STABLE FORMULAS AND THEORIES

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Abstract. Short PG course notes, January–March 2021. (Excluding Examples.)

Contents

Some references: Here are 5 books about stability in model theory, which were used in the preparation of these notes: [Baldwi1988], [Buechl1996], [Casano2011], [Lascar1986], [Pillay1983], [Shelah1990]. For other items used in these notes we refer to the references at the end.

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1. Local Stability

Throughout, *language* refers to a first order language $\mathscr{L}$ of any cardinality. Let $\bar{x}$, $\bar{y}$ be tuples of variables and let $\varphi(\bar{x},\bar{y})$ be a **partitioned $\mathscr{L}$-formula**. This means that apart from φ we also have a distinct partition of the free variables $\bar{x},\bar{y}$ (which might or might not occur in φ). We write $\varphi^{\text{opp}}(\bar{y},\bar{x})$ for the formula $\varphi(\bar{x},\bar{y})$, where the partition of the variables is interchanged.

Let M be an $\mathscr{L}$-structure and let $X=\varphi(M^{\bar{x}}\times M^{\bar{y}})$ be the set defined by φ.[1] For $\bar{a}\in M^{\bar{x}}$, we call the formula $\varphi(\bar{a},\bar{y})$ as well as the set $S_{\bar{a}}=\varphi(\bar{a},M^{\bar{y}})$ the **fibre of φ (resp. X) above $\bar{a}$**. Each such formula/set is called a **fibre (or instance) above an $\bar{x}$-tuple**. If the free variables are clear from the context we will just say "fibre or instance of φ", or "fibre of X".

1.1. **Definition.** [DiScTr2019, section 14.1] If M is an $\mathscr{L}$-structure and $A\subseteq M$ we write

$$\Delta_\varphi(A)=\Delta_{\varphi,\bar{x}}(A)=\{\varphi(\bar{x},\bar{b})\mid \bar{b}\in A^{\bar{y}}\}$$

for the set of fibres of φ above $\bar{y}$-tuples of A, and

$$\Delta^\ell_\varphi(A)=\Delta^\ell_{\varphi,\bar{x}}(A)=\{\bigvee_{i\in I}\bigwedge_{j\in J}\varphi(\bar{x},\bar{b}_{i,j})\mid I,J\text{ finite},\ \bar{b}_{ij}\in A^{\bar{y}}\}$$

$$\Delta^\flat_\varphi(A)=\Delta^\flat_{\varphi,\bar{x}}(A)=\{\bigvee_{i\in I}\bigwedge_{j\in J}\varphi^{\varepsilon_{ij}}(\bar{x},\bar{b}_{i,j})\mid I,J\text{ finite},\ \bar{b}_{ij}\in A^{\bar{y}},\ \varepsilon_{ij}\in\{0,1\}\}$$

Here φ^0 stands for $\neg\varphi$ and φ^1 stands for φ. Hence $\Delta^\ell_\varphi(A)$ is the distributive sublattice of $\mathscr{L}_{\bar{x}}(A)$ generated by the fibres of φ above $\bar{y}$ tuples of A. Furthermore, $\Delta^\flat_\varphi(A)$ is the Boolean envelope of $\Delta^\ell_\varphi(A)$.

If $\bar{\alpha}\in N^{\bar{x}}$ with $N\succ M$ we define the **φ-type** of $\bar{\alpha}$ over A (for M) as $\text{tp}_\varphi(\bar{\alpha}/A)$

$$\text{tp}_\varphi(\bar{\alpha}/A)=\Delta_\varphi(A)\cap\text{tp}(\bar{\alpha}/A).$$

We write $S_\varphi(A)=S_{\varphi,\bar{x}}(A)$ for the set of φ-types, thus

$$S_\varphi(A)=\{p\cap\Delta_\varphi(A)\mid p\in S_{\bar{x}}(A)\}.$$

Recall from [DiScTr2019, Theorem 14.2.5] that $S_\varphi(A)$ is a spectral space, whose patch space is the quotient space of the map $S_{\bar{x}}(A)\longrightarrow S_\varphi(A),\ p\mapsto p\cap\Delta_\varphi(A)$, and whose specialization relation is set theoretic inclusion of φ-types.

Obviously, for each $p\in S_{\bar{x}}(A)$, the set $p\cap\Delta^\flat_\varphi(A)$ is uniquely determined by the φ-type $p\cap\Delta_\varphi(A)$. Hence for each $\mathfrak{p}\in S_\varphi(A)$ we may define

$$\mathfrak{p}^\flat=p\cap\Delta^\flat_\varphi(A),\ \text{where } p\in S_{\bar{x}}(A)\text{ with }p\cap\Delta_\varphi(A)=\mathfrak{p}.$$

More explicitly, $\mathfrak{p}^\flat$ is *the* ultrafilter of $\Delta^\flat_\varphi(A)$ that intersects $\Delta_\varphi(A)$ in $\mathfrak{p}$.

A subset Y of $A^{\bar{y}}$ is called **φ-external** or **φ-externally definable** if there is some $N\succ M$ and some $\bar{\alpha}\in N^{\bar{x}}$ such that

$$Y=\{\bar{b}\in A^{\bar{y}}\mid N\models\varphi(\bar{\alpha},\bar{b})\}.$$

We also denote this set by $\varphi(\bar{\alpha},A^{\bar{y}})$.

1.2. *Remark.* The map $S_\varphi(A)\longrightarrow\{Y\subseteq A^{\bar{y}}\mid Y\text{ is }\varphi\text{-external}\}$ that sends $\mathfrak{p}$ to $\{\bar{b}\in A^{\bar{y}}\mid\varphi(\bar{x},\bar{b})\in\mathfrak{p}\}$ is obviously a bijection.

[1] $M^{\bar{x}}$ is shorthand for M^k, where k is the length of the tuple $\bar{x}$.

1.3. **Definition.** Given subsets A, C of an $\mathscr{L}$-structure M, a type $\mathfrak{p} \in S_\varphi(A)$ is called **definable** over C if there is some $\psi(\bar{y}) \in \mathscr{L}(C)$ such that

$$\text{for all } \bar{b} \in A^{\bar{y}}: \quad \varphi(\bar{x}, \bar{b}) \in \mathfrak{p} \iff \models \psi(\bar{b}).$$ [2]

Such a formula ψ is sometimes written as $d_\mathfrak{p}\bar{x}\varphi(\bar{x}, \bar{y})$ and the expression $d_\mathfrak{p}\bar{x}$ here acts as a *generalized quantifier* (but note that this formula is not an $\mathscr{L}$-formula, only an $\mathscr{L}(C)$-formula)

In what follows we will be talking about theories T of a language $\mathscr{L}$. These always refer to consistent sets of $\mathscr{L}$ sentences. Hence T does not need to be complete, unless this is explicitly stated.

1.4. **Definition.** The formula φ is **stable** for an $\mathscr{L}$-theory T if for all $M \models T$ and every infinite $A \subseteq M$ we have $|S_\varphi(A)| \leq |A|$. Notice that we did not put any restriction on the size of the language.

1.5**. Definition and Remark** The formula φ has the **order property** for an $\mathscr{L}$-theory T if for all $n \in \omega$ there are $M \models T$ and $\bar{a}_0, \dots, \bar{a}_n \in M^{\bar{x}}, \bar{b}_0, \dots, \bar{b}_n \in M^{\bar{y}}$ such that

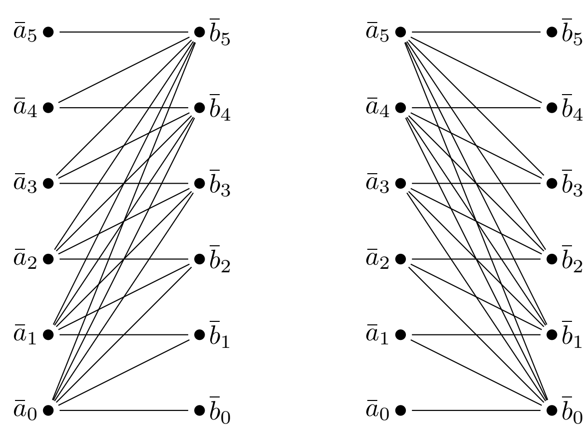
$$\varphi(\bar{a}_i, \bar{b}_j) \iff i \leq j \text{ for all } i, j \in \{0, \dots, n\}.$$

If φ does not have the order property we write $o(\varphi)$ for the largest $n \in \omega$ for which there is a configuration as above.

If φ does have the order property, then by compactness, for every chain I there are $M \models T$ and $\bar{a}_i \in M^{\bar{x}}, \bar{b}_i \in M^{\bar{y}}$ such that

$$\varphi(\bar{a}_i, \bar{b}_j) \iff i \leq j \text{ for all } i, j \in I.$$

Some digestion of the order property: If we consider $\varphi(\bar{x}, \bar{y})$ as a relation $\{\bar{a}_0, \dots, \bar{a}_n\} \longrightarrow \{\bar{b}_0, \dots, \bar{b}_n\}$, then the order property says that the graph of this relation is **bipartite half-complete** and can be pictured for $n = 5$ with the graph to the left of the following drawing:



The drawing to the right shows the order property of φ^{opp}.

[2]Hence if M is a model of T and $\mathfrak{p} = \text{tp}_\varphi(\bar{a}/M)$, then $\mathfrak{p}$ is definable over M, just if the external set $\varphi(\bar{a}, M^{\bar{y}})$ is definable (with parameters from M).

1.6. *Remark.* $\varphi(\bar{x},\bar{y})$ has the order property if and only if for all $n \in \omega$ there are $M \models T$ and $\bar{a}_0,\dots,\bar{a}_n \in M^{\bar{x}}$, $\bar{b}_0,\dots,\bar{b}_n \in M^{\bar{y}}$ such that

$$\varphi(\bar{a}_i,\bar{b}_j) \iff i<j \text{ for all } i\neq j\in\{0,\dots,n\}.$$

(In some sources, this is used as the definition of the order property.)

Proof. This is clear in one direction with the same choice of $\bar{a}_i$, $\bar{b}_i$. Conversely, suppose the condition of the remark holds. Take $\bar{a}_1,\dots,\bar{a}_{2n} \in M^{\bar{x}}$, $\bar{b}_1,\dots,\bar{b}_{2n} \in M^{\bar{y}}$ such that

$$\varphi(\bar{a}_i,\bar{b}_j) \iff i<j \text{ for all } i\neq j\in\{1,\dots,2n\}.$$

Then either, there are n indices $i \in \{1,\dots,2n\}$ such that $\varphi(\bar{a}_i,\bar{b}_i)$, in which case we let $i_1<\dots<i_n$ be such indices and get the order property witnessed for n and the sequences $\bar{a}_{i_1},\dots,\bar{a}_{i_n}$, $\bar{b}_{i_1},\dots,\bar{b}_{i_n}$,

or,

there are n indices $i \in \{1,\dots,2n\}$ such that $\neg\varphi(\bar{a}_i,\bar{b}_i)$, in which case we let $i_1<\dots<i_n$ be such indices and get the order property witnessed for $n-1$ and the sequences $\bar{a}_{i_1},\dots,\bar{a}_{i_{n-1}}$, $\bar{b}_{i_2},\dots,\bar{b}_{i_n}$. □

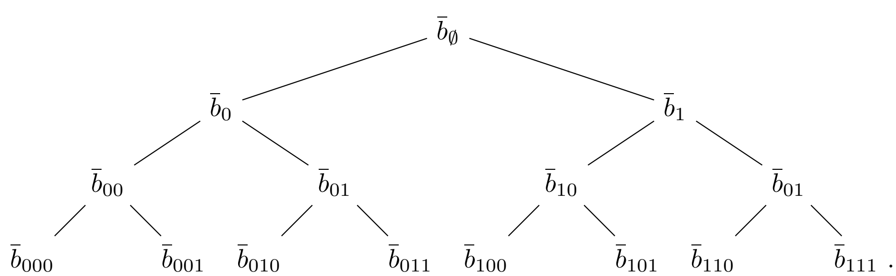
1.7. **Definition.** The partitioned formula $\varphi(\bar{x},\bar{y})$ has the **binary tree property** for T if there is a binary tree $(\bar{b}_s)_{s\in 2^{<\omega}}$ of $\bar{y}$-tuples of parameters in some $M \models T$ such that for all $\sigma \in 2^\omega$ the set

$$\{\varphi^{\sigma(n)}(\bar{x},\bar{b}_{\sigma|_n}) \mid n<\omega\}$$

is consistent. In other words, there are $\bar{a}_\sigma$ for $\sigma \in 2^\omega$ from some elementary extension N of M such that for all $n<\omega$ we have

$$\models \varphi(\bar{a}_\sigma,\bar{b}_{\sigma|_n}) \iff \sigma(n)=1.$$

In pictures: The binary tree $(\bar{b}_s)_{s\in 2^{<\omega}}$ looks like



The binary tree property says that M^n can be subdivided as follows:

$\sigma(0)$: There is some $\bar{b}$ such that M^n can be partitioned into two nonempty sets: $C_1=\varphi(M^n,\bar{b})$ and $C_0=\neg\varphi(M^n,\bar{b})$; this $\bar{b}$ is $\bar{b}_\emptyset$. Hence $C_s=\varphi^{s(0)}(M^n,\bar{b}_{s|0})$, where $s=\sigma|1:1\longrightarrow 2$.

If $\sigma(0)=1$ we are put into C_1, if $\sigma(0)=0$ we are put into C_0.

$\sigma(1)$: For each set X in the partition of the previous item, there is $\bar{b}$ such that $X\cap\varphi(M^n,\bar{b})$ and $X\cap\neg\varphi(M^n,\bar{b})$ are nonempty.

If $\sigma(1)=1$ we are put into $X\cap\varphi(M^n,\bar{b})$, if $\sigma(1)=0$ we are put into $X\cap\neg\varphi(M^n,\bar{b})$.

Since there are 2 choices for sets X in the previous item we get 2 such $\bar{b}$; these are $\bar{b}_0$ and $\bar{b}_1$.

We get a partition of M^n into 4 sets: $C_{00} = C_0 \cap \neg\varphi(M^n, \bar{b}_0)$, $C_{01} = C_0 \cap \varphi(M^n, \bar{b}_0)$, $C_{10} = C_1 \cap \neg\varphi(M^n, \bar{b}_1)$ and $C_{11} = C_1 \cap \varphi(M^n, \bar{b}_1)$. Hence

$$C_s = \bigcap_{i<2} \varphi^{s(i)}(M^n, \bar{b}_{s|i}), \text{ where } s : 2 \longrightarrow 2.$$

$\sigma(2)$: For each set X in the partition of the previous item, there is $\bar{b}$ such that $X \cap \varphi(M^n, \bar{b})$ and $X \cap \neg\varphi(M^n, \bar{b})$ are nonempty.

If $\sigma(1) = 1$ we are put into $X \cap \varphi(M^n, \bar{b})$, if $\sigma(1) = 0$ we are put into $X \cap \neg\varphi(M^n, \bar{b})$.

Since there are 4 choices for sets X in the previous item we get 4 such $\bar{b}$; these are $\bar{b}_{00}$, $\bar{b}_{01}$, $\bar{b}_{10}$ and $\bar{b}_{11}$.

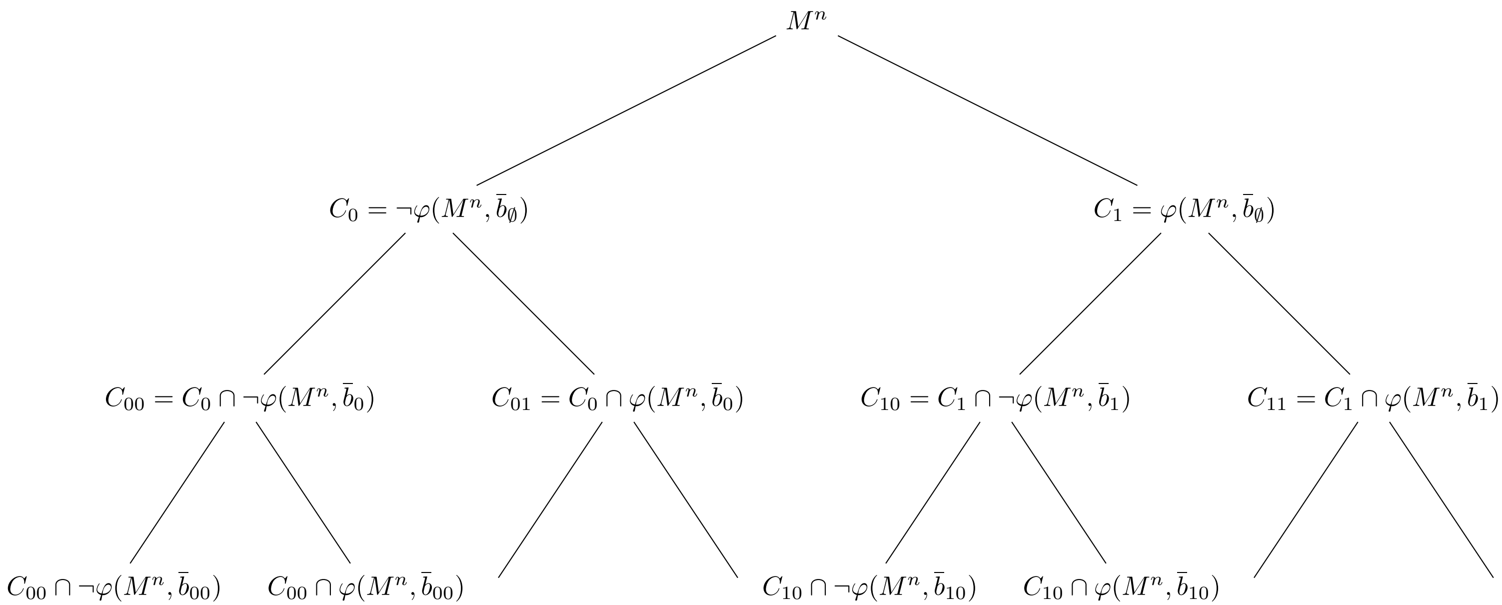
We get a partition of M^n into 8 sets: $C_{000} = C_{00} \cap \neg\varphi(M^n, \bar{b}_{00})$, $C_{001} = C_{00} \cap \varphi(M^n, \bar{b}_{00})$,...

$\vdots$

$\sigma(k)$ At level k we get a partition of M^n into the 2^{k+1} sets

$$C_s = \bigcap_{i \leq k} \varphi^{s(i)}(M^n, \bar{b}_{s|i}), \text{ where } s : k+1 \longrightarrow 2$$

$\vdots$ and so on.


M^n
$C_0 = \neg\varphi(M^n, \bar{b}_\emptyset)$
$C_1 = \varphi(M^n, \bar{b}_\emptyset)$
$C_{00} = C_0 \cap \neg\varphi(M^n, \bar{b}_0)$
$C_{01} = C_0 \cap \varphi(M^n, \bar{b}_0)$
$C_{10} = C_1 \cap \neg\varphi(M^n, \bar{b}_1)$
$C_{11} = C_1 \cap \varphi(M^n, \bar{b}_1)$
$C_{00} \cap \neg\varphi(M^n, \bar{b}_{00})$
$C_{00} \cap \varphi(M^n, \bar{b}_{00})$
$C_{10} \cap \neg\varphi(M^n, \bar{b}_{10})$
$C_{10} \cap \varphi(M^n, \bar{b}_{10})$

1.8. **Characterization of stable formulas** Let $\mathscr{L}$ be any language and let T be an $\mathscr{L}$-theory.[3] The following conditions are equivalent for each partitioned $\mathscr{L}$-formula $\varphi(\bar{x}, \bar{y})$.

(1) φ is stable for T, hence if $M \models T$ and $A \subseteq M$ is infinite, then $|S_\varphi(A)| \leq |A|$.

(1)$^+$ There is an infinite cardinal κ such that for all $M \models T$ and each $A \subseteq M$ with $|A| \leq \kappa$ we have $|S_\varphi(A)| \leq \kappa$.

(1)* There is an infinite cardinal $\kappa \geq |\mathscr{L}|$ such that for all $M \models T$ with $|M| = \kappa$ we have $|S_\varphi(M)| \leq |M|$.

(2) φ does not have the order property (for T).

(2)* Let $\bar{u} = \bar{x}\hat{\ }\bar{y}$, $\bar{v} = \bar{x}'\hat{\ }\bar{y}'$ and let $\psi(\bar{u}, \bar{v})$ be the formula $\varphi(\bar{x}, \bar{y}')$. Then there are no $M \models T$ and $\bar{c}_n \in M^{\bar{u}}$, $n \in \mathbb{N}$ such that $\psi(\bar{c}_n, \bar{c}_k) \iff n \leq k$.

(2)$^+$ The following property holds: Let $M \models T$, $X \subseteq M^{\bar{x}}$, $Y \subseteq M^{\bar{y}}$ (these sets do not need to be definable) and suppose for all finite sets $E \subseteq X$ there is some $\bar{b} \in Y$ with $E \subseteq \varphi(M^{\bar{x}}, \bar{b})$. Then there is a finite subset $F \subseteq Y$ with $X \subseteq \bigcup_{\bar{b} \in F} \varphi(M^{\bar{x}}, \bar{b})$.

If this is the case, then there is a bound on the size of sets F in the implication, which is independent of M, X, Y and such a bound is $o(\varphi)$.

(3) φ does not have the binary tree property (for T).

(4) For all $A \subseteq M \models T$, the Boolean space $(S_\varphi(A))_{\text{con}}$ is scattered.

(4)* There is some $K \in \mathbb{N}$ such that for all $A \subseteq M \models T$, the Boolean space $(S_\varphi(A))_{\text{con}}$ has Cantor-Bendixson rank $\leq K$.

(5) The formula φ^{opp} is stable.

(6) The formula $\neg\varphi$ is stable.

(7) If $\psi(\bar{x}, \bar{y})$ is another stable formula, then $\varphi \wedge \psi$ is stable.

(8) Every $\mathfrak{p} \in S_\varphi(A)$ is definable over A for all $A \subseteq M \models T$.

(9) If M is a model of T, then every $\mathfrak{p} \in S_\varphi(M)$ is definable over M with a definition $d_\mathfrak{p}\bar{x}\varphi(\bar{x}, \bar{y}) \in \Delta^\ell_{\varphi^{\text{opp}}}(M)$.

This in fact is true in a strong uniform way as follows: There is some $K \in \mathbb{N}$ and a lattice combinations $\bigvee\!\!\!\bigvee_{j \leq K} \bigwedge\!\!\!\bigwedge_{i \leq K} \varphi(\bar{x}_{ij}, \bar{y})$ such that for all $M \prec N \models T$, every formula $\gamma(\bar{x})$ with parameters from M and all $\bar{\alpha} \in N^{\bar{x}}$ with $N \models \gamma(\bar{\alpha})$, there are $\bar{a}_{ij} \in \gamma(M^{\bar{x}})$ such that the set $\varphi(\bar{\alpha}, N^{\bar{y}}) \cap M^{\bar{y}}$ is defined in M by

$$\bigvee\!\!\!\bigvee_{j \leq K} \bigwedge\!\!\!\bigwedge_{i \leq K} \varphi(\bar{a}_{ij}, \bar{y}).$$

Proof. Obviously (1) implies (1)$^+$ and (1)*.

Now assume (2) fails, i.e. φ has the order property. We show that (1)$^+$ and (1)* fail, therefore establishing the implications (1)$^+ \Rightarrow$(2) and (1)$^* \Rightarrow$(2).

Let κ be an infinite cardinal. By a theorem of Hausdorff there is a total order I with $|I| = \kappa$ that has at least κ^+ cuts. By compactness, using the order property

[3]Notice that there are no assumptions made here: $\mathscr{L}$ may be uncountable and T may not be complete.

of φ there are $M \models T$ and $\bar{a}_i \in M^{\bar{x}}$, $\bar{b}_i \in M^{\bar{y}}$ for $i \in I$ such that

$$M \models \varphi(\bar{a}_i, \bar{b}_j) \iff i \le j \text{ for all } i, j \in I.$$

Since $|I| = \kappa$, there is a set $A \subseteq M$ of size κ containing the coordinates of all the $\bar{a}_i, \bar{b}_i$. Then for each up-set U of I the set

$$\Psi(U) = \{\neg\varphi(\bar{x}, \bar{b}_j) \mid j \in I \setminus U\} \cup \{\varphi(\bar{x}, \bar{b}_j) \mid j \in U\}$$

is finitely satisfiable in M (if $j_1 < U \ni j_2$, then $N \models \neg\varphi(\bar{a}_{j_2}, \bar{b}_{j_1})) \wedge \varphi(\bar{a}_{j_2}, \bar{b}_{j_2})))$. Obviously $\Psi(U) \neq \Psi(U')$ if $U \neq U'$, showing that $|S_\varphi(A)| > \kappa$. This shows that (1)$^+$ fails.

Now, if $\kappa \ge |\mathscr{L}|$, the Skolem-Löwenheim theorems tell as that we can choose A as a model of T. Therefore, (1)* fails as well.

(2)⇒(1). Suppose $M \models T$ and $A \subseteq M$ is infinite with $|S_\varphi(A)| > |A|$. By 1.2 the set $\mathfrak{S}$ of all φ-external sets is in bijection with $S_\varphi(A)$. Hence $\mathfrak{S} \subseteq \mathcal{P}(A^{\bar{y}})$ has size $> |A| = |A^{\bar{y}}|$ (as A is infinite).

By the Erdős-Makkai theorem [TenZie2012, C.2, page 210], there is an infinite totally ordered set I and $\bar{b}_i \in A^{\bar{y}}$, $Y_i \in \mathfrak{S}$ such that

$$\bar{b}_i \in Y_i \iff i \le j \text{ for all } i, j \in I.$$

Let $N \succ M$ such that $Y_i = \{\bar{b} \in A^{\bar{y}} \mid N \models \varphi(\bar{\alpha}_i, \bar{b})\}$ for some $\bar{\alpha}_i \in N^{\bar{x}}$. Then the equivalence reads as

$$N \models \varphi(\bar{\alpha}_i, \bar{b}_i) \iff i \le j \text{ for all } i, j \in I,$$

showing that φ has the order property.

This shows the equivalence of conditions (1),(1)$^+$,(1)* and (2).

(2) $\iff$ (2)* is a standard application of the compactness theorem.

(2)$^+$⇒(2) Suppose φ does have the order property and let $\bar{a}_i \in M^{\bar{x}}, \bar{b}_i \in M^{\bar{y}}$ for $i < \omega$ with

$$\varphi(\bar{a}_i, \bar{b}_j) \iff i \le j \text{ for all } i, j \in \omega.$$

Take $X = \{\bar{a}_i \mid i < \omega\}$ and $Y = \{\bar{b}_i \mid i < \omega\}$. Obviously the property in (2)$^+$ fails.

(2)⇒(2)$^+$ Let M, X, Y as given in (2)$^+$. Assume first that

(‡) there is no finite subset $F \subseteq Y$ with $X \subseteq \bigcup_{\bar{b} \in F} \varphi(M^{\bar{x}}, \bar{b})$.

We construct $\bar{a}_i \in M^{\bar{x}}, \bar{b}_i \in M^{\bar{y}}$ for $i < \omega$ with

$$\varphi(\bar{a}_i, \bar{b}_j) \iff i \le j \text{ for all } i, j \in \omega$$

by induction on i. Choose $\bar{a}_0 \in X$ arbitrarily. By assumption on X there is some $\bar{b}_0 \in Y$ with $\varphi(\bar{a}_0, \bar{b}_0)$. Suppose we have already constructed $\bar{a}_i \in M^{\bar{x}}, \bar{b}_i \in M^{\bar{y}}$ for $i \le n$ with

$$\varphi(\bar{a}_i, \bar{b}_j) \iff i \le j \text{ for all } i, j \le n$$

By (‡) we have $X \nsubseteq \bigcup_{i \le n} \varphi(M^{\bar{x}}, \bar{b}_i)$, i.e. there is some $\bar{a}_{n+1} \in X$ with $\neg\varphi(\bar{a}_{n+1}, \bar{b}_i)$ for all $i \le n$. By assumption on X and Y there is some $\bar{b}_{n+1} \in Y$ with $\bar{a}_0, \ldots, \bar{a}_{n+1} \in \varphi(M^{\bar{x}}, \bar{b}_{n+1})$. But now we see that

$$\varphi(\bar{a}_i, \bar{b}_j) \iff i \le j \text{ for all } i, j \le n+1.$$

This finishes the induction. Now if (2) holds, i.e. φ does not have the order property, then we see that the induction above must stop at $n = o(\varphi)$. And the reason is that $X \subseteq \bigcup_{i \le n} \varphi(M^{\bar{x}}, \bar{b}_i)$, as claimed by (2)$^+$.

(1)⇒(3). If φ has the binary tree property and we choose parameters $(\bar{b}_s)_{s\in 2^{<\omega}}$ of $\bar{y}$-tuples and $\bar{a}_\sigma \in M^{\bar{x}}$ for $\sigma \in 2^\omega$ in some model M as in 1.7, we see that the set A consisting of all coordinates of all the $\bar{b}_s$ is countable, whereas the φ-types of the $\bar{a}_\sigma$ over A witness $|S_\varphi(A)| \geq 2^{\aleph_0}$.

(3)⇒(2). Suppose φ does have the order property. Choose a total ordering of $2^{\leq\omega}$ such that for all $\sigma \in 2^{\leq\omega}$ we have $\sigma < (\sigma|_n) \iff \sigma(n) = 1$. (For example use the order on the real numbers and the injection $2^{\leq\omega} \longrightarrow \mathbb{R}$ mapping σ to $\sum_{i\in\mathrm{domain}(\sigma)} \frac{2\sigma(i)-1}{3^i}$.) By compactness, there are a model M of T and $\bar{a}_\sigma \in M^{\bar{x}}, \bar{b}_\sigma \in M^{\bar{y}}$ for $\sigma \in 2^{\leq\omega}$ such that

$$M \models \varphi(\bar{a}_\sigma, \bar{b}_\tau) \iff \sigma < \tau \text{ for all } \sigma, \tau \in 2^{\leq\omega}.$$

Then for the binary tree $(\bar{b}_s)_{s\in 2^{<\omega}}$ and all $\sigma \in 2^\omega$ the set

$$\{\varphi^{\sigma(n)}(\bar{x}, \bar{b}_{\sigma|_n}) \mid n < \omega\}$$

is consistent.

(4)⇒(1) Is left as an exercise to the reader: Find an injection $S_\varphi(A) \longrightarrow \Delta^\flat_\varphi(A)$ and then count the codomain.

(3)⇒(4)*. The Cantor-Bendixson rank CB here refers to the one of the patch space of $S_\varphi(A)$. If $\psi(\bar{x})$ is a Boolean combination of formulas from $\Delta_\varphi(A)$ and $\mathrm{CB}(\psi) \geq \alpha$, then for all $\beta < \alpha$ there are $\mathfrak{p}, \mathfrak{q} \in \langle\psi\rangle \subseteq S_\varphi(A)$ of Cantor-Bendixson rank $\geq \beta$ with $\mathfrak{p} \neq \mathfrak{q}$. From $\mathfrak{p} \neq \mathfrak{q}$ we get some $\bar{b} \in A^{\bar{y}}$ such that $\varphi(\bar{x}, \bar{b})$ separates $\mathfrak{p}$ and $\mathfrak{q}$. It follows that $\psi \wedge \varphi(\bar{x}, \bar{b})$ and $\psi \wedge \neg\varphi(\bar{x}, \bar{b})$ have Cantor-Bendixson rank $\geq \beta$.

Consequently, if there is no $K \in \mathbb{N}$ that bounds the Cantor-Bendixson rank of all $S_\varphi(A)$, then one can find binary trees of arbitrary large finite size. By compactness, φ has the binary tree property.

(4)*⇒(4) is a weakening.

[Here is an alternative proof of (1)⇒(4). Suppose $S_\varphi(A)_{\mathrm{con}}$ is not scattered. Then it has a perfect subsets and in the Boolean space $S_\varphi(A)_{\mathrm{con}}$, we can find a countable Boolean algebra B of clopen subsets, such that B has no atoms. Take $A_0 \subseteq A$ such that all members of B are named by Boolean combinations of formulas with parameters from A_0. Obviously then $S_\varphi(A_0)$ has size $\geq 2^{\aleph_0}$, which contradicts (1).]

(2) $\iff$ (5) Property (2) holds for φ just if it holds for φ^{opp}, because if

$$\varphi(\bar{a}_i, \bar{b}_j) \iff i \leq j \text{ for all } i, j \in \{0, \ldots, n\},$$

then with $\bar{c}_i = \bar{b}_{n-i}$, $\bar{d}_i = \bar{a}_{n-i}$ we have

$$\varphi^{\mathrm{opp}}(\bar{c}_i, \bar{d}_j) \iff \varphi(\bar{d}_j, \bar{c}_i) \iff n - j \leq n - i \iff i \leq j \text{ for all } i, j \in \{0, \ldots, n\}.$$

Since we already know that (2) is equivalent to (1), we get the assertion.

(2) $\iff$ (6) Property (2) holds for φ just if it holds for $\neg\varphi$ as is easily verified. Hence we know that all conditions (1)-(6) (and their decorations) are equivalent.

(7)⇒(1) is clear and (1)⇒(7) follows from the fact that there is a surjective map $S_\varphi(A) \times S_\psi(A) \longrightarrow S_{\varphi\wedge\psi}(A)$: We map $(\mathfrak{p}, \mathfrak{q})$ to $r \cap \Delta_{\varphi\wedge\psi}(A)$, if there is a complete type $r \in S_{\bar{x}}(A)$ lying over $\mathfrak{p}$ and $\mathfrak{q}$ (r might then not be unique, we just pick one).

Hence we know that all conditions (1)-(7) (and all their decorations by superscripts) are equivalent.

Items (8) and (9) both imply that all φ-types over models are definable. Hence under either of these assumptions we can count φ-types over a model by counting definable sets over M. Consequently we know that (8)⇒(1) and (9)⇒(1).

(3)⇒(8). For every formula $\vartheta(\bar{x})$ (possibly with parameters from some model), let

$$D(\vartheta)=\max\Big\{k\in\omega \mid \text{there is a binary tree } (\bar{b}_s)_{s\in 2^{<k}} \text{ of } \bar{y}\text{-tuples with entries in } M \text{ such that for all } s\in 2^k \text{ there is an element in } C_s=\vartheta(M^n)\cap\bigcap_{i<k}\varphi^{s(i)}(M^n,\bar{b}_{s|i})\Big\}$$

Since φ does not have the binary tree property, this maximum does indeed exist (use compactness). Now let

$$d=\min\{D(\vartheta)\mid \vartheta\in\mathfrak{p}^\flat\},$$

where $\mathfrak{p}^\flat$ is the ultrafilter of $\operatorname{ba}(\Delta_\varphi(A))$ that intersects $\Delta_\varphi(A)$ in $\mathfrak{p}$. Choose $\vartheta\in\mathfrak{p}^\flat$ with $d=D(\vartheta)$.

Claim. For $\bar{b}\in A^{\bar{y}}$ we have $\varphi(\bar{x},\bar{b})\in\mathfrak{p}\iff d=D(\vartheta\wedge\varphi(\bar{x},\bar{b}))$.
Proof. ⇒. It is clear that $D(\vartheta\wedge\varphi(\bar{x},\bar{b}))\leq D(\vartheta)=d$ and as $\varphi(\bar{x},\bar{b})\in\mathfrak{p}$ the minimality of d implies $d=D(\vartheta\wedge\varphi(\bar{x},\bar{b}))$.

⇐. Suppose $\varphi(\bar{x},\bar{b})\notin\mathfrak{p}$. Then $\neg\varphi(\bar{x},\bar{b})\in\mathfrak{p}^\flat$ and again the minimality of ϑ implies $d=D(\vartheta\wedge\neg\varphi(\bar{x},\bar{b}))$. But then $d=D(\vartheta\wedge\varphi(\bar{x},\bar{b}))$ would imply that $D(\vartheta)\geq d+1$, in contradiction to the choice of ϑ. ◇

By the claim it remains to show that there is a formula $\psi(\bar{y})$ with parameters in A such that

$$d=D(\vartheta\wedge\varphi(\bar{x},\bar{b}))\iff\models\psi(\bar{b}).$$

However this is straightforward, because the property "there is a binary tree $(\bar{b}_s)_{s\in 2^{<d}}$ (of $\bar{y}$-tuples from M) witnessing $d=D(\vartheta\wedge\varphi(\bar{x},\bar{b}))$" can be written as a formula ψ.

Notice that ψ has the same parameters as ϑ and these are all from A; further notice that this ψ introduces new quantifiers, so it is not in $\operatorname{ba}(\Delta_{\varphi^{\mathrm{opp}}}(A))$ (where φ^{opp} is φ with interchanged role of $\bar{x}$ and $\bar{y}$).

(2)⇒ (9) Let $m=o(\neg\varphi)^{\mathrm{opp}}$ as in 1.5. Let

$$\varphi(\bar{\alpha},M^{\bar{y}})=\{\bar{b}\in M^{\bar{y}}\mid\varphi(\bar{x},\bar{b})\in\mathfrak{p}\}$$

be the external φ-fibre above $\bar{\alpha}$ (recall that $\mathfrak{p}=\operatorname{tp}_\varphi(\bar{\alpha}/M)$).

Claim. For each $\gamma(\bar{x})\in\operatorname{tp}(\bar{\alpha}/M)$ there are $\bar{a}_1,\dots,\bar{a}_m\in\gamma(M^{\bar{x}})$ with $\bigcap_{i=1}^m\varphi(\bar{a}_i,M^{\bar{y}})\subseteq\varphi(\bar{\alpha},M^{\bar{y}})$.

Proof. We deploy $(2)^+$. Let $E\subseteq M^{\bar{y}}\setminus\varphi(\bar{\alpha},M^{\bar{y}})$ be finite, i.e., $N\models\bigwedge_{\bar{b}\in E}\neg\varphi(\bar{\alpha},\bar{b})$. Since $M\prec N$, there is some $\bar{a}\in M^{\bar{x}}$ with $M\models\gamma(\bar{a})\wedge\bigwedge_{\bar{b}\in E}\neg\varphi(\bar{a},\bar{b})$. This means that assumption in $(2)^+$ is satisfied for $X=M^{\bar{y}}\setminus\varphi(\bar{\alpha},M^{\bar{y}})$, $Y=\gamma(M^{\bar{x}})$ and the formula $(\neg\varphi)^{\mathrm{opp}}$. Hence by $(2)^+$ we get the claim. ◇

We rewrite the claim: We write $\bar{z}$ for the $m\times|\bar{x}|$-tuple of variables $(\bar{x}_1,\dots,\bar{x}_m)$ and $\bar{c}$ for a $\bar{z}$-tuple $(\bar{a}_1,\dots,\bar{a}_m)$ of elements of M. Further we let

$$\hat{\varphi}(\bar{z},\bar{y}) \text{ be } \varphi(\bar{x}_1,\bar{y})\wedge\ldots\wedge\varphi(\bar{x}_m,\bar{y}).$$

Hence for each $\bar{z}$-tuple $\bar{c} = (\bar{a}_1, \ldots, \bar{a}_m) \in M^{\bar{z}}$, the set $\hat{\varphi}(\bar{c}, M^{\bar{y}})$ is the intersection of the m definable fibres $\varphi(\bar{a}_1, M^{\bar{y}}), \ldots, \varphi(\bar{a}_m, M^{\bar{y}})$ of φ. Now, if $\bar{b}_1, \ldots, \bar{b}_k \in \varphi(\bar{\alpha}, M^{\bar{y}})$ and we apply the claim with $\gamma(\bar{x}) \wedge \varphi(\bar{x}, \bar{b}_1) \wedge \ldots \wedge \varphi(\bar{x}, \bar{b}_k)$ instead of $\gamma(\bar{x})$, we get some $\bar{c} = (\bar{a}_1, \ldots, \bar{a}_m) \in \gamma(M^{\bar{x}})^m$ with $\hat{\varphi}(\bar{c}, M^{\bar{y}}) \subseteq \varphi(\bar{\alpha}, M^{\bar{y}})$ and $\models \varphi(\bar{a}_i, \bar{b}_j)$ for all i, j. This shows that

(†) For all finite sets $E \subseteq \varphi(\bar{\alpha}, M^{\bar{y}})$, there is $\bar{c} \in \gamma(M^{\bar{x}})^m$ with $E \subseteq \hat{\varphi}(\bar{c}, M^{\bar{y}}) \subseteq \varphi(\bar{\alpha}, M^{\bar{b}})$.

Since finite conjunctions of stable formulas are stable by (7) we know that $\hat{\varphi}$ is stable. By (5), also $(\hat{\varphi})^{\mathrm{opp}}$ is stable. But now by $(2)^+$ applied to the sets $X = \varphi(\bar{\alpha}, M^{\bar{y}})$ and $Y = \{\bar{c} \in \gamma(M^{\bar{x}})^m \mid \hat{\varphi}(\bar{c}, M^{\bar{y}}) \subseteq \varphi(\bar{\alpha}, M^{\bar{b}})\}$, we know from (†) that $\varphi(\bar{\alpha}, M^{\bar{b}})$ is the union of at most $o((\hat{\varphi})^{\mathrm{opp}})$ fibres of $\hat{\varphi}$. Hence we may choose $K = \max\{m, o(\hat{\varphi})\}$ (by repeating some of the $\bar{a}_{ij}$ entries if necessary). This shows (9). □

1.9. **Corollary.** *Let $\varphi(\bar{x}, \bar{y})$ be stable for T, let $M \prec N \models T$ and $A \subseteq M \subseteq B \subseteq N$. Then for any $\mathfrak{p} \in S_\varphi(M)$ that is defined over A, the set*

$$\mathfrak{q} = \{\varphi(\bar{x}, \bar{\beta}) \mid N \models d_{\mathfrak{p}}\bar{x}\varphi(\bar{x}, \bar{\beta})\}$$

is in $S_\varphi(B)$ satisfying $\mathfrak{q} \cap \Delta_\varphi(M) = \mathfrak{p}$ and $d_{\mathfrak{q}}\bar{x}\varphi(\bar{x}, \bar{y}) = d_{\mathfrak{p}}\bar{x}\varphi(\bar{x}, \bar{y})$. In particular $\mathfrak{q}$ is again defined over A and it is the unique extension of $\mathfrak{p}$ on B that has the same definition as $\mathfrak{p}$.

Proof. It suffices to show that the set

$$\mathfrak{q} \cup \{\neg\varphi(\bar{x}, \bar{\beta}) \mid \varphi(\bar{x}, \bar{\beta}) \in \Delta_\varphi(B) \setminus \mathfrak{q}\}$$

is finitely satisfiable in N. Otherwise there are $n, m \geq 0$ and $\bar{\beta}_1, \ldots, \bar{\beta}_n, \bar{\beta}'_1, \ldots, \bar{\beta}'_m \in B^{\bar{y}}$ with

$$N \models \bigwedge_{i=1}^{n} d_{\mathfrak{p}}\bar{x}\varphi(\bar{x}, \bar{\beta}_i) \wedge \bigwedge_{j=1}^{m} \neg d_{\mathfrak{p}}\bar{x}\varphi(\bar{x}, \bar{\beta}'_j) \wedge \forall\bar{x}\neg\Bigg(\bigwedge_{i=1}^{n} \varphi(\bar{x}, \bar{\beta}_i) \wedge \bigwedge_{j=1}^{m} \neg\varphi(\bar{x}, \bar{\beta}'_j)\Bigg)$$

Since $d_{\mathfrak{p}}\bar{x}\varphi(\bar{x}, \bar{y}) \in \mathscr{L}(A)$ and $A \subseteq M$ there are also $\bar{b}_i, \bar{b}'_j \in M^{\bar{y}}$ with

$$M \models \bigwedge_{i=1}^{n} d_{\mathfrak{p}}\bar{x}\varphi(\bar{x}, \bar{b}_i) \wedge \bigwedge_{j=1}^{m} \neg d_{\mathfrak{p}}\bar{x}\varphi(\bar{x}, \bar{b}'_j) \wedge \forall\bar{x}\neg\Bigg(\bigwedge_{i=1}^{n} \varphi(\bar{x}, \bar{b}_i) \wedge \bigwedge_{j=1}^{m} \neg\varphi(\bar{x}, \bar{b}'_j)\Bigg)$$

But this means

$$\varphi(\bar{x}, \bar{b}_i) \in \mathfrak{p} \not\ni \varphi(\bar{x}, \bar{b}'_j) \ (1 \leq i \leq n, 1 \leq j \leq m) \text{ and}$$
$$M \models \forall\bar{x}\neg\Bigg(\bigwedge_{i=1}^{n} \varphi(\bar{x}, \bar{b}_i) \wedge \bigwedge_{j=1}^{m} \neg\varphi(\bar{x}, \bar{b}'_j)\Bigg),$$

which is not the case. □

1.10. **Corollary.** *(Heir-coheir symmetry) If $\varphi(\bar{x},\bar{y})$ is stable for T and $M\models T$, then for all $\mathfrak{p}\in S_{\varphi,\bar{x}}(M)$ and $\mathfrak{q}\in S_{\varphi,\bar{y}}(M)$ we have*

$$\mathfrak{q}\models d_{\mathfrak{p}}\bar{x}\varphi(\bar{x},\bar{y}) \iff \mathfrak{p}\models d_{\mathfrak{q}}\bar{y}\varphi(\bar{x},\bar{y})$$

Proof. Firstly, observe that $d_{\mathfrak{p}}\bar{x}\varphi(\bar{x},\bar{y})\in\Delta^{\ell}_{\varphi^{\mathrm{opp}}}(M)$,$d_{\mathfrak{q}}\bar{y}\varphi(\bar{x},\bar{y})\in\Delta^{\ell}_{\varphi}(A)$ by 1.8(9), hence the statement does make sense.

Let $N\succ M$ be $|M|^{+}$-saturated and let $\mathfrak{p}',\mathfrak{q}'$ be the unique extensions of $\mathfrak{p},\mathfrak{q}$ on B that have the same definition as $\mathfrak{p},\mathfrak{q}$ respectively, according to 1.9. By saturation of N we can inductively find $\bar{\alpha}_i\in N^{\bar{x}},\bar{\beta}_i\in N^{\bar{y}}$, $i\in\omega$, starting with $\bar{\alpha}_0$, such that

$$\begin{aligned}\bar{\alpha}_n&\models\mathfrak{p}'\restriction M\bar{\beta}_0\ldots\bar{\beta}_{n-1}\\ \bar{\beta}_n&\models\mathfrak{q}'\restriction M\bar{\alpha}_0\ldots\bar{\alpha}_n.\end{aligned}$$

Then

(a) For all $j>i$ we have

$$N\models\varphi(\bar{\alpha}_j,\bar{\beta}_i)\iff\varphi(\bar{x},\bar{\beta}_i)\in\mathfrak{p}'\iff N\models d_{\mathfrak{p}}\bar{y}\varphi(\bar{x},\bar{\beta}_i)\iff\mathfrak{q}\models d_{\mathfrak{p}}\bar{x}\varphi(\bar{x},\bar{y}),$$

and

(b) for all $j\le i$ we have

$$N\models\varphi(\bar{\alpha}_j,\bar{\beta}_i)\iff\varphi(\bar{\alpha}_j,\bar{y})\in\mathfrak{q}'\iff N\models d_{\mathfrak{q}}\bar{y}\varphi(\bar{\alpha}_j,\bar{y})\iff\mathfrak{p}\models d_{\mathfrak{q}}\bar{y}\varphi(\bar{x},\bar{y}).$$

Now assume that the equivalence claimed in the corollary fails, say $\mathfrak{p}\models d_{\mathfrak{q}}\bar{y}\varphi(\bar{x},\bar{y})$ and $\mathfrak{q}\not\models d_{\mathfrak{p}}\bar{x}\varphi(\bar{x},\bar{y})$. Then we get from (a) and (b) that

$$j\le i\iff N\models\varphi(\bar{\alpha}_j,\bar{\beta}_i)\text{ for all } i,j,$$

which contradicts the assumption that φ does not have the order property. □

1.11. **Corollary.** *If $\varphi(\bar{x},\bar{y})$ is stable for T, and $M\models T$, then for all sets $B\supseteq M$ from some elementary extension N of M there is a spectral section $s:S_{\varphi}(M)\longrightarrow S_{\varphi}(B)$ of the natural map $S_{\varphi}(B)\longrightarrow S_{\varphi}(M)$. Explicitly, for $\mathfrak{p}\in S_{\varphi}(M)$ we have*

$$\begin{aligned}s(\mathfrak{p})&=\{\varphi(\bar{x},\bar{\beta})\mid\text{ the formula from }\Delta^{\ell}_{\varphi}(M)\text{ defining }\varphi(M^{\bar{x}},\bar{\beta})\text{ is in }\mathfrak{p}^{\ell}\}\\&=\{\varphi(\bar{x},\bar{\beta})\mid N\models d_{\mathfrak{p}}\bar{x}\varphi(\bar{x},\bar{\beta})\}.\end{aligned}$$

Proof. By 1.8(1)⇒(9), the inclusion $\Delta_{\varphi}(M)\hookrightarrow\Delta_{\varphi}(B)$ induces a retract $\pi:\Delta^{\ell}_{\varphi}(B)\twoheadrightarrow\Delta^{\ell}_{\varphi}(M)$ mapping a finite lattice combination $\bigvee\!\!\!\bigvee_i\bigwedge\!\!\!\bigwedge_j\varphi(\bar{x},\bar{\beta}_{ij})$ of instances of φ to $\bigcup_i\bigcap_j\varphi(M^{\bar{x}},\bar{\beta}_{ij})$. Hence π is a homomorphism of bounded distributive lattices and by Stone duality it induces a spectral section $s:S_{\varphi}(M)\longrightarrow S_{\varphi}(B)$ of the natural map $S_{\varphi}(B)\longrightarrow S_{\varphi}(M)$. For $\mathfrak{p}\in S_{\varphi}(M)$, we compute

$$\begin{aligned}s(\mathfrak{p})&=\pi^{-1}(\mathfrak{p})\cap\Delta_{\varphi}(M)\\&=\{\varphi(\bar{x},\bar{\beta})\in\Delta_{\varphi}(B)\mid\text{ the formula from }\Delta^{\ell}_{\varphi}(M)\text{ defining }\varphi(M^{\bar{x}},\bar{\beta})\text{ is in }\mathfrak{p}^{\ell}\}\end{aligned}$$

Now observe that the formula from $\Delta^{\ell}_{\varphi}(M)$ defining $\varphi(M^{\bar{x}},\bar{\beta})$ is $d_{\mathfrak{q}}\bar{y}\varphi(\bar{x},\bar{y})$ for $\mathfrak{q}=\mathrm{tp}_{\varphi^{\mathrm{opp}}}(\bar{\beta}/M)$. Hence by 1.10, $\mathfrak{p}\models\varphi(M^{\bar{x}},\bar{\beta})$ is equivalent to $\mathfrak{q}\models d_{\mathfrak{p}}\bar{x}\varphi(\bar{x},\bar{y})$, which just says $N\models d_{\mathfrak{p}}\bar{x}\varphi(\bar{x},\bar{\beta})$ □

1.12. **Theorem.** *(Local separation of parameters) Let $\psi(\bar{x})$ be an $\mathscr{L}$-formula, let $M \models T$ and let $A \subseteq M$ be the set of all entries of tuples from $\psi(M^{\bar{x}})$. If $\varphi(\bar{x}, \bar{y})$ is stable for T and $\bar{b} \in M^{\bar{y}}$, then the set $\varphi(M^{\bar{x}}, \bar{b}) \cap \psi(M^{\bar{x}})$ is definable over A.*

One can apply the theorem also to a parametrically definable set D, defined using parameters $\bar{c}$: Deploy the theorem for the theory $\mathrm{Th}(M, \bar{c})$.

Proof. By 1.8(5),(8), the type $\mathfrak{q} = \mathrm{tp}_{\varphi^{\mathrm{opp}}}(\bar{b}/A)$ is definable over A by some $\mathscr{L}(A)$-formula $d_{\mathfrak{q}}\bar{y}\varphi(\bar{x}, \bar{y}) \in \mathscr{L}(A)$. This means that for all $\bar{a} \in A^{\bar{x}}$ we have $\varphi(\bar{a}, \bar{y}) \in \mathfrak{q} \iff d_{\mathfrak{q}}\bar{y}\varphi(\bar{a}, \bar{y})$. But then for all $\bar{a} \in M^{\bar{x}}$ we have

$$\varphi(\bar{a}, \bar{b}) \wedge \psi(\bar{a}) \iff d_{\mathfrak{q}}\bar{y}\varphi(\bar{a}, \bar{y}) \wedge \psi(\bar{a}),$$

and we may take $d_{\mathfrak{q}}\bar{y}\varphi(\bar{x}, \bar{y}) \wedge \psi(\bar{x})$ as a defining formula for $\varphi(M^{\bar{x}}, \bar{b}) \cap \psi(M^{\bar{x}})$. □

1.13. **Definition.**
An $\mathscr{L}$-theory T is called **stable** if all $\mathscr{L}$-formulas are stable for T.

1.14**. Stably embedded sets** Hence if T is stable, then by 1.12, every parametrically definable subset of a $\emptyset$-definable subset D of M^n can be defined by using parameters from D (which has to be understood as "from the set of entries of tuples from D"). This property of 0-definable sets is sometimes referred to as saying that the set is **stably embedded**.

1.15**. Some stable formulas** Let M be an $\mathscr{L}$-structure that eliminates $\exists^\infty$. For example M could be o-minimal or a p-adically closed field. Let $\varphi(\bar{x}, \bar{y})$ be an $\mathscr{L}$-formula, let $D = \varphi(M^{\bar{x}} \times M^{\bar{y}})$ and let $\pi : M^{\bar{x}} \times M^{\bar{y}} \longrightarrow M^{\bar{x}}$ be the projection.

(1) If φ has the order property, witnessed by $(\bar{a}_{ni}, \bar{b}_{ni})_{i \leq n}$, then $\pi(D)$ is infinite and $\pi|_D$ has infinitely many infinite fibres.
Proof. $\pi(D)$ is infinite because it contains all the $\bar{a}_{ni}$ and $\bar{a}_{ni} \neq \bar{a}_{nj}$ for $i < j \leq n$.
Suppose $\pi|_D$ has only K infinite fibres. Since M eliminates $\exists^\infty$ there is some $N \in \mathbb{N}$ such that all other fibres are of size $\leq N$. Then one of the fibres above $\bar{a}_{K+N+2,1}, \ldots, \bar{a}_{K+N+2,K+1}$ is finite, say the fibre above $\bar{a}_{N+K+2,i}$, $i \in \{1, \ldots, K+1\}$. But this is not true because this fibre contains $\bar{b}_{K+N+2,K+1}, \ldots, \bar{b}_{K+N+2,N+K+2}$. ◇

(2) If M is o-minimal and D is the graph of a function with domain $\pi(D)$, then φ is stable. This follows from (i)

(3) If M is o-minimal and $\dim(D) \leq 1$, then φ is stable. This follows from (i).

2. Heirs and Co-heirs

We work with an arbitrary theory T here without finite models.

2.1. **Definition.** Let $M \prec N \models T$ and $M \subseteq B \subseteq N$. Let $q \in S_{\bar{x}}(B)$ with restriction $p = q \restriction M$ on M. Then q is called an **heir** of p if for every formula $\varphi(\bar{x}, \bar{y}, \bar{a})$ with parameters $\bar{a}$ from M and all $\bar{b} \in M^{\bar{y}}$ with $\varphi(\bar{x}, \bar{b}, \bar{a}) \in q$ there is some $\bar{c} \in M$ with $\varphi(\bar{x}, \bar{c}, \bar{a}) \in p$.

2.2. *Notation.* For each partitioned $\mathscr{L}$-formula $\varphi(\bar{x}, \bar{y})$ let D^{φ} be a new relation symbol of arity $|\bar{y}|$ and let $\mathscr{L}^D$ be the extension of $\mathscr{L}$ by all these new relation symbols.

If M is a structure and $p \in S_{\bar{x}}(M)$ we define the $\mathscr{L}^D$-structure (M, D_p) to be the expansion of M that interprets D^{φ} as

$$D_p^{\varphi}(\bar{a}) \iff \varphi(\bar{x}, \bar{a}) \in p \ (\bar{a} \in M^{\bar{y}}).$$

2.3. **Observation.** *Let $M \prec N$, let $q \in S_{\bar{x}}(N)$ and let $p = q \restriction M$. Then q is an heir of p if and only if (M, D_p) is existentially closed in (N, D_q). This is routine checking. (Notice that for every n, all 0-definable subsets $S \subseteq M^n$ are quantifier free definable without parameters in (M, D_p): If S is defined by $\varphi(\bar{y})$, then S is defined in (M, D_p) by D_p^{φ}.)*

If (M, D_p) is even an elementary substructure of (N, D_q), the q is a called a **strong heir** *of p.*

2.4. *Remark.* By general model theory, using 2.3, if $q \in S_{\bar{x}}(N)$ is an heir of $p \in S_{\bar{x}}(M)$, then there is an elementary extension $N^* \succ N$ and a strong heir r of p on N^* extending q: As (M, D_p) is existentially closed in (N, D_q) there is an elementary extension $\mathscr{N}^*$ of (M, D_p) and an embedding of (N, D_q) into $\mathscr{N}^*$ over M. Then we define $r = \{\varphi(\bar{x}, \bar{c}) \mid \mathscr{N}^* \models D^{\varphi}(\bar{c})\}$ and check that $r \in S_{\bar{x}}(N^*)$ extends q, where $N^* = \mathscr{N}^* \restriction \mathscr{L}$, and that $(N^*, D_q) = \mathscr{N}^*$.

Similarly one checks that each type has an heir on every elementary extension. However we can do better, see 2.10

As a consequence of 2.4 we also get the existence of

2.5**. Heir-Coheir Amalgams** [Hodges1993, Thm. 6.4.3, p. 289] To see this, let $A \prec B, C$ in the notation of [Hodges1993, Thm. 6.4.3, p. 289]. We may assume that $B, C \prec \Omega$ for some resplendent structure Ω of sufficiently high cardinality. By 2.4 there is some $B' \subseteq \Omega$ such that $\mathrm{tp}(B'/C)$ is an heir of $\mathrm{tp}(B/A)$. Take an A-automorphism σ of Ω that restricts to an isomorphism $B' \longrightarrow B$ and choose $C' = \sigma(C)$. Then $A \prec B, C' \prec \Omega$ is an heir-coheir amalgam as claimed in [Hodges1993, Thm. 6.4.3, p. 289]: Let $\psi(\bar{x}, \bar{y})$ be an $\mathscr{L}$-formula, take $\bar{b} \in B^{\bar{x}}$ and $\bar{c}' \in C'^{\bar{y}}$ with $\models \psi(\bar{b}, \bar{c}')$. Then $\models \psi(\sigma^{-1}\bar{b}, \sigma^{-1}\bar{c}')$ and as $\sigma^{-1}\bar{b} \subseteq B'$, $\sigma^{-1}\bar{c}' \subseteq C$ there is some $\bar{a} \in A^{\bar{y}}$ with $\models \psi(\sigma^{-1}\bar{b}, \bar{a})$. As σ is over A we get $\models \psi(\bar{b}, \bar{a})$ as required.

2.6**. Characterization of definable types** [Poizat2000, Theorem 11.7] Let $p \in S_{\bar{x}}(M)$. The following are equivalent.

(i) p is a **definable type**, i.e. all φ-types induced by p are definable, or in the terminology of 2.2: (M, D_p) is a definable expansion of M (but notice that the definition is with parameters!).
(ii) p has exactly one heir on every $B \subseteq M$
(iii) p has at most one strong heir on every $N \supseteq M$.

Proof. (i)⇒(ii). For $\varphi(\bar{x}, \bar{y}) \in \mathrm{Fml}(\mathscr{L})$, take an $\mathscr{L}(M)$-formula $d_p(\bar{y})$ defining $D_p^{\varphi}(M^{\bar{y}})$. Take $N \succ M$ and $M \subseteq B \subseteq N$ and define $q = \{\varphi(\bar{x}, \bar{b}) \mid \varphi(\bar{x}, \bar{y}) \in \mathrm{Fml}(\mathscr{L}),\ N \models d_p(\bar{b})\}$. One verifies that q is the unique heir of p on B.

(ii)⇒(iii) is a weakening.

(iii)⇒(i). If p is not definable, then (M, D_p) is not a definable expansion of M. By the Svenonius definability theorem ([Poizat2000, Theorem 9.2], [Hodges1993, Corollaries 10.5.2,10.5.3, p. 516]) there is an elementary extension $\mathscr{N}$ of the $\mathscr{L}^D$-structure (M, D_p) and an $\mathscr{L}$-automorphism σ of $N = \mathscr{N} \restriction \mathscr{L}$ over M such that σ does not respect the $\mathscr{L}^D$-structure $\mathscr{N}$. Then we define $q = \{\varphi(\bar{x}, \bar{b}) \mid \varphi(\bar{x}, \bar{y}) \in \mathrm{Fml}(\mathscr{L}), \bar{b} \in N^{\bar{y}},\ \mathscr{N} \models D^{\varphi}(\bar{b})\}$. One checks that q is an extension of p and $\mathscr{N} = (N, D_q)$. Consequently q is a strong heir of p.

Let $\mathscr{N}_1$ be the $\mathscr{L}^D$-structure expanding N with $\mathscr{N}_1 \models D^{\varphi}(\bar{b}) \iff \mathscr{N} \models D^{\varphi}(\sigma^{-1}(\bar{b}))$ (i.e. $\varphi(\bar{x}, \sigma^{-1}(\bar{b})) \in q$). Then σ is an $\mathscr{L}^D$-isomorphism $\mathscr{N} \longrightarrow \mathscr{N}_1$ and therefore $\mathscr{N}_1 \succ (M, D_p)$. Now define q_1 for $\mathscr{N}_1$ just like q was defined for $\mathscr{N}$. Again q_1 is an extension of p on N and $\mathscr{N}_1 = (N, D_{q_1})$, thus q_1 is a strong heir of p. On the other hand σ is not an $\mathscr{L}^D$-automorphism of $\mathscr{N}$, which means that $\mathscr{N} \neq \mathscr{N}_1$ and therefore $q \neq q_1$, i.e. p has distinct strong heirs on N. □

2.7. *Remark.* If q is a strong heir of p, then p is definable if and only if q is definable.

Proof. This is clear in one direction. If q is definable, then take an $\mathscr{L}$-formulas $\varphi(\bar{x}, \bar{y})$, $\psi(\bar{y}, \bar{z})$ and some $\bar{b} \in N$ such that $(N, D_q) \models \forall \bar{y}(D^{\varphi}(\bar{y}) \leftrightarrow \psi(\bar{y}, \bar{b}))$. Then $(N, D_q) \models \exists \bar{z}\, \forall \bar{y}(D^{\varphi}(\bar{y}) \leftrightarrow \psi(\bar{y}, \bar{z}))$ and so also $(M, D_p) \models \exists \bar{z}\, \forall \bar{y}(D^{\varphi}(\bar{y}) \leftrightarrow \psi(\bar{y}, \bar{z}))$. This implies that p is definable. □

2.8. **Corollary.** *The following are equivalent for every theory T.*

(i) T is stable.
(ii) Heirs and coheirs coincide.
(iii) Every strong heir is a coheir.

Proof. (i)⇒(ii). If T is stable, then indeed heirs and coheirs are the same thing, as follows from 1.10.

(ii)⇒(iii) is a weakening.

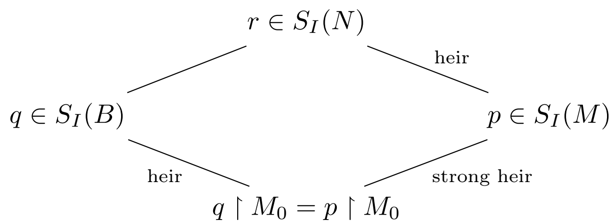
(iii)⇒(i) Suppose T is not stable. Then there is a model M of T and a type $p \in S_{\bar{x}}(M)$ that is not definable. By 2.6, p has two distinct strong heirs on some elementary extension and by 2.7 none of them is definable. Hence we can apply the same reasoning to these strong heirs. In fact the process can be iterated transfinitely: at limit ordinals we take unions of models and strong heirs, which lead again to strong heirs by the elementary chain lemma.

We see that there is an elementary extension N of M such that p has strictly more that $2^{2^{\mathrm{card}(M)}}$ many strong heirs on N. By assumption, then p has also $2^{2^{\mathrm{card}(M)}}$ many coheirs on N. But this is not possible, because the coheirs of p on N are all in the closure of $M^{\bar{x}}$ in $S_{\bar{x}}(N)$ and there are at most $2^{2^{\mathrm{card}(M)}}$ points in that closure. $\square$

2.9. **Weak independence theorem** *(for arbitrary structures) Let $M_0 \prec M \prec N$ and let $M_0 \subseteq B \subseteq N$. Let $p \in S_I(M)$ be a strong heir over M_0 and let $q \in S_I(B)$ be an heir over M_0 with $p \restriction M_0 = q \restriction M_0$.*

Let $\mathscr{N}$ be an elementary extension of (M, D_p) that has N as an elementary $\mathscr{L}$-substructure (the reader should verify that such an $\mathscr{N}$ always exists). Suppose that $\mathrm{tp}^{\mathscr{N}}(B/(M, D_p))$ is an heir over M_0.[4]

Then there is an heir r of p on N containing q.



Proof. We may replace N by the restriction of $\mathscr{N}$ to $\mathscr{L}$ if necessary. Let

$$\Gamma := \{\varphi(\bar{x}, \bar{a}, \bar{z}) \mid \varphi(\bar{x}, \bar{y}, \bar{z}) \in \mathrm{Fml}(\mathscr{L}),\ \bar{a} \in M^{\bar{y}} \text{ and } \varphi(\bar{x}, \bar{a}, \bar{a}_1) \notin p \text{ for all } \bar{a}_1 \in M^{\bar{z}}\}.$$

Notice that for all $\varphi(\bar{x}, \bar{y}, \bar{z}) \in \mathrm{Fml}(\mathscr{L})$ and each $\bar{a} \in M^{\bar{y}}$ we have

$$(+) \qquad \varphi(\bar{x}, \bar{a}, \bar{z}) \in \Gamma \iff (M, D_p) \models \forall \bar{z} \neg D^{\varphi}(\bar{a}, \bar{z}) \iff \mathscr{N} \models \forall \bar{z} \neg D^{\varphi}(\bar{a}, \bar{z}).$$

We only needs to check that the set of $\mathscr{L}_{\bar{x}}(N)$-formulas $q \cup \{\neg\varphi(\bar{x}, \bar{a}, \bar{c}) \mid \varphi(\bar{x}, \bar{a}, \bar{z}) \in \Gamma,\ \bar{c} \in N^{\bar{z}}\}$ is finitely satisfiable in N (because this set also contains p). Otherwise there are some $\mathscr{L}$-formula $\psi(\bar{x}, \bar{v})$, some $\bar{b} \in B^{\bar{v}}$ with $\psi(\bar{x}, \bar{b}) \in q$ and finitely many $\varphi_i(\bar{x}, \bar{a}, \bar{z}) \in \Gamma$ and some $\bar{c} \in N^{\bar{z}}$ with

$$N \models \forall \bar{u} \Big(\psi(\bar{u}, \bar{b}) \to \bigvee_i \varphi_i(\bar{u}, \bar{a}, \bar{c}) \Big).$$

(so here $\bar{u}$ is an $\bar{x}$-tuple of variables). Let $\vartheta(\bar{y}, \bar{v}, \bar{z})$ be the $\mathscr{L}$-formula

$$\forall \bar{u} \Big(\psi(\bar{u}, \bar{v}) \to \bigvee_i \varphi_i(\bar{u}, \bar{y}, \bar{z}) \Big).$$

Using (+) we get

$$\mathscr{N} \models \exists \bar{z} \vartheta(\bar{a}, \bar{b}, \bar{z}) \ \&\ \bigwedge_i \forall \bar{z} \neg D^{\varphi_i}(\bar{a}, \bar{z}),$$

in other words

$$\exists \bar{z} \vartheta(\bar{a}, \bar{y}, \bar{z}) \ \&\ \bigwedge_i \forall \bar{z} \neg D^{\varphi_i}(\bar{a}, \bar{z}) \ \in \ \mathrm{tp}(B/(M, D_p)).$$

Since $\mathrm{tp}(B/(M, D_p))$ is an heir over M_0 there is some $\bar{d} \in M_0^{\bar{y}}$ with

$$(*) \qquad \mathscr{N} \models \exists \bar{z} \vartheta(\bar{d}, \bar{b}, \bar{z}) \ \&\ \bigwedge_i \forall \bar{z} \neg D^{\varphi_i}(\bar{d}, \bar{z}),$$

---

[4]Note that for this condition to hold, we need in the first place that M_0 carries an elementary $\mathscr{L}^D$-substructure of (M, D_p). For that reason we need that p is a strong heir of p_0. Nowhere else in the proof do we need that p is even just an heir over M_0.

in particular $N \models \exists \bar{z} \vartheta(\bar{d}, \bar{b}, \bar{z})$. Consequently q contains $\psi(\bar{x}, \bar{b})$ & $\exists \bar{z} \vartheta(\bar{d}, \bar{b}, \bar{z})$. As q is an heir over its restriction to M_0, there is some $\bar{a}_0 \in M_0^{\bar{y}}$ with

$$\psi(\bar{x}, \bar{a}_0) \wedge \exists \bar{z} \vartheta(\bar{d}, \bar{a}_0, \bar{z}) \;\in\; q \restriction M_0.$$

But then $M_0 \models \exists \bar{z} \vartheta(\bar{d}, \bar{a}_0, \bar{z})$ and so there is some $\bar{a}_1 \in M_0^{\bar{z}}$ with

$$M_0 \models \forall \bar{u} \Big(\psi(\bar{u}, \bar{a}_0) \to \bigvee_i \varphi_i(\bar{u}, \bar{d}, \bar{a}_1) \Big).$$

This says that for any $\bar{u}$-tuple $\bar{\alpha}$ from some elementary extension of M_0 with $\models \psi(\bar{\alpha}, \bar{a}_0)$ we get $\models \bigvee_i \varphi_i(\bar{\alpha}, \bar{d}, \bar{a}_1)$. We now choose $\bar{\alpha}$ to be a realization of p. Then $\models \psi(\bar{\alpha}, \bar{a}_0)$, because $\psi(\bar{x}, \bar{a}_0) \in q \restriction M_0 \subseteq p$. Thus $\models \bigvee_i \varphi_i(\bar{\alpha}, \bar{d}, \bar{a}_1)$. Hence for some i we get $\models \varphi_i(\bar{\alpha}, \bar{d}, \bar{a}_1)$, meaning $\varphi_i(\bar{x}, \bar{d}, \bar{a}_1) \in p$. Consequently $\varphi_i(\bar{x}, \bar{d}, \bar{z}) \notin \Gamma$ and by (+) this means $\mathscr{N} \not\models \forall \bar{z} \neg D^{\varphi_i}(\bar{d}, \bar{z})$. However this contradicts $(*)$. □

2.10**. Existence of heirs** [Poizat2000, Theorem 11.2] *Let $M \subseteq B \subseteq N$, $M \prec N$ and let $q \in S_I(B)$ be an heir of p. Then there is an heir r of p on N containing q.*

Proof. This exactly the content of 2.9 in the case $M_0 = M$. □

3. Rank functions

3.1. List of definitions of (local) rank functions

(i) $R_n^{\varphi}(\psi(\bar{x},\bar{b}))$ (where $\varphi=\varphi(\bar{x},\bar{y})$, with $|x|=n$) is defined in [Pillay1983, 2.8], which is extended to types $p\in S_n(M)$ by $\min\{R_n^{\varphi}(\psi)\mid \psi\in\alpha\}$, and then this is Shelah's $R^n(p,\varphi,2)$. (Sometimes R_n^{φ} is just written as R^{φ} in this book, e.g. in [Pillay1983, 6.19].)

(ii) $R_{\aleph_0}^{\varphi}(\psi)$ is defined in [Pillay1983, 6.13] and extended to types via [Pillay1983, 6.22].

(iii) $D^n(\psi)$ is defined in [Pillay1983, 6.14] and extended to types via [Pillay1983, 6.22].

(iv) $RM^n(\psi)$ is defined in [Pillay1983, 6.16] and extended to types via [Pillay1983, 6.22].

(v) $R_{\varphi}(\psi)$ (resembling Morley rank) is defined in [TenZie2012, Exercise 8.2.10].

(vi) R_{Δ} is defined in [Buechl1996, 5.1.1]

(vii) The (Δ,μ)-rank on formulas is defined in [Baldwi1988, Def. 2.2]

3.2. Local Rank = Local Morley Rank We will be using the definition of $R_{\aleph_0}^{\varphi}(\psi)$ from [Pillay1983, 6.13]. Let T be an $\mathscr{L}$-theory, let $\varphi(\bar{x},\bar{y})$ be an $\mathscr{L}$-formula and let $M\models T$ be $\aleph_0$-saturated. Then for $\psi(\bar{x})\in\mathscr{L}(M)$ we have

$$R_{\aleph_0}^{\varphi}(\psi)=\mathrm{CB}(\pi(\langle\psi\rangle)),$$

where $\pi:S_n(M)\longrightarrow S_{\varphi}^{\flat}(M),\ \pi(p)=p\cap\Delta_{\varphi}^{\flat}(M)$ is the restriction map.

Proof. Firstly, in [Pillay1983, 6.13], the notion "φ-formula" is used. This means "formula of the form $\varphi(\bar{x},\bar{b})$, $\bar{b}\in M^{\bar{y}}$", where M is a monster model. However, for the purpose of that definition, $\aleph_0$-saturation of M suffices. (The reader may check this similar to the proof of [MTofDiffFields, 2.7.8](ii).)

We now check by induction on α that

$$R_{\aleph_0}^{\varphi}(\psi)\geq\alpha\iff\mathrm{CB}(\pi(\langle\psi\rangle))\geq\alpha.$$

Only the induction step needs a proof. Recall from [MTofDiffFields, 2.5.5] for the Boolean space $X=\pi(\langle\psi\rangle)$ that $\mathrm{CB}(\pi(\langle\psi\rangle))\geq\alpha+1$ if and only if there are infinitely many disjoint clopen subsets Z_i of X with $\mathrm{CB}(Z_i)\geq\alpha$. Now each Z_i is a finite union of sets of the form $X\cap V(\Phi)$, where Φ is a finite conjunctions of formulas of the form $\varphi(\bar{x},\bar{b})$ or $\neg\varphi(\bar{x},\bar{b})$ with $\bar{b}\in M^{\bar{y}}$. Since the Cantor-Bendixson rank of a closed set is the maximum of the Cantor-Bendixson ranks of its points, we may assume that each Z_i is equal to $X\cap V(\Phi_i)$ where Φ_i is a finite conjunction of formulas of the form $\varphi(\bar{x},\bar{b})$ or $\neg\varphi(\bar{x},\bar{b})$ with $\bar{b}\in M^{\bar{y}}$.

Now $X\cap V(\Phi_i)=\pi(\langle\psi\wedge\Phi_i\rangle)$. Hence by induction we know that $\mathrm{CB}(Z_i)\geq\alpha\iff R_{\aleph_0}^{\varphi}(\psi\wedge\Phi_i)\geq\alpha$. But now by definition of $R_{\aleph_0}^{\varphi}(\psi)\geq\alpha+1$ in [Pillay1983, 6.13(iii)] we see that $\mathrm{CB}(\pi(\langle\psi\rangle))\geq\alpha+1$ is equivalent to $R_{\aleph_0}^{\varphi}(\psi)\geq\alpha+1$. □

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