\gamma) \models p$. Then $F(\beta) = \gamma \in E$ and $G(\beta) = G \circ F^{-1}(\gamma) \notin E$. Since β realizes p, the Proposition is proved. //

Corollary 4.5. *If T has quantifier elimination and a system of universal axioms, then T^{dense} has quantifier elimination.*

Proof. By Proposition 4.4 and elimination theory. //

The next Proposition shows how we can use the notion 'heir'.

Proposition 4.6. *Let M be o-minimal and let p, q be cuts of M such that q is not definable and q^L is definable in (M, d^p). If p is dense, then $p \sim q$.*

Proof. We may assume that T is model complete. Let α be a realization of p and let N be a prime model of $M\langle\alpha\rangle$. For (M, p^L) is model complete we know that q^L is defined by an existential formula $\varphi(x)$ in the language $\mathscr{L}(\mathcal{D})$, where $\mathcal{D}$ is a unary predicate for the left options of p. For p is dense, α^+ is an heir of p on N, thus $(M, p^L) \subseteq (N, (-\infty, \alpha])$ is existential. Therefore $\varphi(x)$ defines in $(N, (-\infty, \alpha])$ an extension of q on N. Thus q has a definable extension on N (since $(-\infty, \alpha])$ is definable in N). For q is not definable, q is realized in N. But then q must be realized in $M\langle\alpha\rangle$ too, thus $q \sim p$. //

We examine the connection of dense types with the realization rank:

Lemma 4.7. *Let $\bar{\alpha}$ be an n-tuple with $\dim(\bar{\alpha}/M) = n$ such that $t(\bar{\alpha}/M)$ is dense (observe that $M\langle\bar{\alpha}\rangle$ is an elementary extension of M in this case). If β is an element such that $t(\beta/M\langle\bar{\alpha}\rangle)$ is not definable, then $t(\bar{\alpha}\beta/M)$ is a box type.*

Proof. By Proposition 3.10 it is enough to show that $\mathrm{rk}_M(\bar{\alpha}\beta/M) = n + 1$. We prove this by induction on $n + 1$ and suppose that $\mathrm{rk}(\bar{\alpha}/M) = n$. Assume $t(\beta/M)$ is realized in $M\langle\bar{\alpha}\rangle$ by some γ, say $\gamma < \beta$. Since $t(\beta/M\langle\bar{\alpha}\rangle)$ is not definable there is some $\gamma' \in M\langle\bar{\alpha}\rangle$ such that $\gamma < \gamma' < \beta$. But then $(\gamma, \gamma') \cap M = \emptyset$, which is not possible. //

Proposition 4.8. *Let $M \prec M'$ be models of T. Then*

(i) M is dense in M' $\Leftrightarrow$ M' is rigid over M (that is $\mathrm{Aut}\, M'/M = 1$) and there is a transcendence basis of M'/M, which is a basis for the M-dominance of M' over M.

In this case 'M-dominance' coincides with the relation 'algebraic over M'.

(ii) If $\mathrm{rk}_M\, M' < \infty$, then

$$M \text{ is dense in } M' \Leftrightarrow M' \text{ is rigid over } M \text{ and } \mathrm{rk}_M\, M' = \dim_M M'$$

Proof. Let $M \subseteq M'$ be dense. Then the identity is the unique M-automorphism of M'. If $\bar{a} \in M'$ then $\bar{a}$ is the unique realization of $t(\bar{a}/M)$ in M': if $\bar{b} \in M'^n$ is an n-tuple and $a_i < b_i$ for some i, then $a_i < m < b_i$ for some $m \in M$, hence $t(a_i/M) \neq t(b_i/M)$ and $t(\bar{b}/M) \neq t(\bar{a}/M)$.

Consequently, if $a \in M'$, $B \subseteq M'$ and if a is not algebraic over $M \cup B$, then $t(a/M)$ is omitted in $M\langle B\rangle$. That is a is M-indominated over B.

Conversely let M' be rigid over M and let $(b_i)_{i<\lambda}$ be a transcendence basis of M'/M, which is an M-dominance basis of M'/M at the same time (such a basis exist if $\dim_M M' = \mathrm{rk}_M M' < \infty$, hence we prove (i) and (ii) in the moment). It is enough to show that $M\langle b_i \mid i \leq \alpha\rangle / M\langle b_i \mid i < \alpha\rangle$ is rigid for all $\alpha < \lambda$ (if $\dim M'/M = 1$, then the Proposition holds by Corollary 2.6; furthermore we get inductively that $M\langle b_i \mid i < \alpha\rangle$ is a model of T). Otherwise there is some $b' \in M\langle b_i \mid i \leq \alpha\rangle$ with $b_\alpha < b'$ and $t(b_\alpha / M\langle b_i \mid i < \alpha\rangle) = t(b'/M\langle b_i \mid i < \alpha\rangle)$. For M' is rigid over M we have $t(b_\alpha / M\langle b_i \mid i < \lambda,\ i \neq \alpha\rangle) \neq t(b'/M\langle b_i \mid i < \lambda,\ i \neq \alpha\rangle)$ and we find some $c \in M\langle b_i \mid i < \lambda,\ i \neq \alpha\rangle$ such that $b_\alpha \leq c \leq b'$. Therefore $t(b_\alpha/M)$ is realized in $M\langle b_i \mid i < \lambda,\ i \neq \alpha\rangle$; but this is not possible, because $(b_i)_{i<\lambda}$ is a basis for M-dominance over M. //

The last section of this paragraph is on continuous closures of o-minimal expansions of $DOAG$ and is not used in the sequel.

Continuous closures

Lemma 4.9. *Let T be an o-minimal expansion of DOAG. Let $M \prec N$ be models of T such that N contains no infinitesimal element over M. If $t(\beta/M)$ is dense, then $t(\beta/N)$ is dense or definable.*

Proof. Suppose there is some $\gamma \in N, > 0$ with $t(\beta+\gamma/N) = t(\beta/N)$. By assumption we find some $m \in M$ with $0 < m < \gamma$, thus $t(\beta + m/M) = t(b/M)$ and $t(\beta/M)$ is not dense. //

Corollary 4.10. *Let T be an o-minimal expansion of DOAG and let M be a model of T. If B is M-indominated over M and M is dense in $M\langle b\rangle$ for each $b \in B$, then M is dense in $M\langle B\rangle$.*

Proof. We show for $b, b_1, ..., b_n \in B$: if M is dense in $M\langle b_1, ..., b_n\rangle$, then $M\langle b_1, ..., b_n\rangle$ is dense in $M\langle b_1, ..., b_n, b\rangle$.

By Lemma 4.9 it is enough to prove, that $t(b/M\langle b_1, ..., b_n\rangle)$ is not definable: If there is some $\beta \in M\langle b_1, ..., b_n\rangle$, such that b is infinitesimal close to β, then $t(b/M) = t(\beta/M)$; but $b, b_1, ..., b_n$ is indominated over M. //

Proposition 4.11. *Let T be an o-minimal expansion of DOAG. Let $M \prec \mathcal{M}$ be models of T. Then there is a model N of T with $M \prec N \prec \mathcal{M}$, such that:*

(i) M is dense in N.

(ii) If M' is an elementary restriction of $\mathcal{M}$, $M \prec M'$ and if M is dense in M', then there is an elementary embedding $M' \longrightarrow N$ over M.

The embedding in (ii) is unique. If $M \prec N' \prec \mathcal{M}$ and N' has properties (i) and (ii), then there is a unique M-isomorphism $N \longrightarrow N'$.

Proof. Let $X \subseteq \mathcal{M}$ be the set of all $\alpha \in \mathcal{M}$, such that M is dense in $M\langle\alpha\rangle$. Let B be a M-dominance basis of X over M. We claim, that $N := M\langle B\rangle$ has the required properties.

By the above Corollary M is dense in N. Let $M' \prec \mathcal{M}$ be an elementary extension of M, such that M is dense in M'. Let B' be a transcendence basis of M' over M. By Proposition 4.8 the set B' is a basis for M-dominance over M. By the choice of B, every $b' \in B'$ is M-dominated by B. From Proposition 3.7 we know that $t(B'/M)$ is realized in $M\langle B\rangle = N$. Hence $t(M'/M)$ is realized in N and there is an elementary M-embedding $M' \longrightarrow N$.
Both additions are obvious. //

Corollary 4.12. *Let T be an o-minimal expansion of RCF. Let $M \prec \tilde{M} \prec \mathcal{M}$ be models of T, suppose that M is archimedean in $\tilde{M}$ and $\tilde{M}$ is tame in $\mathcal{M}$. We provide $\tilde{M}$ with the topology induced by the ordering of $\tilde{M}$. Let N be the topological closure $\overline{M}$ in this topology. Then $M \prec N \prec \tilde{M} \prec \mathcal{M}$ and N fulfills the conditions (i) and (ii) of Proposition 4.8, both for M and $\tilde{M}$ as well as for M and $\mathcal{M}$. We have*

$$N = \{\alpha \in \tilde{M} \mid M \textit{ is dense in } M\langle\alpha\rangle\}$$

Proof. Let $M \prec N_1 \prec \mathcal{M}$ as in Proposition 4.11 and let $N_1 \prec \tilde{N}_1 \prec \mathcal{M}$, such that N_1 is archimedean in $\tilde{N}_1$ and $\tilde{N}_1$ is tame in $\mathcal{M}$. For $\tilde{N}_1$ and $\tilde{M}$ are isomorphic over M we can suppose that $N_1 \subseteq \tilde{M} = \tilde{N}_1$ (T is an expansion of RCF). Since M is archimedean in $\tilde{M}$, N_1 is contained in $\overline{M} = N$. If $\alpha \in \overline{M}$, then M is dense in $M\langle\alpha\rangle$. If $\alpha \in \tilde{M}$, such that M is dense in $M\langle\alpha\rangle$, then by Lemma 4.9, the set N_1 is dense in $N_1\langle\alpha\rangle$. By choice of N_1 we get therefore $\alpha \in N_1$. This proves $N_1 = N = \{\alpha \in \tilde{M} \mid M \text{ is dense in } M\langle\alpha\rangle\}$. //

Proposition 4.11 applied to a sufficiently large, elementary extension $\mathfrak{M}$ of M yields

Corollary 4.13. *Let T be an o-minimal expansion of DOAG and M be a model of T. Then there is a model $N \succ M$ with:*

(i) *M is dense in N.*
(ii) *If M' is an elementary extension of M and M is dense in M', then there is an elementary M-embedding $M' \longrightarrow N$.*

The embedding in (ii) is unique. N is uniquely determined up to a unique M-isomorphism by conditions (i) and (ii). //

The model N in Corollary 4.13 is the largest elementary extension of M, such that M is dense in N. No type over N is dense.

Because M is dense in N, N can be identified with the set of all 1-types p of M, such that M is dense in the prime model of p.
If T is an RCF-expansion, then we get N by Corollary 4.12 in the following manner: choose $M \prec M_1 \prec \mathfrak{M}$ such that M is archimedean in M_1, M_1 is tame in $\mathfrak{M}$ and $\mathfrak{M}$ is enough saturated. Take

$$N = \{\alpha \in M_1 \mid M \text{ is dense in } M\langle\alpha\rangle\}$$

N is called the **continuous closure** of M (c.f. [PC]).
Finally I want to give an example, where dense cuts and continuous closures may appear:

Let $c < a < b \in \mathbb{R}$ and $f : (c, b) \longrightarrow \mathbb{R}$ be a map (not necessary definable !), analytic in a. Then

$$P := \{F(x, y) \in \mathbb{R}[x, y] \mid \text{there is some } \varepsilon > 0 \text{ such that } F(t, f(t)) \geq 0 \text{ on } (a, a + \varepsilon)\}$$

is an ordering of $\mathbb{R}[x, y]$. Let $T := \sum a_i t^i$ be the Taylor series of f in a. Then $(t, T) \in \mathbb{R}[[t]] \times \mathbb{R}[[t]]$ realizes P (where $\mathbb{R}[[t]]$ is viewed as a subring of the real closed field $\mathbb{R}((t^{\mathbb{Q}}))$).

We identify P with the cut $t(T/\mathbb{R}\langle t\rangle)$ and write $f(a^+)$ for this cut. It is not difficult to see, that $f(a^+)$ is dense (cf the example at the end of §9 for a more conceptual description of cuts of generalized power series). We write $f(a^+)$ for this ordering. Similar we define $f(a^-)$. Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ is analytic. We get a map $f : S_1(\mathbb{R}) \setminus \{\pm\infty\} \longrightarrow S_1(\mathbb{R}\langle t\rangle)$. Note that this extension of f has values in the continuous closure of $\mathbb{R}\langle t\rangle$. This field can be explicitly constructed as a subset of dense cuts of $\mathbb{R}\langle t\rangle$.

Chapter II The invariance group

We work in o-minimal expansions of divisible, ordered, abelian groups in this chapter.

If M is a real closed field and p is a cut of M, then $G := \{a \in M \mid a + p = p\}$ is a convex subgroup of M. G can be viewed as a set of simple, M-definable maps (the maps $x \mapsto x + a$) which fixes p. Crucial is, that G is a set of such functions, which is **definable** in the structure (M, d^p). Certainly G does not know everything about the 'width' of the cut p. But if we apply the same construction to the cut G^+ of $(M^{>0}, \cdot)$ then we get a convex subgroup G^* of $(M^{>0}, \cdot)$ (we'll do this in the next chapter). It turns out that G together with G^* are the only invariants needed to classify the structures (M, d^p) (§16). At least this intention justifies the examination of the 'width' for arbitrary o-minimal expansions of $DOAG$:

§5 Width

Let T be an o-minimal expansion of $DOAG$, let M be a model of T and let p be a cut of M. We define the **width** $W(p)$ of p in the following manner. We choose a realization α of p and a prime model $M\langle p\rangle = M\langle\alpha\rangle$ of p. We define the width

$$W(p) := \{\gamma - \beta \mid \beta, \gamma \in M\langle p\rangle,\ \beta, \gamma \text{ realize } p\}$$

$W(p)$ depends on the choice of the prime model $M\langle p\rangle$. We'll always take this into consideration and if necessary we write $W_\alpha(p)$ instead of $W(p)$ if $\alpha \notin M$ is some element of the selected prime model. We have

$$W_0(p) := W(p) \cap M = \{a \in M \mid a + p = p\}$$

$W_0(p)$ is uniquely determined by the cut p and the group $(M, +)$. We call $W_0(p)$ the **invariance group** of p. We follow R. Rolland ([Rol]) and define

$$\hat{p} := W_0(p)^+$$

We have $W(p) \leq |a - b|$ for all $a, b \in M$ with $a \leq p \leq b$ and from Corollary 2.6 we know:

$$W_0(p) = 0 \Leftrightarrow p \text{ is dense or } p \text{ is } M\text{-bounded and definable}$$
$$W(p) = 0 \Leftrightarrow p \text{ is dense}$$
$$W(p) = M\langle p\rangle \Leftrightarrow W_0(p) = M \Leftrightarrow p = \pm\infty$$

Proposition 5.1. *If p is a cut of M, then $W(p)$ is a convex subgroup of $M\langle p\rangle$.*

Proof. Obviously $W(p)$ is convex and $0 \in W(p) = -W(p)$. Let $\beta, \gamma \in M\langle p\rangle$, $\beta < \gamma$ be realizations of p. We prove $2(\gamma - \beta) \in W(p)$.

Let $F : M \longrightarrow M$ be M-definable such that $F(\beta) = \gamma$. If $2\gamma - \beta \le F(\gamma)$, then $2(\gamma-\beta) \le F(\gamma) - \beta$ and $F(\gamma) - \beta \in W(p)$. Thus we can assume that $F(\gamma) < 2\gamma - \beta$. Furthermore we may assume that $F : M \longrightarrow M$ is bijective. Let $\delta := F^{-1}(\beta) \in M\langle p\rangle$. From $F(F(x)) < 2F(x) - x \in p$ we get $F(F(\delta)) < 2F(\delta) - \delta$, hence $\delta < 2\beta - \gamma$. This gives $2(\gamma - \beta) < \gamma - \delta \in W(p)$. //

Corollary 5.2. *Let p be a cut of M, $\alpha \in M\langle p\rangle$ be a realization of p and $\varepsilon \in W(p)$ such that $\alpha - \varepsilon \not\models p$. Then:*

(i) For all realizations β of p in $M\langle p\rangle$ there is some $\delta \in W(p)$ with $\operatorname{sign}\delta = \operatorname{sign}\varepsilon$ and $\beta - \delta \not\models p$.

(ii) For all realizations β of p in $M\langle p\rangle$ and all $\delta \in W(p)$ with $\operatorname{sign}\delta = \operatorname{sign}\varepsilon$ we have $\beta + \delta \models p$.

Proof. We may assume that $\varepsilon > 0$. For (i) take $\delta = \varepsilon + \beta - \alpha \in W(p)$.

(ii) Suppose there is some realization β of p in $M\langle p\rangle$ and some $\delta > 0$ in $W(p)$ such that $\beta + \delta \not\models p$. We can suppose that $\varepsilon = \delta$. The cut p is realized by $\gamma := \frac{\alpha+\beta}{2}$ and if we write $\varepsilon' := \varepsilon + \frac{\beta-\alpha}{2} \in W(p)$ then $\varepsilon' > 0$, $\gamma + \varepsilon' \not\models p$ and $\gamma - \varepsilon' \not\models p$.
Let $\alpha' < \beta' \in M\langle p\rangle$ be realizations of p in $M\langle p\rangle$ such that $\beta' - \alpha' = 2\varepsilon'$. Assume $|\gamma - \alpha'| \ge \varepsilon'$. Then $\gamma \le \gamma + \varepsilon' \le \alpha'$ or $\alpha' \le \gamma - \varepsilon' \le \gamma$, a contradiction. //

Definition 5.3. If there is some realization α of p in $M\langle p\rangle$ and some $\varepsilon \in W(p)$ with $\alpha - \varepsilon \not\models p$, we define the **signature** of p

$$\operatorname{sign} p := \operatorname{sign}\varepsilon$$

We define $\operatorname{sign} p := 0$ if there is no such ε.

Dense cuts have signature 0. If $a \in M$ we have $\operatorname{sign} a^+ = \operatorname{sign} +\infty = 1$ and $\operatorname{sign} a^- = \operatorname{sign} -\infty = -1$.

Proposition 5.4. *If p is a cut of M, then the following conditions are equivalent:*

(i) $\operatorname{sign} p \neq 0$

(ii) There is a convex subgroup G of M and an element $a \in M$ such that $p = a + G^+$ or $p = a + G^-$.

In this case $G = W_0(p)$ is uniquely determined by (ii) and we have $\operatorname{sign} p = 1$ if and only if $p = a + G^+$.

Proof. (ii)⇒(i) and the supplement. If $p = a + G^+$ and α is a realization of G^+, then $\varepsilon := \alpha \in W(p), > 0$ and $(a + \alpha) - \varepsilon$ is not a realization of p. By definition we have $\operatorname{sign} p = 1$. Similar $\operatorname{sign} p = -1$ if $p = a + G^-$.
Obviously G is contained in $W_0(p)$. If $b \in M$ with $b + (a + G^+) = a + G^+$, then $b + G^+ = G^+$, hence $b \in G$.

(i)⇒(ii) Let $\operatorname{sign} p = 1$, $\varepsilon \in W(p), > 0$ and let α be a realization of p with $\alpha - \varepsilon \not\models p$. Let $a \in M$ with $\alpha - \varepsilon \leq a < \alpha$. Then we have $p = a + W_0(p)^+$: for $\alpha - \varepsilon \leq a < \alpha$ we get $W_0(p) < \varepsilon$. From Corollary 5.2 we know $\alpha + \varepsilon \models p$, hence $\varepsilon \models W_0(p)^+$ and $p = a + W_0(p)^+$.

The same argument works in the case $\operatorname{sign} p = -1$. //

Corollary 5.5. *The signature of p only depends on the ordered group $(M, +, <)$.//*

Corollary 5.6. *If $\operatorname{sign} p \neq 0$, F is an M-definable map with $F(p) = \hat{p}$ and q is an heir of p, then $F(q) = \hat{q}$. Especially $p = a \pm \hat{p}$ implies $q = a \pm \hat{q}$ and $\hat{q}$ is an heir of $\hat{p}$.*

Proof. We take $p = a + \hat{p}$. Then $q - a$ is an heir of $\hat{p}$, hence a convex subgroup of N. By Proposition 5.4 we get $q - a = \hat{q}$ is an heir of $\hat{p}$. Because $F(x + a)$ fixes $\hat{p}$, we get $F(\hat{q} + a) = \hat{q}$. //

Proposition 5.7. *Let $M \prec N$, let p be a cut of M and q a cut of N extending p. Then $W_\alpha(p) \subseteq W_\alpha(q)$ if and only if $F(q) = q$ for every M-definable map $F : M \longrightarrow M$ with $F(p) = p$ (here we choose $N\langle q\rangle = N\langle\alpha\rangle$ for some realization α of q and $M\langle p\rangle = M\langle\alpha\rangle$). In this case $\operatorname{sign} p \neq 0$ implies $signp = \operatorname{sign} q$.*

Proof. First suppose $F(q) = q$ for every M-definable map F with $F(p) = p$. Let $\varepsilon \in W_\alpha(p)$, $\varepsilon > 0$. By Corollary 5.2 there is some realization $\beta \in M\langle p\rangle$ of p such that $\epsilon = |\alpha - \beta|$. Let $F : M \longrightarrow M$ be M-definable and $F(\alpha) = \beta$. Then $F(p) = p$ and by assumption $F(q) = q$; this means $\beta \models q$ and $\varepsilon \in W_\alpha(q)$.

Conversely suppose $W_\alpha(p) \subseteq W_\alpha(q)$. Let $F : M \longrightarrow M$ be M-definable with $F(p) = p$. We assume $\operatorname{sign} q \geq 0$ and $F(x) > x \in p$ (otherwise we take F^{-1} instead of F). If $\beta := F(\alpha)$ then $\varepsilon = \beta - \alpha \in W_\alpha(p)$, $\varepsilon > 0$, hence $\varepsilon \in W_\alpha(q)$ and we get $\beta = \alpha + \varepsilon \models q$. This means $F(q) = q$.

Assume again $W_\alpha(p) \subseteq W_\alpha(q)$ and $\operatorname{sign} q \geq 0$. Let $\epsilon \in W(p)^{>0}$. Then $\alpha + \varepsilon$ is a realization of p, because $\alpha + \varepsilon$ is even a realization of q. //

It follows for all non definable cuts p of M:

$$q \text{ is an heir of } p \text{ iff } W(p) \subseteq W(q)$$

If q is a coheir of p, then $W(p) \subseteq W(q)$ even if p is definable.

If $M \prec N$ are models of T and p is a cut of M, we call a cut q on N a **weak heir** of p, if q extends p and $W_0(p) \subseteq W_0(q)$.

If p is definable and bounded or if p is dense, then all extensions of p are weak heirs of p.

If q is an extension of p, then $W_0(q) \cap M \subseteq W_0(p)$, with equality if q is a weak heir of p. We formulate this fact again in different ways:

Proposition 5.8. *Let $M \prec N$ be models of T, let p be a cut of M and q a cut of N extending p. Then*

(i) If q is a weak heir of p, then $W_0(p) = W_0(q) \cap M$.
(ii) If p is dense or definable, then $W_0(q) \cap M \in \{0, M\}$ for all weak heirs q of p on N.
(iii) If q is a weak heir of p, $q \neq \pm\infty$, then p is definable or dense iff $W_0(q) \cap M = 0$.
(iv) If $M \prec N$ is tame, $\operatorname{sign} q \neq 0$ *and $W_0(q) \cap M_0 = 0$, then p is definable.*
(v) Every heir and every coheir of p is a weak heir of p.

//

The Multiplicative width in real closed fields

Let T be an o-minimal expansion of RCF in the language $\mathscr{L}$, let M be a model of T and p be a cut of M. We define the **multiplicative width**

$$W^*(p) = \{\frac{\gamma}{\beta} \mid \beta, \gamma \in M\langle p\rangle,\ \beta, \gamma \models p\}$$

and the **multiplicative invariance group**

$$W_0^*(p) = W^*(p) \cap M = \{a \in M \mid a \cdot p = p\}$$

Up to this moment all the results, which hold for the width, hold for the multiplicative width too. To be more precise some explanations:

Let $\mathscr{L}^{>0}$ be the signature of ordered groups $\mathscr{L}_0 := \{+, -, <, 0\}$ together with an n-ary predicate R_φ for each $\mathscr{L}$-formula φ with exactly n free variables. If M is a

model of T, then $M^{>0}$ is the $\mathscr{L}^{>0}$-structure with universe $\{a \in M \mid M \models a > 0\}$ and the following interpretation of the non logical symbols:
$M^{>0} \upharpoonright \mathscr{L}_0 := \big((0,\infty), \cdot, ^{-1}, <, 1\big)$, that is the underlying divisible, ordered group of $M^{>0}$ is the multiplicative group of positive elements of M.

If φ is an $\mathscr{L}$-formula with exactly n free variables, then

$$R_\varphi^{M^{>0}} := \varphi[M^n] \cap (0,\infty)^n$$

If we relativize to $(0,\infty)$ we can inductively assign to each $\mathscr{L}^{>0}$-formula a formula from $\mathscr{L}$.

We define $T^{>0} := Th(M^{>0})$. $T^{>0}$ does not depend on the choice of M (we assume always, that T is complete). $T^{>0}$ is an o-minimal expansion of $DOAG$, because $M^{>0}$ is o-minimal. If N is a model of $T^{>0}$, we can construct in an obvious way a model M of T such that $M^{>0} = N$. The transition $M \mapsto M^{>0}$ respects elementary extensions, as well as definable closures: for T is o-minimal, we can recover M-definable subsets of M^n from $M^{>0}$-definable subsets of $(M^{>0})^n$.
If M is a model of T then the n-types of M which contains $\bigwedge\!\!\!\bigwedge x_i > 0$ are in one-to-one correspondence with the n-types of $M^{>0}$. Observe that for these types the notions heir, strong heir and coheir are not touched by switching the point of view: all three notions are fixed if one expands the involved models in a definable way.
We leave the arguments in this way; the transition $T \mapsto T^{>0}$ and vice versa is at no point difficult or tricky.

Note that $W^*(p) = W^*(-p)$ and $\operatorname{sign}^* -p = \operatorname{sign}^* p$, where we define $\operatorname{sign}^* p = \operatorname{sign} \varepsilon$ for $p \in S_1(M)$, if there is a realization α of p and some $\delta \in W^*(p)$, $\delta > 1$ with $\alpha \cdot \delta^{-\varepsilon} \not\models p$ ($\varepsilon \in \{-1, 1\}$). In the other case we define $\operatorname{sign}^* p = 0$. Hence, if $p > 0$ then $\operatorname{sign}^* p$ is the signature of p regarded as type in $S_1(M^{>0})$.

Cuts with signature 0

Again T is an o-minimal expansion of $DOAG$. If p is a cut of M and $W(p)$ is the convex hull of $W_0(p)$, then obviously $\operatorname{sign} p = 0$. We write $\operatorname{co}_B A$ for the convex hull of a subset A of B in B. We say:

The weak signature alternative holds for T, if for all models M of T and each cut p of M:

$$\operatorname{sign} p = 0 \Leftrightarrow W(p) \text{ is the convex hull of } W_0(p)$$

The strong signature alternative holds for T, if for all models M of T and each cut p of M:

$$\operatorname{sign} p = 0,\ F : M \longrightarrow M \text{ definable in } M, \text{ not constant near } p \ \Rightarrow\ \operatorname{sign} F(p) = 0$$

Since $W(p)$ is the convex hull of $W_0(p)$ iff $\hat{p}$ is not realized in $M\langle p\rangle$ the weak alternative follows from the strong one.

If p is a cut of M we say **the strong signature alternative holds for** p if either for all cuts q of M with $p \sim q$ we have $\operatorname{sign} q = 0$ or for all cuts q of M with $q \sim p$ we have $\operatorname{sign} q \neq 0$.

Remark

Both alternatives are useful for classifying cuts. If T is an o-minimal expansion of RCF, then both alternatives imply a good valuation theoretic behaviour of the theory T. The strong alternative can be seen as a replacement for the following Proposition for real closed fields:

Proposition 5.9. *If $R \subseteq S$ are real closed fields, $\dim S/R = 1$, v is a convex valuation on S such that $vS \neq vR$ and $\alpha \in S$, then there is some $a \in R$ such that $v(\alpha - a) \notin vR$.* //

Theorem 5.10. *The strong signature alternative holds for $DOAG$ and for RCF.*

Proof. for $DOAG$:

Let M be a divisible ordered abelian group and let p be a cut of M such that $p = \hat{p}$. Let α be a realization of p and let F be an M-definable map. From $M\langle\alpha\rangle = M \oplus \alpha{\cdot}\mathbb{Q}$ we see that F is piecewise linear, that is $F(x) = a + \frac{n}{m}{\cdot}x$ for some $a \in M$, $\frac{n}{m} \in \mathbb{Q}^*$, in a suitable neighbourhood of p. Hence $\operatorname{sign} F(p) = \pm 1$.

for RCF:

Let M be a real closed field, p be a cut of M with $\operatorname{sign} p = 0$ and α a realization of p. Let A be the convex hull of $\mathbb{Q}$ in M. We can assume that A is proper (if M is archimedean, p is dense). Let B be the convex hull of $\mathbb{Q}$ in $M\langle\alpha\rangle$ and v the corresponding valuation on $M\langle\alpha\rangle$. If $\Gamma_B \neq \Gamma_A$, then by Proposition 5.9 there is some $a \in M$ such that $v(\alpha - a) \notin \Gamma_A$. But this means that $\alpha - a$ realizes a convex subgroup of M, which contradicts $\operatorname{sign} p = 0$. Hence $\Gamma_A = \Gamma_B$. This means again, that no convex subgroup of M is realized in $M\langle\alpha\rangle$. //

Except of Proposition 5.9 the prove does not need further ingredients:

Proposition 5.11. *Let T be an o-minimal expansion of $DOAG$, let M be a model of T and let p be a cut of M with $\operatorname{sign} p = 0$. Then the strong signature alternative holds for p if and only if the extension $M \subseteq M\langle\alpha\rangle$ does not change the rank of the natural valuation (in the sense of ordered, abelian groups).*

The proof is obvious. //

Again a remark: if we take a predicate $\mathcal{D}$ for the left options of p, then it is written in the theory of (M, p^L) if the strong signature alternative holds for p.

We'll see:
If T is a polynomially bounded, o-minimal expansion of RCF then the strong signature alternative holds for T but not for $T^{>0}$. For $T^{>0}$ the weak signature alternative holds. The strong signature alternative does not hold for $Th(\mathbb{R}, exp)$ (§13) and I don't know if the weak one does. But I expect this.

If p is a cut of a model M of T we write p_- and p_+ for the least and the largest extension of p on $M\langle p\rangle$. Especially $p_- < p_+$ always.

Proposition 5.12. *(Properties of the signature)*
Let T be an o-minimal expansion of DOAG, let M be a model of T and let p be a cut of M. Then

(i) *The following conditions are equivalent:*
  (a) $\operatorname{sign} p \geq 0$
  (b) *For all realizations α of p and β of $\hat{p}$ from an elementary extension of M, $\alpha + \beta$ is a realization of p.*
  (c) *$\alpha + \beta$ is a realization of p for all realizations α, β of p, $\hat{p}$ respectively in $M\langle p\rangle$.*
  (d) *If q is the largest extension of p on an extension $N \succ M$, then $\hat{q}$ is the largest extension of $\hat{p}$ on N.*

  If $\operatorname{sign} p \geq 0$, then $p_+ = \alpha + W_\alpha(p)^+$. If $\operatorname{sign} p = 1$, $p = a + \hat{p}$, then $p_- = a + (\operatorname{co} W_0(p))^+$ and $p_+ = a + W_\alpha(p)^+$.

(ii) *The following conditions are equivalent:*
  (a) $\operatorname{sign} p \leq 0$
  (b) *For all realizations α of p and β of $\hat{p}$ from an elementary extension of M, $\alpha - \beta$ is a realization of p.*
  (c) *$\alpha - \beta$ is a realization of p for all realizations α, β of p, $\hat{p}$ respectively in $M\langle p\rangle$.*
  (d) *If q is the least extension of p on an extension $N \succ M$, then $\hat{q}$ is the largest extension of $\hat{p}$ on N.*

  If $\operatorname{sign} p \leq 0$, then $p_- = \alpha - W_\alpha(p)^+$. If $\operatorname{sign} p = -1$, $p = a - \hat{p}$, then $p_- = a + W_\alpha(p)^-$ and $p_+ = a + (\operatorname{co} W_0(p))^-$.

(iii) *If β is a realization of the least extension of $\hat{p}$ on $M\langle\alpha\rangle$, $\alpha \models p$, then both $\alpha + \beta$ and $\alpha - \beta$ are realizations of p.*

(iv) *If α is a realization of p, then $W_\alpha(p)$ is the largest subgroup G' of $M\langle\alpha\rangle$ such that $G' \cap M = W_0(p)$. Thus $W_\alpha(p)^+$ is the largest extension of $\hat{p}$ on $M\langle\alpha\rangle$*

(v) If $\operatorname{sign} p = 0$ *and* $\alpha \in N \succ M$ *is a realization of* p *then we have* $q' = 2\alpha - q$ *for the coheirs* q, q' *of* p *on* N*.*

Proof. (i) (a)⇒(b). Assume $\operatorname{sign} p \geq 0$. If $\alpha + \beta \not\models p$, we have an element $m \in M$ with $\alpha < m < \alpha + \beta$. For $\operatorname{sign} p \geq 0$ we get $m - \alpha > W(p)$. Since $m - \alpha < \beta$ we get $m - \alpha \models \hat{p}$, hence $p = m - \hat{p}$, which contradicts $\operatorname{sign} p \geq 0$.

(b)⇒(c) is a weakening. (c)⇒(a). If $\alpha \models p$ and $\beta \in W(p) \subseteq M\langle\alpha\rangle$, then (c) gives $\alpha + \beta \models p$.

Now we know, that the largest extension of p on $M\langle\alpha\rangle$ is $\alpha + W(p)^+$. The statement for $\operatorname{sign} p = 1$ is obvious.

Clearly $(b) \Leftrightarrow (d)$.

For the assertion (ii) apply (i) to the cut $-p$.

(iii). Because of (i) and (ii) we can suppose that $\operatorname{sign} p = 1$ and $\alpha + \beta \models p$. Let $p = a + \hat{p}$ with $a \in M$ and $m \in M$ with $a < m < p$. We prove $m < \alpha - \beta$: From $m - a < \hat{p}$ we have $m - a < \frac{\alpha - a}{2}$. From the choice of β we get $\beta < \frac{\alpha - a}{2}$, hence $m - a + \beta < \alpha - a$.

(iv) follows immediately from (i) and (ii).

(v) holds for p_- and p_+ by (i) and (ii). The map $x \mapsto 2\alpha - x$ moves the least extension of p_- on N to the largest extension of p_+ on N. These extensions are the coheirs of p on N. //

Corollary 5.13. *Let* $M \prec N$ *be models of* T*, let* p *be a cut of* M *and* q *an extension of* p *on* N*. If* q *is a coheir of* p*, then* $\hat{q}$ *is a coheir of* $\hat{p}$*.*

Proof. The assertion holds clearly if p is definable. If $\operatorname{sign} p = 0$ then $\hat{q}$ is a coheir of $\hat{p}$ by the conditions (d) in (i) and (ii) of the above proposition.
If $p = a + \hat{p}$, then $q - a$ is a coheir of $\hat{p}$ on N, hence $q - a$ is the upper boundary of a convex subgroup (note that $W_0(p) \neq 0$ if p is not definable and $\operatorname{sign} p \neq 0$). //

An example:
If p is a dense cut, then p has heirs q such that $\hat{q}$ is not an heir of $\hat{p}$. We'll see, that this happens regularly in RCF also if p is not dense (see the remark after Proposition 9.10).

Corollary 5.14. *Let* T *be an o-minimal expansion of DOAG, let* M *be a model of* T*, let* p *be a cut of* M *and let* $F : M \longrightarrow M$ *be* M*-definable such that* $F(p) = \hat{p}$*. If* F *is strictly increasing, respectively decreasing near* p*, then* $F(q) = \hat{q}$ *for the largest, respectively the least extension* q *of* p *on* N*.*

Proof. If $\operatorname{sign} p \neq 0$, then $F(q) = \hat{q}$ for each heir q of p by Corollary 5.6.

If $\operatorname{sign} p = 0$, then $\hat{q}$ is the largest extension of $\hat{p}$ on N for both coheirs q of p (Proposition 5.12 (i),(ii) (d)). Thus the assertion holds. //

I want to emphasize again, that I do not know if $\operatorname{sign} p = 0$ and $F(p) = \hat{p}$ can happen at all.

Proposition 5.15. *Let T be an o-minimal expansion of DOAG, let M be a model of T and let p be a cut of M. Then we have for every coheir q of p on N:*

$$\operatorname{sign} q = 0 \ \Leftrightarrow \ \operatorname{sign} p = 0 \textit{ and } p \textit{ is omitted in } N$$

Proof. We may assume that p is not definable.

$\Rightarrow$. If $\operatorname{sign} p \neq 0$, then by Proposition 5.7 we have $\operatorname{sign} q \neq 0$ too.
If $\alpha \in N$ is a realization of p, then $q' = q \restriction M\langle\alpha\rangle$ is a coheir of p on $M\langle\alpha\rangle$. By Proposition 5.12 we have $\operatorname{sign} q' \neq 0$. Since q is a coheir of q' we get $\operatorname{sign} q = \operatorname{sign} q' \neq 0$.

$\Leftarrow$. Assume $\operatorname{sign} p = 0$ and let q be the unique extension of p on N. Let α be a realization of q and $\varepsilon \in N\langle\alpha\rangle$ such that $t(\alpha - \varepsilon/N) \neq q$. We show that $t(\alpha + \varepsilon/N) \neq q$. We may assume that $\varepsilon > 0$. For $t(\alpha - \varepsilon/M) \neq p$ there is some $m \in M$ with $\alpha - \varepsilon < m < \alpha$. Since p has signature 0 there is some $m_0 \in M$ with $m_0 < \alpha - m < 2m_0$ and we get $\alpha < m + 2m_0 < \alpha + m_0 < m + \varepsilon + m_0 < \alpha + \varepsilon$: $p \neq t(\alpha + \varepsilon/M)$. //

Theorem 5.16. *Let T be an o-minimal expansion of DOAG and let M be a model of T. Suppose the weak signature alternative holds for T.*
Let p, q be cuts of M and let $F : M \longrightarrow M$ be an M-definable map such that $F(p) = q$. Suppose the strong signature alternative holds for p. Then:

(i) $\hat{q} \sim \hat{p}$.
(ii) If in addition $\operatorname{sign} p = 0$*, then for each realization α of p and γ of $\hat{p}$ in some T-model $N \succ M$:*

$$|F(\alpha) - F(\alpha + \gamma)| \models \hat{q}$$

and there is a realization b of $\hat{q}$ in $M\langle\gamma\rangle$ such that $|F(\alpha) - F(\alpha + \gamma)| < b$.

Proof. If $\operatorname{sign} p \neq 0$, then $\operatorname{sign} q \neq 0$ by assumption. In this case $\hat{q}$ is realized in $M\langle q\rangle \cong M\langle p\rangle$ and p is realized in $M\langle\hat{p}\rangle$.

Therefore we can assume that $\operatorname{sign} p = \operatorname{sign} q = 0$. Let γ be a realization of $\hat{p}$, α a realization of p and $F : M \longrightarrow M$ an M-definable map, with $F(\alpha) \models q$. Since $\operatorname{sign} t(\gamma/M) = 1$ both p and q are omitted in $M\langle\gamma\rangle$. By Proposition 5.15

$$\operatorname{sign} t(\alpha/M\langle\gamma\rangle) = \operatorname{sign} t(F(\alpha)/M\langle\gamma\rangle) = 0$$

By Proposition 5.12 and $\operatorname{sign} p = 0$ we get $t(\alpha + \gamma/M) = p$, hence $\gamma \in W_0(\alpha/M\langle\gamma\rangle)$. We may assume that F is strictly increasing near p. The map

$$\begin{aligned} \sigma : W_\alpha(\alpha/M\langle\gamma\rangle) &\longrightarrow W_\alpha(F(\alpha)/M\langle\gamma\rangle) \\ \varepsilon \qquad &\longmapsto F(\alpha+\varepsilon) - F(\alpha) \end{aligned}$$

is a strictly increasing bijection (by assumption the weak signature alternative holds for T). The restriction of σ to $W_\alpha(\alpha/M)$ is a strictly increasing bijection onto $W_\alpha(F(\alpha)/M)$. It follows $\sigma\gamma \in W_\alpha(F(\alpha)/M\langle\gamma\rangle)$ and $\sigma\gamma > W_\alpha(q)$. Since $t(F(\alpha)/M\langle\gamma\rangle)$ has signature 0, there is some $b \in M\langle\gamma\rangle$ with

$$\sigma\gamma < b \in W_0(F(\alpha)/M\langle\gamma\rangle)$$

We get $b \in W_0(F(\alpha)/M\langle\gamma\rangle)$ and $b > W_0(q)$. Therefore b realizes $\hat{q}$. //

Proposition 5.17. *Let $M \prec N$ be tame, let p be a cut of M and let q be a cut of N extending p. Let $\mathfrak{m}$ be the group of infinitesimal elements of N over M. Then:*

(i) If p is not definable, then $\mathfrak{m} \subseteq W_0(q)$.
(ii) If p is definable and M-bounded, then $W_0(q) \subseteq \mathfrak{m}$; if $p = a^{\pm}$, $a \in M$ and $W_0(q) = \mathfrak{m}$, then $q = a \pm \hat{q}$ is the coheir of p.
(iii) The following conditions are equivalent:
(a) p is not definable and $W_0(q) = \mathfrak{m}$
(b) $\operatorname{sign} q = 0$ and $W_0(q) = \mathfrak{m}$
(c) p is dense

Proof. Let α be a realization of q.

(i) Let $b \in N^{>0}$ be infinitesimal over M. Then $t(\alpha + b/M) = t(\alpha/M)$, since for some $a \in M$ with $\alpha < a < \alpha + b$ there is (p is not definable) an $a' \in M$ with $0 < a' < a - \alpha < b$. Thus $t(\alpha + b/M) = p$; for p is omitted in N we get $b \in W_0(q)$.

(ii) is obvious.

(iii) (c)⇒(b). By Proposition 5.15 $\operatorname{sign} q = 0$, because p is omitted in N. By (i) we have $\mathfrak{m} \subseteq W_0(q)$. Conversely let $b \in W_0(q)$, $b > 0$. If there is some $a \in M$ with $0 < a < b$, then $\alpha + a$ is a realization of p; but $t(\alpha/M)$ is dense.

(b)⇒(a) is obvious and (a)⇒(c) follows from Proposition 5.8 //

Obtaining cuts of signature 0

A useful tool in the sequel, is an explicit method to produce cuts with signature zero from given cuts.

Lemma 5.18. *Let $M \prec N$ be models of T, let p be a cut of M and let q be an extension of p on N. If $q = a \pm \hat{q}$ with $a \in M$, then $\hat{q}$ extends $\hat{p}$ and $p = a \pm \hat{p}$. Consequently, if $\operatorname{sign} p = 0$ and q is an extension of p with $q = \alpha \pm \hat{q}$, then $\alpha \notin M$.*

Proof. $q - a$ is an extension of $p - a$, thus $p - a$ is the upper/lower boundary of a convex subgroup of M, hence the lemma follows. //

Proposition 5.19. *Let T be an o-minimal expansion of DOAG. Let I be a totally ordered index set, let $M_i \prec M_j$ be models of T $(i \le j \in I)$ and let p_i be a cut of M_i such that $p_i \subseteq p_j$ whenever $i \le j \in I$. Let $M := \bigcup_i M_i \models T$. There is a unique cut of M extending all p_i. We write $\bigcup p_i$ for this cut. We have:*

(i)
$$W_0(\bigcup p_i) = \bigcup_{k\in I}\bigcap_{j\ge k} W_0(p_j) = \bigcap_{k\in I}\bigcup_{j\ge k} W_0(p_j)$$

(ii) If $\hat{p}_j$ does not extend $\hat{p}_i$ for all $i < j$, then
$$W_0(\bigcup p_i) = \bigcap W_0(p_i)$$

(iii) If $\hat{p}_j$ extends $\hat{p}_i$ finally, then
$$\widehat{\bigcup p_i} = \bigcup \hat{p}_i$$

(iv) If $\hat{p}_0 \subseteq \hat{p}_1 \subseteq \hat{p}_2 \subseteq \ldots$ and $\operatorname{sign} p_i = 0$ for all i, then $\operatorname{sign} \bigcup p_i = 0$.

(v) If $\operatorname{sign} \bigcup p_i \neq 0$, then there is some $k \in \mathbb{N}$ such that $\hat{p}_k \subseteq \hat{p}_{k+1} \subseteq \ldots$

Proof. (i) is obvious. (ii) and (iii) are applications of (i). (iv) is implied by (iii) and Lemma 5.18.

(v) Suppose $\bigcup p_i = \alpha + G^+$ with a convex subgroup G of $\bigcup M_i$. Let $\alpha \in M_k$. By Lemma 5.18 we have $p_k = \alpha + \hat{p}_k$ and $p_{k+1} = \alpha + \hat{p}_{k+1}$, hence $\hat{p}_{k+1}$ extends $\hat{p}_k$. //

We describe, in a detailed form a method to produce types of signature 0.

Proposition 5.20. *Let T be an o-minimal expansion of DOAG. Let $M \models T$, let p be a cut of M and let $G \subseteq W_0(p)$ be a convex subgroup of $(M,+)$. Let β be a realization of p and p' be the least extension of G^+ on $M\langle\beta\rangle$. Furthermore let $(\alpha_i)_{0<i<\omega}$ be a left Morley sequence of p'. Let $p_0 = \beta + p'$ and for $i > 0$ let $p_i = \beta + \alpha_1 + ... + \alpha_i + t(\alpha_{i+1}/M\langle\beta,\alpha_1,...,\alpha_i\rangle)$.*

Then
(i) $p \subseteq p_0 \subseteq p_1 \subseteq ...$ *and* $\hat{p}_0 \subseteq \hat{p}_1 \subseteq ...$.
(ii) $\operatorname{sign} \bigcup p_i = 0$ *and* $W_0(\bigcup p_i)$ *is the convex hull of G in $M\langle\beta,\alpha_1,\alpha_2,...\rangle$.*
(iii) $\bigcup p_i = \{\beta + \sum_{i=1}^{n} \alpha_i \mid n \in I\!N\}^+$

Proof. The proof is not difficult, if we write the assertion in another way.

Let $p'_i := t(\alpha_{i+1}/M\langle\beta,\alpha_1,...,\alpha_i\rangle)$ for all $i > 0$. Since $(\alpha_i)_{0<i<\omega}$ is a left Morley sequence, we have that p'_i is the least extension of $p'_0 := p'$ on $M\langle\beta,\alpha_1,...,\alpha_i\rangle$. By definition of p' this means that p'_i is the upper boundary of the convex hull of G in $M\langle\beta,\alpha_1,...,\alpha_i\rangle$. By definition $p'_i = \hat{p}_i$ for all $i \geq 0$. Since p' is the least extension of G^+ on $M\langle\beta\rangle$ we know that p_0 is an extension of p, even if $\operatorname{sign} p = -1$ (Proposition 5.12). The same argument gives $p_i \subseteq p_{i+1}$ for all $i \geq 0$. Let $q = \bigcup p_i$. From Lemma 5.18 we know $\hat{q} = \bigcup p'_i$ and this cut is the upper boundary of the convex hull of G in $M\langle\beta,\alpha_1,\alpha_2,...\rangle$. It remains to show, that $\operatorname{sign} q = 0$.

Suppose $\operatorname{sign} q \neq 0$, say $q = \alpha \pm \hat{q}$ with $\alpha \in M\langle\beta,\alpha_1,...,\alpha_k\rangle$. By Lemma 5.18 we get for all $i \geq k$: $p_i = \alpha \pm p'_i$. We have $p_i = \beta + \alpha_1 + ...\alpha_i + p'_i$, thus $p_i = \alpha + p'_i$ and $|\beta + \alpha_1 + ... + \alpha_i - \alpha| < p'_i, p'_{i+1}$. If we look at the same inequality for $i+1$, the triangle inequality yields $\alpha_{i+1} < p'_{i+1}$. But this is not possible, since p'_{i+1} is the least extension of p'_i and $\alpha_{i+1} \models p'_i$.

Therefore we know (i) and (ii). By definition $\{\beta + \sum_{i=1}^{n} \alpha_i \mid n \in \mathbb{N}\}^+ \leq q$. Suppose equality does not hold. Then there is some $n \in \mathbb{N}$ and $\alpha \in M\langle\beta,\alpha_1,...,\alpha_n\rangle$ such that $\beta + \sum_{i=1}^{n+1} \alpha_i < \alpha < q$. By definition of p_n we get $p_n < \alpha < q$ which is not possible. //

Corollary 5.21. *Let T be an o-minimal expansion of DOAG, let M be a model of T and let p be a cut of M.*

(i) There is an $N \succ M$, $\dim N/M = \aleph_0$ and an extension q of p on N such that $W_0(q) = \operatorname{co}_N W_0(p)$ and $\operatorname{sign} q = 0$.
(ii) There is an $N \succ M$, $\dim N/M = \aleph_0$ and an extension q of p on N such that q is dense.

Proof. For (i), take $G = W_0(p)$ and apply the Proposition. For (ii), take $G = 0$ and apply the Proposition. //

For quantifier elimination results we need

Lemma 5.22. *Let $M \prec N$ be models of T, let q be a cut of N which extends p on M, such that $p > 0$. Let $H = W_0(q)$, $G := H \cap M \neq 0$ such that $G^+ \not\sim p$. Assume that the structure (N, H, q^L) is $|M|^+$-saturated. Then there is some realization $\alpha \in N$ of p such that $q \upharpoonright M\langle\alpha\rangle = \alpha + r$, where r is the unique extension of G^+ on $M\langle\alpha\rangle$.*

Proof. Let $\mathcal{H}$ be a predicate for H and $\mathcal{D}$ be a predicate for q^L. We can describe α with a small set of formulas in $\mathscr{L}(\mathcal{H}, \mathcal{D})$:

For each $m \in M$ with $H < m < q$ choose a realization $\alpha_m \in N$ of p such that $\alpha_m < q < \alpha_m + m$. We define

$$\Phi := \{\alpha_m < x \mid m \in M,\ \mathcal{H} < m < \mathcal{D}\} \cup \{\mathcal{D}(x)\}$$

Because (N, H, q^L) is large, there is some realization α of Φ in N and it is clear that $\hat{q}_\alpha$ extends G^+, where q_α denotes $q \upharpoonright M\langle\alpha\rangle$.

It remains to show $q_\alpha = \alpha + \hat{q}_\alpha$. Certainly $\alpha + \hat{q}_\alpha \leq q_\alpha$, since $\alpha < q_\alpha$. Suppose there is an M-definable map F such that $\alpha + \hat{q}_\alpha < \alpha + F(\alpha) < q_\alpha$. Then $\hat{q}_\alpha < F(\alpha)$ and, because $\hat{q}_\alpha$ is the unique extension of G^+ on $M\langle\alpha\rangle$, we find some $m \in M$ such that $\hat{q}_\alpha < m < F(\alpha)$. This means $m > H$ and $\alpha + m < \alpha + F(\alpha) < q_\alpha$ in contradiction to $\alpha_m + m > q$. Thus the claim is proved. //

Without the assumption $G^+ \not\sim p$ this Lemma is not true in general; this is the reason why a possible quantifier elimination result is not available for the structure (M, p^L) in a small definable expansion of (M, p^L) (cf Proposition 19.1).

§6 The small type $(p, \hat{p})$

T is again an o-minimal expansion of $DOAG$.

Lemma 6.1. *Let $M \prec N$ be models of T, $p \in S_n(M)$ and let q be an heir of p on N. Let $p_i \in S_1(M)$ be the i-th projection of p and q_i be the i-th projection of q $(1 \leq i \leq n)$. Let $\varphi(\bar{x}, \bar{z}, \bar{u})$ be an $\mathscr{L}(M)$-formula, where $\bar{x}$ and $\bar{z}$ are n-tupels of variables.*
If there are $\bar{\alpha} \in N^n$ and $\bar{\beta} \in N^{\bar{u}}$ such that

$$x_1 < \alpha_1 \in \hat{q}_1\ ,\ \ldots\ ,\ x_n < \alpha_n \in \hat{q}_n,\ \varphi(\bar{x}, \bar{\alpha}, \bar{\beta}) \in q$$

then there are $\bar{a} \in M^n$ *and* $\bar{b} \in M^{\bar{u}}$ *such that*

$$x_1 < a_1 \in \hat{p}_1 \ , \ \dots \ , \ x_n < a_n \in \hat{p}_n, \ \varphi(\bar{x}, \bar{a}, \bar{b}) \in p$$

Proof. Choose $\gamma_1, ..., \gamma_n \in N$ with $\gamma_i < x_i < \gamma_i + \alpha_i \in q$. Since q is an heir of p, there are $\bar{a}, \bar{c} \in M^n$ and $\bar{b} \in M^{\bar{u}}$ such that $\varphi(\bar{x}, \bar{a}, \bar{b}) \wedge \bigwedge c_i < x_i < c_i + a_i \in p$. This proves the Lemma. //

Lemma 6.2. *Let* $M \prec N$ *be models of* T*, let* p *be a non definable cut of* M *and let* q *be a cut of* N *extending* p*. Assume* $\hat{q}$ *is an heir of* $\hat{p}$ *and* $\alpha \in N$ *is a realization of* $\hat{p}$ *with* $\alpha < \hat{q}$*. Then*

$$W_0(q) \cap M\langle\alpha\rangle = W_0(q \restriction M\langle\alpha\rangle)$$

Proof. Obviously, we only have to prove $\supseteq$. We suppose that $p > 0$ and that $\hat{p}$ is not definable. Let $F(\alpha) \in M\langle\alpha\rangle, > 0$ such that $F(\alpha) + q \restriction M\langle\alpha\rangle = q \restriction M\langle\alpha\rangle$.

If $t(F(\alpha)/M) < \hat{p}$, there is some $m \in M$ with $F(\alpha) \leq m \in W_0(p) \subseteq W_0(q)$, hence $F(\alpha) + q = q$.

If $t(F(\alpha)/M) > \hat{p}$, then there is some $m \in M$ with $F(\alpha) \geq m \notin W_0(p)$ and we get $F(\alpha) + q \restriction M\langle\alpha\rangle > q \restriction M\langle\alpha\rangle$.

Finally suppose $F(\alpha)$ realizes $\hat{p}$. Then $F(\hat{p}) = \hat{p}$, therefore $F(\hat{q}) = \hat{q}$, since $\hat{q}$ is an heir of $\hat{p}$. For $\alpha < \hat{q}$ we have $F(\alpha) < \hat{q}$ too, that is $F(\alpha) + q = q$. //

Proposition 6.3. *Let* $M \prec N$ *be models of* T*, let* p *be a cut of* M *with* $\operatorname{sign} p = 0$ *and let* q *be a cut of* N *extending* p*, such that* $\hat{q}$ *is an heir of* $\hat{p}$*.*
For each coheir p' *of* p *on some elementary extension of* M *suppose*

$$\operatorname{sign} p' = 0 \Rightarrow p' \not\sim \hat{p}'$$

(*This assumption is only needed if* T *does not have the weak signature alternative*).

Then $(q, \hat{q})$ *is an heir of* $(p, \hat{p})$.

Proof. Because $p \not\sim \hat{p}$, q is an heir of p too.
By Proposition 2.1 we have to show: if $\alpha_1, \alpha_2, \beta_1$ and β_2 are elements of N and $\varphi(v_1, v_2, v_3, v_4) \in \operatorname{Fml} \mathscr{L}(M)$ such that

$$\alpha_1 < x < \alpha_2 \in \hat{q}, \ \beta_1 < y < \beta_2 \in q \text{ and } N \models \varphi(\alpha_1, \alpha_2, \beta_1, \beta_2)$$

then there are a_1, a_2, b_1, b_2 such that

$$a_1 < x < a_2 \in \hat{p},\ b_1 < y < b_2 \in p \text{ and } M \models \varphi(a_1, a_2, b_1, b_2)$$

If α_1 is not a realization of $\hat{p}$, then we know this from Lemma 6.1. Especially, we can suppose that p is not dense.

Thus we assume that α_1 is a realization of $\hat{p}$. By Lemma 6.2 we have $W_0(q) \cap M\langle\alpha_1\rangle = W_0(q \restriction M\langle\alpha_1\rangle)$. Since $p \not\perp \hat{p}$, the only extension $q_1 := q \restriction M\langle\alpha_1\rangle$ of p has again signature 0, thus $q_1 \not\perp \hat{q}_1$. Consequently q is an heir of q_1 too. By Lemma 6.1 applied to q_1 and q, there are M-definable maps F, G_1, G_2 such that

$$\alpha_1 < x < F(\alpha_1) \in \hat{q}_1,\ G_1(\alpha_1) < y < G_2(\alpha_1) \in q_1$$

and

$$M\langle\alpha_1\rangle \models \varphi(\alpha_1, F(\alpha_1), G_1(\alpha_1), G_2(\alpha_1))$$

By assumption $\hat{q}_1$ is an heir of $\hat{p}$. By Corollary 1.10 there is some $r_1 \in S_2(M\langle\alpha_1\rangle)$ with $\hat{q}_1(x) \cup p(y) \subseteq r_1$, which is an heir over M. Since q_1 is the unique extension of p on $M\langle\alpha\rangle$ we get $\hat{q}_1(x) \cup q_1(y) \subseteq r_1$. Since r_1 is an heir over M, there is some $a \in M$ such that

$$a < x < F(a) \in \hat{p},\ G_1(a) < y < G_2(a) \in p$$

and

$$M \models \varphi(a, F(a), G_1(a), G_2(a))$$

//

§7 Application to Divisible Ordered Abelian Groups

We leave the o-minimal expansions of $DOAG$ in the next chapter. Therefore, this is a good moment to use and to illustrate our results by examining (M, d^p), if M is a divisible, ordered, abelian group and p is a cut of M. This section will be used in the sequel only to give easy counter examples.

We work in the language $\mathscr{L} = \{+, -, 0, <\}$ of ordered groups, where we know quantifier elimination and completeness of the theory $DOAG$ of divisible, ordered abelian groups.

Proposition 7.1. *Let $\mathcal{G}$ be a new unary predicate and $DOAG_{pair}$ be the $\mathscr{L}(\mathcal{G})$-theory of divisible, ordered, abelian groups, which picks out a proper (this means $\neq 0$*

and $\neq M$), convex subgroup G of a model of DOAG. Then $DOAG_{pair}$ is complete and has quantifier elimination.

Proof. We first prove model completeness of $DOAG_{pair}$. If $M \prec N$ are models of $DOAG$, p is a non definable cut of M with $p = \hat{p}$ and q is an extension of p on N such that $q = \hat{q}$, then q is an heir of p: By Corollary 2.8 we can suppose that $N = M\langle\alpha\rangle$, where α realizes p. Obviously there are only two convex subgroups H of N such that $H \cap M = W_0(p)$. This shows the assertion.
From the Robinson test we get model completeness of $DOAG_{pair}$.

Now we prove quantifier elimination. An ad hoc construction in advance:

Let U be a torsion free, abelian, ordered group and let $V \subseteq U$ be a convex subgroup (we do not exclude the case $U = V = 0$). Let U_1 be the divisible hull $U \otimes_{\mathbb{Z}} \mathbb{Q}$ of U and V_1 be the convex hull of V in U_1. V_1 is the unique convex subgroup of U_1 such that $V_1 \cap U = V$. Let U_2 be $U_1 \times \mathbb{Q}$ with the antilexicographic ordering, and let V_2 be the convex hull of V_1 in U_2. V_2 is the unique convex subgroup of U_2, which is different from U_2 such that $V_2 \cap U_1 = V_1$ (note that $U_2 \neq 0$). Let U_3 be $U_2 \times \mathbb{Q}$ with the lexicographic ordering, and let V_3 be the convex hull of $V_2 \cup \{\mu\}$ in U_3. V_3 is the unique convex subgroup of U_3, which is proper, such that $V_3 \cap U_2 = V_2$. Only for this proof we define $D(U,V)$ to be the model (U_3, V_3) of $DOAG_{pair}$. From all the uniqueness statements we get:

If (M,G) is a model of $DOAG_{pairs}$, (U,V) is a substructure of (M,G), $\alpha, \beta \in M$ such that $\alpha > G \cup U$ and $\beta \in G$ is positive infinitesimal with respect to $U\langle\alpha\rangle$, then $(U\langle\alpha,\beta\rangle, U\langle\alpha,\beta\rangle \cap G)$ is isomorphic to $D(U,V)$ over U.

Now quantifier elimination follows easily from model completeness with elimination theory.

$DOAG_{pair}$ is complete, since $(0,0)$ is a common substructure of all models. //

Corollary 7.2. *$DOAG_{pair}$ is weakly o-minimal and has unique prime models.*

Proof. Weak o-minimality follows immediately from quantifier elimination.

Let $(M,G) \models DOAG_{pair}$ and let A be a subset of M. If we take U as the convex subgroup generated by A, $V = G \cap U$ and if we apply the ad hoc construction of the proof of Proposition 7.1, we get a model $D(U,V) \prec (M,G)$ extending (U,V). Certainly $D(U,V)$ is a prime model over A and is A-isomorphic to every prime model over A. //

The actual result treats cuts of $M \models DOAG$ with signature 0. The case of a dense cut is done in §4, Proposition 4.4 and Corollary 4.5. Now we examine the remaining candidates:

Theorem 7.3. *Let $\mathcal{D}$ be a new unary predicate and let $DOAG_{zero}$ be the $\mathscr{L}(\mathcal{G},\mathcal{D})$-theory of divisible, ordered, abelian groups, which picks out a proper, convex subgroup G of a model M of $DOAG$, and a set D of left options of a cut p of M with signature 0 such that $W_0(p) = G$. Then $DOAG_{zero}$ is model complete and does not have quantifier elimination. The theory $DOAG_{pair} + \mathcal{D}(0)$ is complete. If we add function symbols $\frac{1}{n}$ to the language, then the obvious definable expansion of $DOAG_{zero}$ has quantifier elimination.*

Proof. Perhaps the proof of this theorem is shorter if we examine the transition to the factor group $M/W_0(D^+)$. But for o-minimal expansions of RCF an analogous method does not work. Note that we'll make a very similar proof for quantifier elimination for certain convex subgroups of real closed fields in §19. Now the proof.

Model completeness is easy if we use §6. Let $M \prec N$ be models of $DOAG$, let p be a cut of M, not dense with $\operatorname{sign} p = 0$ and let q be an extension of p such that $\operatorname{sign} q = 0$ and $\hat{q}$ extends $\hat{p}$. Since $\hat{q}$ is an heir of $\hat{p}$, $(q,\hat{q})$ is an heir of $(p,\hat{p})$ (by Proposition 6.3; we even proved this in the case $\operatorname{sign} q \neq 0$; this more general result will be crucial in o-minimal expansion of RCF). Again the Robinson test yields model completeness of $DOAG_{zero}$.

For quantifier elimination we have to add the functions $\frac{1}{n}$, since otherwise we have non trivial, discrete subgroups of a model of $DOAG$ as substructures (in this case one can not eliminate the quantifier in $\exists x\ x + x = a + b \wedge \mathcal{D}(x)$).

Thus we work in the extended language. Let $(M,G,p),(N,H,q) \models DOAG_{zero}$ very saturated and (U,V,D) be a small common substructure.

Claim There are small models $(M_0,G_0),(N_0,H_0)$ of $DOAG_{pair}$ and cuts p_0, q_0 over M_0, N_0 respectively, such that

$$(U,V,D) \subseteq (M_0,G_0,p_0) \subseteq (M,G,p),$$
$$(U,V,D) \subseteq (N_0,H_0,q_0) \subseteq (N,H,q)$$

and an U-isomorphism $(M_0,G_0,p_0) \longrightarrow (N_0,H_0,q_0)$.

Proof. We can suppose that $p,q > 0$ since this is witnessed in D.

Case 1 $U = 0$.

We take $\alpha \in G,\ \alpha > 0$ and $\beta \in H,\ \beta > 0$, as well as U_M, U_N the substructures generated by α, β respectively. Then $G \cap U_M = U_M = p^L \cap U_M$ and similar for U_N. Thus we can assume that $U \neq 0$. We know that U is a model of $DOAG$ in this case.

Case 2 $U \neq 0$, $V = 0$.

We take $\alpha \in G,\ \alpha > 0$ infinitesimal over U and $\beta \in H,\ \beta > 0$ infinitesimal over U. Then the U-isomorphism, which sends α to β sends $G \cap U\langle\alpha\rangle$ $(\neq 0)$ onto $H \cap U\langle\beta\rangle$ as well as $p^L \cap U\langle\alpha\rangle$ onto $q^L \cap U\langle\beta\rangle$ (α, β are in the invariance groups of p, q respectively). Thus we may suppose that $U \neq 0$ and $V \neq 0$.

Case 3 $V = U \neq 0$

We take $\alpha \in M,\ \alpha > k \cdot p$ for all $k \in \mathbb{N}$ and $\beta \in N,\ \beta > k \cdot q$ for all $k \in \mathbb{N}$. Then the U-isomorphism, which sends α to β sends $G \cap U\langle\alpha\rangle(\neq U\langle\alpha\rangle, \neq 0)$ onto $H \cap U\langle\beta\rangle$ as well as $p^L \cap U\langle\alpha\rangle$ onto $q^L \cap U\langle\beta\rangle$. Thus we have proved the claim.

We want to prove quantifier elimination of $DOAG_{zero}$. Because of the claim and model completeness, it remains to show:

Let $(M, G, p^L), (N, H, q^L) \models DOAG_{zero}$, $p, q > 0$ very saturated and let (U, V, D) be a small common substructure such that $(U, V) \models DOAG_{pair}$. Then there are models (M', G', p'^L) and (N', H', q'^L) of $DOAG_{zero}$ such that

$$(U, V, D) \subseteq (M', G', p'^L) \subseteq (M, G, p^L),$$
$$(U, V, D) \subseteq (N', H', q'^L) \subseteq (N, H, q^L)$$

together with an U-isomorphism $(M', G', p'^L) \longrightarrow (N', H', q'^L)$.

We describe two constructions now, which we'll use alternately to find these structures.

Construction 1. Let $(U, V) \models DOAG_{pair}$ and let (U, V, D) be a substructure of (M, G, p^L). If $V \neq W_0(D^+)$, then $V^+ \not\sim D^+$ and there is a realization $\alpha \in M$ of D^+ such that $p \restriction U\langle\alpha\rangle$ is the cut α+the unique extension of V^+ on $U\langle\alpha\rangle$. Thus $W_0(p \restriction U\langle\alpha\rangle)$ is the convex hull of V in $U\langle\alpha\rangle$.

Proof. If $V^+ \sim D^+$, then $V^+ \sim \hat{D^+}$ (Theorem 5.16). Since definable maps in $DOAG$ are of the form $kx + a$, with $k \in \mathbb{Q}$ and $a \in M$ we get $V = W_0(D^+)$. The remaining part of the claim is exactly Lemma 5.22.

Construction 2. Suppose, that $(U, V) \models DOAG_{pair}$ and let (U, V, D) be a substructure of (M, G, p^L) such that $D^+ = a + V^+$ with $a \in U$. Then there is a realization $\alpha \in M$ of V^+ such that

(a) $G \cap U\langle\alpha\rangle$ is the convex hull of V in $U\langle\alpha\rangle$

(b) $p \restriction U\langle\alpha\rangle = a + r$, where r denotes the largest extension of V^+ on $U\langle\alpha\rangle$.

Proof.

We define the following set of $\mathscr{L}(U)$-formulas:

$$\begin{aligned}\Psi :=& \{x < c \mid c \in U,\ V < c\} \cup \{\neg\mathcal{G}(x)\} \cup \\ & \cup \{\mathcal{D}(a + x + F(x)) \mid F : U \longrightarrow U,\ U\text{-definable},\ F(V^+) = V^+, \\ & \qquad F(x) \geq x \text{ and } F \text{ increasing on } U\}\end{aligned}$$

Let F be a map as in this definition and let $F_1(x) := a + x + F(x)$. Then $F_1(\hat{p}) \leq p$: if $F_1(\hat{p}) > p$ we find some $\xi \in M$ with $0 < \xi < \hat{p}$ such that $F_1([\xi, +\infty)) > \gamma > p$. For $\hat{p}$ is an heir of V^+ we get $F(\xi) < \hat{p}$ and $a + \xi + F(\xi) > p$ which is not possible.

Therefore we know $F_1(\hat{p}) \leq p$, hence $< p$, because $\operatorname{sign} p = 0$. By saturation there is an element $\alpha \in M$, such that $(M, G, p^L) \models \Psi(\alpha)$. This α has the required properties:

(a) $\hat{p} \restriction U\langle\alpha\rangle$ is a coheir of V^+. Because α is a realization of V^+ outside G, it must be the least coheir.
(b) holds by the definition of Ψ.

We prove quantifier elimination of $DOAG_{zero}$ in finding the structures (M', G', p'^L) and (N', H', q'^L) as described above:

We build countable chains

$$(U, V, D) \subseteq (M_0, G_0, p_0^L) \subseteq (M_1, G_1, p_1^L) \subseteq ... \subseteq (M, G, p^L)$$

and

$$(U, V, D) \subseteq (N_0, H_0, q_0^L) \subseteq (N_1, H_1, q_1^L) \subseteq ... \subseteq (N, H, q^L)$$

where (M_i, G_i), $(N_i, H_i) \models DOAG_{pair}$ together with U-isomorphisms

$$\sigma_i : (M_i, G_i, p_i^L) \longrightarrow (N_i, H_i, q_i^L)$$

such that σ_{i+1} extends σ_i and such that $W_0(p_{2i}) = G_{2i}$, $W_0(q_{2i}) = H_{2i}$ are the convex hulls of V in M_{2i}, N_{2i} respectively:

<u>$i = 0$</u> if $V \neq W_0(D^+)$ we choose $\alpha \in M$ as described in Construction 1 and similar $\beta \in N$ as described in Construction 1, as well as $M_0 = U\langle\alpha\rangle$ and $N_0 = U\langle\beta\rangle$. We have no more choice for G_0, p_0, H_0 and q_0. By Construction 1, the map σ_0, which sends α to β and fixes U pointwise has the required properties.

if $V = W_0(D^+)$ and $\operatorname{sign} D^+ \geq 0$ we take $M_0 = U = N_0$.

if $V = W_0(D^+)$ and $D^+ = a - V^+$, we replace the types p, q and D^+ by $2a - p$, $2a - q$ and $2a - D^+$ respectively and proceed with the case $\operatorname{sign} D^+ = 1$. (if the construction is finished we apply again the map $2a - x$ to all cuts p, q, p_i, q_i and to D^+ to get the required chains).

we construct M_{i+1}, N_{i+1} if M_i, N_i are constructed:

<u>i is even</u> if $\operatorname{sign} p_i = 0$ we take $M_{i+1} = M_i$ and $N_{i+1} = N_i$ (note that by the inductive hypothesis $W_0(p_i) = G_i$ and $W_0(q_i) = H_i$, thus we can stop at all).

if $\operatorname{sign} p_i \neq 0$, we see inductively that $\operatorname{sign} p_i = 1$ and similar $\operatorname{sign} q_i = 1$. Now we apply construction 2: choose $\alpha \in M$ as described in construction 2 with respect to M_i and $\beta \in N$ as described in construction 2 with respect to N_i. We choose $M_{i+1} = M_i\langle\alpha\rangle$ and $N_{i+1} = N_i\langle\beta\rangle$. By construction 2, the map σ_{i+1}, which extends σ_i and sends α to β sends G_{i+1} onto H_{i+1} and p_{i+1} to q_{i+1}.

i is odd if $\operatorname{sign} p_i = 0$ we take $M_{i+1} = M_i$ and $N_{i+1} = N_i$ (note that this can only happen if $W_0(p_i) = G_i$ and $W_0(q_i) = H_i$, again we can stop in this case).

if $\operatorname{sign} p_i \neq 0$, we see inductively that $\operatorname{sign} p_i = 1$ and similar $\operatorname{sign} q_i = 1$. Now we apply construction 1 again: choose $\alpha \in M$ as described in construction 1 with respect to M_i and $\beta \in N$ as described in construction 1 with respect to N_i. We choose $M_{i+1} = M_i\langle\alpha\rangle$ and $N_{i+1} = N_i\langle\beta\rangle$. By construction 1, the map σ_{i+1}, which extends σ_i and sends α to β sends G_{i+1} onto H_{i+1} and p_{i+1} to q_{i+1}.

The construction of the chains is completed. We take $M' = \bigcup M_i$ and $N' = \bigcup N_i$. Because G_{2i} is the convex hull of V we have $G' = W_0(p')$ (Proposition 5.19). Furthermore $\operatorname{sign} p' = 0$: since $W_0(p_{2i}) \not\subseteq W_0(p_{2i+1})$ (by construction 2) we can apply (Proposition 5.19 (v)). For the same reason we have $\operatorname{sign} q' = 0$. If we take $\sigma' := \bigcup \sigma_i$ we find the promised isomorphism. //

Chapter III The valuation ring associated to a cut

We work in *o*-minimal expansions of real closed fields in this chapter.

§8 The multiplicative invariance group

Multiplicative and additive, convex subgroups of ordered fields

Let K denote an ordered field in the sequel.

Proposition 8.1. *The map*

$$\left\{\begin{array}{l}\text{convex subgroups } G\\ \text{of } (K,+) \text{ with } 1\notin G\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\text{convex subgroups } H\\ \text{of } (K^{>0},\cdot\ ) \text{ with } 2\notin H\end{array}\right\}$$
$$G \longmapsto 1+G$$

is one-to-one with inverse $H\mapsto H-H=H-1$.

Proof. A unitary, commutative ring A is local if and only if $1+\mathfrak{a}$ is a subgroup of A^* for all proper ideals $\mathfrak{a}$ of A. A convex subgroup G of K with $1\notin G$ is a proper ideal of the convex hull of $\mathbb{Q}$ in K. Important here is the inverse map:

Let H be a convex subgroup of the multiplicative group of K, which does not contain 2. We show that $H-1$ is an additive, convex subgroup of K with $1\notin H-1$ (then $H-H=H-1$ is obvious). Certainly $H-1$ is convex and $1\notin H-1$. Let $\varepsilon\in H-1$, >0. Then $0<2\varepsilon<(1+\varepsilon)^2-1\in H-1$, hence $2\varepsilon\in H-1$. Since $2\notin H$ we have $\frac{\varepsilon^2}{1-\varepsilon}<\varepsilon$, thus $1<\frac{1}{1-\varepsilon}=1+\varepsilon+\frac{\varepsilon^2}{1-\varepsilon}<1+2\varepsilon<(1+\varepsilon)^2\in H$. We get $\frac{1}{1-\varepsilon}\in H$, therefore $-\varepsilon\in H-1$.

If $\varepsilon>0$ with $-\varepsilon\in H-1$, then $1\le 1+\varepsilon\le\frac{1}{1-\varepsilon}\in H$, that is $\varepsilon\in H-1$. //

Proposition 8.2. *The map*

$$\left\{\begin{array}{c}\text{convex valuation rings}\\ \text{of } K\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\text{convex subgroups } H\\ \text{of } (K^{>0},\cdot\ ) \text{ with } 2\in H\end{array}\right\}$$
$$A \longmapsto A^{*>0}$$

is one-to-one with inverse $H\mapsto H-H$.

Proof. If A is a convex valuation ring of K, then $A^*\cap(0,\infty)$ certainly is a convex subgroup of $(K^{>0},\cdot)$, which contains 2. Conversely if H is such a group, then $H+H\subseteq H$, because $2\in H$. A simple calculation shows: $H-H$ is a convex subgroup of $(K,+)$ and the stated maps are inverse to each other. //

If H is a convex subgroup of $(K^{>0}, \cdot)$ with $2 \notin H$ and $X \subseteq H$ generates H as a convex group, then $X - 1$ is a set of generators of the convex group $H - H$:
it is enough to prove the assertion for sets $X = \{h\}$ with $h > 1$. For $2 \notin H$ we have $1 + h + h^2 + ... + h^n \leq 2n + 1$. It follows $h^{n+1} - 1 = (h-1)(1 + h + h^2 + ... + h^n) \leq (h-1)(2n+1)$, that is $H - 1$ is generated by $h - 1$ as a convex group.

Conversely if G is a convex subgroup of $(K, +)$ with $1 \notin G$, and X generates G as a convex group, then $1 + X$ generates the convex group $1 + G$: again it is enough to look at the case $X = \{g\}$ with $g > 0$. In this case $1 + G$ is generated by $1 + g$, since $1 + ng \leq (1+g)^n$.

Let $\mathfrak{o}$ be the convex hull of $\mathbb{Q}$ in K, the valuation ring of the natural valuation v of K with maximal ideal $\mathfrak{m}$ and value group Γ. From Proposition 8.1 we get that $\mathfrak{m}$ and $1 + \mathfrak{m}$ have the same skeleton (in the sense of Ribenboim [Ri]). Obviously this holds for every convex valuation ring of K. We'll never use this, rather we'll use the preceding Propositions.

A convex subgroup of $(K, +)$ is the same as a fractional ideal of $\mathfrak{o}$ (except for K). It is well known that the convex subgroups of $(K, +)$ and the upper segments of Γ are in one-to-one correspondence. If Γ is divisible, this means:

The set of convex subgroups of $(K, +)$ are in one-to-one correspondence with the cuts of Γ.

From now on we assume K to be real closed and write M again.

We define addition, multiplication and the inverse of convex subgroups of $(M, +)$:
If G' is another convex subgroup of $(M, +)$ we define

$$G + G' = \{g + g' \mid g \in G,\ g' \in G'\}$$

and

$$G \cdot G' = \{gg' \mid g \in G,\ g' \in G'\}$$

$G + G'$ and $G \cdot G'$ are convex subgroups of $(M, +)$. The operations $+$ and $\cdot$ on convex subgroups of $(M, +)$ are associative, commutative and distributive. We have $G^n = \{\pm g^n \mid g \in G\}$, since M is real closed. Furthermore we define for each $m \in \mathbb{N}$:

$$G^{\frac{1}{m}} = \{h \in M \mid h^m \in G\}$$

and

$$\frac{1}{G} = \{0\} \cup \{a \in M^* \mid \frac{1}{a} \notin G\}$$

$G^{\frac{1}{m}}$ and $\frac{1}{G}$ are convex subgroups of $(M, +)$. We have

(a) $\frac{1}{G} = \{a \in M \mid a{\cdot}G \subseteq \mathfrak{m}\}$ and $\frac{1}{G^+} = (\frac{1}{G})^+$. Thus we may define $G^{-1} := \frac{1}{G}$. Note that $1 \notin G{\cdot}\frac{1}{G}$.

(b) if $n, m \in \mathbb{N}$, then $(G^n)^{\frac{1}{m}} = (G^{\frac{1}{m}})^n$. We define $G^{\frac{n}{m}} := (G^n)^{\frac{1}{m}}$ and get $(G^{\frac{n}{m}})^+ = (G^+)^{\frac{n}{m}}$.

From this we may define for all $k \in \mathbb{Z} \setminus \{0\}$ and all $m \in \mathbb{N}$:

$$G^{\frac{k}{m}} := (G^{\frac{|k|}{m}})^{\operatorname{sign} k}$$

(a) and (b) shows, that $(G^{\frac{k}{m}})^+ = (G^+)^{\frac{k}{m}}$.

If $1 \notin G$ we define

$$\sqrt{G} := \{a \in M \mid a^n \in G, \text{ for some } n \in \mathbb{N}\} = \bigcup_n G^{\frac{1}{n}}$$

and

$$G^\infty = \bigcap_n G^n$$

The multiplicative invariance group of a cut

In §5 we have defined $W_0^*(p) = \{a \in M \mid a{\cdot}p = p\}$. Observe, that $p = \hat{p}$ if and only if $2 \in W_0^*(p)$.

Lemma 8.3.

$$\{a \in M \mid \textit{there is some } b \in M \textit{ with } a{\cdot}p + b = p\} = \bigcup_{c \in M} W_0^*(p + c)$$

Proof. if $a{\cdot}p + b = p$ and $a \neq 1$ then $a \in W_0^*(p + c)$ with $c = \frac{b}{a-1}$. //

Definition 8.4.

$$\tilde{p} = (W_0^*(p) - W_0^*(p))^+$$

By the Propositions 8.1 and 8.2 the cut $\tilde{p}$ is the upper boundary of a convex subgroup of $(M, +)$ and $1 + \tilde{p}$ is $\hat{p}$ with respect to $(M^{>0}, \cdot)$.

Because $\widehat{-p} = \hat{p}$ and $\widetilde{-p} = \tilde{p}$ we restrict the description of $\tilde{p}$ to cuts $p > 0$.

Proposition 8.5. *(Comparison of $\hat{p}$ and $\tilde{p}$)*
Let p be a cut of M with $p > 0$. Then:

(i) $\tilde{p} = \{a \in M \mid a{\cdot}p \le \hat{p}\}^+$, *in other words*

$$W_0^*(p) - W_0^*(p) = \{a \in M \mid |a|{\cdot}p \le \hat{p}\}$$

(ii) If $p > \hat{p}$, then $p \not\sim \hat{p}$ and $p \not\sim \tilde{p}$ with respect to the multiplicative group of positive elements of M and

(a) $\tilde{p} = \{a \in M \mid a{\cdot}p < \hat{p}\}^+$, *hence* $\tilde{p} = \frac{\hat{p}}{p}$ *(this means that $\tilde{p} = F(r)$ if r is the unique 2-type of the ordered abelian group $(M^{>0}, \cdot, 1, \le)$ with projections p and $\hat{p}$ and F is the $(M^{>0}, \cdot, 1, \le)$-definable map $\frac{x}{y}$; cf Proposition 3.10)*

(b) $\hat{p} = \{a \in M \mid \frac{a}{p} < \tilde{p}\}^+$, *hence* $\hat{p} = p{\cdot}\tilde{p}$.

(c) There are $c \in M^*$ and $\frac{k}{n} \in \mathbb{Q}^*$ such that $\tilde{p} = c{\cdot}(\hat{p})^{\frac{k}{n}}$.

Proof. (i) If $p = \hat{p}$, then (i) is clear. If $p > \hat{p}$, then (i) follows from (ii)(a).

(ii) Because $p > \hat{p}$, we can not have $p = F(\tilde{p})$ with a map $F(x) = a{\cdot}x^q$, $a \in M$ and $q \in \mathbb{Q}^*$, because $\tilde{p}$ is the upper boundary of the convex subgroup $W_0^*(p) - 1$ of $(M, +)$. Each map definable in $(M^{>0}, <, \cdot)$ is piece wise of the above form, thus $p \not\sim \tilde{p}$. The same argument gives $p \not\sim \hat{p}$ with respect to the group $(M^{>0}, \cdot, <)$.

(ii) (a). We know $W_0^*(p) - W_0^*(p) = W_0^*(p) - 1$. Let $a > 0$. If $a{\cdot}p < \hat{p}$, then obviously $(1 + a){\cdot}p = p$. Conversely if $a > 1$, $a{\cdot}p = p$ and $(a-1)p > \hat{p}$, then there is some $0 < a_1 < p$ with $(a-1)a_1 \notin W_0(p)$. If $0 < a_1 < a_2 < p$ with $a_2 + (a-1)a_1 > p$, then $p = ap > aa_2 = a_2 + (a-1)a_2 > a_2 + (a-1)a_1 > p$, a contradiction. Now it is clear that $\tilde{p} = \frac{\hat{p}}{p}$.

(ii) (b). Let α be a realization of p and let γ be a realization of $\hat{p}$. By (i) the element $\frac{\gamma}{\alpha}$ realizes $\tilde{p}$. Because p and $\tilde{p}$ are not equivalent with respect to $(M^{>0}, <, \cdot)$ the element $\gamma = \alpha{\cdot}\frac{\gamma}{\alpha}$ realizes the cut $p{\cdot}\tilde{p}$, hence $\hat{p} = p{\cdot}\tilde{p}$. The description is obvious now.

(ii) (c). Suppose $\hat{p}$ and $\tilde{p}$ are not equivalent with respect to $(M^{>0}, <, \cdot)$. Then $\frac{\hat{p}}{\tilde{p}}$ is a convex subgroup of $(M, <, +)$ and by (ii) we have $p = \frac{\hat{p}}{\tilde{p}}$, which contradicts our assumption $p > \hat{p}$. //

Proposition 8.5 (ii) tells us that in the case $p > \hat{p}$ the multiplicative invariance group of p can be recovered easily from the additive invariance group of p. Later on we need only

Corollary 8.6. *Let T be an o-minimal expansion of RCF and let p be a cut of a model M of T with $p > \hat{p}$. Then there is an M-definable map F with $F(p) = \hat{p}$ if and only if there is an M-definable map F with $F(p) = \tilde{p}$. For short:*

$$p > \hat{p} \;\Rightarrow\; (p \sim \hat{p} \Leftrightarrow p \sim \tilde{p})$$

Corollary 8.7. *Let M be a real closed field and let p, q be cuts of M with $\hat{q} = \hat{p} < p < q$.*
(i) $\tilde{q} \leq \tilde{p}$.
(ii) $\tilde{q} < \tilde{p} \Leftrightarrow$ *there is* $c \in M$ *with* $p < c\hat{p} < q$.

Proof. (i) follows immediately from Proposition 8.5 (ii) (a).
(ii) Let α, β and γ be realizations of p, q and $\hat{p}$ respectively. If $\tilde{q} < d < \tilde{p}$ for some $d \in M$, then $\frac{\gamma}{\beta} < d < \frac{\gamma}{\alpha}$, thus $c := \frac{1}{d}$ fulfills $p < c\hat{p} < q$.
Conversely, if $p < c\hat{p} < q$, we see in the same way, that $\frac{1}{c}$ lies between $\tilde{q}$ and $\tilde{p}$. //

§9 The valuation to a convex group

In this section we examine the invariance group G^* of a cut p of $(M^{>0}, \cdot)$ (M real closed) if p is the upper boundary of a convex subgroup G of $(M, +)$. The group G^* is the set of positive units of a convex valuation ring $A(G)$ of M. It is remarkable, that in the case $G \neq 0, M$ the convex valuation ring $A(G)$ is the only proper one which can be defined in the structure (M, G) (§18). Thus $A(G)$ turns out to be a good invariant for G, and finally for all cuts p, if we combine the constructions of §5,§8 and §9.

Definition 9.1. Let G be a convex subgroup of $(M, +)$. We define

$$A(G) = \{a \in M \mid a \cdot G \subseteq G\}$$

Certainly $A(G)$ is a convex valuation ring of M. We write $\Gamma(G)$, v_G and $\mathfrak{m}(G)$ for the corresponding value group, valuation and maximal ideal of $A(G)$ (obviously it is not possible to write $v(G)$ instead of v_G).

Proposition 9.2. (*Properties of* $A(G)$)
(i) $A(G)$ *is the largest subring* R *of* M*, such that* G *is an* R*-module. We have* $A(G) = M \Leftrightarrow G = 0$ *or* $G = M$.
(ii) $A(G)^* = \{a \in M \mid a \cdot G = G\}$, *if* $0 \neq G$.
(iii) $\mathfrak{m}(G) = \{a \in M \mid a \cdot G \subsetneqq G\} = G \cdot \frac{1}{G} = \{\frac{a}{b} \mid a \in G,\ b \notin G\}$, *if* $0 \neq G \neq M$.
(iv) $G = \mathfrak{m}(G) \Leftrightarrow 1 \notin G$ *and* $G = G^{\frac{1}{2}}$.
(v) $G = A(G) \Leftrightarrow 1 \in G$ *and* $G = G^{\frac{1}{2}}$.
(vi) $A(a \cdot G) = A(G)$ *for all* $a \in M^*$.

(vii) *If* $0 \le a \le b$, *then*

$$v_G(a) = v_G(b) \Leftrightarrow \text{there is no } c \in M \text{ such that } a \in c{\cdot}G \text{ and } b \notin c{\cdot}G$$

(viii) *If* $G \neq \sqrt{G}$ *and* $1 \notin G$, *then* $G^\infty \subsetneqq G \subsetneqq \sqrt{G}$ *and* $G^\infty \subsetneqq \sqrt{G}$ *is a direct specialization in the spectrum of* $A(G)$.

(ix) *If* L *is an ordered field extending* M *and* H *is a convex subgroup of* L *with* $M \cap H = G$, *then* $A(H) \cap M \subseteq A(G)$. $A(H)$ *lies over* $A(G)$ *if and only if for all* $a \in M$, $a > 0$ *with* $a{\cdot}G = G$ *we have* $a{\cdot}H = H$.

(x) $W_0^*(G^+) = A(G)^{*>0}$ *(=the positive units of* $A(G)$ *).*

Proof. From the definition. //

If $1 \notin G$, then $A(\sqrt{G})$ is the largest valuation ring B of M, such that G is contained in the maximal ideal of B. By (vi), this valuation ring can be much bigger than $A(G)$. The advantage of $A(G)$ towards $A(\sqrt{G})$ is the definability in the structure (M, G).

Proposition 9.3. *Let* M *be a real closed field and let* A *be a proper, convex valuation ring of* M *with maximal ideal* $\mathfrak{m}$ *and valuation* $v : M^* \longrightarrow \Gamma$. *Then the map*

$$\begin{aligned} S_1(\Gamma) \setminus \Gamma &\longrightarrow \{\text{fractional ideals of } A \text{ in } M\} \\ \xi \quad &\longmapsto \{a \in M \mid v(a) > \xi\} \end{aligned}$$

is one-to-one with inverse

$$G \mapsto \{v_G(a) \mid a \in G\}^-$$

If $\xi \in S_1(\Gamma)$ *is a cut with corresponding fractional ideal* G, *then the sequence*

$$0 \longrightarrow W_0(\xi) \longrightarrow \Gamma \longrightarrow \Gamma_G \longrightarrow 0$$

is exact. Especially $A = A(G)$ *if and only if* ξ *is dense or definable.*

Proof. The bijection is well known. To prove exactness of the sequence let $\xi \in S_1(\Gamma)$ be a cut with corresponding fractional ideal G. Then $A \subseteq A(G)$ and $vW_0^*(G^+) = W_0(\xi)$. From $\mathrm{Ker}(\Gamma \longrightarrow \Gamma_G) = A(G)^*/A^* = vW_0^*(G^+)$ we get the claim. //

Definition 9.4. If $A(G) \neq M$ we define ξ_G to be the cut of $\Gamma(G)$, which corresponds to the fractional ideal G of $A(G)$.

By Proposition 9.3, ξ_G is dense or definable. We have $\operatorname{sign}^* G^+ = -\operatorname{sign} \xi_G$, hence ξ_G is definable if and only if $\operatorname{sign}^* G^+ \neq 0$

$\operatorname{sign}^* G^+ = -1$ if and only if $G = b\cdot\mathfrak{m}(G)$ for some $b \in M$ if and only if $\xi_G = v_G(b)^+$.

$\operatorname{sign}^* G^+ = 1$ if and only if $G = b\cdot A(G)$ for some $b \in M$ if and only if $\xi_G = v_G(b)^-$.

Lemma 9.5. *Let α realize G^+ in a real closed field $N \supseteq M$ and let H be a convex subgroup of N with $H \cap M = G$, such that $A(H)$ lies over $A(G) \neq M$. Then*

$$\operatorname{sign}\xi_G \geq 0,\ \alpha \in H \ \Rightarrow\ v_H(\alpha) \notin v_H(M)$$

$$\operatorname{sign}\xi_G \leq 0,\ \alpha \notin H \ \Rightarrow\ v_H(\alpha) \notin v_H(M)$$

Proof. If $a \in M$, $a > 0$ with $v_H(a) = v_H(\alpha)$, then $\frac{a}{\alpha}\cdot H = H$. If $\alpha \in H$ we get $a \in G$ and $\{v_G(a) \mid a \in G\}$ has a least element. If $\alpha \notin H$ we get $a \notin G$ and $\{v_G(a) \mid a > G\}$ has a largest element. //

Definition 9.6. Let M be a real closed field and let p be a cut of M. We define:

$$A_p := A(W_0(p))$$

$$\Gamma_p := \Gamma(W_0(p))$$

$$\mathfrak{m}_p := \mathfrak{m}(W_0(p))$$

$$v_p := v_{W_0(p)}$$

$$\xi_p := \xi_{W_0(p)}$$

For the following reason the definition makes sense not only for cuts p with $p = \hat{p}$: We'll show that the application of a semi algebraic function to an arbitrary cut p does not change the ring A_p (§11). I want to say as well, that p and $\hat{p}$ are not always linked by a semi algebraic function.

Proposition 9.7. *(Movements of $\hat{p}$)*
Let M be real closed and let p be a cut of M with $A_p \neq M$ (this means p is neither definable nor dense). Then:

(i) *If $\mathfrak{p}$ is a prime ideal different from $\mathfrak{m}_p$ in A_p and $b \in M$ such that $b > \mathfrak{p}$, then there is some $c \in M$ such that*

$$\mathfrak{p}^+ < c\cdot\hat{p} < b$$

(ii) *ξ_p is definable if and only if there is some $c \in M$ such that $\mathfrak{m}_p \subseteq c\cdot W_0(p) \subseteq A_p$.*

(iii) *If A is a convex valuation ring of M, then*

$$A_p \subsetneqq A \Leftrightarrow \textit{there is some } c \in M \textit{ with } \mathfrak{m}_A \subsetneqq c\cdot W_0(p) \subsetneqq A$$

Proof. We write G for the invariance group of p.

(i) Since $\mathfrak{p} \subsetneqq \mathfrak{m}_p$ we may suppose that $v_p(b) > 0$. Because $\mathfrak{p}$ is prime, we have $\mathfrak{p} < b^2 < a$. Now Proposition 9.2 (vii) gives the claim.

(ii) If ξ_p is definable, then $G = c{\cdot}\mathfrak{m}_p$ or $G = c{\cdot}A(G)$ with some $c \in M$. Conversely, if there is some $c \in M$ with $\mathfrak{m}_p \subseteq c{\cdot}G \subseteq A(G)$, then the right options of ξ_G have infimum $v_G(c)$; hence ξ_G is definable.

(iii) $\Leftarrow$. Take $\mathfrak{m}_A < a \in c{\cdot}W_0(p) < b \in A$. By Proposition 9.2 (vii) we have $v_p(a) > v_p(b)$. Since a, b are units of A, the valuation ring A_p can not contain A.

"$\Rightarrow$" follows from (i) for $\mathfrak{p} = \mathfrak{m}_A$ and $b = 1$. //

The next Proposition collects information about the location of $W_0^*(p+c)$.

Proposition 9.8. (*Location of* $W_0^*(p+c)$)
Let M be a real closed field and let p be a non definable cut of M. Then:

(i) *If $c_0 \in M$ with $c_0 > \hat{p}$, then*

$$\bigcup_{\substack{c\in M\\ c_0>c+p>\hat{p}}} W_0^*(p+c) = 1 + \mathfrak{m}_p$$

(ii) *If $\hat{p} = b{\cdot}\mathfrak{m}_p$ with $b \in M,\ b > 0$, then there is some $c \in M$ with $\hat{p} < p+c < b{\cdot}A_p^+$; for all these c's we have $W_0^*(p+c) = 1 + \mathfrak{m}_p$.*

(iii) *If $\operatorname{sign}\xi_p < 1$, then $W_0^*(p+c) \neq 1 + \mathfrak{m}_p$ for all $c \in M$.*

(iv) $|p| \leq A_p^+ \Leftrightarrow 1 + W_0(p) \subseteq W_0^*(p)$

Proof. We can assume that p is neither definable nor dense.

(i) $\subseteq$. Let $c + p \neq \hat{p}$. We can assume that $p = c + p > 0$. Suppose there is some $a \in W_0^*(p)$ with $a > 1 + \mathfrak{m}_p$, that is $\frac{1}{(a-1)}{\cdot}W_0(p) \subseteq W_0(p)$. For $p \neq \hat{p}$ there is some $m \in M$ with $0 < m < p < 2m$. From $m \notin W_0(p)$ we get $(a-1)m \notin W_0(p)$ and there is some $b \in M$ with $m < b < p < b+(a-1)m$. But $b+(a-1)m < b+(a-1)b = ab < p$, since $a{\cdot}p = p$: contradiction.

(i) $\supseteq$. Let $a \in \mathfrak{m}_p$, $a > 0$. For $\frac{1}{a} \notin A_p$ there is some $b \in W_0(p)$, $b > 0$ with $\frac{b}{a} \notin W_0(p)$. We can assume $\frac{2b}{a} < c_0$. Let $d < p$ with $d + \frac{b}{a} > p$. We show $1 + a \in W_0^*(p+c)$ and $c_0 > p + c > \hat{p}$ for $c = \frac{b}{a} - d$:

$(1+a){\cdot}(p+c) = p+c$: let $m \in M$ with $d < m < p$. We have to show $(1+a)(m+c) < p+c$: $(1+a)(m+c) = m+c+a(m+\frac{b}{a}-d) = m+c+b+am-ad < m+c+b+b$, since $ap < b+ad$. From $b \in W_0(p)$ we get $m + 2b < p$, hence $m + 2b + c < p + c$.

$p + c > \hat{p}$ holds, because $\hat{p} < \frac{b}{a} < p - d + \frac{b}{a} = p + c$

$p+c<c_0$ holds, because $p-d<\frac{b}{a}$, hence $p+c=p-d+\frac{b}{a}<\frac{2b}{a}<c_0$.

(ii) By (i) there is even some $c\in M$ with $\hat{p}<p+c<b$. If $\hat{p}<p+c<b\cdot A_p^+$ we write $q:=\frac{1}{b}(p+c)$ and get $1+\mathfrak{m}_p=1+W_0(q)\subseteq W_0^*(q)$, since $q<a$ for some $a\in A_p=A_q$. From $W_0^*(q)=W_0^*(p+c)$ we get the claim.

(iii) Let $\hat{p}<b<p$ with $b\in M$. It is enough to prove $1+\mathfrak{m}_p\not\subseteq W_0^*(p)$. If $b{\cdot}\mathfrak{m}_p\subseteq W_0(p)$, then $v_p(b)$ is the infimum but not the minimum of $\{v(a)\mid a\in W_0(p)\}$. That is $\operatorname{sign}\xi_{\hat{p}}=1$. Thus, by assumption, $b{\cdot}\mathfrak{m}_p\not\subseteq W_0(p)$.
Let $a\in\mathfrak{m}_p$ with $a\cdot b>\hat{p}$. We have $1+a\notin W_0^*(p)$, since for some $c\in M$ with $b<c<p$: $c+ab>p$. Finally $(1+a)c=c+ac>c+ab>p$.

(iv) if $0<a<p<A_p^+$, $b\in W_0(p)$, $b>0$ then $(1+b)a=a+ba<p$, since $ba\in W_0(p)$. Conversely if $A_p<a<p$, $b\in W_0(p)$, $b>0$ with $ab>W_0(p)$, then there is some $a<c<p$ with $c+ab>p$. Thus $(1+b)c=c+cb>c+ab>p$ and $(1+b)\cdot p>p$. //

We translate some results from §5:

Proposition 9.9. *Let T be an o-minimal expansion of RCF, let p be a cut of a model M of T and $N\succ M$. Then*

(i) If $\operatorname{sign}p\neq 0$ and q is an heir of p on N, then A_q lies over A_p.

(ii) If q is a coheir of p on N, then A_q is a coheir of A_p, especially A_q lies over A_p.

(iii) If $\operatorname{sign}^* p=0$ and α is a realization of p from N, then

$$q'=\alpha^2\cdot\frac{1}{q}$$

if q and q' denotes the coheirs of p on N.

(iv) Let $p=\hat{p}$ be non definable.
(a) If ξ_p is definable and q is an heir of p on N, then ξ_q extends ξ_p and $\operatorname{sign}\xi_q=\operatorname{sign}\xi_p$.
(b) If ξ_p is dense and q,q' are the coheirs of p on N then $A_q=A_{q'}$. If α is a realization of p in N, then ξ_q and $\xi_{q'}$ are definable, more precisely: $\operatorname{sign}\xi_q=1$ if and only if q is the least extension of p on N.

Proof. (i) is Corollary 5.6 we know that $\hat{q}$ is an heir of $\hat{p}$. In particular $W_0^*(p)\subseteq W_0^*(q)$.

(ii) is Corollary 5.13 applied to p and then to $\hat{p}$ and $T^{>0}$.

(iii) is Proposition 5.12 (v) applied to $T^{>0}$.

(iv) is again Proposition 5.12 applied to $T^{>0}$. //

Obtaining cuts with $\operatorname{sign} p = \operatorname{sign}^* \hat{p} = 0$

Proposition 9.10. *Let T be an o-minimal expansion of RCF.*

Let $M \models T$ and let p be a cut of M. Then there is an elementary extension N of M and an extension q of p on N such that $\hat{q}$ extends $\hat{p}$, A_q is the convex hull of A_p, $\operatorname{sign} q = 0$ and $\operatorname{sign}^* \hat{q} = 0$.

Proof. We construct a chain $M_0 \prec M_1 \prec M_2 \prec ...$ of models of T, together with cuts p_i of M_i, such that:
(a) $p_0 \subseteq p_1 \subseteq p_2 \subseteq ...$, $\hat{p}_0 \subseteq \hat{p}_1 \subseteq \hat{p}_2 \subseteq ...$ and A_{p_i} lies over A_p $(i < \omega)$.
(b) $\operatorname{sign} p_{2i+1} = 0$ for all i .
(c) $\operatorname{sign}^* \hat{p}_{2i+2} = 0$ and $A^+_{p_{2i+2}}$ is the least extension of A_p on M_{2i+2} for all i

Take $M_0 := M$ and $p_0 := p$. Suppose $M_0, ..., M_i$ and $p_0, ..., p_i$ are constructed, and (a)-(c) holds up to i. We define M_{i+1} and p_{i+1} in the following way:

Case 1 i is even.
By Corollary 5.21 there is some $M_{i+1} \succ M_i$ and a cut p_{i+1} extending p_i such that $\operatorname{sign} p_{i+1} = 0$ and $\hat{p}_{i+1}$ is the least extension of $\hat{p}_i$. By Proposition 9.9 (ii), $A_{p_{i+1}}$ lies over A_{p_i}, hence over A_p.

Case 2 i is odd.
Let α be a realization of p_i and let r be the least extension of $\hat{p}_i$ on $M\langle\alpha\rangle$. Then $\alpha + r$ is an extension of p_i and A_r lies over A_{p_i} (Proposition 9.9 (ii)), hence over A_p. From Proposition 5.20 applied to $T^{>0}$, r and the convex hull of A_p in $M_i\langle\alpha\rangle$ we get some $M_{i+1} \succ M_i\langle\alpha\rangle$ and an extension r_1 of r on M_{i+1} such that $\operatorname{sign}^* r_1 = 0$ and $1 + \tilde{r_1}$ is the least extension of A^+_p $(1 + \tilde{r_1}$ is the type $\hat{r}_1$ with respect to the theory $T^{>0})$. Especially r_1 is the upper boundary of a convex subgroup of $(M_{i+1}, +)$ and $A^+_{r_1}$ is the least extension of A^+_p on M_{i+1}. Now we take $p_{i+1} = \alpha + r_1$

Thus the construction is completed. Let $N = \bigcup M_i$ and $q = \bigcup p_i$. By Proposition 5.19 (iii) and (a) we have $\hat{q} = \bigcup \hat{p}_i$. By Proposition 5.19 (iii) applied to $T^{>0}$ and (c) we have $A_q = \bigcup A_{p_i}$ (note that $A_{p_{2i+2}}$ lies over $A_{p_{2i}}$). By Proposition 5.19 (iv) and (b) we get $\operatorname{sign} q = 0$. By Proposition 5.19 (iv) applied to $T^{>0}$ and (c) we get $\operatorname{sign}^* \hat{q} = 0$. //

Remark
Now it is not difficult to see that there are cuts $p \subseteq q$, such that q is an heir of p and $\hat{q}$ is not an heir of $\hat{p}$ and such that the invariance group of p can be arbitrary large.

We give an example to illustrate our definitions for real closed field:

Let M_0 be a real closed field and $\Gamma \models DOAG$. We work in the real closed field $M_0((x^\Gamma))$ of generalized power series. Let A be the subring $M_0[x^\gamma \mid \gamma \in \Gamma]$ of $M_0((x^\Gamma))$.

Let λ be an ordinal number and for each $i < \lambda$ let $a_i \in M_0,\ a_i \neq 0,\ \gamma_i \in \Gamma$, such that the map $i \mapsto \gamma_i$ preserves the ordering. We define for the element $\alpha = \sum_{i<\lambda} a_i x^{\gamma_i}$ in the case $\lambda \geq \omega$:

$$\begin{aligned} \rho\alpha = &\ \text{the cut of } \Gamma \text{ with left options} \\ &(\rho\alpha)^L = \{\gamma \in \Gamma \mid \text{there is some } i < \omega \text{ with } \gamma < \gamma_i\} \end{aligned}$$

Thus, $\rho\alpha$ is a 1-type over Γ, which is not realized. If $\lambda < \omega$, we define $\rho\alpha = +\infty$ ($\in S_1(\Gamma)$). Let

$$B := \{\alpha \in M_0((x^\Gamma)) \mid \rho\alpha = +\infty\}$$

(1) We have for $\alpha \in M_0((x^\Gamma))$ with infinite support and arbitrary $\beta \in M_0((x^\Gamma))$:

$$\begin{aligned} \operatorname{ord}\beta > \rho\alpha &\Leftrightarrow \alpha \text{ and } \alpha+\beta \text{ generate the same cut of } A \\ &\Leftrightarrow \alpha \notin B,\ \alpha \text{ and } \alpha+\beta \text{ generate the same cut of } B \end{aligned}$$

Proof. Let $\operatorname{ord}\beta > \rho\alpha$. Let $I := \rho\alpha^L \cap \operatorname{supp}\alpha$ and let $\gamma_0 < \gamma_1 < \dots$ be an enumeration of I. We can write

$$\alpha = \sum_{i<\omega} a_i x^{\gamma_i} + \alpha_1$$

with $\operatorname{ord}\alpha_1 > \rho\alpha$ and $a_i \in M_0,\ a_i \neq 0$. If $d \in M_0((x^\Gamma))$ with $\alpha - \beta \leq d \leq \alpha + \beta$, then $d_\gamma = a_\gamma$ for all $\gamma \in \rho\alpha^L$ (since $\operatorname{ord}\beta > \rho\alpha$). It follows $\rho d < \infty$, hence $d \notin B$.

Conversely suppose $\operatorname{ord}\beta < \rho\alpha < +\infty$. Then there is some $\xi \in \Gamma$ with $\operatorname{ord}\alpha, \operatorname{ord}\beta < \xi < \rho\alpha$. The set $(-\infty, \xi) \cap \operatorname{supp}\alpha$ is non empty and finite, hence $d := \sum_{\substack{\gamma \in \operatorname{supp}\alpha \\ \gamma < \xi}} a_\gamma x^\gamma \in A$. If γ denotes $\operatorname{ord}\beta$, then $d + \frac{\beta_\gamma}{2} x^\gamma \in A$ is properly between α and $\alpha + \beta$. //

(2) B is a subring of $M_0((x^\Gamma))$. If $H(T_1, ..., T_n) \in M_0[[T_1, ..., T_n]]$ is a power series and $\alpha_1, ..., \alpha_n \in B$ with $\operatorname{ord}\alpha_i > 0$ then $H(\alpha_1, ..., \alpha_n) \in B$.

Proof.
If $T, S \subseteq \Gamma$ are of order type ω and cofinal in Γ, then $S + T$ is of order type ω and cofinal in Γ too. Therefore B is a subring of $M_0((x^\Gamma))$.

Remark

If $\Gamma = \mathbb{Q}$, $S = \mathbb{N}$ and $T = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\}$, then $S + T$ is of order type $\omega{\cdot}\omega$.

If $S \subseteq \Gamma^{>0}$ is of order type ω and cofinal, and $m{\cdot}S := \{s_1 + ... + s_m \mid s_i \in S\}$ $(m \in \mathbb{N})$, then

$\bigcup\limits_{m=1}^{\infty} m{\cdot}S$ is of order type ω and cofinal, since $m{\cdot}\min S \to \infty$ $(m \to \infty)$

Let $H(T_1, ..., T_n) \in M_0[[T_1, ..., T_n]]$ and $\alpha_1, ..., \alpha_n \in B$ with $S_i := \operatorname{supp} \alpha_i > 0$. Then it makes sense to build $H(\alpha_1, ..., \alpha_n) \in M_0((x^{\mathbb{Q}}))$ and

$$\operatorname{supp} H(\alpha_1, ..., \alpha_n) \subseteq \{0\} \cup \bigcup_{1 \le i_1 < ... < i_k \le n} \Big(\bigcup_{m=1}^{\infty} m{\cdot}S_{i_1} + \ldots + \bigcup_{m=1}^{\infty} m{\cdot}S_{i_k}\Big)$$

is of order type ω and cofinal, hence $H(\alpha_1, ..., \alpha_n) \in B$. //

(3) Γ is archimedean if and only if the field of quotients of A is contained in B. In this case B is the unique, maximal dense real closed subfield in $M_0((x^\Gamma))$, which contains A, as constructed in §4. Furthermore if $A \subseteq N \subseteq B$ is a real closed field $\alpha \in M_0((x^\Gamma)) \setminus N$ and $p := t(\alpha/N)$ then

$W_0(p) = W(p) = 0$, hence p is dense iff $\alpha \in B$. More general:

$$W(p) = \{\varepsilon \in N\langle\alpha\rangle \mid \operatorname{ord} \varepsilon > \rho\alpha\}$$

$W(p)$ is the convex hull of $\{x^\gamma \mid \gamma > \rho\alpha\} \subseteq A$. Especially p has signature 0.

Proof. if Γ is archimedean, then the real closure M of A in $M_0((x^\Gamma))$ is the union of all $M_0((x^{\Gamma_0}))$, where Γ_0 runs through all finitely generated subgroups of Γ. Thus M is contained in B. Conversely if $\gamma \in \Gamma$, $\gamma > 0$ such that $n\gamma$ is not cofinal in Γ, then $\frac{1}{1-x^\gamma} \notin B$.

Suppose now, Γ is archimedean. In order to prove, that B is a real closed field, it is enough to show that $N\langle\alpha\rangle \subseteq B$ for all real closed fields N with $M \subseteq N \subseteq B$ and all $\alpha \in B$: By (1) we know that $t(\alpha/N)$ is dense. Therefore, if $\beta \in N\langle\alpha\rangle$ then the type $t(\beta/N)$ is dense too, hence $\beta \in B$ (by (1)). By (1) we know that B is the unique, maximal dense real closed subfield of $M_0((x^\Gamma))$, which contains A.

§10 T-convex rings

Lou van den Dries and Adam Lewenberg generalize in the paper [vdD-Lew],the Cherlin-Dickmann result on model completeness of a real closed field with a convex valuation:

If T is an o-minimal expansion of RCF and $M_0 \prec M$ are models of T, then (M, A) is model complete if A is the convex hull of M_0 in M.

The proof consists of two main steps:

1. Give an axiomatization of 'convex hull of an elementary substructure of M' in the language $\mathscr{L}$ augmented by a new unary predicate. This leads to the notion 'T-convex valuation ring'.

2. Prove for a proper T-convex valuation ring A of a model M of T and a realization α of A^+: The only T-convex valuation rings B of $M\langle\alpha\rangle$ lying over A are 'the coheirs' of A^+ on $M\langle\alpha\rangle$ (This is an explicit statement of [vdD-Lew], namely the Main Lemma there).

By Corollary 2.8 the quoted theorem follows from 1. and 2.
The problem for a possible generalization to a reasonable definable expansion of the structure (M, A) for an arbitrary convex valuation ring is, that we have no replacement for 1., which allows us to prove 2.

Summary of T-convexity from the paper "T-convexity and tame extensions" [vdD-Lew]

In this paragraph T always denotes an o-minimal expansion of RCF.

Definition 10.1. Let M be a model of T. A convex valuation ring A of M is called T-convex, if A is closed under all 0-definable, continuous maps $M \longrightarrow M$.

If $A \subseteq M$ is T-convex, then $\mathrm{cl}\,\emptyset \subseteq A$. If $M_0 \prec M$ and if A is the convex hull of M_0 in M, then A is T-convex.

Proposition 10.2. *For all convex valuation rings $A \subseteq M \models T$ the following conditions are equivalent:*

(i) A is T-convex.

(ii) If $M_0 \prec M$, $M_0 \subseteq A \subseteq M$ and $F : M^n \longrightarrow M$ is a continuous, M_0-definable map, then $F(A^n) \subseteq A$.

(iii) If $F : M \longrightarrow M$ is continuous and 0-definable with $F(-x) = F(x)$, F strictly increasing on $[0, +\infty)$ and $\lim_{x\to+\infty} F(x) = +\infty$, then $F(A) \subseteq A$.

Lemma 10.3. *Let $M \prec N$ be models of T and let A be a T-convex subring of N, which contains M. If $\alpha \in A$ and $\alpha > M$, then $M\langle\alpha\rangle \subseteq A$.*

Theorem 10.4. *Let $A \subseteq M \models T$ be T-convex and $M_0 \prec M$ with $M_0 \subseteq A$. Then:*

M_0 is maximal definable closed in A if and only if the map $M_0 \longrightarrow A \longrightarrow \kappa_A := A/\mathfrak{m}_A$ is onto.

In this case we have:

(i) A is the convex hull of M_0 in N and M_0 is tame in N.

(ii) If $M_0' \subseteq A$ is maximal definable closed, then there is a unique $\mathscr{L}$-isomorphism $M_0 \longrightarrow M_0'$, which commutes with the residue maps.

Thus $M_0 \longrightarrow \kappa_A$ induces an $\mathscr{L}$-structure on κ_A, which is independent of the choice of M_0. κ_A is again a model of T. We always consider κ_A to be this expansion of the field κ_A.

Lemma 10.5. *Let $A \subseteq M \models T$ be T-convex and $M \prec N$. If $B \subseteq N$ is a T-convex ring of N lying over A, then the canonical morphism $\kappa_A \longrightarrow \kappa_B$ is an elementary map of models of T and we have $\dim \kappa_B/\kappa_A \leq \dim N/M$.*

Definition 10.6. Let $\mathcal{O}$ be a new unary predicate and T_{convex} be the $\mathscr{L}(\mathcal{O})$-theory which says that $\mathcal{O}$ is a proper T-convex valuation ring.

Proposition 10.7.

(i) If T is universally axiomatized, then each substructure (M_0, A_0) of a model of T_{convex} has a unique prime model.

(ii) If (M, A) is a model of T_{convex} and N is an elementary extension of M, then the heirs of A^+ on N are exactly the cuts B^+ for some T-convex valuation ring B of N lying over A.

Theorem 10.8. *T_{convex} is complete and weakly o-minimal. If T is model complete, then T_{convex} is model complete, too. If T has quantifier elimination and is universally axiomatized, then T_{convex} has quantifier elimination.*

Additions

Let M be an o-minimal expansion of RCF.

Lemma 10.9. *If $X \subseteq M$ is convex, $0 \in X$ and $M_0 \prec M$ is maximal definably closed with $M_0 \cap X = 0$, then M_0 is tame in M. Such an M_0 exists iff X is infinitesimal over* $\operatorname{cl}\emptyset$.

Proof. If $\alpha \in M \setminus M_0$, then $M_0\langle\alpha\rangle$ is an elementary restriction of M and there is an M_0-definable map F with $F(\alpha) \in X, \neq 0$. Since X is convex and $0 \in X$, $F(\alpha)$ is infinitesimal over M_0, hence α is definable over M_0. //

Definition 10.10. Let G be a convex subgroup of $(M,+)$, such that G is infinitesimal over $\operatorname{cl}\emptyset$. We define

$$B^T(G) = \text{the convex hull of all } M_0 \prec M \text{ with } G \cap M_0 = 0$$

We have $G \neq \{0\} \;\Rightarrow\; B^T(G) \neq M$, because for each $g \in G, \neq 0$ the element $\frac{1}{g}$ is not in $B^T(G)$. $B^T(G)$ is the union of the T-convex rings $\operatorname{co}_M M_0$, where $M_0 \cap G = 0$, hence $B^T(G)$ is T-convex, too. $B^T(G)$ is the largest T-convex ring of M with $G \subseteq \mathfrak{m}_{B^T(G)}$. If $M_0 \prec M$ is maximal with $M_0 \cap G = 0$, then M_0 is maximal in $B^T(G)$ and vice versa.

Obviously $A(G) \subsetneqq B^T(G)$ and $A(G) \supsetneqq B^T(G)$ are possible in general.

If $T = RCF$ and G is a convex subgroup of M with $1 \notin G$, then

$$\mathfrak{m}_{B^T(G)} = \sqrt{G}$$

and

$$B^T(G) = A(\sqrt{G}) = \{a \in M \mid a \cdot \sqrt{G} \subseteq \sqrt{G}\}$$

If M is a model of T and G is a convex subgroup of $(M,+)$ with $G \cap \operatorname{cl}\emptyset = 0$ we define:

$$\sqrt[T]{G} := \{F(g) \mid g \in G,\; F : M \to M \text{ 0-definable, continuous with } F(0) = 0\}$$

Lemma 10.11. *If $G \cap \operatorname{cl}\emptyset = 0$, then*

$$\sqrt[T]{G} = \mathfrak{m}_{B^T(G)}$$

We have $G = \sqrt[T]{G}$ if and only if for all $a \in M$ the following condition holds: if a is infinitesimal over $\operatorname{cl}\emptyset$ and $a \notin G$, then $G \cap \operatorname{cl} a = 0$.

Proof. Suppose the condition in the addition holds for G. Then we have $G = \mathfrak{m}_{B^T(G)}$, since each $a \in \mathfrak{m}_{B^T(G)}$ is infinitesimal over $\operatorname{cl}\emptyset$. For $G \cap \operatorname{cl} a \neq 0$ the element a is in G.

Certainly $\sqrt[T]{G}$ is a convex subgroup of $(M,+)$, which contains G. By the mean value theorem $\sqrt[T]{G}$ is convex. If $M_0 \prec M$ with $M_0 \cap G = 0$, then $M_0 \cap \sqrt[T]{G} = 0$, too. Therefore we have $B^T(\sqrt[T]{G}) = B^T(G)$. It follows $\sqrt[T]{G} \subseteq \mathfrak{m}_{B^T(\sqrt[T]{G})} = \mathfrak{m}_{B^T(G)}$. Let $a \in M,\ a > 0$ be infinitesimal over $\operatorname{cl}\emptyset$ and $\operatorname{cl} a \cap \sqrt[T]{G} \neq 0$. We prove that $a \in \sqrt[T]{G}$. Let $F : M \longrightarrow M$ be 0-definable, $g \in G$ and $G : M \longrightarrow M$ be 0-definable and continuous such that $G(0) = 0$ and $F(a) = G(g)$. We may suppose that $g, G(g) > 0$. The elements a, g and $G(g)$ are positive and infinitesimal over $\operatorname{cl}\emptyset$. Therefore

$$\lim_{\substack{x \to 0 \\ x \in \operatorname{cl}\emptyset, x > 0}} F(x) = 0$$

There is some $b \in \operatorname{cl}\emptyset,\ > 0$, such that $F|_{(0,b]} : (0,b] \longrightarrow (0, F(b)]$ is an increasing homeomorphism. The map

$$\begin{aligned} F_1 : M &\longrightarrow M \\ x &\longmapsto \begin{cases} x & \text{if } x \leq 0 \\ F(x) & \text{if } x \in (0,b] \\ F(b) + (x-b) & \text{if } x > b \end{cases} \end{aligned}$$

is 0-definable, continuous and bijective with $F_1(a) = F(a)$. F_1^{-1} is continuous too and we have $F_1^{-1}(0) = 0$. Thus $a = F_1^{-1}(G(g)) \in \sqrt[T]{G}$. This shows $\sqrt[T]{G} = \mathfrak{m}_{B^T(G)}$.

Finally suppose $G = \mathfrak{m}_{B^T(G)}$. If $a \in M \backslash G$ is infinitesimal over $\operatorname{cl}\emptyset$, then $\frac{1}{a} \in B^T(G)$ is infinite over $\operatorname{cl}\emptyset$. By Lemma 10.3 $\operatorname{cl} a \subseteq A$, hence $G \cap \operatorname{cl} a = 0$. //

Lemma 10.12. *Let $M \prec N$ be models of T, let G be a convex subgroup of $(M,+)$ with $G \cap \operatorname{cl}\emptyset = 0$ and let H be the convex hull of G in N. Then $\sqrt[T]{H}$ is the convex hull of $\sqrt[T]{G}$ in N.*

Proof. Let $0 < h < g \in G$, $h \in H$ and $F : N \longrightarrow N$ be 0-definable and continuous with $F(0) = 0$. We have to show that $F(h)$ is in the convex hull of $\sqrt[T]{G}$: Since $F|_{[0,g]} : [0,g] \longrightarrow M$ is M-definable, F has a maximum and a minimum on $[0,g]$. Let $a, b \in [0,g] \subseteq M$ with $F(a) \leq F([0,g]) \leq F(b)$. We have $a, b \in G$, hence $F(a), F(b) \in \sqrt[T]{G}$ and, since $F(a) \leq F(h) \leq F(b)$, $F(h)$ is in the convex hull of $\sqrt[T]{G}$ in N. //

Proposition 10.13. *Let $M \prec N$ be models of T and let A be a convex valuation ring of M.*

(i) If B^+ is an heir of A^+ and A is T-convex then B is T-convex too.
(ii) If p is a cut of M with $A = A_p$ and A is T-convex, then A_q is T-convex for each coheir q of p on N.

Proof. (i) if A is T-convex and $F : N \longrightarrow N$ is 0-definable with $F(b) > B$ for some $b \in B$, then $b < x \wedge F(b) > x \in B^+$. Since B^+ is an heir of A^+ there is some $a \in M$ with $a \in A$ and $F(a) > A$.

(ii) if q is a coheir of p, then A_q is a coheir of A_p (Proposition 9.9). Thus (ii) is implied by (i) if $A \neq M$. If $A = M$, then B is the convex hull of M in N, which is T-convex. //

Polynomially bounded theories

Definition 10.14. Let T be an o-minimal expansion of RCF. The theory T is called polynomially bounded if the following two conditions hold:

(i) If M is a model of T and $F : M \longrightarrow M$ is an M-definable map, then there is some $a \in M$ and some $n \in \mathbb{N}$ such that $|F(x)| < x^n$ for all $x > a$.
(ii) There is an archimedean model of T.

In [vdD-Lew] the authors define "polynomially bounded" without condition (ii). In our situation, all convex valuation rings appear, so we have to restrict to the condition (ii) if we want to have

Proposition 10.15. *The following conditions are equivalent:*

(i) T is polynomially bounded.
(ii) All convex valuation rings of all models of T are T-convex.
(iii) There is a model of T_{convex} with value group isomorphic to $\mathbb{R}$.

The proof is again in [vdD-Lew] //

Remark on the value group
If T is an o-minimal expansion of RCF and A is a proper T-convex valuation ring of a model of T with value group Γ, then M induces a weak o-minimal structure on Γ. With this structure, Γ is o-minimal if and only if T is polynomially bounded. (All this is from [vdD-Lew]).

We never use that Γ is o-minimal if T is polynomially bounded.

Furthermore we do not really use a result of Chris Miller [Mi], which I want to repeat now.

Theorem 10.16. *(Growth Dichotomy)*
If $\mathcal{R}$ is an o-minimal expansion of the real closed field $\mathbb{R}$, then $\mathcal{R}$ is polynomially bounded or $\mathcal{R}$ defines the exponential $\exp(x)$. //

§11 Rigidity of convex valuation rings

We prove that the valuation ring A_p of a cut p can not be changed by an M-definable map F to $A_{F(p)}$, if M is a polynomially bounded, o-minimal expansion of RCF. We'll use this, to prove that even A_p differs not from A_q for any cut q, which is definable in (M, d^p) (Theorem 18.2). Obviously $\exp(A) \neq A$ in general, in real exponential fields. In spite of this, the valuation ring A_p is a bit located in arbitrary o-minimal expansions of RCF.

Definition 11.1. Let T be an o-minimal expansion of $DOAG$. Let $M \prec N$ be models of T and $X \subseteq M^k$. We define the hull $H(X)$ of X in N to be

$$H(X) := \{\bar{b} \in N^k \mid \text{there is some } \bar{a} \in X \text{ such that } b_i - a_i \text{ is infinitesimal over } M \ (1 \leq i \leq k)\}$$

Theorem 11.2. (*David Marker*)
Let $M \prec N$ be a tame extension of models of T and let $F : N^k \longrightarrow N$ be an N-definable map. Then there is a partition $M^k = D \,\dot\cup\, C_1 \,\dot\cup ... \dot\cup\, C_n$ of M^k in M-definable cells $D, C_1, ..., C_n$, with

$\dim D < k$ *and open C_i, such that for all $i \in \{1, ..., n\}$ exactly one of the following conditions hold:*

(i) F is positive infinite over M on $H(C_i)$.
(ii) F is negative infinite over M on $H(C_i)$.
(iii) there is some M-definable map $G : M^k \longrightarrow M$ such that $F(\bar{b}) - G(\bar{b})$ is infinitesimal over M for all $\bar{b} \in H(C_i)$.

Proof. [Ma]. //

We need only the information of the behavior of F on the C_i in this theorem (not on all of $H(C_i)$).

Proposition 11.3. *Let $M \prec N$ be a tame extension of models of T, $p \in S_k(M)$ and $p' \in S_1(M)$. Let q, q' be extensions of p, p' on N respectively. Assume:*
(a) q is a coheir of p and $\dim p = k$.
(b) p' is not definable.

If there is an N-definable map $F : N^k \longrightarrow N$ with $F(q) = q'$, then there is an M-definable map $G : M \longrightarrow M$ such that $G(p) = p'$.

Proof. We write $\mathfrak{m}$ for the group of infinitesimal elements of N over M. We write again $D, C_1, ..., C_n$ for the elements of a partition of M^k as in Theorem 11.2. For $\dim p = k$ we have $p \in \langle C_i \rangle$ for some i. We write $C = C_i$. Since q is a coheir of p, there is a net $(\bar{a}_i)$ in M^k with $\lim_i \bar{a}_i = q$ (in $S_k(N)$). $q' = F(q)$ is M-bounded, hence $F(\bar{a}_i)$ is M-bounded finally and it follows that $F|_C$ is M-bounded by the choice of the C_i in Theorem 11.2. By Theorem 11.2 there is an M-definable map $G : M^k \longrightarrow M$ such that $F(\bar{b}) - G(\bar{b}) \in \mathfrak{m}$ for all $\bar{b} \in C$. We show that $G(p) = p'$. Assume $G(p) \neq p'$, say $G(p) < p'$. There is an M-definable cell $Z \subseteq C$ and some $c \in M$, $c < p'$ with $p \in Z$ and $G(Z) \leq c$. Because p' is not definable, we can assume $G(Z) < c$. By Proposition 5.17 (i) we have $\mathfrak{m} \subseteq W_0(q')$. Since $G(\bar{a}) - F(\bar{a}) \in \mathfrak{m}$ for all $\bar{a} \in Z$, we have $F(\bar{a}_i) < c$ finally. Thus $F(q) \leq c < q'$, contradiction. //

If p' is definable, an analogou of this Proposition is of less sense, because all the values of F can be inside some fixed $a + \mathfrak{m}$ with $a \in M$.

Corollary 11.4. *Let $M \prec N$ be tame and $p, p' \in S_1(M)$. Let q, q' be coheirs of p, p' on N respectively. If $q \sim q'$ then p is definable iff p' is definable.* //

Theorem 11.5. *(Rigidity of convex valuation rings)*

Let T be an o-minimal expansion of RCF and let p and q be cuts of a model M of T.

If $q \sim p$ and A_q is T-convex then $A_q \subseteq A_p$

Proof. Case 1 $q \sim p$, $\operatorname{sign} \xi_q = 1$, A_q T-convex $\Rightarrow A_q \subseteq A_p$.
Preliminary remark: If $F : M \longrightarrow M$ is M-definable with $F(q) = p$, then p is definable iff $F(p)$ is definable and p is dense iff $F(p)$ is dense. Only in these cases $A_p = M$ or $A_q = M$. We suppose that $A_p, A_q \subsetneq M$. For $\operatorname{sign} \xi_q = 1$ we have $W_0(q) = c{\cdot}\mathfrak{m}_q$ with some $c \in M$ and we may suppose $W_0(q) = \mathfrak{m}_q$. Now $0 < q + d < 1$ for some $d \in M$ and we may assume that $0 < q < 1$ in addition.

Proof of Case 1:
Suppose $A_p \subsetneqq A_q$. From Proposition 9.7 (iii) we get some $c \in M$ with $\mathfrak{m}_q \subsetneqq W_0(cp) \subsetneqq A_q$. Thus there is some $d \in M$ with $c\hat{p} < c\hat{p}+d < A_q^+$. Therefore we can suppose, that $|p| < A_q^+$ and $W_0(p) \not\subseteq \mathfrak{m}_q$. Let $M_0 \prec M$ be tame in M, such that A_q is the convex hull of M_0 in M and let $p_0 := p \restriction M_0$. From $\mathfrak{m}_q \subsetneqq W_0(p)$ it follows (Proposition 5.17): p_0 is neither definable, nor dense (since $p < A_q^+$, p is M_0-bounded).
Let $q_0 := q \restriction M_0$. Again by Proposition 5.17, q_0 is dense or definable. If q_0 is dense, then q is a coheir of q_0. If q_0 is definable, then q is a coheir of q_0, since $W_0(q) = \mathfrak{m}_q$.

Since q is a coheir of q_0, p is a coheir of p_0 and p_0 is not definable, we get $q_0 \sim p_0$ from Proposition 11.3. But q_0 is dense or definable and p_0 is neither definable nor dense: contradiction. We proved Case 1.

Case 2 $q \sim p,\ q \sim \hat{q},\ A_q$ T-convex $\Rightarrow A_q \subseteq A_p$
For $q \sim \hat{q}$ we may assume $q = \hat{q}$. If $\operatorname{sign} \xi_q = -1$, then $A_q = A_{\frac{1}{q}}$, $\operatorname{sign} \xi_{\frac{1}{q}} = 1$ and we can apply Case 1. Assume $\operatorname{sign} \xi_q = 0$. Let α be a realization of q. The cuts q_- and $F(q_-)$ are coheirs of q and p on $M\langle\alpha\rangle$ respectively. By Proposition 9.9 (iv) we have $\operatorname{sign} \xi_{q_-} = 1$ and by Proposition 9.9 (ii) the convex valuation rings $A_{q_-}^+$ and $A_{F(q_-)}^+$ are coheirs of A_q^+ and A_p^+ respectively. We know from Proposition 10.13 that A_{q_-} is T-convex, too. Now case 1 gives $A_{q_-} \subseteq A_{F(q_-)}$. Since A_{q_-} lies over A_q and $A_{F(q_-)}$ lies over A_p we have proved case 2.

Case 3 $q \sim p,\ q \not\sim \hat{q},\ A_q$ T-convex $\Rightarrow A_q \subseteq A_p$
Let α be a realization of p. Let r be a coheir of p on $M\langle\alpha\rangle$ and let $F : M \longrightarrow M$ be M-definable with $F(p) = q$. $F(r)$ is a coheir of q, hence the unique extension (unique since $q \not\sim \hat{q}$) of $\hat{q}$ on $M\langle\alpha\rangle$ is $\widehat{F(r)}$. Thus $A_{F(r)}^+$ is a coheir of A_p^+ (Proposition 9.9). Similar we get that A_r lies over A_p. By Proposition 5.12, $\operatorname{sign} F(r) \neq 0$ and $\operatorname{sign} r \neq 0$. By Proposition 10.13 the ring $A_{F(r)}$ is T-convex. Finally Case 2 gives $A_{F(r)} \subseteq A_r$, hence $A_q \subseteq A_p$. //

We can not improve the theorem to equality if T is not polynomially bounded:
Take $T = Th(\mathbb{R}, \exp)$ and μ a positive infinitesimal element over $\mathbb{R}$. Let A be the convex hull of $\mathbb{R}$ in $\mathbb{R}\langle\mu\rangle$, $q = A^+$ and $p = \exp(\frac{1}{\mu} \cdot q)$.

The remaining part of this paragraph treats a sequence of equivalent formulations and consequences of this theorem, which are central arguments in the sequel. Especially in the polynomially bounded case.

Corollary 11.6. *If A_q is a T-convex valuation ring of M, q is a cut of M, G is a convex subgroup of M and $F : M \longrightarrow M$ is M-definable with $F(q) = G^+$, then G is a fractional ideal of A_q.*

Proof. If G is not a fractional ideal of A_q, then $A(G) \subsetneqq A_q$. //

For further consequences it is convenient to reformulate Theorem 11.5 in an odd way:

Theorem 11.7. *Let $\bar{\alpha}\hat{\ }\beta$ be an $(n+1)$-tuple from an elementary extension of M and let $F : M^{n+1} \longrightarrow M$ be M-definable. Take $\gamma = F(\bar{\alpha}, \beta)$ and assume*

$A_{\beta/M\langle\bar{\alpha}\rangle}$ lies over $A_{\beta/M}$ and $A_{\gamma/M\langle\bar{\alpha}\rangle}$ lies over $A_{\gamma/M}$

Then we have

$$A_{\beta/M\langle\bar{\alpha}\rangle}\ T\text{-convex} \ \Rightarrow\ A_{\gamma/M} \supseteq A_{\beta/M}$$

and

$$A_{\gamma/M\langle\bar{\alpha}\rangle}\ T\text{-convex} \ \Rightarrow\ A_{\gamma/M} \subseteq A_{\beta/M}$$

Proof. By Theorem 3 we even have $A_{\beta/M\langle\bar{\alpha}\rangle}$ T-convex $\Rightarrow$ $A_{\gamma/M\langle\bar{\alpha}\rangle} \supseteq A_{\beta/M\langle\bar{\alpha}\rangle}$ and $A_{\gamma/M\langle\bar{\alpha}\rangle}$ T-convex $\Rightarrow$ $A_{\gamma/M\langle\bar{\alpha}\rangle} \subseteq A_{\beta/M\langle\bar{\alpha}\rangle}$. //

By definition, polynomially bounded theories are exactly those o-minimal expansions of RCF, such that each convex valuation ring over each model of T is T-convex. Thus

Corollary 11.8. *Let T be a polynomially bounded, o-minimal expansion of RCF. Let $\bar{\alpha}\hat{\ }\beta$ be an $(n+1)$-tuple from an elementary extension of M and $F : M^{n+1} \longrightarrow M$ be M-definable. Let $\gamma = F(\bar{\alpha}, \beta)$ and assume that $t(\gamma/M\langle\bar{\alpha}\rangle)$ and $t(\beta/M\langle\bar{\alpha}\rangle)$ are coheirs over M. Then*

$$A_{\gamma/M} = A_{\beta/M}$$

Proof. By Proposition 9.9 the T-convex valuation rings $A_{\gamma/M\langle\bar{\alpha}\rangle}$ and $A_{\beta/M\langle\bar{\alpha}\rangle}$ lies over $A_{\gamma/M}$ and $A_{\beta/M}$ respectively. //

Theorem 11.9. *Let T be a polynomially bounded, o-minimal expansion of RCF and let M be a model of T. If B is a basis for the M-realization rank of N over M, then*

$$\{A_{\beta/M} \mid \beta \in B\} = \{A_{\alpha/M} \mid \alpha \in N \setminus M\}$$

Proof. We have to show the following: if $p_1, ..., p_k$ are cuts of M, such that $(p_1, ..., p_k)$ is a box type and $F : M^k \longrightarrow M$ is M-definable, not constant near $(p_1, ..., p_k)$, then

$$A_{F(p_1,...,p_k)} \in \{A_{p_1}, ..., A_{p_k}\}$$

If $k = 1$, then $A_{F(p_1)} = A_{p_1}$ by Theorem 11.5, because T is polynomially bounded.

Assume that $k > 1$. Let $q := F(p_1, ..., p_k)$. Because M-dominance is a weak dependence relation (§3) we may assume that $(p_1, ..., p_{k-1}, q)$ is a box type, too. Let $\alpha_1, ..., \alpha_{k-1}$ be realizations of $p_1, ..., p_{k-1}$ respectively. Then neither p_k nor q is realized in $M\langle\alpha_1, ..., \alpha_{k-1}\rangle$ and the above Corollary gives the result. //

Especially we get for cuts $p_1, ...p_k$ of M with $A_{p_i} \neq A_{p_j}$ $(i \neq j)$, that $(p_1, ..., p_k)$ is a box type (obvious induction on k).

Corollary 11.10. *Let T be a polynomially bounded, o-minimal expansion of RCF and let $M \prec N$ be models of T with $\mathrm{rk}_M(N/M) = 1$. Then for all $\alpha, \beta \in N \setminus M$ we have*

$$A_{\alpha/M} = A_{\beta/M}$$

//

If $\dim N/M$ is finite, then $\mathrm{rk}_M\, N/M \leq \dim N/M$ is finite and by Theorem 11.9 there are finitely many convex valuation rings of M associated to the extension $M \prec N$. Therefore it is convenient to look at the structure $(M, A_1, ..., A_k)$ if A_i are T-convex valuation rings of M (this structure is model complete relative $\mathscr{L}$ also if T is not polynomially bounded; see §18)

§12 Location of $F'(p)$

We are going to prove the weak signature alternative for polynomially bounded, *o*-minimal expansions of RCF. A key step of the proof needs the main result of this section, which holds for arbitrary *o*-minimal expansions of RCF: If p is a cut of signature 0 and F is an M-definable map with $F(p) = \hat{p}$, then $F'(p) - 1 = \mathfrak{m}_p^{\pm}$, where F' is the derivative of F.

Let T be an *o*-minimal expansion of RCF. If M is a model of T and $F : M \longrightarrow M$ is an M-definable map, then there is a finite subset E of M, such that F is k-times differentiable on $M \setminus E$. If F is not differentiable in a, we define $F'(a) = 1$. Now the derivative F' of F is again M-definable. We put the proof of this to Appendix A.

Lemma 12.1. *Let p be a cut of M and let $F : M \longrightarrow M$ be M-definable. Let $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ be nets in M with $a_i \neq b_i$.*

If $\lim_i a_i = \lim_i b_i = p$ *in* $S_1(M)$*, then* $F'(p) = \lim_i \frac{F(b_i)-F(a_i)}{b_i-a_i}$

Proof. If F is linear near p, then the assertion holds clearly. Thus we may assume, that F is non linear, hence $F'(p)$ is not isolated. If (c,d) is a neighbourhood of $F'(p)$, then there is a neighbourhood (a,b) of p, such that F is differentiable, F' is continuous on (a,b) and $F'(x) \in (c,d)$ for all $x \in (a,b)$. From some index i_0 on, a_i and b_i are in (a,b). By the differential mean value theorem we must have $\frac{F(b_i)-F(a_i)}{b_i-a_i} \in (c,d)$. //

Therefore, if $c \in W_0(p)$ and $a \nearrow p$ then

$$\frac{F(a+c)-F(a)}{c} \begin{cases} \nearrow F'(p) & \text{if } F''(p) > 0 \\ \searrow F'(p) & \text{if } F''(p) < 0 \end{cases}$$

With $G(x) := F(x+c) - F(x)$ this is $F'(p) = \frac{1}{c}G(p)$. It is at least dangerous to write

$$F'(p) = \frac{1}{c}(F(p+c) - F(p))$$

Proposition 12.2. *If p is a non definable cut of M and $F(p) = p$, then*

$$H^- \leq F'(p) \leq H^+$$

where $H := \bigcup_{c \in M} W_0^*(p+c)$

Proof. Let α be a realization of p. We work in $M\langle\alpha\rangle$. We have $H = \{F'(\alpha) \mid F(p) = p\} \cap M$ (Lemma 8.3). If F is linear near p, then $F'(p) \in H$. Thus we may assume that $F'(\alpha)$ is not in M. Furthermore we assume $F(\alpha) > \alpha$ (if $F(\alpha) < \alpha$ we take the analogous proof).

<u>Case 1:</u> $F''(p) < 0$.
If $a < p < b$ are close to p, then the straight line g through the points $(a, F(a))$ and $(b, F(b))$ fulfills $x \leq g(x) \leq F(x)$ for all $x \in (a,b)$ (since $F''(\alpha) < 0$). We get $g(p) = p$ and $\frac{F(b)-F(a)}{b-a} = g'(p) \in H$. By Lemma 12.1 the assertion follows.

<u>Case 2:</u> $F''(p) > 0$.
<u>Case 2.1:</u> $F'(p) > 1$.
If $a \in M$, $a < p$ is close to p, then the straight line g through the point $(a, F(a))$ with gradient $F'(a)$ fulfills $x \leq g(x)$ near p (since $F'(p) > 1$) and $g(x) < F(x)$ near p (since $F''(p) > 0$). Hence $g(p) = p$ again. For $F'(a) = g'(p) \in H$ converges in $S_1(M)$ to $F'(p)$, if $a \in M$ converges to p, we get the assertion.

<u>Case 2.2:</u> $F'(p) < 1$.
Is similar Case 1 for some $a > p$ close to p. //

Corollary 12.3. *Let* $\operatorname{sign} p = 0$ *and let* $F : M \longrightarrow M$ *be* M*-definable. If* $F(p) = p$*, then*

$$\mathfrak{m}_p^- \leq F'(p) - 1 \leq \mathfrak{m}_p^+$$

PROOF.
By Proposition 9.8 we have $H = 1 + \mathfrak{m}_p$, since $p + c \neq \hat{p}$ for all $c \in M$. //

Proposition 12.4. (T *o-minimal* RCF*-expansion*)
Let $\operatorname{sign} p = 0$ *and let* $F : M \longrightarrow M$ *be* M*-definable. If* $F(p) = \hat{p}$*, then* $F'(p) = \mathfrak{m}_p^+$ *or* $F'(p) = \mathfrak{m}_p^-$*. Consequently* A_p *is realized in* $M\langle p\rangle$*,*

$$\frac{1}{|F'|} \text{ maps } p \text{ to } A_p^+$$

.

I don't know if this proposition is empty, because I don't know one o-minimal expansion of RCF and a cut p in this theory, with $\operatorname{sign} p = 0$ and $p \sim \hat{p}$.

PROOF. We assume $p > 0$. Suppose $\mathfrak{m}_p^- \neq F'(p) \neq \mathfrak{m}_p^+$.

For $\operatorname{sign} p \geq 0$ the map $x + F(x)$ does not move the type p. By the above Corollary we get $\mathfrak{m}_p^- < F'(p) < \mathfrak{m}_p^+$. Let $a \in \mathfrak{m}_p$ with $0 < |F'(p)| < a$. For $1 + \mathfrak{m}_p = H$ ($H = \bigcup_{c \in M} W_0^*(p + c)$ again), there is some $c \in M$ with $1 + a, 1 - a \in W_0^*(p + c)$ and $p + c > \hat{p} > 0$. We replace $F(x)$ by $F(x - c)$ and p by $p + c$ and may assume that $1 + a, 1 - a \in W_0^*(p)$.

<u>Claim</u> there is a linear map $G : M \longrightarrow M$ with $G(p) = p$ and a neighbourhood (c, d) of p with $G(x) \geq x + F(x)$ in (c, d).

<u>Case 1</u> $F'(p) > 0$
Since $F'(p) > 0$, we find a neighbourhood $[c, d]$ of p, $c > 0$, where F is strictly increasing and differentiable, such that $F(c) \in W_0(p)$ and $F'|_{(c,d)} < a$. Take $G(x) = F(c) + (1 + a)\cdot x$. Then $G(x) > x + F(x)$ in $[c, d]$, because $G(c) > F(c) + c$ and $G' > F' > 0$ in (c, d). We have $G(p) = F(c) + (1 + a)\cdot p = p$ by the choice of c and a.

<u>Case 2</u> $F'(p) < 0$
For $F'(p) < 0$ there is some neighbourhood $[c, d]$ of p, where F is strictly decreasing and differentiable, such that $F(d) \in W_0(p)$ and $F'|_{(c,d)} > -a$. We choose $d > p$ such that $a\cdot d \in W_0(p)$: this is possible since $a\cdot p \leq \hat{p}$ (Proposition 8.5 (i)); hence, for $\operatorname{sign} p = 0$, we have $a\cdot p < \hat{p}$.

Let $G(x) = F(d)+ad+(1-a)x$. Then $G(x) \geq x+F(x)$ on $[c,d]$, since $G(d) = F(d)+d$ and $G' < F' < 0$ in (c,d). We have $G(p) = F(d) + ad + (1-a)\cdot p = p$ by the choice of d and a.

Now the claim is proved and we have a linear map G with $G(p) = p$ and $G(x) - x > F(x)$ in (c,d). This means $G_1(p) = \hat{p}$ if $G_1(x)$ denotes $G(x) - x$. Since G_1 is linear it is not possible, that $\operatorname{sign} p = 0$. //

We knew for each cut p of M and each M-definable map F: $F(p) = p \ \Rightarrow\ F$ is increasing in a neighbourhood of p. We sharpened this remark here, in giving the possible locus of $F'(p)$.

I want to make an additional remark about the map $\frac{1}{F'}$, where this map comes up in another consideration of 1-dimensional extensions.

If $N \succ M$ are models of T, $\dim N/M = 1$, then each $\alpha \in N \setminus M$ defines a derivative d_α on N via

$$d_\alpha F(\alpha) := F'(\alpha)$$

The field of constants of (N, d_α) is M. If $\beta \in N$ is another element not in M, then $d_\beta = d_\beta\alpha\cdot d_\alpha$. In other words: if $F, G : M \longrightarrow M$ are M-definable, G is not constant near $t(\alpha/M)$, then $d_{G(\alpha)}F(\alpha) = \frac{F'(\alpha)}{G'(\alpha)}$.

If $F : M \longrightarrow M$ is M-definable and non linear near $t(\alpha/M)$, then $F(\alpha) = \frac{H'(\alpha)}{G'(\alpha)}$ for the maps $H = x - \frac{F}{F'}$ and $G = -\frac{1}{F'}$. Thus F has the 'integral' $x - \frac{F}{F'}$ with respect to the derivative $d_{-\frac{1}{F'}(\alpha)}$. For example, $\frac{1}{\alpha}$ has the integral 2α with respect to the derivative d_{α^2}.

§13 The strong signature alternative is not true if exp is definable

We'll prove the strong signature alternative for polynomially bounded expansions of RCF in the next chapter. This alternative implies a good valuation theory for T. Here we give two examples, where the strong signature alternative is not true.

Example: The strong signature alternative is not true for the theory $RCF^{>0}$

Proof. Let R be a real closed field, let t be infinitesimal, positive over R and let N be the real closure of $R(t)$. Let $\mathscr{L}$ be the language of ordered groups together with a unary function symbol f. Let M be the $\mathscr{L}$-structure $(N^{>0}, \cdot, <, F)$, where

$F(a) := 1 + a$. M is o-minimal, for M is definable in N. Of course we write the addition in M multiplicatively. The symbol $+$ is reserved for addition in N. Let v be the canonical R-valuation of N and $G := \{a \in N \mid v(a) > \sqrt{2}\}$. Let $p := G^+$ and $q := 1 + p \in S_1(M)$. Then $q = (1 + G)^+$ and $1 + G$ is a subgroup of M, hence $W_0(q) = 1 + G$ and $q = W_0(q)^+$. But $\operatorname{sign} p = 0$ (sign is the signature with respect to M) since ξ_G is dense (note that $A(G)$ is the convex hull of R in N).

Because the exponential maps the additive group of $\mathbb{R}$ isomorphically to the multiplicative group of $\mathbb{R}^{>0}$ it is not difficult to see, that the strong signature alternative can not be true for $Th(\mathbb{R}, \exp)$:

Lemma 13.1. *Let M be a non archimedean, real closed field and $EXP : M \longrightarrow M$ such that (M, EXP) is a model of $Th(\mathbb{R}, exp)$. Let $v : M \longrightarrow \Gamma\infty$ be a convex valuation on M (not necessary T-convex). Suppose there is some dense $\xi \in S_1(\Gamma)$. Let $G \subseteq M$ be the corresponding fractional ideal of A. Then* $\operatorname{sign} LOG(G^+) = 0$.

PROOF. Because $\operatorname{sign}^*(G^+) = 0$ and LOG is an isomorphism. //

Note that we can provide such a situation by general model theory: the valuation in the above example is not assumed to be T-convex; hence we can apply Robinsons consistency theorem:

Let $\mathscr{L}_1$ be the language $\mathscr{L}$ of ordered rings together with a unary predicate $\mathcal{G}$ and let $\mathscr{L}_2$ be the language of $(\mathbb{R}, exp)$. Take a real closed field N together with a convex subgroup G of $(N, +)$ such that ξ_G is dense. Then $Th(\mathbb{R}, exp) \restriction \mathscr{L} = Th(N, G) \restriction \mathscr{L}$. From Robinsons consistency theorem we get a structure $(M, EXP, \mathfrak{G})$, such that $(M, EXP) \models (\mathbb{R}, exp)$ and $(M, \mathfrak{G}) \models (N, G)$.

We want to give a more conceptual counterexample:

Let T be an o-minimal expansion of RCF and let A be a proper T-convex valuation ring of a model M of T. By Corollary 5.21(i) applied to $T^{>0}$ and $p := A^+$ there is an elementary extension N of M and a cut q of N which extends p such that $\operatorname{sign}^* q = 0$ and such that $W_0^*(q)$ is the convex hull of $W_0^*(p) = A^{*>0}$. Especially q is the upper boundary of a convex subgroup of $(N, +)$ and A_q is T-convex. Now apply the Lemma:

$$\operatorname{sign} LOG(q) = 0$$

Chapter IV Definability of Cuts

We work in polynomially bounded, o-minimal expansions of real closed fields, in this chapter. Let me explain why this assumption is needed in the sequel and how the results of the first three chapters are used now.

In §16 we give model completeness results for reasonable expansions of (M, p^L) $(M \models T,\, p \in S_1(M))$ by applying Robinsons test for model completeness. Therefore we want to have a description of the heirs q of p on elementary extensions N of M, in terms of extension properties of simple subsets of M, which are definable in $Th(M, d^p)$. In the moment we have two key informations:

(1) If $p = A^+$ for some proper T-convex valuation ring A of M, then q is an heir of p if and only if $q = B^+$ for some T-convex valuation ring B of N (cf §10). Note that " T-convex " is axiomatizable in $Th(M, A)$.

(2) If p is not definable, then q is an heir of p if and only if $W(p) \subseteq W(q)$ (Proposition 5.7, for arbitrary o-minimal expansions of $DOAG$).

Furthermore we have the cut $\hat{p}$ associated to the cut p. Our strategy is to reduce the question " is q an heir of p " to a question about the extension $\hat{p} \subseteq \hat{q}$.

If $\operatorname{sign} p \neq 0$, that is $p = a \pm \hat{p}$ for some $a \in M$, then q is an heir of p if and only if $q = a \pm \hat{q}$ and if $\hat{q}$ is an heir of $\hat{p}$ (Proposition 14.4). It remains to look at the case when $\operatorname{sign} p = 0$. Suppose the weak signature alternative holds for T (that is $W(p)$ is the convex hull of $W_0(p)$ if $\operatorname{sign} p = 0$). Then q is an heir of p iff $\hat{q}$ extends $\hat{p}$; thus the reduction is successful if we know the weak signature alternative.

If T is polynomially bounded we can prove the weak signature alternative for T as well as for $T^{>0}$ in §14. Applying the above arguments first to the cut p and the theory T and then to the cut $\hat{p}$ and the theory $T^{>0}$ we can reduce the question " is q an heir of p " to the question " is A_q an heir of A_p ". Because T is polynomially bounded (1) gives the simple answer: " if and only if A_q extends A_p ".

The proof of both weak signature alternatives uses:

(a) the location of $F'(p)$ (§12) which holds for arbitrary expansions of RCF.

(b) the rigidity of convex valuation rings (§11) which fails for many cuts if an exponential is definable in T (the Remark before Theorem 14.4 explains in detail why our methods does not work for expansions of $Th(\mathbb{R}, exp)$.

§14 The weak signature alternative

In this paragraph T is always a polynomially bounded, o-minimal expansion of RCF. I want to repeat, that for us "polynomially bounded" means that there is an archimedean model, too.

Proposition 14.1. *Let M be a model of T and let G be a convex subgroup of $(M,+)$. We write $p := G^+$.*

(i) If $\operatorname{sign}^* G^+ = 0$, then there is no M-definable map F with $F(G^+) = A(G)^+$. For the cut p this means: $W^*(p)$ is the convex hull of $W_0^*(p)$.

(ii) If $\operatorname{sign}^* G^+ \neq 0$ and if q_1, q_2 are different heirs of G^+ on $N \succ M$, then there is no N-definable map F with $F(q_1) = q_2$.

Proof. Suppose first, that ξ_G is dense and α is a realization of G^+. If $F(p) = A(G)^+$ for some M-definable map F, then $B = F(p_-)$ and $C = F(p_+)$ are different, convex valuation rings of $M\langle p\rangle$. By Proposition 9.9 (iii), p_- and p_+ are equivalent (via $x \mapsto \frac{\alpha^2}{x}$). Thus we can find an $M\langle\alpha\rangle$-definable map F_1 with $F_1(B^+) = C^+$. This contradicts the rigidity of convex valuation rings (T is polynomially bounded).

Now suppose ξ_G is definable, say $G = a \cdot A(G)$. We have $q_1 = a \cdot A(q_1)$ and $q_2 = a \cdot A(q_2)$. If q_1 and q_2 are equivalent over N, then $A(q_1) = A(q_2)$ (rigidity of convex valuation rings), hence $q_1 = q_2$, a contradiction. //

Corollary 14.2. *If p is a cut of M with $p = \hat{p}$ and $\operatorname{sign}^* p = 0$ and if q is an extension of p with $q = \hat{q}$ on $N \succ M$, then q is an heir of p if and only if A_q lies over A_p.*

Proof. For $p = \hat{p}$, the ring A_q lies over A_p, if q is an heir of p. Conversely suppose A_q lies over A_p. A_q lies over A_p means $W_0^*(p) \subseteq W_0^*(q) \subseteq W^*(q)$. By Proposition 14.1, $W^*(G^+)$ is the convex hull of $W_0^*(G^+)$. Hence $W^*(p) \subseteq W^*(q)$ and q is an heir of p. //

If $p = A_p^+$ and q is an extension of p such that A_q lies over A_p then q need not be an heir of p, for example if $q = \alpha + (\mathrm{co}_N\, A_p)^+$ for some realization α of p in N.

Theorem 14.3. *The weak signature alternative holds for polynomially bounded, o-minimal expansions of RCF, that is:*

If $p \in S_1(M)$ and there is some M-definable map $F : M \longrightarrow M$ with $F(p) = \hat{p}$, then $p = a + \hat{p}$ or $p = a - \hat{p}$ for some $a \in M$.

Proof. Suppose $p \sim \hat{p}$ and $\operatorname{sign} p = 0$. From $\operatorname{sign} p = 0$ it follows $q \sim q'$ for different coheirs q, q' on $N \succ M$. Since $p \sim \hat{p}$ it follows $r \sim r'$ for both coheirs r, r' of $\hat{p}$ on N. By Proposition 14.1 (ii) we get $\operatorname{sign}^* \hat{p} = 0$.

On the other hand $\operatorname{sign} p = 0$ and $p \sim \hat{p}$ implies $\hat{p} \sim A_p^+$: here Proposition 12.4 is needed. By Proposition 14.1 (i) the cut ξ_p must be definable, a contradiction. //

Remark

If T is an arbitrary o-minimal expansion of RCF, where Proposition 14.1 holds, then, with the same proof, we get the above Corollary and Theorem 14.3.
In the the proof of Proposition 14.1 we need only a relative version of the "rigidity of convex valuation rings" as follows:

> If $M \prec N$ are models of T with $\dim N/M = 1$ and A is a convex valuation ring of M, which is realized in N, then $B^+ \not\sim C^+$ for the coheirs B^+ and C^+ of A^+ on N.

If T is an expansion of $(\mathbb{R}, \exp)$ then even this condition is not fulfilled. Look at the following example:

Let A_0 be a proper convex valuation ring of a model M of T and let G be a convex subgroup of M with $1 \in G$, $A(G) = A_0$ and $\operatorname{sign}^* G = 0$ (at the end of §13 we construct such a situation where A_0 is T-convex in addition). Take $A := EXP(G)$. By Proposition 9.9 (iii) the coheirs of G^+ on $M\langle\alpha\rangle$ $(\alpha \models G^+)$ are equivalent. Hence the coheirs of A^+ on $M\langle\alpha\rangle$ are equivalent too.

Theorem 14.4. *If T is polynomially bounded, then the weak signature alternative holds for $T^{>0}$. This means for the theory T:*

If p is a cut of a model M of T and $q := W_0^*(p)^+$ with $p \sim q$, then there is some $a \in M$, $a \neq 0$, such that $p = a \cdot q$ or $p = \frac{a}{q}$.

Proof. Let $\operatorname{sign}^* p = 0$. We prove that $p \not\sim q$. If $\operatorname{sign} p = 0$, then we we have $p \not\sim \hat{p}$. By Corollary 8.6 we have $p \not\sim \tilde{p}$, thus $p \not\sim q = 1 + \tilde{p}$.

If $p = a + G^+$ with $a > 0$ and $G := W_0(p)$, then $p = a \cdot (1 + \frac{1}{a} G^+)$ and we may assume that $p = 1 + G^+$. Because $\operatorname{sign}^* p = 0$ we must have $1 \in G$ (Proposition 8.1). Thus $p = G^+$ and $\operatorname{sign}^* G^+ = 0$. By Proposition 14.1 we know that $q = A_p^+$ is not realized in $M\langle p\rangle$.

If $p = a + G^-$ with $a > 0$, we take the same proof. If $p = a + G^-$ with $a < 0$, we replace p by $-\frac{1}{a}p$ and get a cut p of the form $-1 + G^+$ and $p > 0$. Again we have $1 \in G$ and the theorem follows. //

We prove a theorem of R. Rolland ([Ro]) about the movements of cuts p with $p = \hat{p}$ if we apply semialgebraic maps to p. We never use this result, but it is adapted very well to our context. The original prove has less connection with the one we give here.

Theorem 14.5. *If M is a real closed field, G is a proper convex subgroup of $(M, +)$ and $F : M \longrightarrow M$ is M-definable not constant near G^+, then there are $a, b \in M$ and $\frac{n}{m} \in \mathbb{Q}^*$ with $F(G^+) = a + b(G^+)^{\frac{n}{m}}$.*

Proof. We write $p = G^+$ and $q = F(p)$. By the strong signature alternative for RCF we have $\operatorname{sign} q \neq 0$. Thus we may suppose, that $q = H^+$ for a convex subgroup H of $(M, +)$ and prove $q = b \cdot p^{\frac{n}{m}}$ for some $b \in M$ and $\frac{n}{m} \in \mathbb{Q}^*$. By the rigidity of convex valuation rings (11.5) we may define $A := A_p = A_q$. Let $\mathfrak{m}$ be the maximal ideal of A and let $v : M \longrightarrow \Gamma \cup \{\infty\}$ be the valuation of A. By Proposition 14.1 we have that ξ_G is dense if and only if ξ_H is dense.

Let α be a realization of p. If ξ_G is definable, then there is some $c \in M$ such that $G^+ = cA$ or $G^+ = c\mathfrak{m}^+$. The same is true for H^+. Thus $q = bp^n$ with $n \in \{+1, -1\}$ for some $b \in M$. Now assume that ξ_G is dense. By Lemma 9.5 we know that $w(\alpha), w(F(\alpha)) \notin \Gamma$, if w denotes the unique real valuation extending v on $M\langle\alpha\rangle$. Because $\dim_{\mathbb{Q}} \Gamma_w/\Gamma = 1$ we know that $t(w(\alpha)/\Gamma)$ and $t(w(F(\alpha)/\Gamma)$ are equivalent in the sense of $DOAG$. Therefore there is some $\gamma \in \Gamma$ and some $\frac{n}{m} \in \mathbb{Q}$ with $t(w(F(\alpha))/\Gamma)) = \gamma + \frac{n}{m} \cdot t(w(\alpha)/\Gamma)$. If $b \in M, b > 0$ with $v(b) = \gamma$ then $q = b \cdot p^{\frac{n}{m}}$. //

§15 The small type $(p, \hat{p}, A_p^+)$

Let T be a polynomially bounded, o-minimal expansion of RCF. Let M be a model of T and let p be a cut of M with $\operatorname{sign} p = \operatorname{sign}^* \hat{p} = 0$. Then $(p, \hat{p}, A_p^+)$ is a box type. If q is a cut of $N \succ M$ extending p such that $\hat{q}$ is an extension of $\hat{p}$ and A_q lies over A_p, then $(q, \hat{q}, A_q^+)$ is an heir of $(p, \hat{p}, A_p^+)$.

Proof. $(p, \hat{p}, A_p^+)$ is a box type:
Let γ be a realization of $\hat{p}$. For $p \not\sim \hat{p}$, p has a unique extension r on $M\langle\alpha\rangle$ and this extension has signature 0 again (Proposition 5.15). For $\hat{p} \not\sim A_p^+$, A_p has a unique extension B on $M\langle\alpha\rangle$ and we have $B = A_r$. From $\operatorname{sign} r = 0$ and $\operatorname{sign}^* \hat{r} \neq 0$ we get $r \not\sim B^+$. Let α be a realization of r. Then B^+ is not realized in $M\langle\alpha, \gamma\rangle$. Because B is the unique extension of A_p, A_p^+ is not realized in $M\langle\alpha, \gamma\rangle$. Thus $(p, \hat{p}, A_p^+)$ is a box type.

Now assume $p > 0$. By Corollary 14.2, $\hat{q}$ is an heir of $\hat{p}$. By Proposition 6.3, $(q, \hat{q})$ is an heir of $(p, \hat{p})$

We have to show: If $\varphi(v_1, ..., v_6)$ is an $\mathscr{L}(M)$-formula and $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in N$ such that

$$\alpha_1 < x < \alpha_2 \in q,\ \beta_1 < y < \beta_2 \in \hat{q},\ \gamma_1 < z < \gamma_2 \in A_q^+,\ N \models \varphi(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2)$$

then there are $a_1, a_2, b_1, b_2, c_1, c_2 \in M$ such that

$$a_1 < x < a_2 \in p,\ b_1 < y < b_2 \in \hat{p},\ c_1 < z < c_2 \in A_p^+,\ M \models \varphi(a_1, a_2, b_1, b_2, c_1, c_2)$$

If γ_1 is not a realization of A_p, we know this from Lemma 6.1 applied to the theory $T^{>0}$, because $(q, \hat{q})$ is an heir of $(p, \hat{p})$.

Suppose γ_1 is a realization of A_p^+ and let $p_1 := q \restriction M\langle\gamma_1\rangle$ be the unique extension of p on $M\langle\gamma_1\rangle$. By Lemma 6.2 applied to $T^{>0}$, A_q lies over A_{p_1}. For $\operatorname{sign}^* \hat{p_1} = 0$ too, $\hat{q}$ is an heir of $\hat{p_1}$. Again, $(q, \hat{q})$ is an heir of $(p_1, \hat{p_1})$. By Lemma 6.1 applied to the theory $T^{>0}$, and the 2-types $(q, \hat{q})$ and $(p_1, \hat{p_1})$, there are M-definable maps H, G_1, G_2, F_1, F_2 with

$$F_1(\gamma_1) < x < F_2(\gamma_1) \in p_1,\ G_1(\gamma_1) < y < G_2(\gamma_1) \in \hat{p_1},\ \gamma_1 < z < H(\gamma_1) \in A_{p_1}^+$$

and

$$M\langle\gamma_1\rangle \models \varphi(F_1(\gamma_1), F_2(\gamma_1), G_1(\gamma_1), G_2(\gamma_1), \gamma_1, H(\gamma_1))$$

By Corollary 1.9 there is a 3-type $r(x, y, z)$ over $M\langle\gamma_1\rangle$ with $A_{p_1}^+(z) \cup p(x) \cup \hat{p}(y) \subseteq r$, such that r is an heir over M. For p_1, $\hat{p_1}$ are the unique extensions of p, $\hat{p}$ respectively on $M\langle\gamma_1\rangle$ we get $p_1(x) \cup \hat{p_1}(y) \subseteq r$. This proves the claim. //

§16 Model complete expansions of (M, p^L)

Let T be a polynomially bounded, o-minimal theory in the language $\mathscr{L}$ and let $\mathcal{G}$ be a new unary predicate.

Let T_{group} be the $\mathscr{L}(\mathcal{G})$-theory

$$T\ \ \&\ \ \mathcal{G} \text{ is a proper convex subgroup of } (M, +)$$

Again, 'proper' means $\neq 0$ and $\neq M$.

Remark

Let M be a model of T and let G be a convex subgroup of $(M,+)$; let $p := G^+$. Then the structures (M,G) and (M,p^L) are bidefinable with quantifier free formulas:
$x \in G \Leftrightarrow x \in p^L \wedge -x \in p^L$ and $x \in p^L \Leftrightarrow x < 0 \vee x \in G$.
Thus we may switch the point of view as we want.

Definition 16.1. We call a convex subgroup G of $(M,+)$ (M real closed) **residually definable**, if ξ_G is definable. If ξ_G is dense, we call G **residually dense**.

Residually definable groups

Let $\mathfrak{e}$ be a new constant with respect to $\mathscr{L}(\mathcal{G})$. We define two $\mathscr{L}(\mathcal{G},\mathfrak{e})$-theories $T_{valring}$ and $T_{maxideal}$:

$$T_{valring} = T_{group} \;\&\; \mathfrak{e}\cdot\mathcal{G} = A(\mathcal{G}) \;\&\; A(\mathcal{G}) \text{ is proper}$$

and

$$T_{maxideal} = T_{group} \;\&\; \mathfrak{e}\cdot\mathcal{G} = \mathfrak{m}(\mathcal{G}) \;\&\; A(\mathcal{G}) \text{ is proper}$$

Some explanations: Obviously $A(G)$ and $\mathfrak{m}(G)$ are definable in (M,G), if G is a convex subgroup of $(M,+)$; hence we can find an $\mathscr{L}(\mathcal{G},\mathfrak{e})$-formula, which expresses $\mathfrak{e}\cdot\mathcal{G} = A(\mathcal{G})$ and $\mathfrak{e}\cdot\mathcal{G} = \mathfrak{m}(\mathcal{G})$.

Theorem 16.2.

$T_{valring}$ and $T_{maxideal}$ are model complete relative $\mathscr{L}$.

Proof. This follows easily from the model completeness of T_{convex}. //

Residually dense groups

Let $\mathcal{G}^*$ be a new predicate with respect to $\mathscr{L}(\mathcal{G})$ and let $T_{res,dense}$ be the $\mathscr{L}(\mathcal{G},\mathcal{G}^*)$-theory

$$T_{res,dense} = T_{group} \;\&\; [\forall u\; u\cdot\mathcal{G} \neq A(\mathcal{G}) \wedge u\cdot\mathcal{G} \neq \mathfrak{m}(\mathcal{G})] \;\&\; \mathcal{G}^* = W_0^*(\mathcal{G}^+)$$

That is: a model of $T_{res,dense}$ is a triple (M,G,G^*), where G is a proper convex subgroup of $(M,+)$ with $\text{sign}^*\, G^+ = 0$ and G^* is the group $W_0^*(G^+)$.

Remark

If (M,G,G^*) is a model of $T_{res,dense}$ we may identify the reduct (M,G^*) with the $\mathscr{L}_{convex}$-structure $(M,A(G))$ since G^* and $A(G)$ are bidefinable with quantifier free formulas in the language of ordered fields. Note that we always assume, that RCF is formulated in a language where $^{-1}$ is a function symbol.

Theorem 16.3. *The theory $T_{res,dense}$ is model complete relative $\mathscr{L}$. If M is a model of T and G is a proper convex subgroup, then the appropriate definable expansion of (M,G) is a model of exactly one of the theories $T_{valring}$, $T_{maxideal}$ or $T_{res,dense}$.*

Proof. Let $(M,G,G^*) \subseteq (N,H,H^*)$ be models of $T_{res,dense}$. We have to show that the extension is existential.

Let p be the type G^+, p^* be the type G^{*+}, let q be the type H^+ and q^* be the type H^{*+}. We work in the theory $T^{>0}$: we have $\operatorname{sign}^* p = 0$. By §14 we know that the weak signature alternative holds for $T^{>0}$. In $T^{>0}$, the groups G^* and H^* are the invariance groups of p and q. Thus q is an heir of p. From §10 we know that q^* is an heir of p^*.

By §6 we get a 2-type r over $N^{>0}$, which extends $q(x)$ and $q^*(y)$, such that r is an heir over $M^{>0}$. If we look at the situation from the structures M and N we know that r is an heir over M, too. But this means that $(M,G,G^*) \subseteq (N,H,H^*)$ is existential. //

I want to emphasize, that for a model M of T and a cut p of M, the $\mathscr{L}^{defc}$-theory $Th(M,d^p)$ is not model complete in the language $\mathscr{L}^{defc}$ in general ! For example if $p = G^+$ for some residually dense subgroup G of a model M of RCF. We'll show in §19 that $T_{res,dense}$ together with the axiom $1 \notin \mathcal{G}$ is complete and has quantifier elimination. Now take a real closed field $N \supseteq M$ and a residually dense subgroup H of N such that $H \cap M = G$ and such that $A(H)$ lies not over $A(G)$ (we may assume that $A(G)$ is not the convex hull of $\mathbb{Q}$). Then $(M,d^p) \subseteq (N,d^q)$ is not existential (where q denotes the cut H^+) and $(M,d^p) \equiv (N,d^q)$.

Cuts of signature 0, with residually definable invariance group

We keep the notations as above. Let $\mathcal{D}$ be a new unary predicate with respect to $\mathscr{L}(\mathcal{G},\mathcal{G}^*,\mathfrak{e})$ and let $T^{\text{sign}=0}$ be the $\mathscr{L}(\mathcal{D})$-theory

$$T \;\&\; \mathcal{D} \text{ is the set of left options of a cut } p \text{ with } \operatorname{sign} p = 0$$

Note that the sentence "$\mathcal{D}$ is the set of left options of a cut p with $\operatorname{sign} p = 0$" is axiomatizable in $\mathscr{L}(\mathcal{D})$ with a single sentence in the language of ordered rings together with the symbol $\mathcal{D}$.

Let $T^{\text{sign}=0}_{valring}$ be the $\mathscr{L}(\mathcal{D},\mathcal{G},\mathfrak{e})$-theory

$$T^{\text{sign}=0} \;\&\; T_{valring} \;\&\; \mathcal{G} = W_0(\mathcal{D}^+)$$

and $T^{\text{sign}=0}_{maxideal}$ the $\mathscr{L}(\mathcal{D},\mathcal{G},\mathrm{c})$-theory

$$T^{\text{sign}=0} \ \&\ T_{maxideal} \ \&\ \mathcal{G} = W_0(\mathcal{D}^+)$$

Theorem 16.4.
The theories $T^{\text{sign}=0}_{valring}$ and $T^{\text{sign}=0}_{maxideal}$ are model complete relative $\mathscr{L}$.

Proof. We prove the theorem for $T^{\text{sign}=0}_{valring}$; the proof for $T^{\text{sign}=0}_{maxideal}$ is literally the same.

Let (M,D,G,c) and (M_1,D_1,G_1,c_1) be models of $T^{\text{sign}=0}_{valring}$ with $M \prec M_1$ and $(M,D,G,c) \subseteq (M_1,D_1,G_1,c_1)$. We show that $(M,D,G,c) \subseteq (M_1,D_1,G_1,c_1)$ is existential. First we have $c = c_1$. Let $p \in S_1(M)$ and $p_1 \in S_1(M_1)$ with left options D and D_1. For $c = c_1$ we know (§6), that $\hat{p}_1$ is an heir of $\hat{p}$. By §6 there is some $r \in S_2(M_1)$ with $\hat{p}_1(x) \cup p_1(y) \subseteq r$, such that r is an heir over M. Especially $(M,D,G,c) \subseteq (M_1,D_1,G_1,c_1)$ is existential. //

Cuts of signature 0, with residually dense invariance subgroups

Let $T^{\text{sign}=0}_{res,dense}$ be the $\mathscr{L}(\mathcal{D},\mathcal{G},\mathcal{G}^*)$-theory

$$T^{\text{sign}=0} \ \&\ T_{res,dense} \ \&\ \mathcal{G} = W_0(\mathcal{D}^+)$$

Theorem 16.5. *The theory $T^{\text{sign}=0}_{res,dense}$ is model complete relative $\mathscr{L}$.*

Proof. Let (M,D,G,G^*) and (M_1,D_1,G_1,G_1^*) be models of $T^{\text{sign}=0}_{res,dense}$ with $M \prec M_1$ and $(M,D,G,G^*) \subseteq (M_1,D_1,G_1,G_1^*)$. We show that the latter extension is existential.

Let $p \in S_1(M)$ and $p_1 \in S_1(M_1)$ with left options D and D_1. $T^{\text{sign}=0}_{res,dense}$ says that $\operatorname{sign} p = \operatorname{sign} p_1 = 0$ as well as ξ_p and ξ_{p_1} are dense. Because of $(M,D,G,G^*) \subseteq (M_1,D_1,G_1,G_1^*)$ we know that $\hat{p}_1$ extends $\hat{p}$ and A_{p_1} lies over A_p. By §15 there is some $r(x,y,z) \in S_3(M_1)$ with $p_1(x) \cup \hat{p}_1(y) \cup A^+_{p_1}(z) \subseteq r$, such that r is an heir over M. Especially $(M,D,G,G^*) \subseteq (M_1,D_1,G_1,G_1^*)$ is existential. //

We summarize:

Theorem 16.6.
If M is a model of T and p is a cut of M with $\operatorname{sign} p = 0$, then a definable expansion of (M,p^L) is a model of exactly one of the theories

T^{dense}, $T^{\text{sign}=0}_{valring}$, $T^{\text{sign}=0}_{maxideal}$ or $T^{\text{sign}=0}_{res,dense}$. All these theories are model complete relative $\mathscr{L}$. //

Corollary 16.7. *If M is a model of T, p is a cut of M and Z is a subset of M^k definable in (M, p^L), then there is an $\mathscr{L}(M)$-formula $\varphi(\bar{x}, \bar{y})$ and a quantifier free formula $\chi(\bar{x}, \bar{y})$ in the language $\mathscr{L}(\mathcal{D}, \mathcal{G}, \mathcal{G}^*)$ with parameters from M such that Z is defined by*

$$\exists \bar{y}\ \varphi(\bar{x}, \bar{y}) \wedge \chi(\bar{x}, \bar{y})$$

Proof. By elimination theory (Appendix B) //

Weak o-minimality of $Th(M, d^p)$

Proposition 16.8. *Let T be an o-minimal expansion of DOAG, let M be a model of T and let $p_1, ..., p_n$ be cuts of M. Let $\mathcal{D}_1, ..., \mathcal{D}_n$ be new predicates for the left options of $p_1, ..., p_n$ respectively and $\psi(x, y_1, ..., y_l, z_1, ..., z_k)$ be a quantifier free formula in the signature $\mathscr{L}(\mathcal{D}_1, ..., \mathcal{D}_n)$. Then there is some $N \in \mathbb{N}$ such that for each $\bar{b} \in M^k$ the set*

$$\{a \in M \mid (M, p_1^L, ..., p_n^L) \models \exists \bar{y}\ \psi(a, \bar{y}, \bar{b})\}$$

is a finite union of at most N convex subsets of M.

Proof. We can assume that $l = k$ and that no p_i is definable. A quantifier free formula $\psi(x, \bar{y}, \bar{z})$ in the language $\mathscr{L}(\mathcal{D}_1, ..., \mathcal{D}_n)$ is a finite disjunction of formulas of the form

$$\varphi(x, \bar{y}, \bar{z}) \wedge \bigwedge_{\substack{i=1,\dots,n \\ j=1,\dots,m}} F_{ij}(x, \bar{y}, \bar{z}) \in \mathcal{D}_i \wedge G_{ij}(x, \bar{y}, \bar{z}) \notin \mathcal{D}_i$$

for a quantifier free $\mathscr{L}$-formula $\varphi(x, \bar{y}, \bar{z})$ and 0-definable maps $F_{ij}, G_{ij} : M^{2k+1} \longrightarrow M$. It is enough to prove the Proposition for these formulas.
By cell decomposition there is some $N \in \mathbb{N}$ such that for all $\bar{c}, \bar{d} \in M^n$ and all $\bar{b} \in M^k$ the set

$$Z^{\bar{b}}_{\bar{c},\bar{d}} := \{(a, \bar{a}') \in M^{k+1} \mid M \models \varphi(a, \bar{a}', \bar{b}) \wedge \bigwedge_{\substack{i=1,\dots,n \\ j=1,\dots,m}} F_{ij}(a, \bar{a}', \bar{b}) < c_i \wedge G_{ij}(a, \bar{a}', \bar{b}) > d_i\}$$

is a union of at most N definably connected, M-definable subsets of M^{k+1}. Thus, for each $\bar{c}, \bar{d}$ with $c_i < p_i < d_i$ the set $pr\ Z^{\bar{b}}_{\bar{c},\bar{d}}$ is the union of at most N convex subsets of M, if pr denotes the projection onto the x-coordinate. For $(Z^{\bar{b}}_{\bar{c},\bar{d}})_{c_i < p_i < d_i}$ is upward, we get that $pr \bigcup_{c_i < p_i < d_i} Z^{\bar{b}}_{\bar{c},\bar{d}}$ is also the union of at most N convex subsets of M. But the latter set is the one we have to describe. //

Theorem 16.9. *Let T be a polynomially bounded, o-minimal expansion of RCF and let $\mathcal{D}$ be a new unary predicate. Let $T_{\mathcal{D}}$ be the $\mathscr{L}(\mathcal{D})$-theory which extends T and says that $\mathcal{D}$ is the set of left options of a cut of M. Then $T_{\mathcal{D}}$ is weakly o-minimal: if $\varphi(x, \bar{z})$ is an $\mathscr{L}(\mathcal{D})$-formula, then there is some $N \in \mathbb{N}$ such that for all models (M, D) of $T_{\mathcal{D}}$ and each $\bar{b} \in M^{\bar{z}}$, the set $\varphi(M, \bar{b})$ is the union of at most N convex subsets of M.*

Proof. We assume T to have quantifier elimination. If M is a model of T and p is a cut of M, then, by the Theorems 16.3 and 16.6, a subset of M, definable in (M, d^p) with parameters from M, can be defined uniformly by an existential formula in a language $\mathscr{L}(\mathcal{D}_1, ..., \mathcal{D}_4)$, where $\mathcal{D}_i$ are predicates of left options of some cuts of M. Now the theorem follows easily from Proposition 16.8. //

Because (M, d^p) is a definable expansion of (M, p^L) we get:

Corollary 16.10. *If T is polynomially bounded, M is a model of T and p is a cut of M, then $Th(M, d^p)$ is weakly o-minimal.* //

Model Companions

Let T be the (necessary unique) model companion of an $\mathscr{L}$-theory T_0 (that is: T is model complete and each model of T_0 has an embedding in a model of T), such that each T_0-model is an ordered field. We want to prove that some of the model complete theories of this paragraph are model companions of certain expansions of T_0. Corollary 5.21 and Proposition 9.10 (obtaining cuts with $\operatorname{sign} p = \operatorname{sign}^* \hat{p} = 0$) together with Proposition 16.13 below gives all the methods needed.

Let G be an abelian ordered group. Then G acts on the set of cuts of G via $g + \xi := (g + \xi^L, g + \xi^R)$. The stabilizer of ξ under this action is called the invariance group of ξ and is denoted by $W_0(\xi)$. Certainly $W_0(\xi)$ is a convex subgroup of G. The upper boundary of $W_0(\xi)$ is denoted by $\hat{\xi}$.

$$\hat{\xi} := W_0(\xi)^+$$

Lemma 16.11. *Let $G \subseteq H$ be an extension of abelian ordered groups and let ξ be a cut of G (that is $\xi = (\xi^L, \xi^R)$, where $\xi^L \cup \xi^R = G$ and $\xi^L < \xi^R$). Let η be an extension of ξ on H. Then*

(i) $W_0(\eta) \cap G \subseteq W_0(\xi)$

(ii) If η is the least or the largest extension of ξ (that is $\eta^L = \operatorname{co}_H \xi^L$ or $\eta^R = \operatorname{co}_H \xi^R$), then

$$W_0(\eta) \cap G = W_0(\xi)$$

PROOF. If $g \in G$ and $g + \eta = \eta$, then $g + \xi^L \subseteq (g + \eta^L) \cap G \subseteq \eta^L \cap G = \xi^L$. This proves (i). For (ii) let $g \in W_0(\xi)$ be positive. If $\eta^L = \mathrm{co}_H\, \xi^L$ then $g + \eta^L = \eta^L$. If $\eta^R = \mathrm{co}_H\, \xi^R$ then $-g + \eta^R = \eta^R$. In any case $g + \eta = \eta$. //

REMARK
If G is an abelian ordered group and ξ is a cut of G such that $0 \in \xi^L + \xi^L \subseteq \xi^L$, then the following conditions are equivalent:

(i) if $H \supseteq G$ is an abelian ordered group, then the largest extension of ξ on H is the upper boundary of a convex subgroup of H.
(ii) the largest extension of ξ on $G \otimes_{\mathbb{Z}} \mathbb{Q}$ is the upper boundary of a convex subgroup of H.

If $n \cdot G$ is dense in G for some $n \in \mathbb{N}$, then both conditions are fulfilled.

We give an example where these conditions are not fulfilled. Let $\omega > \mathbb{R}$ be an infinite element and let $K := \mathbb{Q}(\omega)$. Let $G := (K^{>0}, \cdot, 1, \leq)$ and $H := (K(\sqrt{\omega})^{>0}, \cdot, 1, \leq)$. Let U be the convex hull of $\mathbb{Q}$ in K and let $\xi := U^+$. Then $U < \sqrt{\omega} < \xi^R$ but $\omega = (\sqrt{\omega})^2 \in \xi^R$. Note that G and H are dense ordered in this example.

Lemma 16.12. *Let $G \subseteq H$ be ordered abelian groups, let ξ be a cut of G and let $h \in H$ such that $\xi^L < h < \xi^R$. Then*

$$h \pm [\mathrm{co}_H\, W_0(\xi)]^+ \text{ lies over } \xi$$

PROOF. If $g \in G$ is positive with $g + \xi = \xi$ then $h < \xi^R \subseteq \xi^R - g$, hence $h + g < \xi^R$. This proves that $h + \mathrm{co}_H\, W_0(\xi)^+$ extends ξ. The same argument with negative g proves that $h - \mathrm{co}_H\, W_0(\xi)^-$ extends ξ. //

Proposition 16.13. *Let $K \subseteq K'$ be ordered fields, let ξ be a cut of K and suppose there is some $\beta \in K'$ such that $\xi^L < \beta < \xi^R$. Then there is a cut η of K, such that*

$$(K, \xi^L, W_0(\xi), A_\xi) \subseteq (K', \eta^L, W_0(\eta), A_\eta)$$

and such that $W_0(\eta)$ is the convex hull of $W_0(\xi)$ on K'.

PROOF. We take $\eta := \beta + \mathrm{co}_{K'}\, W_0(\xi)$ (with respect to the abelian ordered group $(K, +, 0, \leq)$. By Lemma 16.12 we know that η extends ξ. Since $\hat{\eta}$ is the least extension of $\hat{\xi}$ with respect to the ordered abelian group $(K^{>0}, \cdot, 1, \leq)$ we know that $W_0(\hat{\eta})$ lies over $W_0(\hat{\xi})$. But this means that A_η lies over A_ξ. //

Theorem 16.14. *Let T be a polynomially bounded, o-minimal expansion of RCF, which is the model companion of the $\mathscr{L}$-theory T_0. Assume that T_0 expands the theory of ordered fields (for example if $T = RCF$ and T_0 is the theory of ordered fields we have this situation). Let $\mathscr{D}, \mathscr{G}$ and $\mathscr{O}$ be new unary predicates with respect to $\mathscr{L}$.*

(i) *T^{dense} is the model companion of the $\mathscr{L}(\mathscr{D})$-theory T_0 together with the axiom*

$$\mathscr{D} \text{ is the set of left options of a cut}$$

(ii) *$T_{res,dense}$ is the model companion of the $\mathscr{L}(\mathscr{G}, \mathscr{O})$-theory T_0 together with the axiom*

$$\mathscr{G} \text{ is a proper convex subgroup and } \mathscr{O} = \{a \mid a{\cdot}\mathscr{G} \subseteq \mathscr{G}\}$$

(iii) *$T_{res,dense}^{\text{sign}=0}$ is the model companion of the $\mathscr{L}(\mathscr{D}, \mathscr{G}, \mathscr{O})$-theory T_0 together with the axiom*

$$\mathscr{D} \text{ is the set of left options of a cut, } \mathscr{G} = \{a \mid a + \mathscr{D} \subseteq \mathscr{D}\} \text{ and } \mathscr{O} = \{a \mid a{\cdot}\mathscr{G} \subseteq \mathscr{G}\}$$

Proof. By the Theorems 16.3 and 16.6 all these theories are model complete.

If K is a model of T_0 and ξ is a cut of M, then by Proposition 16.13 there is a model M of T and a cut p of M such that $(K, \xi^L, W_0(\xi), A_\xi) \subseteq (M, p^L, W_0(p), A_p)$.

(i) By Corollary 5.21 there is an extension $N \succ M$ and a dense cut q of N such that $(M, p^L) \subseteq (N, q^L)$. This gives (i).

(ii) and (iii). By Proposition 9.10 there is a model $(N, q^L, W_0(q), A_q)$ of $T_{res,dense}^{\text{sign}=0}$ which extends $(M, p^L, W_0(p), A_p)$. This gives (ii) and (iii). //

Note that T^{dense} and $T_{res,dense}$ are model completions if T has quantifier elimination and a universal system of axioms (Corollary 4.5 and Theorem 19.2) .

§17 The Box Theorem

Let T be a polynomially bounded, o-minimal expansion of RCF (with archimedean prime model).

Theorem 17.1. *Let A be a convex valuation ring of a model M of T and let q be a non definable cut over M, such that q^L is definable in (M, A). Then there are $a, b \in M$ such that*

$$q = a + b{\cdot}A^+ \text{ or } q = a + b{\cdot}\mathfrak{m}_A^+$$

Proof. Because q is not definable, A is proper.

Claim If B is a proper convex valuation ring of M, definable in (M, A), then $B = A$.
Proof of the Claim:
Suppose, that B is definable in (M, A) and $B \neq A$. Let $\varphi(x, \bar{y})$ be an $\mathscr{L}(\mathcal{O})$-formula, such that

$T_{convex} \vdash$ there is some $\bar{y}$, such that $\{x \mid \varphi(x, \bar{y})\}$ is a convex valuation ring different from $\mathcal{O}$.

Let $\mathcal{P}$ be the definable closure of $\emptyset$ and $\mu > \mathcal{P}$. The only proper convex valuation ring of $\mathcal{P}\langle\mu\rangle$ is the convex hull of $\mathcal{P}$ in $\mathcal{P}\langle\mu\rangle$ (here we need, that $\mathcal{P}$ is archimedean). But T_{convex} is complete, a contradiction.

We prove the theorem:
Suppose first that $\operatorname{sign}^* \hat{q} = 0$. We replace q by $\hat{q}$ and find a contradiction as follows: Let $\varphi(x, \bar{v})$ be an $\mathscr{L}_{convex}$-formula and $\bar{a} \in M^{\bar{v}}$ such that $W_0(q)$ is defined by $\varphi(x, \bar{a})$ in (M, A). Let α be a realization of q and let B be the unique convex valuation ring of $N := M\langle\alpha\rangle$ lying over A. From $(M, A) \prec (N, B)$ we know that $\varphi(x, \bar{a})$ defines a convex subgroup H of N, which is residually dense and such that $H \cap M = W_0(q)$. Let $q_1 := H^+$. By the Claim it follows $A(H) = B$. Because $\operatorname{sign} \xi_q = 0$ we know that q_1 is an heir, thus a coheir of q on N. But this is not possible, because both coheirs of q on M are residually definable.

We have proved that $\operatorname{sign}^* \hat{q} \neq 0$.

Suppose $\operatorname{sign} q = 0$. Let $\varphi(x, \bar{v})$ be an $\mathscr{L}_{convex}$-formula and $\bar{a} \in M^{\bar{v}}$ such that q^L is defined by $\varphi(x, \bar{a})$ in (M, A). Let α be a realization of q and $N := M\langle\alpha\rangle$. If q is dense, then $q \not\sim A^+$. If q is not dense, then A_q is proper, $\hat{q} \sim A_q^+$ and, by the claim, $A = A_q$. Therefore we have $q \not\sim A^+$. Let B be the unique convex valuation ring of N lying over A.

Case 1 q is dense.
From $(M, A) \prec (N, B)$ we know that $\varphi(x, \bar{a})$ defines a dense cut q_1 of N, which extends q. But this is not possible, since q has only definable extensions on N.

Case 2 q is not dense.
Then $A = A_q$. Let $b \in M$ and $m \in \{+1, -1\}$ such that $\hat{q} = b \cdot (A^+)^m$. From $(M, A) \prec (N, B)$ we know that $\varphi(x, \bar{a})$ defines a cut q_1 of N, which extends q, such that $\hat{q}_1 = b \cdot (B^+)^m$ and such that $\operatorname{sign} q_1 = 0$ (all three assertions are definable in (M, A)). For $\operatorname{sign} q = 0$ and $\hat{q}_1$ extends $\hat{q}$, we see that q_1 is an heir, thus a coheir of q on N. Again this is not possible, since both coheirs of p on N have signature $\neq 0$.//

We arrive at our main result:

Definition 17.2. If M is a real closed field and p is a cut of M, we define the p-**box** to be

$$\operatorname{box}(p) := \begin{cases} p & \text{if } p \text{ is definable.} \\ p & \text{if } p \text{ is dense.} \\ p & \text{if } \hat{p} \text{ is not definable, } \operatorname{sign} p \neq 0 \text{ and } \operatorname{sign}^* \hat{p} \neq 0 \\ (p, A_p^+) & \text{if } \operatorname{sign} p \neq 0 \text{ and } \operatorname{sign}^* \hat{p} = 0 \\ (p, \hat{p}) & \text{if } \hat{p} \text{ is not dense, } \operatorname{sign} p = 0 \text{ and } \operatorname{sign}^* \hat{p} \neq 0 \\ (p, \hat{p}, A_p^+) & \text{if } \operatorname{sign} p = 0 \text{ and } \operatorname{sign}^* \hat{p} = 0 \end{cases}$$

$\operatorname{box}(p)$ is a box type if M is a polynomially bounded, o-minimal expansion of RCF with archimedean prime model. Sometimes we refer to the p-boxes as a set. We wrote the definition in this odd form, since the three cases 'definable', 'dense' and 'valuation ring' are three sorts of cuts which are (and have to be treated) very different from each other, which have the best model theory and which can be seen as the "atomic sorts".

Theorem 17.3. (*Box Theorem*)
Let T be a polynomially bounded, o-minimal expansion of RCF with archimedean prime model. If p, q are cuts of M and q is not definable, but definable in (M, d^p), then there is some $r \in \operatorname{box} p$ such that $q \sim r$. Thus $q \sim p$, $q \sim \hat{p}$ or $q \sim A_p^+$.

Proof. We know that p is not definable. By Proposition 4.5 we can suppose that p is not dense too.

Suppose $q \not\sim r$ for each member r of $\operatorname{box} p$. Then p has a unique extension p_1 on $M\langle\alpha\rangle$, if α is a realization of q. Furthermore it follows that neither $\hat{p}$ nor A_p^+ is realized in $M\langle\alpha\rangle$. Thus $\hat{p}_1$ and $A_{p_1}^+$ are the unique extensions of $\hat{p}$ and A_p^+ respectively. Especially $\operatorname{sign} p_1 = \operatorname{sign} p$, $\operatorname{sign}^* \hat{p}_1 = \operatorname{sign}^* \hat{p}$ (Propositions 5.7 and 5.15) and $\operatorname{box} p_1$ is the unique extension of $\operatorname{box} p$ on $M\langle\alpha\rangle$. Let $n := \operatorname{card} \operatorname{box} p = \operatorname{card} \operatorname{box} p_1$ ($\in \{1, 2, 3\}$). Let $(\alpha_1, ..., \alpha_n)$ be a realization of $\operatorname{box} p_1$. Since $\operatorname{box} p_1$ is a box type we have $\operatorname{rk}_{M\langle\alpha\rangle}(\alpha_1, ..., \alpha_n)/M\langle\alpha\rangle = n$. Thus $\operatorname{rk}_M(\alpha, \alpha_1, ..., \alpha_n)/M = n+1$, which means that $(q, \operatorname{box} p)$ is a box type.

Now we reduce the situation to the case $\operatorname{sign} p \neq 0$ and $\operatorname{sign}^* \hat{p} \neq 0$, which contradicts Theorem 17.1. We may assume that T has quantifier elimination.

Reduction 1. Suppose $\operatorname{sign} p = \operatorname{sign}^* \hat{p} = 0$.
Let α be a realization of p, let q_1 denote the unique extension of q on $M\langle\alpha\rangle$ and let p_1 be a coheir of p. Then $\hat{p}_1$ is the unique extension of $\hat{p}$ on $M\langle\alpha\rangle$ and A_{p_1} is the unique valuation ring lying over A_p. We know that $(q_1, \hat{p}_1, A_{p_1}^+)$ is again a box type. By Theorem 16.5 the theory of $(M, p^L, W_0(p), A_p)$ is model complete. Hence there is

a quantifier free formula $\chi(x,\bar{u},\bar{v})$ in the language $\mathscr{L}(\mathcal{D},\mathcal{G},\mathcal{G}^*)$ and $\bar{a} \in M^{\bar{v}}$ such hat q^L is defined by $\varphi(x) := \exists \bar{u}\ \chi(x,\bar{u},\bar{a})$. Because $(M,p,\hat{p},A_p^+) \subseteq (M\langle\alpha\rangle, p_1, \hat{p}_1, A_{p_1}^+)$ is existential (§15) we know that $\varphi(x)$ defines the unique extension q_1 of q on $M\langle\alpha\rangle$. Thus q_1 is definable in $(M\langle\alpha\rangle, d^{p_1})$, hence definable in $(M\langle\alpha\rangle, d^{\hat{p}_1})$ (because $\operatorname{sign} p_1 \neq 0$). Finally we get $\operatorname{sign} \hat{p}_1 \neq 0,\ \operatorname{sign}^* \hat{p}_1 = 0$.

Similar arguments in the other cases shows that we can do the promised reduction. Explicitly:

Reduction 2. Suppose $\operatorname{sign} p \neq 0$ and $\operatorname{sign}^* \hat{p} = 0$
Because $\operatorname{sign} p \neq 0$ the cut q is definable in $(M, d^{\hat{p}})$ and we may assume that $p = \hat{p}$. Let α be a realization of p, let p_1 be a coheir of p on $M\langle\alpha\rangle$ and let q_1 denote the unique extension of q on $M\langle\alpha\rangle$. Then A_{p_1} is the unique valuation ring of $M\langle\alpha\rangle$ lying over A_p and $q_1 \not\sim A_{p_1}^+$. Let $\chi(x,\bar{u},\bar{v})$ be a quantifier free formula in the language $\mathscr{L}(\mathcal{G},\mathcal{G}^*)$ and $\bar{a} \in M^{\bar{v}}$ such hat q^L is defined by $\varphi(x) := \exists \bar{u}\ \chi(x,\bar{u},\bar{a})$. Because $(M,p,A_p^+) \subseteq (M\langle\alpha\rangle, p_1, A_{p_1}^+)$ is existential we know that $\varphi(x)$ defines the unique extension q_1 of q on $M\langle\alpha\rangle$. Thus q_1 is definable in (M, A_{p_1}) and $q \not\sim A_{p_1}^+$ in contradiction to Theorem 17.1.

Reduction 3: Suppose $\operatorname{sign} p = 0$ and $\operatorname{sign}^* \hat{p} \neq 0$.
By our assumption at the beginning of the proof, p is not dense, therefore the valuation ring A_p is proper and $\hat{p}$ is not definable. Let α be a realization of p, let p_1 be a coheir of p on $M\langle\alpha\rangle$ and let q_1 denote the unique extension of q on $M\langle\alpha\rangle$. Then $(q, \hat{p}_1)$ is again a box type.

Let $\chi(x,\bar{u},\bar{v})$ be a quantifier free formula in the Language $\mathscr{L}(\mathcal{D},\mathcal{G})$ and $\bar{a} \in M^{\bar{v}}$ such that q^L is defined by $\varphi(x) := \exists \bar{u}\ \chi(x,\bar{u},\bar{a})$. Because $(M,p,\hat{p}) \subseteq (M\langle\alpha\rangle, p_1, \hat{p}_1)$ is existential we know that $\varphi(x)$ defines the unique extension q_1 of q on $M\langle\alpha\rangle$.
Again we get that q_1 is definable in $(M\langle\alpha\rangle, A_{p_1})$ and $q_1 \not\sim A_{p_1}^+$, in contradiction to Theorem 17.1. //

§18 Applications of the Box Theorem

Theorem 18.1. *Let T be a polynomially bounded, o-minimal expansion of RCF, let M be a model of T and let p, q be cuts of M with $p \sim q$. Then $\hat{p} \sim \hat{q}$,*

$$\operatorname{sign} p = 0 \Leftrightarrow \operatorname{sign} q = 0$$

and

$$\operatorname{sign}^* \hat{p} = 0 \Leftrightarrow \operatorname{sign}^* \hat{q} = 0$$

Therefore the sort of the p-box as stated in the definition, can not be changed by an M-definable map. Especially the strong signature alternative holds for T.

Proof. We first prove the strong signature alternative for T:

Suppose there is a model M of T, a cut p of M and a convex subgroup G of $(M, +)$ such that $\operatorname{sign} p = 0$ and $p \sim G^+$. By Proposition 9.10 there is an $N \succ M$ and an extension r of p such that $\hat{r}$ extends $\hat{p}$, A_r lies over A_p, $\operatorname{sign} r = 0$ and $\operatorname{sign}^* \hat{r} = 0$. Let $G^+ = F(p)$ with $F : M \longrightarrow M$, M-definable. For $\operatorname{sign} p = 0$, r is an heir of p (weak signature alternative). Thus $F(r)$ is an heir of $F(p) = G^+$. Especially $F(r)$ is the upper boundary of a convex subgroup of $(N, +)$. Let α be a realization of $F(r)$ and β be a realization of $A_{F(r)}$. By the Box Theorem $r, \hat{r}$ and A_r^+ are realized in $M\langle \alpha, \beta \rangle$, since r is definable in $(M, d^{F(r)})$. By §15 $(r, \hat{r}, A_r^+)$ is a box type. But this is not possible, since $\dim M\langle \alpha, \beta \rangle / M \leq 2$.

Now Theorem 5.16 says that $\hat{p} \sim \hat{q}$ is implied by $p \sim q$.

The theorem holds if p is definable or dense. If $\operatorname{sign} p \neq 0$ and $\operatorname{sign}^* \hat{p} \neq 0$, then the theorem holds by Theorem 17.1. From the Box Theorem we get that $\operatorname{sign} q = \operatorname{sign}^* \hat{q} = 0$ if $q \sim p$ and $\operatorname{sign} p = \operatorname{sign}^* \hat{p} = 0$. //

The next theorem is a strengthening of the rigidity of convex valuation rings in the polynomially bounded case:

Theorem 18.2. *Let T be a polynomially bounded, o-minimal expansion of RCF, let M be a model of T and let p be a cut of M. If q is a non definable cut of M, definable in (M, d^p), then*
(i) $A_q = A_p$.
(ii) $\hat{q} \sim \hat{p}$ *or* $\hat{q} \sim A_p^+$.
(iii) If $\operatorname{sign} q = 0$ *then* $p \sim q$.

Proof. (i) follows from the Box Theorem and the rigidity of convex valuation rings.
(ii) and (iii) follows from the Box Theorem and the strong signature alternative.//

Reduction of convex valuations to the canonical valuations

An extension $M \prec N$ of polynomially bounded models induces on M the valuation rings $A_{\alpha/M}$, where $\alpha \in N \setminus M$. By Theorem 11.9 the number of these valuation rings is less or equal to the realization rank of N over M, especially less or equal to the dimension of N over M. "Reduction" in the title above means that we can reduce some questions concerning arbitrary convex valuations v on N to questions about convex valuations w on N such that $w|M$ has a valuation ring $A_{\alpha/M}$ as above. In the moment the reduction is fruitful in the realization rank one case (cf Theorem 18.10).

Lemma 18.3. *Let $M \prec N$ be models of T, let M' be a continuous closure of M in N and let $\alpha \in N \setminus M'$. Then $A_{\alpha/M} = M$ iff $t(\alpha/M')$ is definable.*

Proof. Let $p = t(\alpha/M)$ and $q = t(\alpha/M')$. Note that $A_p = M$ means that p is dense or definable. If p is definable, then p is omitted in M', thus q is a coheir of p. By 9.9 we know that A_q is a coheir of A_p, hence $A_q = M'$. Since M' is a continuos closure of M in N the cut q must be definable.

Conversely assume that q is definable. If p is realized in M', then p is dense. If p is omitted in M', then again A_q lies over A_p. But $A_q = M'$, hence $A_p = M$. //

Definition 18.4. We call an extension $M \prec N$ of models of T **almost tame** if some continuous closure of M in N is tame in N.

In this case, by Lemma 18.3, each continuous closure of M in N is tame in N:

Corollary 18.5. *If M' is a continuous closure of M in N, then the following conditions are equivalent:*

(i) For all $\alpha \in N \setminus M$ the valuation ring $A_{\alpha/M}$ is equal to M, that is: $M \prec M\langle\alpha\rangle$ is dense or tame.

(ii) M' is tame in N. //

Lemma 18.6. *Let $M \prec N$ be models of T and let A be a convex valuation ring of M. Assume that $A \subseteq A_{\alpha/M}$ for all $\alpha \in N \setminus M$.*

(i) If B is a convex valuation ring of N lying over M, then $\kappa_A \prec \kappa_B$ is almost tame.

(ii) If B is the convex hull of A in N, then $\kappa_A \prec \kappa_B$ is dense.

Proof. We may assume that $\kappa_A \prec M$, $\kappa_B \prec N$ and that these extensions are tame.

(i) Let $\alpha \in \kappa_B \setminus \kappa_A$. and let $p = t(\alpha/\kappa_A)$. We have to show that $A_p = \kappa_A$. If p is realized in M, then p is definable. If p is omitted in M then the unique extension

$q = t(\alpha/M)$ of p on M is a coheir of p, hence A_q lies over A_p. Since $A \subseteq A_q$ we get $A_p = \kappa_A$.

(ii) follows immediately from (i) since $\kappa_A \prec \kappa_B$ is archimedean. //

Lemma 18.7. *If* $\mathrm{rk}_M(N/M) = 1$, *B is not the convex hull of A in N and if B is the largest convex valuation ring of N, lying over A then the value group Γ_A is equal to the value group Γ_B.*

Proof. We may assume that A is proper. Suppose there is some $\alpha \in N$ such that $v_B(\alpha) \notin \Gamma_A$. Since A^+ is realized in N (B is not the convex hull of A) and $\mathrm{rk}_M(N/M) = 1$ we know that there are $a, b \in M$ and $k \in \{+1, -1\}$ such that $q := t(\alpha/M) = a + b\cdot(A^+)^k$ (by Theorem 17.1). Because B is the largest extension of A there is some $\beta \in B$ which realizes A^+ such that $\alpha = a + b\beta^k$. Let R be the real closure of $M(\beta)$ and $C = B \cap R$. Then $\kappa_A \subseteq \kappa_C$ is proper (since $\beta \in C \setminus \mathrm{co}_R A$) and $v_C(a + b\beta^k) \notin \Gamma_A$. But for the one dimensional extension of real closed fields $M \subseteq R$ we know that either the residue fields or the valuation groups of A and C respectively must be equal. Thus we have a contradiction. //

Lemma 18.8. *Let $M \subseteq N$ be ordered fields, let V, A be convex valuation rings of M and let W, B be convex valuation rings of N lying over V, A respectively.*
(i) If $V \subsetneq A$, W is the convex hull of V in N and $\kappa_A = \kappa_B$ then $\kappa_V = \kappa_W$.
(ii) If $A \subsetneq V$ and $\Gamma_A = \Gamma_B$ then $\Gamma_V = \Gamma_W$.

Proof. (i) We show that $V/\mathfrak{m}_V \longrightarrow W/\mathfrak{m}_W$ is surjective:
Let $w \in W$, $w > \mathfrak{m}_W$ and $a \in A$ such that $a - w \in \mathfrak{m}_B$. By assumption there is some $v \in V$ such that $w < v$. Thus $0 < a < v + (a - w) < v + 1 \in V$, hence $a \in V$ and $a - w \in \mathfrak{m}_W$.

(ii) follows from the commutative diagram with surjective columns:

$$\begin{array}{ccc} M^*/A^* & \longrightarrow & N^*/B^* \\ \downarrow & & \downarrow \\ M^*/V^* & \longrightarrow & N^*/W^* \end{array}$$

//

Lemma 18.9. (*Reduction Lemma*)
Let $M \prec N$ be models of T and let A be a convex valuation ring of M such that $A \subseteq A_{\alpha/M}$ for all $\alpha \in N \setminus M$. Let B be the convex hull of A in N and let W be a convex valuation ring of N such that $V := W \cap M \neq A$. If $\Gamma_A = \Gamma_B$, then $\Gamma_V = \Gamma_W$.

Proof. We write v for the valuation induced by W. By (ii) of the lemma above we may assume that $V \subsetneq A$. We have the following commutative diagram:

$$\begin{array}{ccc} A^*/V^* & \xrightarrow{i} & B^*/W^* \\ \downarrow & & \downarrow \\ M^*/V^* & \xrightarrow{\varphi} & N^*/W^* \\ \downarrow & & \downarrow \\ M^*/A^* & \xrightarrow{\psi} & N^*/B^* \end{array}$$

By assumption ψ is onto. By Lemma 18.6 (ii) we know that $\kappa_A \prec \kappa_B$ is a dense extension. Therefore the value group A^*/V^* of $V \bmod \mathfrak{m}_A \subseteq \kappa_A$ is the same as the value group B^*/W^* of $W \bmod \mathfrak{m}_B \subseteq \kappa_B$. This means that i is an isomorphism. The commutative diagram now gives that φ is onto, which means $\Gamma_V = \Gamma_W$. //

Theorem 18.10.
Let $\mathrm{rk}_M(N/M) = 1$ (*this means: if* p *and* q *are cuts of* M *which are realized in* N *then* $p \sim q$) *and let* W *be a convex valuation ring of* N *with valuation* v*. If* W *is the convex hull of* $W \cap M$ *and* $v(N) \neq v(M)$ *then for all* $\alpha \in N \setminus M$ *there is some* $a \in M$ *such that* $v(\alpha - a) \notin v(M)$.

Proof. We write $V := W \cap M$. We already know:

(i) The Theorem holds if $A = M$, hence we can assume that A is proper.
(ii) $M \prec N$ is not dense, because $\Gamma_V \neq \Gamma_W$.
(iii) The Theorem holds if $M \prec N$ is tame.

We prove the Theorem. Let A be the convex valuation ring of M according to the realization rank 1 extension $M \prec N$. By (ii) and (iii) we can assume that A is proper, that is $M \prec N$ is neither dense nor tame. Let $p := t(\alpha/M)$.

Case 1: $A = V$ Choose $\beta \in N$ such that $v(\beta) \notin v(M)$ and let $q := t(\beta/M)$. We may assume that $0 < \beta < 1$. Because $v(\beta) \notin v(M)$ we have $q = \hat{q}$. Since T is polynomially bounded the strong signature alternative (18.1) implies that $\operatorname{sign} p \neq 0$ (we have $p \sim q$ and $\operatorname{sign} q = 1$). This means that $p = a \pm \hat{p}$ for some $a \in M$ and we may replace p by $\hat{p}$. Thus we assume that α realizes the upper boundary of a proper convex subgroup of M. We show that $v(\alpha) \notin v(M)$.

Case 1.1: $\operatorname{sign}^* p = 0$ Then B is the unique convex valuation ring of N, lying over A (14.4). Let r be a coheir of p on N. Then $A_r = B$ and by Thesis 9.5 we get $v(\alpha) \notin v(M)$.

Case 1.2: $\operatorname{sign}^* p \neq 0$ Thus $p = a \cdot A^+$ or $p = a \cdot \mathfrak{m}_A^+$ for some $a \in M$. Hence $V^+ = A^+$ is realized in N and we can suppose that α realizes V^+. By assumption W is the convex hull of $V = A$ and it is clear that $v(\alpha) \notin v(M)$.

Case 2: $A \neq V$ By Lemma 18.9, we get from $\Gamma_V \neq \Gamma_W$ that $V \subsetneq A$ as well as $\Gamma_A \neq \Gamma_B$ if B denotes the convex hull of A in N. By Case 1 there is some $a \in M$ such that $v_B(\alpha - a) \notin v_B(M)$ if v_B denotes the valuation of B. If we look at the commutative diagram of the proof of Lemma 18.9 we get at once that $v(\alpha - a) \notin v(M)$ //

Corollary 18.11. *If* $\dim N/M = 1$ *and* v *is a convex valuation of* N *such that* $v(N) \neq v(M)$ *then for all* $\alpha \in N \setminus M$ *there is some* $a \in M$ *such that* $v(\alpha - a) \notin v(M)$.

Proof. Let W be the valuation ring of v and $V = W \cap M$. If W is not the convex hull of V, then W is in addition the largest convex valuation ring of N lying over V, since $\dim N/M = 1$. But this contradicts Lemma 18.7. Thus W is the convex hull of V and we can apply the Theorem. //

Corollary 18.12. *If* $\dim N/M = 1$ *and* W *is a convex valuation of* N *lying over* V, *then* $\kappa_V = \kappa_W$ *or* $\Gamma_V = \Gamma_W$.

Proof. By Corollary 18.11 we can reduce the situation to the case of real closed fields, where we know the assertion. //

Counting heirs

With the weak signature alternatives for T and $T^{>0}$ we can count the heirs of a cut on an elementary extension roughly. Proposition 18.13 below is formulated with left Morley sequences (cf end of §2) in order to give at least a small piece of a uniform information.

Let $M \prec N$ be models of T and let p be a cut of M. We write

$$\operatorname{heir}(p, N) := \{q \in S_1(N) \mid q \text{ is an heir of } p\}$$

By definition, $\operatorname{heir}(p, N)$ is a closed subset of $S_1(N)$.

Proposition 18.13.

(i) If $\dim N/M$ *is finite and* $p \sim A_p^+$ *is not definable, then*

$$\operatorname{card}\operatorname{heir}(p, N) = 1 + \max\{\operatorname{card}\{\alpha_1, ..., \alpha_n\} \mid (\alpha_1, ..., \alpha_n) \textit{ is a left Morley sequence of } p \textit{ in } M\}$$

(ii) Let $p \not\sim A_p^+$ *and assume that* p *is not definable.*

(a) If $\operatorname{card}\operatorname{heir}(p, N) \geq 3$, *then* $\operatorname{heir}(p, N)$ *has no isolated points.*

(b)

$$\operatorname{card}\operatorname{heir}(p,N) = \begin{cases} 1 & \textit{if } p \textit{ is omitted in } N \\ 2 & \textit{if } p \textit{ is realized in } N \textit{ and there is no} \\ & \textit{left Morley sequence of } p \textit{ in } N \textit{ of length 2} \\ \geq 2^{\aleph_0} & \textit{if } p \textit{ is realized in } N \textit{ and there is a} \\ & \textit{left Morley sequence of } p \textit{ in } N \textit{ of length 2} \end{cases}$$

Proof. Clearly (i) holds, since $\dim N/M$ is finite.

(ii) (b) follows immediately from (a). Therefore it remains to show:
If $\operatorname{card}\operatorname{heir}(p,N) \geq 3$ and q is an heir of p on N, then q is not isolated in $\operatorname{heir}(p,N)$.

Because $\operatorname{card}\operatorname{heir}(p,N) \geq 3$, there are realizations α,β of p in N such that the interval (α,β) contains an heir of p. Thus $t(\beta/M\langle\alpha\rangle)$ is the largest extension of p on $M\langle\alpha\rangle$.

Suppose first, that $\operatorname{sign} p = 0$. Let $G := \operatorname{co}_N W_0(p)$. We have $W_0(\beta/M\langle\alpha\rangle) = W_\alpha(p)$. If q is the largest extension of p on N, then $G \subsetneqq W_0(q)$ and $\alpha + G^+ \neq q$ converges to q if α is a realization of p which converges to q. For $\operatorname{sign} p = 0$, we know that $\alpha + G^+$ is an heir of p. The same argument works if q is the least extension of p on N. If q is neither the least nor the largest extension of p on N and $\operatorname{sign} q \leq 0$, we can choose realizations α of p in N, $\alpha < q$ arbitrary close to q. Now $\alpha + G^+$ is an heir of p, different from q and between α and q.

If $\operatorname{sign} p \neq 0$, we can suppose that $p = \hat{p}$. By assumption $\operatorname{sign}^* p = 0$. If we apply the above proof to $T^{>0}$ we get (a). //

Heirs of box types

Let T be an o-minimal, polynomially bounded expansion of RCF.

This last application can be seen as an attempt to get some informations of heirs for types $p \in S_2(M)$. We only have tiny results. Suppose $M \prec N$ are models of T and $p_1, ..., p_k$ are cuts of M. Suppose furthermore that q_i is an heir of p_i on N for every i. We are interested in conditions which guarantee a type $r \in S_k(N)$, such that $q_1(x_1) \cup ... \cup q_k(x_k) \subseteq r$ and such that r is an heir of $r \restriction M$.

The chief difficulty is to give conditions in the case where $(p_1, ..., p_k)$ is a box type.

Proposition 18.14. *Let $M \prec N$ be models of T, let p be a cut of M and q an extension of p on N. Let $M \prec M_1 \prec N$.*

(i) If q is an heir of p, p is not dense and no member of $\operatorname{box}(p)$ *is realized in M_1 then q is also an heir of $q \restriction M_1$.*

(ii) If p is dense, p is omitted in M_1 and $M \subseteq M_1$ is archimedean, then q is an heir of $q \restriction M_1$.

Proof. (ii) follows from Lemma 4.9. Note that q need not be an heir of $q \restriction M_1$ if p is dense and omitted in M_1.

(i) We write q_1 for $q \restriction M_1$. The assertion is clear if p is definable. Thus we can assume that $\hat{p}$ is not definable. In any case $\hat{p}$ is omitted in M_1 and $q_1 := q \restriction M_1$ is an heir of p. Therefore $W_0(q_1)$ is the convex hull of $W_0(p)$ in M_1 and we get $W_0(q_1) \subseteq W_0(q)$. A similar argument shows that A_q lies over A_{q_1}. If $\operatorname{sign} p = 0$ then $\operatorname{sign} q_1 = 0$ and q is an heir of q_1. Hence we assume $\operatorname{sign} p$ to be non zero. If $\operatorname{sign}^* \hat{p} = 0$, then $\operatorname{sign}^* \hat{q}_1 = 0$ and q is a heir of q_1. If $\operatorname{sign}^* \hat{p}$ is not 0, say $\hat{p} = c \cdot A_p^+$, then $\hat{q}_1 = c \cdot A_{q_1}^+$ and $\hat{q} = c \cdot A_q^+$, thus q is an heir of q_1. //

For example box(p) is omitted in M_1 if $A_p \neq A_{\alpha/M}$ for all $\alpha \in M_1$: neither p, nor $\hat{p}$ or A_p^+ is realized in M_1.

Theorem 18.15. *Let $M \prec N$ be models of T, let $p_1, ..., p_k$ be cuts of M and let $q_1, ..., q_k \in S_1(N)$ be heirs of $p_1, ..., p_k$ respectively. Assume $A_{p_i} \neq A_{p_j}$ $(i \neq j)$. Then there is a type $r \in S_k(N)$ such that $q_i(x_i) \subseteq r$ $(1 \leq i \leq k)$, which is an heir over M.*

Proof. Preliminary notes:

(1) From Theorem 11.9 we know that $(p_1, ..., p_k)$ is a box type.
If $\alpha \in N$ is a realization of p_1 and $p_i' := q_i \restriction M\langle\alpha\rangle$, we still have $A_{p_i'} \neq A_{p_j'}$ $(i \neq j)$, because p_i' is a coheir of p_i for all i, that is $A_{p_i'}$ lies over A_{p_i}. By Proposition 18.14, q_i is an heir of p_i' for all $i \geq 2$.

(2) We apply (1) to the p_i' and get: if β is a realization of p_1' in N, then $A_{p_i''} \neq A_{p_j''}$ for all $i \neq j,\ i, j \geq 2$, if p_i'' denotes the unique extension of p_i' on $M\langle\alpha, \beta\rangle$ $(2 \leq i \leq k)$.

(3) If $\gamma \in M\langle\alpha, \beta\rangle$, then $A_{\gamma/M\langle\alpha\rangle}$ is the valuation ring $A_{p_1'}$, which lies over A_{p_1} is different from $A_{p_i'}$ $(i \geq 2)$. By Proposition 18.14 again, q_i is an heir of p_i'' for all $i \geq 2$.

We prove the theorem by induction on k. $k = 1$ is obvious. Assume we have proved the Proposition for k; Let $p, p_1, ..., p_k \in S_1(M)$ with mutually distinct valuation rings and $(q, q_1, ..., q_k)$ an extension of $(p, p_1, ..., p_k)$ on N. We have to show the following:

If $\bar{\alpha}, \bar{\beta} \in N^k$, $\gamma, \delta \in N$ and $\varphi(u, v, \bar{w}, \bar{z})$ is a $(2k+2)$-formula in the language $L(M)$, such that

$$\gamma < x < \delta \in q,\ \alpha_i < y_i < \beta_i \in q_i\ (1 \leq i \leq k) \text{ and } N \models \varphi(\gamma, \delta, \bar{\alpha}, \bar{\beta})$$

then there are $\bar{a}, \bar{b} \in M^k$, $c, d \in M$ such that

$$c < x < d \in p,\ a_i < y_i < b_i \in p_i\ (1 \leq i \leq k) \text{ and } M \models \varphi(c, d, \bar{a}, \bar{b})$$

We may assume that δ and γ are realizations of p or elements of M. Moreover we may assume that δ or γ is a realization of p (because if not, we can take the induction hypothesis). We can suppose as well, that α_i or β_i is a realization of p_i $(1 \leq i \leq k)$.

Case 1 γ and δ is a realization of p.
By the preliminary remark (3), we know that we can use the induction hypothesis for $M\langle\gamma,\delta\rangle \prec N$ and the types $q_1 \upharpoonright M\langle\gamma,\delta\rangle, ..., q_k \upharpoonright M\langle\gamma,\delta\rangle$. Therefore we may assume that $N = M\langle\gamma,\delta\rangle$. Because q is an heir of p there is some $r \in S_{k+1}(N)$, which contains $q(x)$ and $p_1(y_1) \cup ... \cup p_k(y_k)$. But the p_i's only have one extension to N by the preliminary remark. This shows that r has the desired properties.

Case 2 γ or δ is not a realization of p.
Similar and more easy than case 1. //

Corollary 18.16. *Let $\mathscr{L}(\mathcal{O}_1, ..., \mathcal{O}_k)$ be the language $\mathscr{L}$ together with k new relation symbols and let T^* be the $\mathscr{L}(\mathcal{O}_1, ..., \mathcal{O}_k)$-theory T together with the statement, that $\mathcal{O}_1, ..., \mathcal{O}_k$ are mutually distinct convex valuation rings. Then T^* is model complete relative $\mathscr{L}$.* //

If T is an arbitrary expansion of RCF then the Corollary is still true if we replace 'convex' by 'T-convex': Theorem 18.15 (with the same proof) is true for the types $A_1^+, ..., A_k^+$ if all A_i are T-convex.

§19 Quantifier elimination

We search quantifier elimination results for the model complete theories of §16 which describes the structures (M, d^p) in the case T is universally axiomatized and has quantifier elimination. By §10 and §4 we have quantifier elimination for T_{convex} and T^{dense} in this case. First we give a negative result:

Proposition 19.1. *Let $\mathscr{L}$ be a language, where the theory RCF admits an universal axiomatization. Let T be the $\mathscr{L}_{convex}(\mathcal{D})$-theory, which extends RCF_{convex} and says the following things:*

(a) $\mathcal{D}$ is the set of left options of a cut with signature 0.
(b) $\mathcal{O}$ is the invariance group of $\mathcal{D}^+$.

Then T does not have quantifier elimination: the formula

$$\exists u\ u > 0 \wedge \neg\mathcal{D}(u) \wedge \mathcal{O}(x\cdot u)$$

is not equivalent modulo T to a quantifier free formula.

Proof. Let M_0 be a real closed field and A_0 be a proper convex valuation ring of M_0. Let $A_0 < b \in M_0$ and $p_0 := b\cdot A_0^+$. Because $A_0 \subseteq W_0(p_0)$ there is an extension p_1 of p_0 on some $M_1 \succ M_0$ such that $\operatorname{sign} p_1 = 0$ and $A_1 := W_0(p)$ is the convex hull of A_0 in M_1 (Corollary 5.21). Thus A_1 is a proper convex valuation ring of N. Let B_1 be the largest extension of A_0 on M_1. Again Corollary 5.21 applied to $q := b\cdot A_1^+$ and $B_1 \subseteq b\cdot A_1 = W_0(q)$ there is some $M_2 \succ M_1$ and an extension p_2 of q on M_2 such that $\operatorname{sign} p_2 = 0$ and $A_2 := W_0(p_2)$ is the convex hull of B_1 in M_2.

Thus we have the models (M_1, A_1, p_1^L) and (M_2, A_2, p_2^L) of T and the common substructure (M_0, A_0, p_0^L) with $(M_0, A_0) \models RCF_{convex}$. Because $\frac{1}{b}p_0 = A_0^+$, we know that $\frac{1}{b}p_1$ extends A_0^+. Since p_1 is not an heir of p_0 we must have $A_1^+ < \frac{1}{b}p_1 < B_1^+$. Because $q < p_1$ we get $\frac{1}{b}p_2 < A_2^+$. Finally we have

$$(M_1, A_1, p_1^L) \not\models \exists u\ u > \mathcal{D} \wedge \frac{1}{b}u \in \mathcal{O}$$

and

$$(M_2, A_2, p_2^L) \models \exists u\ u > \mathcal{D} \wedge \frac{1}{b}u \in \mathcal{O}$$

Thus $\exists u\ u > 0 \wedge \neg\mathcal{D}(u) \wedge \mathcal{O}(x\cdot u)$ is not equivalent to a quantifier free formula modulo T. //

Now it is not difficult to see, that quantifier elimination for the remaining theories can hold only for $T_{res,dense}$ if T is reasonable. Indeed:

Theorem 19.2. *Let T be an o-minimal, polynomially bounded expansion of RCF. Suppose T has quantifier elimination and is universally axiomatized. Then $T_{res,dense}$ has quantifier elimination. The theory $T_{res,dense}\&\mathcal{D}(1)$ as well as the theory $T_{res,dense}\&\neg\mathcal{D}(1)$ is complete.*

Proof. We work in the language $\mathscr{L}_{convex}(\mathcal{G}) = \mathscr{L}(\mathcal{O}, \mathcal{G})$. We follow the same concept as for the proof of quantifier elimination for $DOAG_{zero}$ in §7.

Let $(M, A, G), (N, B, H) \models T_{res,dense}$ very saturated and (U, V, W) be a small common substructure.

Claim

There are small models $(M_0, A_0), (N_0, B_0)$ of T_{convex} and convex subgroups G_0, H_0 of M_0, N_0 respectively, such that

$$(U,V,W) \subseteq (M_0,A_0,G_0) \subseteq (M,A,G),$$
$$(U,V,W) \subseteq (N_0,B_0,H_0) \subseteq (N,B,H)$$

and an U-isomorphism $(M_0,A_0,p_0) \longrightarrow (N_0,B_0,q_0)$.

PROOF. For T has a universal systems of axioms and T has quantifier elimination, U is an elementary substructure of M and N. We can suppose that $1 \in G, H$ since this is witnessed in W.

Suppose that $V = U$. By saturation and because T is polynomially bounded, there are small models $M_1 \prec M$ and $N_1 \prec N$ containing U such that $M_1 \subseteq G$, $N_1 \subseteq H$ and such that U is not archimedean in M_1, N_1 respectively. We take $\alpha \in M_1$ $\alpha > U$ and $\beta \in N_1$, $\beta > U$. The U-isomorphism, which sends α to β sends $A \cap U\langle\alpha\rangle (\neq U\langle\alpha\rangle)$ onto $B \cap U\langle\beta\rangle$. This gives claim 1.

We want to prove quantifier elimination of $T_{res,dense}$. Because of the claim and model completeness, it remains to show:

Let $(M,A,G),(N,B,H) \models T_{res,dense}$ with $1 \in H, G$ very saturated and let (U,V,W) be a small common substructure such that $(U,V) \models T_{convex}$. Then there are models (M',A',G') and (N',B',H') of $T_{res,dense}$ such that

$$(U,V,W) \subseteq (M',A',G') \subseteq (M,A,G),$$
$$(U,V,W) \subseteq (N',B',H') \subseteq (N,B,H)$$

together with an U-isomorphism $(M',A',G') \longrightarrow (N',B',H')$.

We describe two constructions now, which we'll use alternately to find these structures.

Construction 1 Let $(U,V) \models T_{convex}$ and let (U,V,W) be a substructure of (M,A,G). If $V \neq A(W)$, then $V^+ \not\sim W^+$ and there is a realization $\alpha \in M$ of W^+ such that $G \cap U\langle\alpha\rangle$ is the convex hull of V in $U\langle\alpha\rangle$.

PROOF. If $V^+ \sim W^+$, then $A(W) = V$ (Rigidity of convex valuation rings). The remaining part of the claim is exactly Lemma 5.22 applied to $T^{>0}$.

Construction 2 Suppose, that $(U,V) \models T_{convex}$ and let (U,V,W) be a substructure of (M,A,G) such that $W^+ = a{\cdot}V^+$ with $a \in U$. Then there is a realization $\alpha \in M$ of V^+ such that

(a) $A \cap U\langle\alpha\rangle$ is the convex hull of V in $U\langle\alpha\rangle$

(b) $G \cap U\langle\alpha\rangle = a{\cdot}r$, where r denotes the largest extension of V^+ on $U\langle\alpha\rangle$.

PROOF.

We define the following set of $\mathscr{L}(\mathcal{O},\mathcal{G})$-formulas with parameters from U:

$$\begin{aligned}\Psi :=&\{x < c \mid c \in U,\ V < c\} \cup \{\neg\mathcal{O}(x)\} \cup \\ &\cup \{\mathcal{G}(a + x\cdot F(x)) \mid F : U \longrightarrow U,\ U\text{-definable},\ F(V^+) = V^+, \\ &\qquad F(x) \geq x \text{ and } F \text{ increasing on } U\}\end{aligned}$$

Let F be a map as in this definition and let $F_1(x) := a\cdot x\cdot F(x)$. Then $F_1(A^+) \leq G^+$: if $F_1(A^+) > G^+$ we find some $\xi \in M$ with $0 < \xi \in A$ such that $F_1([\xi, +\infty)) > \gamma > G$. For A^+ is an heir of V^+ we get $F(\xi) < A$ and $a\cdot\xi\cdot F(\xi) > G$ which is not possible.

Therefore we know $F_1(A^+) \leq G^+$, hence $< G^+$, because $\operatorname{sign}^* G^+ = 0$. By saturation there is an element $\alpha \in M$, such that $(M, A, G) \models \Psi(\alpha)$. This α has the required properties:

(a) $A \cap U\langle\alpha\rangle$ is a coheir of V^+. Because α is a realization of V^+ outside A, it must be the least coheir.
(b) holds by the definition of Ψ.

We prove quantifier elimination of $T_{res,dense}$ in finding the structures (M', A', G') and (N', B', H') as described above:

We build countable chains

$$(U, V, W) \subseteq (M_0, A_0, G_0) \subseteq (M_1, A_1, G_1) \subseteq ... \subseteq (M, A, G)$$

and

$$(U, V, W) \subseteq (N_0, B_0, H_0) \subseteq (N_1, B_1, H_1) \subseteq ... \subseteq (N, B, H)$$

where $(M_i, A_i),\ (N_i, B_i) \models T_{convex}$ together with U-isomorphisms

$$\sigma_i : (M_i, A_i, G_i) \longrightarrow (N_i, B_i, H_i)$$

such that σ_{i+1} extends σ_i and such that $A(G_{2i}) = A_{2i}$, $A(H_{2i}) = B_{2i}$ are the convex hulls of V in M_{2i}, N_{2i} respectively:

$\underline{i = 0}$ if $V \neq A(W)$ we choose $\alpha \in M$ as described in Construction 1 and similar $\beta \in N$ as described in Construction 1, as well as $M_0 = U\langle\alpha\rangle$ and $N_0 = U\langle\beta\rangle$. We have no more choice for A_0, G_0, B_0 and H_0. By Construction 1, the map σ_0, which sends α to β and fixes U pointwise has the required properties.

if $V = A(W)$ and $\operatorname{sign}^* W^+ \geq 0$ we take $M_0 = U = N_0$.

if $V = A(W)$ and $W^+ = a\cdot\mathfrak{m}_V$, we replace the groups G, H and W by $\frac{a^2}{G^+}$, $\frac{a^2}{H^+}$ and $\frac{a^2}{W^+}$ respectively and proceed with the case $\operatorname{sign}^* W^+ = 1$. (if the construction is finished we apply again the map $\frac{a^2}{x}$ to all groups G, H, G_i, H_i and to W to get the required chains).

we construct M_{i+1}, N_{i+1} if M_i, N_i are constructed in the following way:

<u>i is even</u> if $\operatorname{sign}^* G_i^+ = 0$ we take $M_{i+1} = M_i$ and $N_{i+1} = N_i$ (note that by the inductive hypothesis $A(G_i) = A_i$ and $A(H_i) = B_i$, thus we can stop at all).

if $\operatorname{sign}^* G_i^+ \neq 0$, we see inductively that $\operatorname{sign}^* G_i = 1$ and similar $\operatorname{sign}^* H_i = 1$. Now we apply construction 2: choose $\alpha \in M$ as described in construction 2 with respect to M_i and $\beta \in N$ as described in construction 2 with respect to N_i. We choose $M_{i+1} = M_i\langle\alpha\rangle$ and $N_{i+1} = N_i\langle\beta\rangle$. By construction 2, the map σ_{i+1}, which extends σ_i and sends α to β sends A_{i+1} onto B_{i+1} and G_{i+1} onto H_{i+1}.

<u>i is odd</u> if $\operatorname{sign}^* G_i = 0$ we take $M_{i+1} = M_i$ and $N_{i+1} = N_i$ (note that this can only happen if $A(G_i) = A_i$ and $A(H_i) = B_i$, again we can stop in this case).

if $\operatorname{sign}^* G_i^+ \neq 0$, we see inductively that $\operatorname{sign}^* G_i^+ = 1$ and similar $\operatorname{sign}^* H_i^+ = 1$. Now we apply construction 1 again: choose $\alpha \in M$ as described in construction 1 with respect to M_i and $\beta \in N$ as described in construction 1 with respect to N_i. We choose $M_{i+1} = M_i\langle\alpha\rangle$ and $N_{i+1} = N_i\langle\beta\rangle$. By construction 1, the map σ_{i+1}, which extends σ_i and sends α to β sends A_{i+1} onto B_{i+1} and G_{i+1} onto H_{i+1}.

The construction of the chains is completed. We take $M' = \bigcup M_i$ and $N' = \bigcup N_i$. Because A_{2i} is the convex hull of V we have $A' = A(G')$ (Proposition 5.19 applied to $T^{>0}$).
Furthermore $\operatorname{sign}^* G'^+ = 0$: since $A(G_{2i}) \not\subseteq A(G_{2i+1})$ (by construction 2) we can apply (Proposition 5.19 (v)). For the same reason we have $\operatorname{sign}^* H' = 0$. If we take $\sigma' := \bigcup \sigma_i$ we find the promised isomorphism.

Thus we know that $T_{res,dense}$ has quantifier elimination. Because the prime model of T is archimedean we have the prime structure $(\mathcal{P}, \mathcal{P}, \mathcal{P})$ of the theory $T_{res,dense}\&\mathcal{D}(1)$ and the prime structure $(\mathcal{P}, \mathcal{P}, 0)$ of the theory $T_{res,dense}\&\neg\mathcal{D}(1)$. Certainly both theories are consistent, hence complete. //

Appendix A The derivative of a definable map

In this appendix we prove:

Proposition A.1. *Let M be an o-minimal expansion of a real closed field and let $F(x,\bar{y}) : M^{n+1} \longrightarrow M$ be a 0-definable map. Then for each $k \in \mathbb{N}$ there is some $K \in \mathbb{N}$ such that for all $\bar{a} \in M^n$ there is a subset $Z \subseteq M$ with* $\operatorname{card} Z \leq K$ *and such that $F(x,\bar{a})$ is C_k on $M \setminus Z$.*

If $F : M \longrightarrow M$ is an M-definable map, differentiable on (a,b), then the tangents on the graph of F at points $x \in (a,b)$ has the same approximation properties as usual.

Certainly the proof is not difficult. I have not found a source for this Proposition and I guess that it is known.

My prove is a combination of the 'definable function Theorem' ([PS]) with the following

Lemma A.2. *Let $F : [a,b] \longrightarrow M$ be a continuous, M-definable map and $t := \frac{F(b)-F(a)}{b-a}$. Then:*

(i) there is some $x \in [a,b)$ such that $\frac{F(y)-F(x)}{y-x} \leq t$ for all $y \in (x,b)$.
(ii) there is some $x \in [a,b)$ such that $\frac{F(y)-F(x)}{y-x} \geq t$ for all $y \in (x,b)$.
(iii) there is some $x \in (a,b]$ such that $\frac{F(y)-F(x)}{y-x} \leq t$ for all $y \in (a,x)$.
(iv) there is some $x \in (a,b]$ such that $\frac{F(y)-F(x)}{y-x} \geq t$ for all $y \in (a,x)$.

PROOF. (i) We write $c := F(a)$ and $d := F(b)$. Suppose (i) does not hold.

Let $f(x) = tx + (c - ta)$ the line trough (a,c) and (b,d). Let $x_1 \in (a,b)$ such that $\frac{F(x_1)-F(a)}{x_1-a} > t$ and $r = F(x_1) - f(x_1)$. We have $r > 0$:

$F(x_1) > F(a)+t(x_1-a) = c+t(x_1-a)$ and $r = F(x_1)-f(x_1) > c+t(x_1-a)-f(x_1) = c + t(x_1 - a) - tx_1 - (c - ta) = 0$

By o-minimality the supremum of all $x \in [a,b]$ such that $F(x) - f(x) \geq r$ exists; we write s for this supremum. For F and f are continuous on $[a,b]$, we even have $F(s) - f(s) \geq r$. We prove $s = b$ (in contradiction to $f(b) = F(b)$): suppose $s < b$. Take $s' \in (s,b)$ such that $\frac{F(s')-F(s)}{s'-s} \geq t$. Then

$F(s') \geq F(s)+t(s'-s) \geq f(s)+r+t(s'-s) = ts+(c-ta)+r+t(s'-s) = f(s')+r$, in contradiction to the choice of s.

We proved (i). (ii) is (i) for $-F(x)$. (iii) is (i) for $-F(b+a-x)$ and (iv) is (i) for $F(b+a-x)$. //

We prove Proposition A.1:

Because the theory of M is o-minimal it is enough to prove: if $F : M \longrightarrow M$ is M-definable, then F is differentiable up to finitely many points; for the derivative of F is definable again, the statement 'k-times differentiable' is only a cosmetic procedure.

Suppose there is an interval $[a,b]$ such that $F|[a,b]$ is continuous and a strictly increasing homeomorphism onto $[F(a),F(b)]$, such that F is not differentiable in all $x \in (a,b)$.
If $x \in (a,b)$ let $G_x(y) := \frac{F(y)-F(x)}{y-x}$ defined on $(a,b) \setminus \{x\}$. For $G_x(x^{\pm})$ is definable, the interval (a,b) is decomposed into M-definable subsets $(a,b) = C_1 \cup C_2 \cup C_3$, where

$C_1 = \{x \in (a,b) \mid G_x(x^+) = +\infty\}$
$C_2 = \{x \in (a,b) \mid G_x(x^-) = +\infty\}$ and
$C_3 = \{x \in (a,b) \mid G_x(x^-) \neq G_x(x^+),\ \text{and}\ G(x^+), G(x^-) \neq +\infty\}$

If C_1 contains an interval $[c,d]$, then we apply Lemma A.2.(i) to get a contradiction. If C_2 contains an interval $[c,d]$, then we apply Lemma A.2.(iii) to get a contradiction.

Therefore we can suppose that C_3 contains an interval (a',b'). For each $x \in (a',b')$ let $h_{\pm}(x)$ be the element of M such that $G_x(x^{\pm})$ is infinitely close to $h_{\pm}(x)$. Certainly h_- and h_+ are M-definable. Let $[c,d] \subseteq (a',b')$ be an interval, such that h_- and h_+ are continuous on $[c,d]$. For $h_-(x) \neq h_+(x)$ ($x \in [c,d]$) we know by the mean value theorem $h_- < h_+$ or $h_- > h_+$ on $[c,d]$, say $h_- < h_+$. By eventually shrinking $[c,d]$ we can suppose that there is some $r \in M$ such that $h_- < r < h_+$ and such that $\frac{F(d)-F(c)}{d-c} > r$. By Lemma A.2.(iv) there is some $x \in (c,d]$ such that $\frac{F(y)-F(x)}{y-x} \geq \frac{F(d)-F(c)}{d-c}$ ($y \in (c,x)$), hence $\frac{F(y)-F(x)}{y-x} > r$ ($y \in (c,x)$). But this means $h_-(x) \geq r$, a contradiction. //

Appendix B Elimination theory used

We describe some model theory of A. Robinson in this appendix.

Propositions B.1 and B.2 are in the model theoretic air. We only give hints instead of proofes.

Proposition B.1. *Let Δ be a base of $\mathscr{L}$-formulas, let T be an $\mathscr{L}$-theory and let $\varphi(\bar{x})$ be an $\mathscr{L}$-formula with n free variables $\bar{x} = (x_1, ..., x_n)$. Suppose one of the following conditions holds for $\varphi(\bar{x})$:*

(A) For all T-models M, N and $\bar{a} \in M^n$, $\bar{b} \in N^n$ we have:
if for every $\delta(\bar{x}) \in \Delta$ $[M \models \delta(\bar{a}) \Rightarrow N \models \delta(\bar{b})]$ holds, then
$[M \models \varphi(\bar{a}) \Rightarrow N \models \varphi(\bar{b})]$ holds too.

or

(B) Δ is boolean. For all T-models M, N and $\bar{a} \in M^n$, $\bar{b} \in N^n$ we have:
if for every $\delta(\bar{x}) \in \Delta$ $[M \models \delta(\bar{a}) \Leftrightarrow N \models \delta(\bar{b})]$ holds, then
$[M \models \varphi(\bar{a}) \Rightarrow N \models \varphi(\bar{b})]$ holds.

Then there is a formula $\delta(\bar{x}) \in \Delta$ such that $T \vdash \forall \bar{x}(\delta(\bar{x}) \leftrightarrow \varphi(\bar{x}))$.

Proof. We use the following lemma on boolean algebras, which follows easily from the fact that the spectrum of a boolean algebra is a boolean space:

Lemma B.2. *(abstract elimination lemma)*
Let A be a boolean Algebra, E a sublattice of A and $Y, Z \subseteq \operatorname{Spec} A$ be closed sets.
Suppose that for $y \in Y$ and $z \in Z$ there is a $g \in E$ with $y \in D(g)$ and $z \in D(\neg g)$.
Then there is a $h \in E$ with $Y \subseteq D(h)$ and $Z \subseteq D(\neg h)$. //

To prove Proposition B.1 with assumption (A) take:
(i) A the boolean algebra of $\mathscr{L}$-formulas in the free variables $x_1, ..., x_n$ modulo T-equivalence.
(ii) E the distributive sublattice of A of formulas from Δ in the free variables $x_1, ..., x_n$
(iii) Y the n-types of T, which contain φ and $Z := S_n(T) \setminus Y$.

To prove Proposition B.1 with assumption (B), apply (A) to the smallest base of formula, which contains Δ and which is closed under "$\neg$". //

If we assume further properties on Δ we can weaken the assumption (A):

Proposition B.3. *Let Δ be a base of $\mathscr{L}$-formulas, let T be an $\mathscr{L}$-theory and let $\varphi(\bar{x})$ be an $\mathscr{L}$-formula with n variables $\bar{x} = (x_1, ..., x_n)$. Suppose the following conditions hold for Δ:*

(i) each quntifier free $\mathscr{L}$-formula is in Δ.
(ii) if $\delta(x_1, ..., x_n) \in \Delta$ and $t_1, ..., t_n$ are $\mathscr{L}$-terms, such that x_i is free for t_i in $\delta(x_1, ..., x_n)$, then $\delta(t_1, ..., t_n) \in \Delta$.

Further suppose the following condition holds:

if $M, N \models T$ and $\bar{a} \in M^n$ is an n-tupel, which generates the same substructure U in M and in N, such that for all $\delta(\bar{x}) \in \Delta$ we have

$$\bar{c} \in U^n,\ M \models \delta(\bar{c}) \Rightarrow N \models \delta(\bar{c}).$$

Then $M \models \varphi(\bar{a}) \Rightarrow N \models \varphi(\bar{a})$.

Then there is a formula $\delta(\bar{x}) \in \Delta$ such that $T \vdash \forall \bar{x}(\delta(\bar{x}) \leftrightarrow \varphi(\bar{x}))$. //

Definition and Corollary
Let $\mathscr{L} \subseteq \mathscr{L}'$ be signatures and let T' be an $\mathscr{L}'$-theory. Then the following conditions are equivalent:

(i) T' is **model complete relative** $\mathscr{L}$, that is:
If M', N' are models of T' with $M' \subseteq N'$ and $M' \restriction \mathscr{L} \prec N' \restriction \mathscr{L}$, then $M' \subseteq N'$ is existential.

(ii) each $\mathscr{L}'$-formula is equivalent to a finite disjunction of formulas of the form $\exists \bar{y}\ \varphi(\bar{x}, \bar{y}) \wedge \chi(\bar{x}, \bar{y})$, where φ is an $\mathscr{L}$-formula and χ is a quantifier free $\mathscr{L}'$-formula.

Outline of the proof:

Clearly (ii) implies (i). Assume (i) holds.

Claim If M', N' are models of T' with $M' \subseteq N'$ and $M' \restriction \mathscr{L} \prec N' \restriction \mathscr{L}$, then $M' \subseteq N'$ preserves formulas of quantification degree $n \in \mathbb{N}$.
This is proved by induction on n, where $n = 1$ holds by assumption.

Now we apply Proposition B.3 to the base of formulas generated by the formulas $\exists \bar{y}\ \varphi(\bar{x}, \bar{y}) \wedge \chi(\bar{x}, \bar{y})$, where φ is an $\mathscr{L}$-formula and χ is a quantifier free $\mathscr{L}'$-formula.//

Obviously T' is model complete if $T \restriction \mathscr{L}$ is model complete and T' is model complete relative $\mathscr{L}$.

Index of definitions and settings

$\mathbb{N}$	are the natural numbers $1, 2, 3, ...$ and ω are the natural numbers $0, 1, 2, ...$
p^L, p^R	denotes the set of left, right options of a cut p: $p = (p^L, p^R)$.
X^+, X^-	denotes the upper, lower boundary of the set X. Thus X^+ is the cut with right options $\{a \mid a > X\}$

an extension of a cut p is a 1-type extending p

p_- and p_+	denotes the least and the largest extension of a cut p on $M\langle p\rangle$
$W_0(p)$	$:= \{a \mid a + p = p\}$ is the invariance group of the cut p and $W(p)$ is the width of p (definition on page 37).
$W_0^*(p)$	$:= \{a \mid a{\cdot}p = p\}$ is the multiplicative invariance group of the cut p.
$A(G)$	is the valuation ring $\{a \mid a{\cdot}G \subseteq G\}$ for a convex subgroup G and $A_p := A(W_0(p))$ for a cut p.
$\hat{p}$	is the cut $W_0(p)^+$ and $\tilde{p}$ is the cut $(W_0^*(p) - W_0^*(p))^+$
ξ_G, ξ_p	are defined on pages 62 and 63
$p \sim q$	means that there is an M-definable map F, such that $F(p) = q$
$\operatorname{sign} p$	is defined on page 38

the weak and the strong signature alternative are defined on page 41

Δ	denotes a base of formulas (definition on page 7)
(M, d^p)	is defined on page 6
$\mathscr{L}^{\mathrm{def}\ c}$	is defined on page 6

$\operatorname{cl} A$	is the definable closure of a set A
$M\langle\bar{\alpha}\rangle$	is the definable closure of $M \cup \{\alpha_1, ..., \alpha_n\}$ and $M\langle p\rangle$ is $M\langle\bar{\alpha}\rangle$ if $\bar{\alpha} \models p$
dim	is dimension relative to a fixed o-minimal theory
$M^{>0}$	is a structure, not only the universe $\{a \in M \mid a > 0\}$; definition on page 41
$T^{>0}$	is the theory of $M^{>0}$
A-dominance	is defined on page 24

left (right) Morley sequences are defined on page 21

rk — is the realization rank, defined in §3

box(p) — is the p-box (definition on page 96)

box types — are defined on page 27

polynomially bounded theories have by definition an archimedean prime model

a weak heir — of a cut p is an extension q of p with $W_0(p) \subseteq W_0(q)$

residually dense groups are defined on page 88

T_{convex} — is the theory of T-convex, proper valuated models of T in the language $\mathscr{L}_{convex} := \mathscr{L} \cup \{\mathcal{O}\}$, where $\mathcal{O}$ is a predicate for the valuation ring.

T^{dense} — denotes the theory of a dense cut in the language $\mathscr{L}(\mathcal{D})$, where $\mathcal{D}$ is a predicate for the left options of a cut.

$T_{valring}$ and $T_{maxideal}$ are defined on page 88

$T_{res,dense}$ — is defined on page 89

$T^{\text{sign}=0}_{valring}$, $T^{\text{sign}=0}_{maxideal}$ and $T^{\text{sign}=0}_{res,dense}$ are defined on page 90

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