p-ADICALLY CLOSED RINGS

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This is an extended abstract of work in progress on a new class of rings called *p-adically closed rings*. These generalise the notion of a p-adically closed field to commutative rings and serve as rings of sections of what one might call abstract p-adic functions associated to an arbitrary commutative ring. What we have in mind is an approach to the topology of p-adic sets parallel to the real case where Niels Schwartz in [Schw] has developed abstract semi-algebraic spaces and functions; these are certain ringed spaces whose affine models have so-called *real closed rings* as rings of sections. A direct parallel approach in the p-adic case seems difficult and we still do not have a good algebraic description of p-adically closed rings. We hope to be able to generalise Luc Bélair's work [Be91, Be95], where local p-adically closed rings are studied, to obtain such an explicit description.

Instead we take a different path, following the model theoretic approach to real closed rings from [Tr07, section 2] (which also has to some extent a category theoretic counterpart, cf. [SchwMa, section 12]). Finally this note explains how one should define abstract semi-algebraic functions in the p-adic case (see the conclusion 7 below) and lays the algebraic grounds for the development of abstract p-adic spaces. The final version of the paper will also treat the finite rank case, i.e. we will study p-adically closed rings of finite p-rank.

The prototype of a p-adically closed ring is the ring of continuous definable function $K^n \longrightarrow K$ for a p-adically closed field K.

Here the formal definition, which we can give only implicitly in the moment. Justification and purpose of the affair follow afterwards.

Definition 1. Let A be a commutative unital ring. A *p-adic structure on A* is a collection $\mathscr{F}$ of functions $f_A : A^n \to A$ for each continuous 0-definable (in the language of rings) function $f : \mathbb{Q}_p^n \to \mathbb{Q}_p$ and each $n \in \mathbb{N}$ such that the following hold true:

(i) The structure expands the ring structure of A, i.e.: If $f : \mathbb{Q}_p^2 \longrightarrow \mathbb{Q}_p$ is addition or multiplication in $\mathbb{Q}_p$ then $f_A : A^2 \longrightarrow A$ is addition or multiplication in A, respectively; if $f : \mathbb{Q}_p \longrightarrow \mathbb{Q}_p$ is the identity or the constant function 0 or the constant function 1 in $\mathbb{Q}_p$ then $f_A : A \longrightarrow A$ is the identity or the constant function 0 or the constant function 1 in A.

(ii) The following composition rule holds for functions from $\mathscr{F}$:

$$[f \circ (f_1, \ldots, f_n)]_A = f_A \circ (f_{1,A}, \ldots, f_{n,A}),$$

where $f \in \mathscr{F}$ is of arity n and each $f_i \in \mathscr{F}$ is of arbitrary arity.

A *p-adically closed ring* is a commutative unital ring A for which there exists a p-adic structure on A. Observe that the Null ring is also p-adically closed.

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For example, the ring A of all continuous definable (with or without parameters) functions $\mathbb{Q}_p^d \longrightarrow \mathbb{Q}_p$, where $d \in \mathbb{N}$ is fixed, is a p-adically closed ring. A p-adic structure is given as follows: For each $n \in \mathbb{N}$ and every 0-definable continuous function $f : \mathbb{Q}_p^n \longrightarrow \mathbb{Q}_p$, let $f_A : A^n \longrightarrow A$ be the composition with f. Trivially, the collection $\mathscr{F}$ of all such maps f_A is a p-adic structure on A.

Since every (continuous) 0-definable map $\mathbb{Q}_p^n \longrightarrow \mathbb{Q}_p$ operates naturally on every p-adically closed field K, also K is a p-adically closed ring. Indeed also the converse is true, i.e. every p-adically closed ring which is a field is a p-adically closed field (but this is not obvious).

We want to underline that the ring $\mathcal{O}_K$ of integral elements of a p-adically closed field is *not* a p-adically closed ring, since all p-adically closed rings different from the null ring contain the henselisation of $\mathbb{Q}$ in $\mathbb{Q}_p$ (given by the 0-definable constant functions). Nevertheless this arithmetic part of the theory can be incorporated after some localization theory for p-adically closed rings is developed (this will not be explained in this summary).

Our initial theorem on p-adically closed rings says that the implicit definition above can be made explicit (although we still do not have a good explicit algebraic definition yet).

Theorem 2. *Let A be a p-adically closed ring. Then there is a unique p-adic structure $\mathscr{F}$ on A and every function from $\mathscr{F}$ is 0-definable in the ring A (by an $\exists$-formula). Moreover the class of p-adically closed ring is first order axiomatizable (in the language of rings) by $\forall\exists$-sentences.*

If A is a p-adically closed ring and $f : \mathbb{Q}_p^n \longrightarrow \mathbb{Q}_p$ is continuous, 0-definable, then by 2, we may denote by f_A the function $A^n \longrightarrow A$ given by the unique p-adic structure on A. One should think of f_A as the base change of f to A.

As an easy but important consequence of theorem 2 we obtain that the structures on p-adically closed rings are respected by all ring homomorphisms and that p-adically closed rings form a variety in the sense of universal algebra:

Theorem 3. *Let $\varphi : A \longrightarrow B$ be a ring homomorphism between p-adically closed rings. Then φ respects the p-adic structures, i.e. for all continuous, 0-definable $f : \mathbb{Q}_p^n \longrightarrow \mathbb{Q}_p$ we have*

$$\varphi(f_A(a_1, ..., a_n)) = f_B(\varphi(a_1), ..., \varphi(a_n)) \ (a_1, ..., a_n \in A).$$

The category `PCR` *of p-adically closed rings together with ring homomorphisms has arbitrary limits and colimits (which in general are different from those in the category of commutative rings, e.g. fibre sums of p-adically closed rings are not the tensor products of rings).*

The next theorem says that the most basic operations in commutative ring theory stay inside p-adically closed rings:

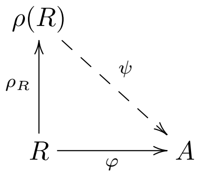
Theorem 4. *(Algebraic properties of p-adically closed rings)*
Let A be a p-adically closed ring. Then

(i) *A is a reduced ring.*
(ii) *For every radical ideal I of A (i.e. A/I is a reduced ring), the ring A/I is p-adically closed.*
(iii) *For every multiplicatively closed subset S of A, the classical localisation $S^{-1} \cdot A$ is p-adically closed.*

(iv) *For every prime ideal of A, the quotient field of $A/\mathfrak{p}$ is a p-adically closed field; in particular, a p-adically closed ring which is a field is a p-adically closed field.*

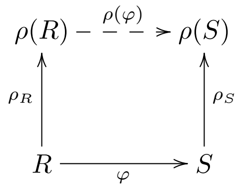
The most important feature of p-adically closed rings, or better the category `PCR` of p-adically closed rings is the existence of a p-adic closure of every ring (where 'ring' always means commutative and unital):

Theorem 5. *Let R be a ring. There is a p-adically closed ring $\rho(R)$ and a ring homomorphism $\rho_R : R \longrightarrow \rho(R)$ such that for each other p-adically closed ring A and each ring homomorphism $\varphi : R \longrightarrow A$ there is a unique ring homomorphism $\psi : \rho(R) \longrightarrow A$ making the diagram*



commutative. Of course, the pair $(\rho(R), \rho_R)$ is uniquely determined up to isomorphism by this condition.

Thus, if $\varphi : R \longrightarrow S$ is a ring homomorphism between arbitrary rings then there is a unique ring homomorphism $\rho(\varphi) : \rho(R) \longrightarrow \rho(S)$ making the diagram



commutative. Note that by Theorem 3 we have $\rho(\rho(R)) = \rho(R)$ and $\rho_{\rho(R)}$ is the identity. In terms of category theory, theorem 5 then says that ρ is a functor $\rho :$ `CommRings` $\longrightarrow$ `PCR` which is an idempotent reflector and the adjoint morphism of R is $\rho_R : R \longrightarrow \rho(R)$.

Warning. p-adic closures of rings are constructed for *pure* rings here, not for rings equipped with some valuation. For example if K is a field then the p-adic closure of the ring K is a certain von Neumann regular ring where the residue fields are the p-adic closures of (K, v) and v runs through the p-adic valuations of K (if there is no p-adic valuation on K then the p-adic closure of K is the Null ring). There is no conflict with the traditional notion of p-adic closures, since fields never had p-adic closures, only p-valued fields have p-adic closures.

The ℓ-adic spectrum $\ell\text{-}\operatorname{Spec} R$ of a ring R is the *spectral space* whose points are pairs $(\mathfrak{p}, (P_n)_{n\in\mathbb{N}})$ with $\mathfrak{p} \in \operatorname{Spec} R$ (the Zariski spectrum of R) and for some p-valuation v of the quotient field $\operatorname{qf}(A/\mathfrak{p})$ of A at $\mathfrak{p}$, P_n is the set of all elements of $\operatorname{qf}(A/\mathfrak{p})$ which are n-th powers in the p-adic closure of $(\operatorname{qf}(A/\mathfrak{p}), v)$.

We skip the definition of the topology of $\ell\text{-}\operatorname{Spec} R$ and refer to [Ro86, Be90, BS] instead. To see an example, if $R = \mathbb{Q}_p[x_1, ..., x_n]$ then $\ell\text{-}\operatorname{Spec} R$ is bijective (but not homeomorphic) to the n-types of the field $\mathbb{Q}_p$.

We can show that the passage from R to its p-adic closure $\rho(R)$ transforms $\ell\text{-}\operatorname{Spec} R$ into $\operatorname{Spec} \rho(R)$:

Theorem 6. *If A is a p-adically closed ring then the* support map

$$\mathrm{supp}:\ell\text{-}\operatorname{Spec} A \longrightarrow \operatorname{Spec} A$$

defined by $\mathrm{supp}(\mathfrak{p},(P_n)_{n\in\mathbb{N}})=\mathfrak{p}$ *is an homeomorphism.*

If R is an arbitrary ring then the natural map $\ell\text{-}\operatorname{Spec}\rho(R)\longrightarrow\ell\text{-}\operatorname{Spec} R$ is an homeomorphism as well. Hence we get a natural homeomorphism

$$\operatorname{Spec}\rho(R)\longrightarrow\ell\text{-}\operatorname{Spec} R.$$

Conclusion 7. *Let R be a ring. By theorem 6 the space* $\operatorname{Spec}\rho(R)$ *is the correct space for studying topological aspects of p-adic phenomenons of R. By theorem 4 and 3 the affine scheme* $\operatorname{Spec}\rho(R)$ *has p-adically closed stalks, p-adically closed residue fields and all rings of sections of open sub-schemes are p-adically closed. Hence the arithmetic associated to p-adic-topological aspects of R (and $\ell\text{-}\operatorname{Spec} R$) is entirely encoded in the scheme* $\operatorname{Spec}\rho(R)$. *In this sense $\rho(R)$ is the correct ring of 'abstract p-adic functions'.*

References

[Be88] Bélair L., Substructures and uniform elimination for p-adic fields, Annals of Pure and Applied Logic 39 (1988), pp. 1–17.

[Be90] Bélair L., Spectre p-adique: aspects topologiques et géométriques, Séminaire sur les structures algébriques ordonnées, Vol II, no. 33 (1990), pp.151-163.

[Be91] Bélair L., Anneaux p-adiquement clos et anneaux de fonctions définissables, Journ. Symb. Logic 56 (1991), pp. 539–553.

[Be95] Bélair L., Anneaux de fonctions p-adiques, Journ. Symb. Logic 60 (2) (1995), pp. 484–497.

[BS] Bröcker L.; Schinke J.-H.; On the L-adic spectrum. Schriftenreihe des Mathematischen Instituts der Universität Münster, 2. Serie, 1986. iv+45 pp.

[De84] Denef J., The rationality of the Poincaré series asssociated to the p-adic points on a variety, Inventiones Mathematicae 77, 1-23 (1984).

[De86] Denef J., p-adic semi-algebraic sets and cell decomposition, Journal fur die Reine und. Angewandte Mathematik, vol. 369 (1986), pp. 154–166.

[vdD84] van den Dries L., Algebraic theories with definable skolem functions, Journ. Symb. Logic 49 (1984), pp. 625–629.

[vdDSco88] van den Dries L., Scowcroft P.; On the structure of semialgebraic sets over p-adic fields; J. Symbolic Logic 53 (1988), no. 4, 1138–1164

[GT07] Guzy N., Tressl M., (p,d)-adically closed spaces, in preparation.

[Ma76] Macintyre A., On definable subsets of p-adic fields, Journ. Symb. Logic 41 (1976), pp. 605–610.

[PrRo] Prestel A., Roquette P., Formally p-adic fields, Lecture Notes in Mathematics 1084.

[Ri] Ribenboim P., Théorie des valuations, Presse de l'Université de Montréal, 1969.

[Ri2] Ribenboim P., Equivalent forms of Hensel's Lemma, Expo. Math. 3 (1985), pp. 3–24.

[Ro86] Robinson E., The p-adic spectrum, Journal of Pure and Applied Algebra 40 (1986), pp 281–296.

[Ro87] Robinson E., The geometric theory of p-adic fields, Journal of Algebra 110 (1987), pp 158–172.

[Schw] N. Schwartz; The basic theory of real closed spaces; Mem. Am. Math. Soc., vol 397 (1989).

[SchwMa] Schwartz N., Madden J.J.; Semi-algebraic Function Rings and Reflectors of Partially Ordered Rings; Lecture Notes in Mathematics 1712.

[Tr07] Tressl M., Super real closed rings, Fundamenta Mathematicae 194 (2) (2007), pp. 121–177.

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