DEDEKIND CUTS OF ORDERED ABELIAN GROUPS AND FIELDS

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Abstract. A starting point of the theory.

Contents

1. Invariance groups in ordered abelian groups

By a **cut** ξ of a totally ordered set X we mean a pair $\xi = (\xi^L, \xi^R)$, where $\xi^L \cup \xi^R = X$ and $\xi^L < \xi^R$. If S is a subset of X, then the **upper edge** of S, denoted by S^+ is defined as the cut ξ of X with $\xi^R = \{x \in X \mid S < x\}$. Similarly, the **lower edge** of S is defined and denoted by S^-. In particular, the upper edge of $\emptyset$ is $(\emptyset, X)$ (also denoted by $-\infty$) and the upper edge of X itself is $(X, \emptyset)$ (also denoted by $+\infty$). The **principal cuts** of X are defined to be $+\infty$, $-\infty$ and all the cuts x^+, x^- where $x \in X$. If $X \subseteq Y$ are totally ordered, then a cut ξ of X is **realized** by $y \in Y$ if $\xi^L < y < \xi^R$; if there is no such y we say that ξ is **omitted** in Y. A cut η of Y **extends** ξ if $\xi^L = \eta^L \cap X$ and $\xi^R = \eta^R \cap X$.

Let G be an abelian ordered group. If ξ is a cut of G, then $-\xi$ denotes the cut $(-\xi^R, -\xi^L)$. If $S \subseteq G$, then $-(S^+) = (-S)^-$. Further, G acts on the set of its cuts via $g + \xi := (g + \xi^L, g + \xi^R)$. We write $g - \xi$ for $g + (-\xi)$. The stabilizer of ξ under the action is called the **invariance group** of ξ and is denoted by $G(\xi)$. Clearly $G(\xi)$ is a convex subgroup of G. The upper edge of $G(\xi)$ is denoted by

$$\hat{\xi} := G(\xi)^+.$$

If $G \subseteq H$ is an extension of abelian ordered groups and $h \in H \setminus G$, then we write $G(\alpha/G)$ for the invariance group of the cut of H that is realized by h.

1.1. **Lemma.** *Let G be an abelian ordered group and let U be a convex subgroup of G. Then the following are equivalent.*

(i) U^+ is realized in the divisible hull $G \otimes_{\mathbb{Z}} \mathbb{Q}$.
(ii) G/U has a smallest positive element.

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(iii) *there is some* $g \in G$ *with* $U^+ = g + U^-$.
(iv) *there is some* $g \in G, g > U$ *such that for all* $g_0 \in G$ *with* $U < 2g_0$ *we have* $2g_0 > g$. *In other words, there is some* $g \in G$ *with* $g > U$ *such that no* $g_0 \in G$ *satisfies* $U < 2g_0 \leq g$.
(v) *There is some* $g \in G$ *with* $g > U$ *which is not the sum of two elements* $> U$.
(vi) *there is some abelian ordered group* $H \supseteq G$, *such that the largest extension of* U^+ *on* H *is not the upper edge of a convex subgroup of* H.

If this is the case, then for each $g \in G$ *the following are equivalent:*

- $U^+ = g + U^-$.
- $\frac{g}{2}$ *realizes* U^+.
- $\frac{g}{n}$ *realizes* U^+ *for all* $n \geq 2$.
- $g \bmod U$ *is the smallest positive element of* G/U.
- $g > U$ *and for all* $g_0 \in G$ *with* $U < 2g_0$ *we have* $2g_0 > g$.
- $g > U$ *is not the sum of two elements* $> U$.

Note that it may happen that (i) holds, but the convex hull of U *in some* $H \supseteq G$ *is not the only convex subgroup of* H *lying over* U. *For example if* $\mathbb{Z} = G \subseteq \mathbb{Q}((t^{\mathbb{Q}}))$ *and* $U = \{0\}$. *Then the infinitesimal elements of* $\mathbb{Q}((t^{\mathbb{Q}}))$ *witness this.*

Proof. Let $g \in G$, $g > 0$.

(i)⇒(ii). If U^+ is realized in $G \otimes_{\mathbb{Z}} \mathbb{Q}$, then clearly there is some $g_0 \in G$, such that $g_0/2$ realizes U^+.

Suppose $g/2$ realizes U^+ and $0 < h \bmod U < g \bmod U$ for some $h \in G$. Then $h > U$ and $g - h > U$, thus $h > g/2$ and $g - h > g/2$. But $g - h > g/2$ implies $2g - 2h > g$, thus $g > 2h$, a contradiction.

(ii)⇒(iii). If $g \bmod U$ is the smallest positive element of G/U and $h \in G$, $h > U$, then $g - u \leq h$ for some $u \in U$. In other words $U^+ = g + U^-$.

(iii)⇒(iv). If $U^+ = g + U^-$ and $g_0 \in G$ with $U < 2g_0$, then $g_0 > U$ and $g - u \leq g_0$ for some $u \in U$. Thus $2g_0 \geq g + g_0 - u > g$.

(iv)⇒(v). An element g as in (iv) cannot be the sum of two elements $g_1, g_2 > U$ because if $g_1 \leq g_2$ we had $2g_1 \leq g$.

(v)⇒(i). If $g > U$ is not the sum of two elements $> U$, then $g/2$ realizes U^+: Otherwise there is $h > U$ with $h \leq g/2$, thus $2h \leq g$. But then $g = h + (g - h)$ and $g - h \geq h > U$, a contradiction.

(i)⇒(vi). Take $H = G \otimes_{\mathbb{Z}} \mathbb{Q}$.

(vi)⇒(i). Take $h \in H$ realizing U^+ such that $2h$ is not a realization of U^+. Hence there is some $g \in G$ with $U < g \leq 2h$ and U^+ is realized by $g/2$.

Hence we know that (i)-(vi) are equivalent. Moreover, our proof shows the second set of equivalences. □

1.2. **Corollary.** *Let* G *be an abelian ordered group and let* U *be a convex subgroup of* G. *The convex hull of* U *in* $G \otimes_{\mathbb{Z}} \mathbb{Q}$ *is the unique convex subgroup of* $G \otimes_{\mathbb{Z}} \mathbb{Q}$, *lying over* U.

Proof. This is clear if U^+ is not realized in $G \otimes_{\mathbb{Z}} \mathbb{Q}$ and follows from 1.1 in the other case. □

1.3. **Definition.** Let G be an ordered abelian group and let ξ be a cut of G. We define the **signature** of ξ as

$$\operatorname{sign}\xi := \begin{cases} 1 & \text{if } \xi = g+\hat{\xi} \text{ for some } g\in G \text{ and } \hat{\xi} \text{ is omitted in } G\otimes_{\mathbb{Z}}\mathbb{Q}.\\ -1 & \text{if } \xi = g-\hat{\xi} \text{ for some } g\in G \text{ and } \hat{\xi} \text{ is omitted in } G\otimes_{\mathbb{Z}}\mathbb{Q}.\\ 0 & \text{if there is no } g\in G \text{ with } \xi=g+\hat{\xi} \text{ or } \xi = g-\hat{\xi}.\\ \infty & \text{otherwise.}\end{cases}$$

Observe that by 1.1(i)⇔(iii), the first two cases in this definition cannot occur simultaneously. Therefore $\operatorname{sign}\xi$ is a well defined element of $\{-1,0,1,\infty\}$.

If $G\subseteq H$ is an extension of abelian ordered groups and $h\in H\setminus G$, then we write $\operatorname{sign}(\alpha/G)$ for the signature of the cut of H that is realized by h.

1.4. *Remarks.*

(i) If ξ is a cut of G, $g\in G$ and U is a convex subgroups of G with $\xi = g+U^+$ or $\xi = g-U^+$ (which is equal to $g+U^-$), then obviously $U = G(\xi)$.

(ii) If G is divisible by n for some $n\in\mathbb{N}$, $n\geq 2$, then no edge of a subgroup of G is realized in $G\otimes\mathbb{Q}$, in particular no cut of G has signature ∞. This follows immediately from the equivalent conditions characterizing realisations in 1.1.

1.5. **Corollary.** *Let G be an abelian ordered group and let ξ be a cut of G. If $\operatorname{sign}\xi = 0$, then $\hat{\xi}$ is omitted in $G\otimes_{\mathbb{Z}}\mathbb{Q}$.*

Proof. Let $U := G(\xi)$ and suppose U^+ is realized in $G\otimes_{\mathbb{Z}}\mathbb{Q}$. By 1.1, there is some $g\in G$, such that $g>U$ and $g \bmod U$ is the least positive element in G/U. As $g>G(\xi)$ there is some $h\in G$ with $h<\xi<h+g$. We claim that $\xi = h+\hat{\xi}$. Clearly $h+\hat{\xi}\leq\xi$. Conversely let $g_1\in G$ with $h+\hat{\xi}<g_1$. Then $g_1-h>\hat{\xi}$, thus $(g_1-h)\bmod U>0$ in G/U. So $(g_1-h)\bmod U\geq g\bmod U$ and there is some $u\in U$ with $g_1-h\geq g-u$. It follows $g_1\geq h+g+u>\xi+u=\xi$. □

Hence by 1.5, the signature of a cut ξ is ∞ if and only if $\hat{\xi}$ is realized in $G\otimes_{\mathbb{Z}}\mathbb{Q}$.

In general, there are cuts ξ of G with $\operatorname{sign}\xi=0$ which are realized in $G\otimes_{\mathbb{Z}}\mathbb{Q}$. For example if $n\in\mathbb{N}$, $n\geq 2$ and G is the additive group of the localizatin of $\mathbb{Z}$ at n. Then for any prime p, which does not divide n, the cut ξ of G realized by $\frac{1}{p}$, has signature 0: G is dense in $\mathbb{Q}$ and divisible by n.

1.6. *Example.* Here is an example, where the signature is ∞. Let $\omega>\mathbb{R}$ be an infinite element and let $K:=\mathbb{Q}(\omega)$. Let $G:=(K^{>0},\cdot,1,\leq)$ and $H:=(K(\sqrt{\omega})^{>0},\cdot,1,\leq)$. Let U be the convex hull of $\mathbb{Q}$ in K and let $\xi:=U^+$. Then $U<\sqrt{\omega}<\xi^R$ but $\omega=(\sqrt{\omega})^2\in\xi^R$. Note that G and H are densely ordered in this example. Related to this example, also see 2.3 (and 2.4), and 2.5.

1.7. **Observation.** *Let G be an abelian ordered group and let ξ be a cut of G. The following are equivalent.*

(i) For all $\xi<a$ there is some $\xi<b<a$ with $2b-a<\xi$.

(ii) $\operatorname{sign}(\xi)\in\{0,1\}$.

(iii) $\xi\neq g-\hat{\xi}$ for all $g\in G$.

Proof. (ii) and (iii) are equivalent by definition and 1.1.

(iii)⇒(i). Let $U=G(\xi)$ and pick $\xi<a$. Since $\xi\neq a-\hat{\xi}$, there is some $c\in G$ with $\xi<c<a-\hat{\xi}$. Since $a-c>U$, there is some $b>\xi$ with $b-(a-c)<\xi$. By shrinking b if necessary we may assume that $b \le c$. Then $\xi < b \le c < a$ and

$$2b - a = b - (a - c) + b - c \le b - (a - c) < \xi.$$

(i)⇒(iii). Assume $\xi = g - \hat{\xi}$ for some $g \in G$. Then $\xi < g$ and by (i) there is some $b \in G$ with $g - \hat{\xi} < b < g$ and $2b - g < \xi$. But then $g - \hat{\xi} < b < g$ implies $b - g \in U$ and so $g - \hat{\xi} < b$ implies $\xi = g - \hat{\xi} < b + (b - g) = 2b - g$, which is impossible. □

1.8. **Proposition.** *Let $G \subseteq H$ be an extension of abelian ordered groups and let ξ be a cut of G. Then*

(i) If η is an extension of ξ on H, then $G(\eta) \cap G \subseteq G(\xi)$.

(ii) If η is the least or the largest extension of ξ on H then $\hat{\eta}$ is the least or the largest extension of $\hat{\xi}$ on H.

(iii) If ξ is omitted in H and η is the unique extension of ξ on H, then $\hat{\eta}$ is the largest extension of $\hat{\xi}$ on H. If in addition $\operatorname{sign} \xi = 0$, then $\operatorname{sign} \eta = 0$, too.

(iv) Let $\operatorname{sign} \xi = 0$ and let η_1, η_2 be the least and the largest extension of ξ on H. Then $\hat{\eta}_1 = \hat{\eta}_2$ is the largest extension of $\hat{\xi}$on H and for every realization h of ξ in H we have $\eta_1 = h - \hat{\eta}_1$, $\eta_2 = h + \hat{\eta}_2$ and $\eta_2 = 2h - \eta_1$.

(v) Let $\operatorname{sign} \xi = 1$ and let η_1, η_2 be the least and the largest extension of ξ on H. Then $\hat{\eta}_1$ is the least extension of $\hat{\xi}$ on H and $\hat{\eta}_2$ is the largest extension of $\hat{\xi}$ on H.

Moreover, if $g \in G$ with $\xi = g + \hat{\xi}$, then $\eta_1 = g + \hat{\eta}_1$ is of signature 1 and $\eta_2 = g + \hat{\eta}_2$ is of signature 1.

(vi) Let $\operatorname{sign} \xi = -1$ and let η_1, η_2 be the least and the largest extension of ξ on H. Then $\hat{\eta}_1$ is the largest extension of $\hat{\xi}$ on H and $\hat{\eta}_2$ is the least extension of $\hat{\xi}$ on H.

(vii) Let $\operatorname{sign} \xi = \infty$ and let $g \in G$ such that $g \bmod G(\xi)$ is the least positive element of $G/G(\xi)$. Let η_1, η_2 be the least and the largest extension of ξ on H. Then $\hat{\eta}_1 = \hat{\eta}_2$ is the least extension of $\hat{\xi}$ on H and $\hat{\eta}_2 = g - \hat{\eta}_1$.

Further, there is some $g_0 \in G$ such that $\xi = g_0 + \hat{\xi} = g_0 + g - \hat{\xi}$ and for each such g_0 we have $\eta_1 = g_0 + V^+$ and $\eta_2 = g_0 + g + V^-$, where V is the convex hull of $G(\xi)$ in H.

Proof. If $g \in G$ and $g + \eta = \eta$, then $g + \xi^L \subseteq (g + \eta^L) \cap G \subseteq \eta^L \cap G = \xi^L$. This proves (i).

Claim. If η is the least or the largest extension of ξ on H, then $\hat{\eta}$ extends $\hat{\xi}$.

Proof. Let $g \in G(\xi)$ be positive. If $\eta^L = \operatorname{conv.hull}_H \xi^L$ then $g + \eta^L = \eta^L$. If $\eta^R = \operatorname{conv.hull}_H \xi^R$ then $-g + \eta^R = \eta^R$. In any case $g + \eta = \eta$. This proves the claim. □

(iii). Take some $h \in H$ with $h + \eta > \eta$. Since ξ is omitted in H, there is $g_1 \in G$ with $g_1 < \xi$ such that $h + g_1 > \eta$. Since ξ is omitted in H there is $g_2 \in G$ with $h + g_1 \ge g_2 > \eta$. Hence $h \ge g_2 - g_1 > G(\xi)$ and h cannot be a realization of $\hat{\xi}$. Now the claim implies that $\hat{\eta}$ is the largest extension of $\hat{\xi}$ on H.

Suppose now that ξ is omitted in H and $\operatorname{sign} \xi = 0$. Suppose $\eta = h + \hat{\eta}$ for some $h \in H$. As ξ is omitted in H, there is some $g \in G$ with $h \le g < \xi$. Thus $\eta = g + \hat{\eta}$ and this implies that $\xi = g + \hat{\xi}$, a contradiction.

(iv). By 1.5, and 1.1(v), the largest extension of $\hat{\xi}$ on H is the upper edge of a convex subgroup H_0 of H. By the claim we know that $\hat{\eta}_i$ extends $\hat{\xi}$. Let $h_0 \in H_0$

be a realization of $\hat{\xi}$. Since $\operatorname{sign}\xi = 0$, and $k{\cdot}h_0 \in H_0$ for all $k \in \mathbb{Z}$, ξ is omitted in the subgroup $G(h_0)$ generated by G and h_0 of H. Let η be the unique extension of ξ on $G(h_0)$. Hence η_1, η_2 are the least and the largest extension of η on H. By the claim we know that $h_0 + \eta = \eta$. By the claim applied to η we get $h_0 + \eta_i = \eta_i$ $(i = 1, 2)$. This shows that $\hat{\eta}_1 = \hat{\eta}_2 = H_0^+$.

Now let $h \in H$ be a realization of ξ. We already know $h + H_0^+ \leq \eta_2$. Suppose there is some $h_1 \in H$ with $h + H_0 < h_1 < \eta_2$. Then $h_1 - h$ is not a realization of $\hat{\xi}$. Take $g \in G$ with $h_1 - h \geq g > \hat{\xi}$. Then $h + g \leq h_1$, but $h + g$ does not realize ξ, a contradiction.

(v) and (vi) are immediate consequences of 1.1.

(vii). We write $U = G(\xi)$. Using 1.5, we know that $\xi = g_1 + U^+$ or $\xi = g_1 + U^-$ for some $g_1 \in G$. By 1.1 we know that $U^+ = g + U^-$. Hence if $\xi = g_1 + U^+$, then we may choose $g_0 = g_1$ and get $\xi = g_0 + U^+$. If $\xi = g_1 + U^-$, then $\xi = g_1 + (-g + U^+)$ and we may choose $g_0 = g_1 - g$, thus $\xi = g_0 + U^+$.

Hence there i some g_0 as claimed and for the rest of the proof of (vii) we may thus assume that $g_0 = 0$, hence $\xi = U^+ = g + U^-$ with $U = G(\xi)$. It is then clear that η_1 is the upper edge of the convex hull V of U in H. and that $\eta_2 = g + V^-$. Thus (vii) follows.

(ii) follows by the descriptions of the invariance groups of the least and the largest extension of ξ on H in (iv)-(vii). ☐

1.9. **Definition.** Let $G \subseteq H$ be totally ordered abelian groups and let ξ be a cut of G. We define

$$R_H(\xi) = \{h \in H \mid h \text{ realizes } \xi\}.$$

Hence $R_H(\xi) \subseteq H \setminus G$ is the set of realizations of ξ in H. Further we define

$$G_H(\xi) = R_H(\xi) - R_H(\xi) = \{h_1 - h_2 \mid h_1, h_2 \in R_H(\xi)\}.$$

1.10. **Corollary.** *Suppose $R_H(\xi) \neq \emptyset$.*

(i) If $\operatorname{sign}(\xi) \neq \infty$, then $G_H(\xi)$ is the largest convex subgroup of H lying over $G(\xi)$.

(ii) If $\operatorname{sign}(\xi) = \infty$, then there is some $g \in G$ such that $\hat{\xi} = g - \hat{\xi}$ and for each such g we have

$$G_H(\xi) = \{h \in H \mid |h| < g + G(\xi)\}.$$

*Notice that if H contains a realization of $\hat{\xi}$ from the divisible hull of G, then $G_H(\xi)$ is **not** a convex subgroup of H.*

Proof. (i). Let W be the largest convex subgroup of H lying over $G(\xi)$.

If $\operatorname{sign}\xi = 0$ and $h \in R_H(\xi)$, then $R_H(\xi) = h + W$ by 1.8(iv) and so $G_H(\xi) = R_H(\xi) - R_H(\xi) = W$.

If $\operatorname{sign}\xi = 1$ and $h \in R_H(\xi)$, then by 1.8(v) we have $R_H(\xi) \cap [h, +\infty)_H = h + W^{\geq 0}$. Hence

$$G_H(\xi) = \bigcup_{h \models \xi} R_H(\xi) \cap [h, +\infty)_H - R_H(\xi) \cap [h, +\infty)_H = W.$$

If $\operatorname{sign}\xi = -1$, apply the previous case and $G_H(\xi) = G_H(-\xi)$.

(ii). If $\operatorname{sign}(\xi)=\infty$, then let V be the convex hull of $G(\xi)$ in H and take g, g_0 as in 1.8(vii). By 1.8(vii) we then have $R_H(\xi)=g_0+\{h\in H\mid V<h<g+V\}$. It is then straightforward to see that

$$G_H(\xi)=R_H(\xi)-R_H(\xi)=\{h\in H\mid |h|<g+V\}.$$

□

1.11. **Definition.** Let $f:X\longrightarrow Y$ be a monotone map between totally ordered sets let η be a cut of Y. Then clearly $(f^{-1}(\eta^L),f^{-1}(\eta^R))$ is a cut of X, which we denote by $f^{-1}(\eta)$. Hence by definition $f^{-1}(\eta)^L=f^{-1}(\eta^L)$ and $f^{-1}(\eta)^R=f^{-1}(\eta^R)$.

1.12. **Lemma.** *Let $f:G\longrightarrow H$ be a homomorphism between totally ordered abelian groups.*

(i) If $S_1,S_2\subseteq G$, then $f(S_1+S_2)=f(S_1)+f(S_2)$.
(ii) If $T_1\subseteq f(G)$ and $T_2\subseteq H$, then $f^{-1}(T_1+T_2)=f^{-1}(T_1)+f^{-1}(T_2)$.

Notice that the sets S_i and T_i considered here may also be empty, because the complex operation induced by addition of G on the powerset of G is defined as $S_1+S_2=\{g\in G\mid \exists s_1\in S_1,s_2\in S_2: g=s_1+s_2\}$. In particular $\emptyset+S=S+\emptyset=\emptyset$.

Proof. (i) is clear. To see (ii), take $g\in G$. If $g_i\in f^{-1}(T_i)$ with $g=g_1+g_2$, then $f(g)=f(g_1)+f(g_2)\in T_1+T_2$, thus $g\in f^{-1}(T_1+T_2)$. Conversely, if $g\in f^{-1}(T_1+T_2)$, then there are $t_1\in T_1,t_2\in T_2$ with $f(g)=t_1+t_2$. As $T_1\subseteq f(G)$, there is some $g_1\in G$ with $f(g_1)=t_1$. Then $g=g_1+g_2$ with $g_2=g-g_1$ and $f(g_2)=f(g)-f(g_1)=t_1+t_2-t_1=t_2$, confirming $g\in f^{-1}(T_1)+f^{-1}(T_2)$. □

1.13. **Proposition.** *Let G be a totally ordered abelian group and let U be a convex subgroup of G.*

(i) If η is a cut of G/U, then $G(\pi^{-1}(\eta))=\pi^{-1}(G(\eta))$ and $\operatorname{sign}(\pi^{-1}(\eta))=\operatorname{sign}(\eta)$.
(ii) If ξ is a cut of G, then $\xi=\pi^{-1}(\eta)$ for some cut η of G/U if and only if $U\subseteq G(\xi)$.

Proof. (i) To verify $G(\pi^{-1}(\eta))=\pi^{-1}(G(\eta))$ it suffices to check that for $g\in G$ we have

$$g+\pi^{-1}(\eta^L)=\pi^{-1}(\eta^L)\iff \pi(g)+\eta^L=\eta^L.$$

Since π is surjective we know that $\eta^L=\pi(\pi^{-1}(\eta^L))$.

$\Rightarrow$. $\pi(g)+\eta^L=\pi(g)+\pi(\pi^{-1}(\eta^L))\overset{1.12(i)}{=}\pi(g+\pi^{-1}(\eta^L))=\pi(\pi^{-1}(\eta^L))=\eta^L$.

$\Leftarrow$. $\pi^{-1}(\eta^L)=\pi^{-1}(\pi(g)+\eta^L)\overset{1.12(ii)}{=}\pi^{-1}(\pi(g))+\pi^{-1}(\eta^L)=g+U+\pi^{-1}(\eta^L)=g+\pi^{-1}(\eta^L)$.

Now we show $\operatorname{sign}(\pi^{-1}(\eta))=\operatorname{sign}(\eta)$. Firstly $\operatorname{sign}(\eta)=\infty\iff\hat{\eta}$ is realized in $(G/U)\otimes\mathbb{Q}$ iff $(G/U)/G(\eta)$ has a smallest positive element iff $G/\pi^{-1}(G(\eta))$ has a smallest positive element iff $\operatorname{sign}(\pi^{-1}(\eta))=\infty$. Hence we may assume that $\operatorname{sign}(\eta),\operatorname{sign}(\pi^{-1}(\eta))\neq\infty$. But then $\operatorname{sign}(\pi^{-1}(\eta))=\operatorname{sign}(\eta)$ is immediate from 1.5 and definition 1.3.

(ii) is an easy exercise. □

1.14. **Proposition.** *Let $G\subseteq H$ be totally ordered abelian groups and let U be a convex subgroup of H. Let $\pi:H\longrightarrow H/U$ be the natural map and let $h\in H$ with $\pi(h)\notin\pi(G)$. Let η be the cut of $\pi(G)$ realized by $\pi(h)$. Let ξ be the cut of G realized by h. Let π_0 be the restriction of π to G. Then*

(i) $\xi^L=\pi_0^{-1}(\eta^L)$ and $\xi^R=\pi_0^{-1}(\eta^R)$.

(ii) $G(\xi) = \pi_0^{-1}(G(\eta))$.
(iii) $\operatorname{sign}(\xi) = \operatorname{sign}(\eta)$.

Proof. (i) To see $\xi^L = \pi_0^{-1}(\eta^L)$, we need to show that $g < h \iff \pi(g) < \pi(h)$ for all $g \in G$. This is obvious, since π is order preserving and $\pi(g) = \pi(h)$ cannot occur by assumption. Similarly $\xi^R = \pi^{-1}(\eta^R)$.
(ii) We assume that $\eta^L \neq \emptyset$, otherwise we proceed with $-\xi$ and $-\eta$. For $g \in G$ we clearly have

$$(*) \qquad \pi_0(g + \pi_0^{-1}(\eta^L)) = \pi_0(g) + \eta^L \text{ and } \pi_0^{-1}(\pi_0(g) + \eta^L) = g + \pi_0^{-1}(\eta^L).$$

Hence

$$g \in G(\xi) \iff g + \xi^L = \xi^L \overset{\text{by }(i)}{\iff} g + \pi_0^{-1}(\eta^L) = \pi_0^{-1}(\eta^L) \iff$$
$$\overset{(*)}{\iff} \pi_0(g) + \eta^L = \eta^L \iff g \in \pi_0^{-1}(G(\eta)).$$

(iii) follows (i),(ii) and 1.13. (Exercise) □

2. Cuts in ordered fields, $G(\xi)$ and $V(\xi)$

2.1. **Lemma.** *Let K be an ordered field, let G be the multiplicative group of positive elements of K. If H is a convex subgroup of G with $2 \notin H$, then H^+ is omitted in $G \otimes_{\mathbb{Z}} \mathbb{Q}$.*

Recall from 1.6 that the assumption $2 \notin H$ cannot be dropped.

Proof. Suppose H^+ is realized in $G \otimes_{\mathbb{Z}} \mathbb{Q}$. By 1.1, there is some realization γ of H^+ in the real closure R of K, such that $\gamma^2 \in K$. Since $2 \notin H$, $H - 1$ is a convex subgroup of $(K, +, \leq)$. Since $(K, +)$ is divisible, $3{\cdot}(\gamma - 1)$ realizes $(H - 1)^+$. Since $1 \leq 1 + \gamma \leq 3$ we have $(\gamma - 1) \leq (\gamma - 1)(\gamma + 1) \leq 3{\cdot}(\gamma - 1)$, hence also $(\gamma - 1)(\gamma + 1)$ realizes $(H - 1)^+$. But this is impossible, since $(\gamma - 1)(\gamma + 1) = \gamma^2 - 1 \in K$. □

2.2. **Definition.** If K is an ordered field and ξ is a cut of K then we define the **multiplicative invariance group** of ξ, written as $G^*(\xi)$, as the invariance group of $|\xi|$ w.r.t. $(K^{>0}, \cdot, \leq)$. Explicitly we have

$$G^*(\xi) = \{a \in K \mid a{\cdot}\xi = \xi\}$$

(This also applies if $\xi < 0$).
The **multiplicative signature** of ξ is defined as

$$\operatorname{sign}^* \xi := \text{ the signature of } |\xi| \text{ w.r.t. } (K^{>0}, \cdot, <).$$

If $K \subseteq L$ is an extension of ordered fields and $\alpha \in L \setminus K$, then we write $G^*(\alpha/K)$ and $\operatorname{sign}^*(\alpha/K)$ for the multiplicative invariance group and the multiplicative signature of the cut of K that is realized by α.

2.3. **Corollary.** *Let K be an ordered field and let $\xi > 0$ be a cut of K with $\operatorname{sign}^*(\xi) = \infty$. Then $\xi = \hat{\xi}$ and $\xi > \mathbb{Q}$ or $\xi^{-1} > \mathbb{Q}$.*

Proof. By 2.1 we have $2 \in G^*(\xi)$, which is equivalent to $\xi = \hat{\xi}$.
If $\xi > 1$, then $2 \in G^*(\xi)$ also implies $\xi > \mathbb{Q}$. If $\xi < 1$, then as $\operatorname{sign}^*(\xi^{-1}) = \operatorname{sign}^*(\xi)$ we get $\xi^{-1} > \mathbb{Q}$. □

The multiplicative signature is only a new invariant for cuts with $|\xi| = \hat{\xi}$:

2.4. **Proposition.** *If ξ is a cut of an ordered field with $\xi > \hat{\xi}$, then*

$$\operatorname{sign} \xi = \operatorname{sign}^* \xi.$$

Proof. By 2.3 we know that $\operatorname{sign}^* \xi \neq \infty$. Since $(K, +)$ is divisible, also $\operatorname{sign} \xi \neq \infty$.

Let $G := G(\xi)$. If $\operatorname{sign} \xi = 1$, then there is some $a \in K$, $a > G$ with $\xi = a + G^+$. Since $1 \notin \frac{1}{a} \cdot G$, $1 + \frac{1}{a} \cdot G$ is a convex subgroup of $(K^{>0}, \cdot, \leq)$. Hence $\xi = a \cdot (1 + \frac{1}{a} \cdot G^+) = a \cdot (1 + \frac{1}{a} \cdot G)^+$ has multiplicative signature 1.

If $\operatorname{sign} \xi = -1$, then there is some $a \in K$, $a > G$ with $\xi = a + G^-$. Since $1 \notin \frac{1}{a} \cdot G$, $1 + \frac{1}{a} \cdot G$ is a convex subgroup of $(K^{>0}, \cdot, \leq)$. Hence $\xi = a \cdot (1 + \frac{1}{a} \cdot G^-) = a \cdot (1 + \frac{1}{a} \cdot G)^-$ has multiplicative signature -1.

If $\operatorname{sign}^* \xi = 1$, then $\xi = a \cdot H^+$ for a convex subgroup H of $(K^{>0}, \cdot, \leq)$ and some $a > 0$. Since $\xi > \hat{\xi}$, $2 \notin H$ and $G := H - 1$ is a convex subgroup of $(K, +, \leq)$. Hence $\xi = a \cdot (1 + G^+) = a + (a \cdot G)^+$ has signature 1.

If $\operatorname{sign}^* \xi = -1$, then $\xi = a \cdot H^-$ for a convex subgroup H of $(K^{>0}, \cdot, \leq)$ and some $a > 0$. Since $\xi > \hat{\xi}$, $2 \notin H$ and $G := H - 1$ is a convex subgroup of $(K, +, \leq)$. Hence $\xi = a \cdot (1 + G^-) = a + (a \cdot G)^-$ has signature -1.

Hence we know that $\operatorname{sign} \xi = 1 \iff \operatorname{sign}^* \xi = 1$ and $\operatorname{sign} \xi = -1 \iff \operatorname{sign}^* \xi = -1$. This shows the proposition. □

2.5. *Example.* Let K be an ordered field and let $\alpha \notin K$, $\alpha > 0$ be an element from the real closure of K such that $\alpha^n \in K$. Suppose $1 \leq \alpha \leq n$ for some $n \in \mathbb{N}$. Let ξ be the cut of K realized by α. Then $\operatorname{sign} \xi = \operatorname{sign}^* \xi = 0$.

Proof. Since $1 \leq \alpha \leq n$, we have $\xi > \hat{\xi}$, hence by 2.4, $\operatorname{sign}^* \xi = \operatorname{sign} \xi \neq \infty$. Since ξ is realized in the divisible hull of the multiplicative group of positive elements of K, ξ cannot have signature 1 (otherwise $\frac{1}{b} \cdot \xi$ would be the upper edge of a convex subgroup of $K^{>0}$, realized in the divisible hull of $K^{>0}$). The same argument shows that $\operatorname{sign}^* \xi \neq -1$. Hence $\operatorname{sign}^* \xi = 0$. □

The multiplicative invariance group can be computed from the additive invariance group, provided $|\xi| > \hat{\xi}$: Let K be an ordered field. Firstly, recall from [6, proof of (3.5)]:

The set of convex subgroups of $(K, +, \leq)$ that do not contain 1 is in bijection with the convex subgroups of $(K^{>0}, \cdot, \leq)$ that do not contain 2. The bijection is given by $G \mapsto 1 + G$. Moreover we have

2.6. **Proposition.** *Let K be an ordered field and let ξ be a cut of K with $|\xi| > \hat{\xi}$. There is some $c \in K$ such that*

$$G^*(\xi) = c \cdot G(\xi) + 1 \ (= \{c \cdot a + 1 \mid a \in G(\xi)\}).$$

Proof. By [6, (3.5)]. This is included here for completeness:

We may assume that $\xi > \hat{\xi}$. Let $H := G^*(\xi)$. Since $\xi > \hat{\xi}$ we have $2 \notin H$.

Claim 1. $H - 1$ is a convex subgroup of $(K, +, \leq)$.

$H - 1$ is convex, since H is convex. Hence we only have to show that $2 \cdot (H - 1) \subseteq H - 1$ and $H - 1 = -(H - 1)$. Let $\varepsilon \in H - 1$, $\varepsilon > 0$. Then $0 < 2\varepsilon < (1 + \varepsilon)^2 - 1 \in H - 1$, hence $2\varepsilon \in H - 1$. Since $2 \notin H$ we have $\frac{\varepsilon^2}{1 - \varepsilon} < \varepsilon$, thus $1 < \frac{1}{1 - \varepsilon} = 1 + \varepsilon + \frac{\varepsilon^2}{1 - \varepsilon} < 1 + 2\varepsilon \in H$. We get $\frac{1}{1 - \varepsilon} \in H$, therefore $-\varepsilon \in H - 1$.

If $\varepsilon > 0$ with $-\varepsilon \in H - 1$, then $1 < 1 + \varepsilon < \frac{1}{1 - \varepsilon} \in H$, that is $\varepsilon \in H - 1$.

Claim 2. $H - 1 = \{a \in K \mid |a| \cdot \xi < \hat{\xi}\} = \{a \in K \mid |a| \cdot \xi \leq \hat{\xi}\}$.

The second equality holds since $\xi > \hat{\xi}$. To see the first equality we may assume that $a > 0$. If $a\cdot\xi < \hat{\xi}$, then easily $(1+a)\cdot\xi = \xi$. Conversely take $h \in H$ and assume $(h-1)\cdot\xi > \hat{\xi}$.

First suppose $h > 1$. Then there is some $0 < h_1 < \xi$ with $(h-1)h_1 \notin G(\xi)$, hence there is some $h_2 \in K$, $0 < h_1 \leq h_2 < \xi$ with $h_2 + (h-1)h_1 > \xi$. It follows $\xi = h\xi > hh_2 = h_2 + (h-1)h_2 \geq h_2 + (h-1)h_1 > \xi$, a contradiction.

This argument shows that $h > 1$ and $h\cdot\xi = \xi$ imply $(h-1)\cdot\xi \leq \hat{\xi}$, thus $(h-1)\cdot\xi < \hat{\xi}$. On the other hand, if $0 < h < 1$ and $h\cdot\xi = \xi$ then by claim 1 we have $1-h = -(h-1) \in H-1$, whence $(2-h)\cdot\xi = \xi$ and $2-h > 1$. By what we have just proved it follows $(1-h)\cdot\xi < \hat{\xi}$.

Now we prove the proposition. Let $q := H^+ - 1$. By claim 1 it is enough to find some $c \in K$ with $q = c\cdot\hat{\xi}$. By elementary real algebra, there is an ordered field L containing K and realizations α, γ of $\hat{\xi}$ and ξ respectively. By claim 2 we know that $\beta := \frac{\alpha}{\gamma}$ realizes q. Let G' be the convex hull of $G(\xi)$ in $K(\alpha,\gamma)$ and let α' be a realization of G'^+ from an ordered field extension of L. Note that $\alpha' \leq \alpha$. Certainly $\frac{\alpha'}{\beta}$ is a realization of $\frac{1}{\beta}\cdot G'^+$, hence of U^+, where $U := K \cap \frac{1}{\beta}\cdot G'$. Since U is a convex subgroup of $(K,+,\leq)$ and $\xi > \hat{\xi}$, the element $\gamma = \frac{\alpha}{\beta}$ is not a realization of U^+, hence $\frac{\alpha'}{\beta} \leq a \leq \frac{\alpha}{\beta}$ for some $a \in K$. As α and α' realize $\hat{\xi}$ it follows that $a\cdot\beta$ realizes $\hat{\xi}$. Since β realizes q this means $q = \frac{1}{a}\cdot\hat{\xi}$. □

2.7. **Definition.** Let K be an ordered field and let G be a convex subgroup of $(K,+,\leq)$. The **invariance ring** of G is defined as

$$V(G) := \{a \in K \mid a\cdot G \subseteq G\}.$$

If ξ is a cut of K, then the invariance ring of ξ is defined as the invariance ring of $G(\xi)$:

$$V(\xi) = V(G(\xi)).$$

We also write V_ξ for $V(\xi)$.

If $K \subseteq L$ is an extension of ordered fields and $\alpha \in L \setminus K$, then we write $V(\alpha/K)$ for the invariance ring of the cut of K that is realized by α.

2.8. *Remark.* Let G be a convex subgroup of $(K,+,\leq)$. Obviously $V(G)$ is a convex subring of K and the set of units of $V(G)$ is

$$V(G)^\times = \{a \in K \mid a\cdot G = G\}.$$

It follows that the set of positive units of $V(G)$ is the multiplicative invariance group of the upper edge G^+ of G:

$$V(G)^{\times>0} = G^*(G^+).$$

We write $\mathfrak{m}(G)$ and $\mathfrak{m}(\xi)$ for the maximal ideal of $V(G)$, $V(\xi)$, respectively.

2.9. **Proposition.** *Let $K \subseteq L$ be ordered fields and let ξ be a cut of K. Let η be the least or the last extension of ξ on L. Then $\hat{\eta}$ is the least or the largest extension of $\hat{\xi}$ on L and V_η^+ is the least or the largest extension of V_ξ^+ on L. In particular*

$$(K,\xi^L,G(\xi),V_\xi) \subseteq (L,\eta^L,G(\eta),V_\eta).$$

Moreover, if η_1,η_2 are the least and the largest extension of ξ on L and $V_{\eta_1} = V_{\eta_2}$ (for example if L/K is algebraic), then either $\hat{\eta}_1 = \hat{\eta}_2$ or there is some $a \in L$ with $\hat{\eta}_1 = a/\hat{\eta}_2$.

Proof. Everything except the additions follows from 1.8(ii). So let $V_{\eta_1} = V_{\eta_2}$. This means that the invariance groups of $\hat{\eta}_1$ and $\hat{\eta}_2$ w.r.t. $(K^{>0}, \cdot)$ are the same. By 1.8(v),(vi), the multiplicative signature of $\hat{\xi}$ is either 0 or ∞ and the proposition follows from 1.8 (iv),(vii) applied to $\hat{\xi}$ and $(K^{>0}, \cdot, <)$. □

2.10. **Lemma.** *Let K be an ordered field and let G be a convex subgroup of $(K, +, \leq)$. The following are equivalent.*

(i) $\operatorname{sign}^* G^+ = -1$ *or* $\operatorname{sign}^* G^+ = \infty$

(ii) There is some $a \in K$ such that $G = a \cdot \mathfrak{m}(\xi)$.

Proof. The cut $\mathfrak{m}(G)^+$ is the lower edge of the multiplicative invariance group $V(G)^{\times>0}$ of G^+. Hence if $G = a \cdot \mathfrak{m}(\xi)$, then $\operatorname{sign}^* G^+ = -1$ or $\operatorname{sign}^* G^+ = \infty$. Conversely, $\operatorname{sign}^* G^+ = -1$ and $\operatorname{sign}^* G^+ = \infty$ imply that $G^+ = a \cdot G^*(\xi)^-$ for some $a > 0$. □

2.11. **Definition.** Let K be an ordered field with real closure R and let ξ be a cut of K. We define the **degree of** ξ to be the infimum of all $d \in \mathbb{N}$ such that ξ is realized by some $\alpha \in R$, with $[K(\alpha) : K] = d$. If ξ is not realized in R, we define the degree of ξ to be ∞. We write $\deg \xi \in \mathbb{N} \cup \{\infty\}$ for the degree of ξ.

A realization α of ξ in some ordered field extension L of K is called **ξ-generic** if $[K(\alpha) : K] = \deg \xi$.

An element α of some ordered field extension L of K is called **K-generic**, or **generic over** K, if $\alpha \in K$ or if α is ξ-generic for the cut ξ of K realized by α.

2.12. *Example.* Here is an example of an irreducible polynomial $f(T)$ over an ordered field K with two roots from the real closure, realizing the same cut over K. Let $K = k(X)$ where k is an arbitrary ordered field and $X > k$. Let ξ be the cut of K realized by $\sqrt{X}$. Then both $\sqrt{X} + \sqrt[4]{X}$ and $\sqrt{X} - \sqrt[4]{X}$ realize ξ. Moreover both elements are roots of the minimal polynomial f of $\sqrt{X} + \sqrt[4]{X}$ over $k(X)$. We compute f: We have

$$p(T) := (T - (\sqrt{X} + \sqrt[4]{X}))(T - (\sqrt{X} - \sqrt[4]{X})) = (T - \sqrt{X})^2 - \sqrt{X}$$

On the other hand

$$q(T) := (T - (-\sqrt{X} + i\sqrt[4]{X}))(T - (-\sqrt{X} - i\sqrt[4]{X})) = (T + \sqrt{X})^2 + \sqrt{X}$$

Then

$$\begin{aligned} p(T) \cdot q(T) &= (T^2 + X - (2T\sqrt{X} + \sqrt{X}))(T^2 + X + (2T\sqrt{X} + \sqrt{X})) = \\ &= (T^2 + X)^2 - (2T\sqrt{X} + \sqrt{X})^2 = (T^2 + X)^2 - X(2T + 1)^2 = \\ &= T^4 + 2T^2X + X^2 - 4XT^2 - 4XT - X = \\ &= T^4 - 2XT^2 - 4XT + X^2 - X \end{aligned}$$

It is clear that no proper polynomial factor of f has coefficients in $K = k(X)$, so $f(T) = p(T)q(T) = T^4 - 2XT^2 - 4XT + X^2 - X$ is irreducible over K and vanishes in the realizations $\sqrt{X} + \sqrt[4]{X}$ and $\sqrt{X} - \sqrt[4]{X}$ of ξ.

Observe that $\deg \xi \geq 2$ for all cuts ξ of ordered fields.

2.13. **Lemma.** *Let $K \subseteq L$ be ordered fields and let ξ be a cut of K. Let $\alpha, \beta \in L$ be realizations of ξ and let $f(T), g(T) \in K[T]$ be polynomials with $f(T)/g(T) \notin K$. If $\deg f(T), \deg g(T) < \deg \xi$ then the cut determined by $f(\alpha)/g(\alpha)$ over K is equal to the cut determined by $f(\beta)/g(\beta)$ over K.*

Proof. Observe that the statement makes sense, since $f(\alpha)/g(\alpha), f(\beta)/g(\beta) \notin K$ by the degree assumption and $f(T)/g(T) \notin K$. Clearly we may assume that L is real closed. Suppose there is some $a \in K$ with $f(\alpha)/g(\alpha) < a < f(\beta)/g(\beta)$. Since $\deg g < \deg \xi$, g does not have zeroes in the closed interval determined by α and β in L. By the mean value property for real closed fields, there is some $\gamma \in L$ between α and β with $f(\gamma)/g(\gamma) = a$. Hence γ is a zero of $h(T) := a\cdot g(T) - f(T)$ and γ realizes ξ. Since $\deg h < \deg \xi$ this is not possible. □

2.14. **Definition.** Let K be an ordered field and let ξ be a cut of K. Let $h(T) \in K(T) \setminus K$, such that there are $f(T), g(T) \in K[T]$ with $\deg f(T), \deg g(T) < \deg \xi$ and $h(T) = \frac{f(T)}{g(T)}$. We define the cut $h(\xi)$ of K to be the cut determined by $f(\alpha)/g(\alpha)$, where α is a realization of ξ in some ordered field extension $L \supseteq K$, By 2.13, this makes sense.

2.15. **Definition.** Let K be an ordered field with real closure R and let ξ be a cut of K. Let $s : R \longrightarrow R$ be semi-algebraic. We say that s is **strictly increasing in** ξ if for all realizations $\alpha < \beta$ from any ordered field extension L of R we have $s(\alpha) < s(\beta)$.

We say that s is **strictly decreasing in** ξ if for all realizations $\alpha < \beta$ from any ordered field extension L of R we have $s(\alpha) > s(\beta)$.

We say that s is **strictly monotonic in** ξ if s is strictly decreasing or strictly increasing in ξ.

We say that s is **constant in** ξ if s is constant on all realizations of ξ in any ordered field extension L of R.

We say that s is **defined at** ξ, if for all realizations α, β of ξ from some real closed field, $s(\alpha)$ and $s(\beta)$ induce the same cut of K. In this case we may define $s(\xi)$ to be this cut.

Note that if ξ is omitted in R, then s is constant or strictly monotonic in ξ. Note also that a polynomial with coefficients in K is in general neither constant nor strictly monotonic nor defined in a given cut of K.

2.16. **Definition.** Let K be an ordered field with real closure R. A map $s : R \longrightarrow R$ is called **piecewise K-rational** if there is a decomposition of $R = I_1 \dot\cup ... \dot\cup I_k$ into intervals with endpoints in $K \cup \{\pm\infty\}$ such that for each j there is some $Q \in K(T)$ without poles on I_j such that $s|_{I_j} = Q|_{I_j}$. In particular $s(K) \subseteq K$.

2.17. **Lemma.** *Let K be an ordered field with real closure R and let ξ be a cut of K. Let $s : R \longrightarrow R$ be piecewise K-rational.*

- *(i) If ξ is principal then ξ is omitted in R and either $s(\eta) \in K$ or $s(\eta) \restriction K$ is a principal cut of K, where η is the unique extension of ξ on R.*
- *(ii) If ξ is a non principal cut of $(K, +)$ and s is strictly monotonic in ξ, then there are a piecewise K-rational, strictly monotonic homeomorphism $t : R \longrightarrow R$ and elements $a < \xi < b$ in K such that $s|_{[a,b]} = t|_{[a,b]}$ is a K-rational map on $[a, b]$ (so equal to some $Q \in K(T)$ on $[a, b]$).*

  If in addition, s is defined in ξ, then

$$s(\xi) = \begin{cases} s(\xi^L \cap [a, +\infty))^+ = s(\xi^R \cap (-\infty, b])^- & \text{if } s \text{ is increasing in } [a, b], \\ s(\xi^R \cap (-\infty, b])^+ = s(\xi^L \cap [a, +\infty))^- & \text{if } s \text{ is decreasing in } [a, b]. \end{cases}$$

Proof. (i). We may assume that $\xi > K$. Clearly ξ is omitted in K. Let $a := \lim_{t\to+\infty} s(t) \in R\cup\{+\infty\}$. If $a = +\infty$, then $s(\eta) = +\infty$ and we are done. If $a \in R$ then, as s is piecewise K-rational, $a \in K$ and $s(\eta) = a^+$ or $s(\eta) = a^-$. In any case, (i) holds.

(ii). We assume that s is strictly increasing in ξ. As s is piecewise K-rational, there are $a < \xi < b$ and some $Q(T) \in K[T]$ such that $s|_{(a,b)} = Q|_{(a,b)}$. Let p be a cut of R, lying over ξ. As s is strictly increasing in ξ we must have $Q' > 0 \in p$ (here we consider p as a 1-type over R, observe that p is not realized, hence $Q' = 0 \notin p$). In particular, if p_1, p_2 are the least and the largest extension of ξ on R, we have $Q' > 0 \in p_1, p_2$. Since ξ is not principal, we can shrink the interval (a, b) such that $Q' > 0$ on $(a, a_1)\cup(b_1, b)$ for some $a_1, b_1 \in R$ with $p_1 < a_1, b_1 < p_2$. But then, since s is strictly increasing in ξ, Q must be strictly increasing in $(a, b) \subseteq R$. Now a map t as claimed can easily be patched together.

Finally assume that s is also defined at ξ. The only thing we need to show is that there are no elements $c \in K$ between $s(\xi^L\cap[a, +\infty))$ and $s(\xi^R\cap(-\infty, b])$. Say s is increasing in ξ and suppose

$$s(\xi^L \cap [a, +\infty)) \le c \le s(\xi^R \cap (-\infty, b]).$$

Since ξ is non-principal and s is strictly increasing in $[a, b]$ we have

$$s(\xi^L \cap [a, +\infty)) < c < s(\xi^R \cap (-\infty, b]).$$

Take realizations α and β of the cuts c^-, c^+ from some real closed field S. Since s is strictly increasing and continuous in $[a, b]_S$, there are $\alpha_0, \beta_0 \in [a, b]_S$ with $s(\alpha_0) = \alpha$, $s(\beta_0) = \beta$, $\xi^L \cap [a, +\infty) < \alpha_0$ and $\beta_0 < \xi^R \cap (-\infty, b]$. But then α_0, β_0 realize ξ, whereas $s(\alpha_0) < c < s(\beta_0)$, i.e. s is not defined at ξ. □

2.18. **Definition.** A cut ξ of an ordered abelian group is called **dense** if ξ is not principal and $G(\xi) = \{0\}$.

2.19. **Corollary.** *Let K be an ordered field with real closure R, let ξ be a cut of K omitted in R and let η be the unique extension of ξ on R. Let $s : R \longrightarrow R$ be piecewise K-rational and nonconstant in ξ. Then $s(\eta)$ is the unique extension of $s(\eta) \restriction K$ and*

(i) ξ is principal if and only if $s(\eta) \restriction K$ is principal.
(ii) ξ is dense if and only if $s(\eta) \restriction K$ is dense.

Proof. All statements hold true by 2.17(i), if ξ is principal. So we assume that ξ is not principal. As ξ is omitted in R, s is strictly monotonic and defined in ξ. Say s is strictly increasing in ξ. By 2.17(ii) we may assume that s is a strictly increasing homeomorphism $R \longrightarrow R$. Then $s(\eta) = (s(\eta^L), s(\eta^R))$ and $s(\eta) \restriction K = s(\xi^L)^+ = s(\xi^R)^-$. Consequently $s(\eta)$ is the unique extension of $s(\eta) \restriction K$ and (i) holds.

Now we prove (ii). From the o-minimal case we know that η is dense is and only if $s(\eta)$ is dense. But ξ is dense if and only if η is dense (by (i) and since $\hat{\eta}$ lies over $\hat{\xi}$). As $s(\eta) \restriction K$ is omitted in R it follows that $s(\eta) \restriction K$ is dense if and only if $s(\eta)$ is dense. Altogether we get (ii). □

2.20. **Theorem.** *Let $K \subseteq L$ be ordered fields and let ξ be a cut of K. Then*

(i) If $f, g \in K[T]$ with $1 \le \deg f + \deg g \le \deg \xi$ and g has no zero in ξ, then f/g is strictly monotonic on the realizations of ξ in L. In particular there is at most one zero of f realizing ξ.

(ii) *If $f \in K[T]$, $1 \leq \deg f < \deg \xi$ and L is real closed, then f maps the realizations of ξ in L onto the realizations of $f(\xi)$ in L.*

(iii) *Let $\alpha, \beta \in L$, $\alpha \neq \beta$ be algebraic with minimal polynomial μ_α, μ_β respectively. If α, β are ξ-generic, then either μ_α and μ_β are strictly increasing on the realizations of ξ in L or μ_α and μ_β are strictly decreasing on the realizations of ξ in L. Moreover μ_α and μ_β are coprime.*

Proof. (i). As $0 \leq \deg(gf' - g'f) < \deg \xi$, $(f/g)'$ does not have zeroes in the realizations of ξ in the real closure of L. Hence (i) follows.

(ii). We assume that L is the real closure of K first. If ξ is omitted in L then (ii) holds by 2.19. So we may assume that ξ is realized in L, hence ξ is not principal. By 2.13, f maps the realizations of ξ in L into the realizations of $f(\xi)$ in L. By (i), f is strictly monotonic in ξ. Hence by 2.17, there are $a < \xi < b$ in K and a piecewise K-rational, strictly monotonic homeomorphism $t : R \longrightarrow R$ such that $f|_{[a,b]} = t|_{[a,b]}$. Thus every realization of $f(\xi)$ is the image of a realization of ξ in L.

Now let L be an arbitrary real closed field extending K and let R be the real closure of K in L. Let η_1, η_2 be the least and the largest extension of ξ on R. Let η_1', η_2' be the least and the largest extension of ξ on L. By what we have shown, $f(\eta_1)$ and $f(\eta_2)$ are the least and the largest extension of $f(\xi)$ on R - possible in the reverse order. Since f is strictly monotonic in ξ we get that $f(\eta_1')$ and $f(\eta_2')$ are the least and the largest extension of $f(\xi)$ on L - possible in the reverse order. This proves (ii).

(iii). Suppose $\alpha < \beta$ and suppose $\mu_\alpha'(\alpha) > 0 > \mu_\beta'(\beta)$. By (i), μ_α is strictly increasing in ξ and μ_β is strictly decreasing in ξ. Since $\mu_\alpha(\alpha) = \mu_\beta(\beta) = 0$ there must be some γ in the real closure of L with $\alpha < \gamma < \beta$ such that $\mu_\alpha(\gamma) = \mu_\beta(\gamma)$. But then $[K(\gamma) : K] \leq \deg(\mu_\alpha - \mu_\beta) < \deg \mu_\alpha = \deg \mu_\beta = \deg \xi$, a contradiction.

The same argument gives a contradiction if $\mu_\alpha'(\alpha) < 0 < \mu_\beta'(\beta)$. By (i) it follows that μ_α and μ_β are strictly increasing on the realizations of ξ in L or μ_α and μ_β are strictly decreasing on the realizations of ξ in L.

By (i), μ_α and μ_β are coprime. □

2.21. **Corollary.** *Let K be an ordered field with real closure R and let ξ be a cut of K. Let $\alpha \in R$ be a ξ-generic realization of ξ. Then for every polynomial $g \in K[T]$ of degree $< \deg \xi$ the least and the largest extension of ξ on R is mapped via g onto the least and largest extension (possibly in reverse order) of the cut induced by $g(\alpha)$ on K.*

Proof. By 2.19 and 2.20(ii). □

2.21 says: ξ can be moved to all cuts of K, realized in some $K(\alpha)$, α ξ-generic, in such a way that the movement is perfectly witnessed in R. By 2.9, we can then use the theory of cuts of real closed fields. Here an example

2.22. **Definition.** If $K \subseteq L$ is an extension of ordered fields, then we write

$$V_{L/K} = \bigcap_{\alpha \in L \setminus K} V(\alpha/K).$$

2.23. **Proposition.** *Let $K \subseteq L$ be ordered fields and let ξ be a cut of K. Let $\alpha \in L$ be a ξ-generic realization of ξ. If $L = K(\alpha)$, then*

$$V_{L/K} = V(\xi).$$

Proof. First recall from [8, Thm. 5.1], that this is true if K is real closed (and actually not difficult to prove directly in that case).

Case 1. ξ is not realized in the real closure R of K

Let η be the unique extension of ξ on R and let $\beta \in L \setminus K$. As α is transcendent over K. Take a rational map $f : K \longrightarrow K$ with $\beta = f(\alpha)$. By 2.19, $f_R(\eta)$ is the unique extension of $f(\xi)$ on R. From the real closed field case, we know that $V(f_R(\eta)) = V(\eta)$. From 2.9 we know that $V(f_R(\eta))$ lies over $V(f(\xi))$. Thus $V(\beta/K) = V(f(\xi)) = V(\xi)$.

Case 2. ξ is realized in the real closure R of K.

By assumption $\alpha \in R$ is a generic realization of ξ. Let $\beta \in L \setminus K$ and take a polynomial $f(T) \in K[T]$ of degree $< \deg(\xi)$ such that $\beta = f(\alpha)$. By 2.21, the least and the largest extension η_1, η_2 of ξ on R is mapped via f onto the least and largest extension (possibly in reverse order) of the cut induced by $f(\alpha) = \beta$ on K. Again, from the real closed case we know that $V(f_R(\eta_i)) = V(\eta_1)$. Again 2.9 shows that $V(f_R(\eta_i))$ lies over $V(f(\xi))$. □

2.24. *Example.* Without the genericity, 2.23 fails: Let K be an ordered field. If ξ and η are cuts of K realized in the real closure of K, then $V(\xi) \neq V(\eta)$ in general. To see an example let K_0 be any ordered field and let x, y be from an ordered field extension with $K_0 < x$ and $K_0(x) < y$. Take $K = K_0(x, y)$, ξ the cut of K realized by $\alpha := \sqrt{x}$ and η the cut of K realized by $\beta := \sqrt{y}$. Then ξ and η are upper edges of distinct convex valuation rings of K.

2.25. *Example.* In general the ξ-generics are not convex: To see an example (also cf. 2.24) let K_0 be any ordered field and let x, y be from an ordered field extension with $K_0 < x$ and $K_0(x) < y$. Take $K = K_0(x, y)$ and let ξ be the cut of K realized by $\alpha := \sqrt{x}$. Let $\beta := \sqrt{y}$. Then $\alpha, \alpha+1$ and $\alpha + 1/\beta$ are realizations of ξ. $\alpha, \alpha+1$ are ξ-generic, but $\alpha + 1/\beta$ is not.

3. Dense cuts and the order completion.

3.1. **Proposition.** *Let K be an ordered field and let ξ be a cut of K. Then the following are equivalent.*

(i) ξ is dense, i.e. ξ is not principal and $G(\xi) = 0$.
(ii) There is an ordered field extension of K such that ξ has a unique realization in that field.

If this is the case, then ξ has at most one realization α in every ordered field extension L of K which is archimedean over K and K is dense in $K(\alpha)$.

Proof. (ii)⇒(i) is easy.

Now suppose ξ is dense. Then ξ can be realized in an ordered extension field of K, which is archimedean over K (if ξ is realized in the real closure R of K we can take the real closure; if ξ is omitted in the real closure, then the unique extension of ξ on R is again dense, hence K is archimedean in $R <$ a realization of $\xi >$). If L is any ordered field extension of K, archimedean over K and $\alpha < \beta$ are from K, then there is some $a \in K$ with $0 < a < \beta - \alpha$. As $G(\xi) = 0$, α and β cannot realize ξ at the same time. This shows that (i) implies (ii) and it remains to show that K is dense in $K(\alpha)$ if $\alpha \models \xi$ and $K(\alpha)$ is archimedean over K.

Case 1. ξ is omitted in the real closure R of K. As ξ is dense, the unique extension of ξ on R is dense, too. In particular R is archimedean in $R\langle\alpha\rangle$. Then K

is archimedean in $R\langle\alpha\rangle$. Take rational functions $f,g\in K(T)$ such that $f(\alpha)<g(\alpha)$, both not in K. Take some $a\in K$ such that $f(\alpha)+a<g(\alpha)$. By 2.19(ii), the cut defined by $f(\alpha)$ over K is dense, too. Hence $g(\alpha)$ cannot define the same cut as $f(\alpha)$ over K and there must be some $a\in K$ with $f(\alpha)<a<g(\alpha)$ as desired.

Case 2. ξ is realized in R, by r say. Then r is the unique realization of ξ in R and $\alpha-r$ is infinitesimal over R. Let μ be the minimal polynomial of r over K. If $\alpha\neq r$, then $\mu(\alpha)\neq 0$ is infinitesimal over R, in contradiction to our assumption that K is archimedean in $K(\alpha)$. Hence $\alpha=r\in R$ is the unique realization of ξ on R. In particular $[K(\alpha):K]=\deg\xi$. Take polynomials $f,g\in K[T]$, $\deg f,\deg g<[K(\alpha):K]$ such that $f(\alpha)<g(\alpha)$, both not in K. By 2.20(ii) we know that $f(\alpha)$ is the unique extension of the cut of $f(\alpha)$ over K. Hence there is some $a\in K$ with $f(\alpha)<a<g(\alpha)$. □

Observe that K need not be archimedean in $K(\alpha)$ if α realizes a dense cut over K. For example if ε is infinitesimal, $K=\mathbb{Q}$ and $\alpha=\sqrt{2}+\varepsilon$. Then $\alpha^2-2=2\sqrt{2}\varepsilon+\varepsilon^2\in K(\alpha)$ is infinitesimal over K.

3.2. **Corollary.** *Let $K\subseteq L\subseteq M$ be ordered fields and let $X\subseteq M$. If K is dense in $K(x)$ for all $x\in X$ and if K is archimedean in M then L is dense in $L(X)$.*

Proof. We work inside the real closure $\bar{M}$ of M. We may assume that X is finite, $X=\{x_1,...,x_n\}$ and we do an induction on the cardinality of X. First let $X=\{x\}$, $x\notin L$, let ξ be the cut of x over K and let η be the cut of x over L.

By assumption, K is archimedean in $L(x)$. Hence by 3.1, x is the unique realization of ξ in $L(x)$. Thus x is the unique realization of η in $L(x)$. Again by 3.1, L is dense in $L(x)$.

So we know the corollary in the case $n=1$. Now suppose K is dense in $K(x)$ for every $x\in X$ and K is dense in $K(y)$. By induction L is dense in $L(X)$, hence K is archimedean in $L(X)$. So from the case $n=1$ we get that $L(X)$ is dense in $L(X\cup\{y\})$, thus L is dense in $L(X\cup\{y\})$. □

If K is an ordered subfield of an ordered field M and K is archimedean in M, then by 3.2, for all fields $K\subseteq L_1,L_2\subseteq M$ with the property that K is dense in L_1 and in L_2, K is also dense in the compositum $L_1{\cdot}L_2\subseteq M$. Applying Zorn's lemma therefore shows that there is a largest subfield L of M such that K is dense in L.

For an ordered field K we may now define the **dense closure** (also called the **continuous closure** or the **completion**) of K as follows: Let $\overline{K}$ be the real closure of K and let $\hat{\overline{K}}$ be the completion of $\overline{K}$ (see [7, section 3]). Notice that K is archimedean in $\hat{\overline{K}}$. We define the dense closure $\hat{K}$ of K as

$$\hat{K}=\text{ the largest subfield of }\hat{\overline{K}}\text{ that contains }K\text{ as a dense subfield.}$$

3.3. **Proposition.** *If $K\subseteq L$ are ordered fields and K is dense in L, then there is a unique K-embedding of ordered fields $L\longrightarrow\hat{K}$.*

Proof. Uniqueness is clear. To see existence of such an embedding, let Ω be the real closure of L. Then there is a K-embedding of $\overline{K}$ into Ω and we may assume that $\overline{K}\subseteq\Omega$. Since K is archimedean in Ω we know from 3.2 that $\overline{K}$ is dense in

$\overline{K}{\cdot}L$. It follows that $\overline{K}$ is dense in $\overline{K{\cdot}L}$ (see the description of S in [7, Cor 3.2])). By [7, Cor 3.3], there is an embedding

$$\varphi : \overline{K{\cdot}L} \longrightarrow \hat{\overline{K}}$$

over $\overline{K}$. Hence the restriction of φ to L maps L onto a subfield of $\hat{\overline{K}}$ that contains K as a dense subfield. By definition of $\hat{K}$ we therefore have $\varphi(L) \subseteq \hat{K}$. □

3.4. **Corollary.** [3]
If $K \subseteq L$ are ordered fields and K is dense in L, then the real closure $\overline{K}$ of K is dense in the real closure $\overline{L}$ of L.

Proof. By 3.3 we may assume that $L \subseteq \hat{K}$. Now we have

$$K \subseteq L \subseteq \hat{K} \subseteq \hat{\overline{K}}$$

and $\hat{\overline{K}}$ is real closed. Consequently

$$\overline{K} \subseteq \overline{L} \subseteq \hat{\overline{K}},$$

i.e. $\overline{K}$ is dense in $\overline{L}$. □

4. Convex valuations on realizations of cuts

4.1. **Proposition.** *If V is a convex valuation ring of an ordered field K, then the convex hull W of V in the real closure R of K is the unique convex valuation ring of R with $W \cap K = V$.*

*Observe that this does **not** mean that V^+ is omitted in R.*

Proof. For a more general reference see [1] (it says that on an algebraic extension of fields there cannot be a proper inclusions between valuations extending the same valuation of the base field).

Take $\alpha \in R$ with $\alpha > V$. It suffices to show that for some $d \in \mathbb{N}$ and some $a \in K$ we have

$$V < a < \alpha^d.$$

Let w be the valuation belonging to the convex hull W of V in the real closure R. Since α is algebraic over K, there are $i > j$ and $c_i, c_j \in K^\times$ with $w(c_i\alpha^i) = w(c_j\alpha^j)$, hence

$$(*) \qquad w(\alpha^{i-j}) = w(\frac{c_j}{c_i}).$$

We take $d = i - j + 1$ and $a = |\frac{c_j}{c_i}|$. As $\alpha > V$ we have $w(\alpha) < 0$ and by $(*)$ also $w(a) = w(\frac{c_j}{c_i}) < 0$. Since $a > 0$, this means $V < a$. On the other hand $w(\alpha^d) = w(\alpha\cdot\alpha^{i-j}) = w(\alpha)+w(a) < w(a)$, which implies $a < \alpha^d$ as w is compatible with the order. □

Recall from 4.1 that every convex valuation ring V of an ordered field K has a unique extension to a convex valuation ring of the real closure R of K, namely the convex hull of V in R.

Throughout this section we fix

- ordered fields $K \subseteq L$
- a convex valuation ring W of L which is the convex hull of $V := W \cap K$. The maximal ideal of W is denoted by $\mathfrak{m}$ and the residue map $W \longrightarrow W/\mathfrak{m}$ is denoted by λ.
- A cut ξ of K and a realization $\alpha \in L$ of K.

4.2. **Lemma.** *If $a, b \in K$ and $w(a\alpha - b) \notin w(K)$, then* $\operatorname{sign}(\xi) \neq 0$.

Proof. Clearly $a \neq 0$. Since $\operatorname{sign}\xi$ is invariant under the map $ax + b$, we may assume that $\alpha > 0$ and $w(\alpha) \notin w(K)$. Then for all $c \in K$ with $0 \le c < \alpha$ we have $w(\alpha) > w(c) = w(2c)$ and therefore $2c < \alpha$. Thus α realizes the upper edge of a convex subgroup of $(K, +, \le)$. □

4.3. **Lemma.** *If G is a convex subgroup of $(K, +, \le)$, $\xi = G^+$ and $V \subseteq V(\xi)$, then $w(\alpha) \notin w(K)$.*

Proof. Suppose $a \in K$, $a > 0$ with $w(\alpha) = w(a)$. Then $w(\frac{\alpha}{a}) = 0$ and so $\frac{\alpha}{a}$ and $\frac{a}{\alpha}$ are in the convex hull of V. By assumption, $\frac{\alpha}{a}$ and $\frac{a}{\alpha}$ are in the convex hull of $V(\xi)$. Take $b \in V(\xi)$ with

$$\frac{\alpha}{a}, \frac{a}{\alpha} < b.$$

Since $\frac{a}{\alpha} < b$ we have

$$a = \frac{a}{\alpha}\cdot\alpha < b\cdot\alpha \models G^+,$$

because $b \in V(\xi)$ (and $b \geq 1$). Hence $a \in G$. But this contradicts

$$\alpha = a \cdot \frac{\alpha}{a} < a \cdot b \overset{b \in V(\xi)}{\in} G < \alpha.$$

□

4.4. **Corollary.** *If* $\operatorname{sign}(\xi) \neq 0$ *and* $V \subseteq V(\xi)$*, then there is some* $a \in K$ *with* $w(\alpha - a) \notin w(K)$*.*

4.5. **Lemma.** *If* $\alpha \in W$ *is a realization of* ξ *and* $1 \in G(\xi)$*, then* $\lambda(\alpha) \notin \lambda(V)$*.*

Proof. Otherwise there is some $a \in V$ with $\alpha - a \in \mathfrak{m}$, hence $\alpha = a + \mu$ for some $\mu \in \mathfrak{m}$. Since $1 \in G(\xi)$, α and $\alpha + 2$ realize the same cut of K, which contradicts

$$\alpha = a + \mu < a + 1 < a + \mu + 2 = \alpha + 2.$$

□

4.6. **Corollary.** *If* $V(\xi) \subsetneq V$*, then there are* $a, b \in K$ *such that* $a\alpha + b \in W$ *and* $\lambda(a\alpha + b) \notin \lambda(V)$*.*

Proof. Since $V(\xi) \subsetneq V$, there is some $a \in K$ with $V(\xi)^+ \leq a \cdot \hat{\xi} < V^+$. Since $\widehat{a\xi} = a\hat{\xi}$, there is some $b \in K$ with

$$(*) \qquad V(\xi)^+ \leq a \cdot \hat{\xi} \leq a\xi + b < V^+.$$

Then $a\alpha + b \in W$ and $1 \in V(\xi) \subseteq aG(\xi) = G(a\xi) = G(a\xi + b)$. Hence 4.5 applies. □

4.7. **Lemma.** *If* $\alpha \in W$*,* $\operatorname{sign}(\xi) = 0$*,* $G(\xi) = \mathfrak{m}(\xi)$ *and* $V(\xi) \subseteq V$*, then* $\lambda(\alpha) \notin \lambda(V)$*.*

Proof. Say $\alpha > 0$. Assume there is some $a \in K$ with $\alpha - a \in \mathfrak{m}$. We may assume that $a = 0$, otherwise we continue to work with $\xi - a$ and $\alpha - a$. Thus we may assume that $\alpha \in \mathfrak{m}$. Since $\operatorname{sign}(\xi) = 0$, α does not realize $(\mathfrak{m} \cap K)^+$. As $\alpha \in \mathfrak{m}$ this means $0 < \xi < (\mathfrak{m} \cap K)^+$. But this contradicts $G(\xi) = \mathfrak{m}(\xi) \supseteq \mathfrak{m} \cap K$. □

4.8. **Corollary.** *If* $\operatorname{sign}(\xi) = 0$*,* $\operatorname{sign}^*(\hat{\xi}) \in \{-1, \infty\}$ *and* $V(\xi) \subseteq V$*, then there are* $a, b \in K$ *such that* $a\alpha + b \in W$ *and* $\lambda(a\alpha + b) \notin \lambda(V)$*.*

Proof. This is true if $V(\xi) \subsetneq V$ by 4.6. So assume $V(\xi) = V$. As $\operatorname{sign}^*(\hat{\xi}) \in \{-1, \infty\}$, there is some $a \in K$ with $a \cdot G(\xi) = \mathfrak{m}(\xi)$ (see 2.10). As $a \cdot G(\xi) = G(a \cdot \xi)$ we have $\operatorname{sign}(a\xi) = 0$ and $\mathfrak{m}(a\xi) = \mathfrak{m} \cap K$. Consequently there is some $b \in K$ with $0 < a\xi + b < 1$, in particular $a\alpha + b \in W$. Now 4.7 applies to $a\xi + b$. □

4.9. **Lemma.** *If* $\alpha \in W$ *with* $\lambda(\alpha) \notin \lambda(V)$ *then* $V(\xi) \subseteq V$ *and if* $V(\xi) = V$*, then* $G(\xi) = \mathfrak{m} \cap K$ *and the cut of* $\lambda(V)$ *realized by* $\lambda(\alpha)$ *has invariance group* $\{0\}$*.*

Proof. We use 1.14. Let η be the cut of $\lambda(V)$ determined by $\lambda(\alpha)$. Let λ_0 be the restriction of λ to V. By 1.14(ii) applied to $V \subseteq W$ and $\mathfrak{m}$ we have $G(\xi) = \lambda_0^{-1}(G(\eta))$, which contains $\mathfrak{m} \cap K = \lambda_0^{-1}(0)$.
If $G(\xi) = \mathfrak{m} \cap K$, then $V(\xi) = V$ and $G(\eta) = \{0\}$.

Otherwise, $G(\eta) = \lambda(G(\xi)) \neq \{0\}$ and so $V(\eta) \neq \lambda(V)$. Take $a \in V$ with $\lambda(a) > V(\eta)$ and $b \in V$, $b > 0$ with $\lambda(b) \in G(\eta)$ and $\lambda(a)\lambda(b) > G(\eta)$. Then $b \in \lambda_0^{-1}(G(\eta)) = G(\xi)$ and $a \cdot b > \lambda_0^{-1}(G(\eta)) = G(\xi)$. Hence $a \notin V(\xi)$. As $a \in V$ this shows $V(\xi) \subsetneq V$. □

4.10. **Conclusion** The fact that $a\alpha+b$ does not have a new value and $a\alpha+b$ does not have a new residue w.r.t. K and W, for all $a,b\in K$, is determined by $\operatorname{sign}\xi$, $\operatorname{sign}^*\hat{\xi}$ and the position of V w.r.t. $V(\xi)$. In fact we have the following table:

Let $K\subseteq L$ be ordered fields, let ξ be a cut of K realized by α and suppose $L=K(\alpha)$. Let V be a convex valuation ring of K and let W be the convex hull of V in L.

	$V\subsetneq V(\xi)$	$V=V(\xi)$	$V(\xi)\subsetneq V$
ξ principal	$\Gamma_V\neq\Gamma_W$	$\Gamma_V\neq\Gamma_W$	not possible
ξ dense, L/K archimedean	immediate	$\kappa_V\neq\kappa_W$	not possible
$\operatorname{sign}\xi\neq 0$	$\Gamma_V\neq\Gamma_W$	$\Gamma_V\neq\Gamma_W$	$\kappa_V\neq\kappa_W$
$\operatorname{sign}\xi=0,\ \operatorname{sign}^*\hat{\xi}\in\{-1,\infty\}$	linear immediate	$\kappa_V\neq\kappa_W$	$\kappa_V\neq\kappa_W$
$\operatorname{sign}\xi=0,\ \operatorname{sign}^*\hat{\xi}\in\{0,1\}$	linear immediate	linear immediate	$\kappa_V\neq\kappa_W$

Here, "linear immediate" stands for the property

> For all $a,b\in K$, $w(a\alpha+b)\in\Gamma_V$ and, if $a\alpha+b\in W$, then $\lambda(a\alpha+b)\in\kappa_V$"

Proof. The first two rows are clear and the last column follows from 4.6. Using 4.4, also the third column follows.

So we are left with the following sub-table:

	$V\subsetneq V(\xi)$	$V=V(\xi)$	
$\operatorname{sign}\xi=0,\ \operatorname{sign}^*\hat{\xi}\in\{-1,\infty\}$	linear immediate	$\kappa_V\neq\kappa_W$	
$\operatorname{sign}\xi=0,\ \operatorname{sign}^*\hat{\xi}\in\{0,1\}$	linear immediate	linear immediate	

If $\operatorname{sign}\xi=0,\ \operatorname{sign}^*\hat{\xi}\in\{-1,\infty\}$ and $V(\xi)=V$, then $\kappa_W\neq\kappa_V$ by 4.8.

The three remaining cases are linear immediate by 4.2 and 4.9.

□

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