DIFFERENTIAL WEIL DESCENT AND DIFFERENTIALLY LARGE FIELDS

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ABSTRACT. A differential version of the classical Weil descent is established in all characteristics. It yields a theory of differential restriction of scalars for differential varieties over finite differential field extensions. This theory is then used to prove that in characteristic 0, *differential largeness* (a notion introduced here as an analogue to largeness of fields) is preserved under algebraic extensions. This provides many new differential fields with minimal differential closures. A further application is Kolchin-density of rational points in differential algebraic groups defined over differentially large fields.

CONTENTS

1. INTRODUCTION

In 1959, André Weil introduced his method of restricting scalars for a finite separable field extension $K \subseteq L$, cf. [Wei82, §1.3]. It says that scalar extension, seen as a functor from K-algebras to L-algebras, has a left adjoint, which sends an L-algebra D to a K-algebra $W(D)$, the Weil descent (aka Weil restriction) of D from L to K. The construction has been vastly generalised by Grothendieck, see [Gro95].

We establish a similar descent for differential algebras with respect to a given extension of differential rings $A \subseteq B$, where B is finitely generated and free as an A-module. Here a differential ring A is a commutative unital ring equipped with a finite set of derivations $A \longrightarrow A$. If D is a differential B-algebra with commuting

derivations, its descent $W^{\mathrm{diff}}(D)$ is a differential A-algebra in commuting deriva-
tions, see 3.4. This is deduced from our first main result, which concerns rings and
algebras with a single derivation:

Theorem A. (see 3.2 and 3.3)
Let $d : A \longrightarrow A$ be a derivation of a ring A and let (B, δ) be a differential (A, d)-
algebra. Assume that B is finitely generated and free as an A-module.

(i) Let (D, ∂) be a differential (B, δ)-algebra. Then there is a unique derivation
∂^W on the classical Weil descent $W(D)$ such that $(W(D), \partial^W)$ is a differential
(A, d)-algebra and the unit of the adjunction at D (given by the classical Weil
descent), namely the map $W_D : D \longrightarrow W(D) \otimes B$, is a differential (B, δ)-
algebra homomorphism $(D, \partial) \longrightarrow (W(D) \otimes B, \partial^W \otimes \delta)$.

(ii) If B is a subring of D and the inclusion is the structure morphism of D as
a B-algebra, then the assignment $\partial \mapsto \partial^W$ is a Lie-ring and an A-module
homomorphism.

We expose several applications of the differential Weil descent. A major one ad-
dresses a method to produce differential fields (in commuting derivations and of
characteristic 0), which possess a minimal differential closure (or in Kolchin's ter-
minology, *constraint closure*). First examples of such fields were given by Singer in
[Sin78a]. He showed that for every closed ordered differential field K in one deriva-
tion (cf. [Sin78b]), the algebraic closure $K[i]$ is differentially closed and minimal
over K. The only other known examples are fixed fields of models of $\mathrm{DCF}_{0,m}$ A
the theory differentially closed fields with a generic differential automorphism, see
[LS16].
    A key notion in this task is *differentially large field*, introduced here in §5. Recall
from [Pop96] that a field is *large* if it is existentially closed in its Laurent series field
$K((t))$; equivalently, if every smooth curve defined over K with a K-rational point,
has infinitely many K-rational points. Basic examples of large fields are local fields
and fraction fields of local henselian rings, like the quotient field of the power series
ring $K[[T_1, \ldots, T_n]]$ for any field K. Large fields make a remarkable appearance in
the work of Pop [Pop96], where he studies finite split embedding problems for the
absolute Galois groups of a large field; for instance, he shows that if K is large then
every finite group is regularly realizable as a Galois group over $K(t)$.
    Our cue to define differentially large fields comes from the Uniform Companion
theory UC_m of differential fields of characteristic zero with m commuting deriva-
tions, introduced in [Tre05]. We fix a set $\Delta = \{\delta_1, \ldots, \delta_m\}$ of commuting derivations
and assume that our fields have characteristic zero. We say that a differential field
$K = (K, \Delta)$ is differentially large if K is large as a field and is a model of the theory
UC_m. The theory UC_m has rather elaborate axioms which involve somewhat deep
differential-algebraic terminology, such as autoreduced and characteristic sets (see
Section 4). However, in Theorems 5.2 and 5.5 below, we give several more practical
and rather accessible characterisations of differential largeness. For instance, we
prove that a large and differential field K is differentially large if and only if it
is existentially closed in every differential field extension L, provided K is existen-
tially closed in L as a field. Concrete examples of differentially large fields for which
our results are novel contain differentially large fields expanding real closed fields

(denoted by CODF_m), p-adically closed fields or pseudo-finite fields of characteristic 0, see [Tre05, §8] for details. A more general mechanism on how to produce differentially large field is given in 5.6.

We give another more geometric characterisation of differential largeness in terms of smooth points of certain algebraic varieties associated to differential varieties – the so-called "jets"–, see Theorem 5.2 part (iii). In §8 we give a further characterisation, an algebraic-geometric one in the spirit of the Pierce-Pillay axioms for ordinary differentially closed fields [PP98], see 8.4.

In analogy to the algebraic case, where it is known that algebraic extensions of large fields are again large (see [Pop96, Proposition 1.2]), we will be using the differential Weil descent and results from §5 to prove:

Theorem B. (cf. 6.1) Every algebraic extension of a differentially large field is again differentially large (with the uniquely induced derivations).

It follows quickly that the algebraic closure of a differentially large field is in fact differentially closed. This is used in a forthcoming paper by Aschenbrenner, Chernikov et al. to show that the theory of differentially closed fields has a distal expansion. As a further consequence, differentially large fields have *minimal* differential closures. Finally, Theorem B together with results in §5 will be used in Theorem 7.1 to show that the K-rational points of a connected differential algebraic group G defined over a differentially large field K are Kolchin-dense in G.

Outlook. We expect many more applications of the differential Weil descent in connection with differentially large fields. For instance, we expect that differentially large fields will make a notable appearance in differential Galois cohomology and the parameterised Picard-Vessiot theory for linear differential equations. This will potentially be in the form of finiteness results for cohomology groups of linear differential algebraic groups over differentially large fields with some additional properties (e.g., a differential analogue of Serre's property (F), see [Ser94, Ch. III, §4] used in the classical finiteness theorems for linear algebraic groups).

2. Classical Weil Descent for Algebras

In this section we review the classical construction of Weil descent of scalars for algebras, see for example [BLR90, §7.6], [MS10, §2] and [Gro95]. For our purposes we need certain explicit formulas, so we give details.

Convention. Throughout, we assume our rings and algebras to be commutative and unital; ring and algebra homomorphisms are meant to be unital as well.

Let A be a ring and let B be an A-algebra. For each A-algebra C, the scalar extension by B is the B-algebra $C \otimes_A B$ with structure map $b \mapsto 1 \otimes b$ [1]. This assignment has a natural extension to a covariant functor $F : A\text{-}\mathtt{Alg} \longrightarrow B\text{-}\mathtt{Alg}$. The functor F has a right adjoint $B\text{-}\mathtt{Alg} \longrightarrow A\text{-}\mathtt{Alg}$ given by restricting scalars. If B is finitely generated and free as an A-module, then F also has a left adjoint W, called **Weil descent**, or **Weil restriction**. We start with a reminder on left adjoints in general, ready made for use later on.

2.1. **Fact.**   [ML98, Thm 2, p.83, Cor. 1,2, p.84]
Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a covariant functor between categories $\mathcal{C}$ and $\mathcal{D}$.

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[1] As a general reference for tensor products, specifically in the category of algebras we refer to [Mat89, Appendix A]

(i) The following are equivalent.
   (a) F has a left adjoint W, i.e., $W : \mathcal{D} \longrightarrow \mathcal{C}$ is a covariant functor such that
       for all $D \in \mathcal{D}$ the functor $\mathrm{Hom}_{\mathcal{D}}(D, F(\_)) : \mathcal{C} \longrightarrow \mathtt{Sets}$ is represented by
       $W(D)$ meaning that the functors $\mathrm{Hom}_{\mathcal{C}}(W(D), \_)$ and $\mathrm{Hom}_{\mathcal{D}}(D, F(\_))$
       are isomorphic [2].
   (b) For each $D \in \mathcal{D}$ there are $W(D) \in \mathcal{C}$ and a $\mathcal{D}$-morphism $W_D : D \longrightarrow$
       $F(W(D))$ such that the following condition holds:

       (†): For every $C \in \mathcal{C}$ and each morphism $f : D \longrightarrow F(C)$, there is
            a unique $\mathcal{C}$-morphism $g : W(D) \longrightarrow C$ such that the following
            diagram commutes

$$
\begin{array}{ccc}
F(W(D)) & \xrightarrow{\;F(g)\;} & F(C) \\
{\scriptstyle W_D}\big\uparrow & \nearrow{\scriptstyle f} & \\
D & &
\end{array}
$$

       In other words, W_D gives rise to a bijection

$$\tau(D,C) : \mathrm{Hom}_{\mathcal{C}}(W(D),C) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(D, F(C)), \; g \longmapsto F(g) \circ W_D.$$

(ii) If (i)a holds, then for every such functor W, all $D \in \mathcal{D}$ and each isomorphism
     $\tau(D, \_) : \mathrm{Hom}_{\mathcal{C}}(W(D), \_) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(D, F(\_))$ as in (i)a, the choice $W(D)$
     and $W_D = \tau(D, W(D))(\mathrm{id}_{W(D)})$ satisfy property (†) of (i)b.

     The assignment $D \mapsto W_D$ is a natural transformation $\mathrm{id}_{\mathcal{D}} \longrightarrow F \circ W$ and
     is called the **unit of the adjunction**; W_D is called the **component** at D of
     that unit.

     Similarly, for each $C \in \mathcal{C}$ the morphism $F_C : W(F(C)) \longrightarrow C$ that is sent
     to $\mathrm{id}_{F(C)}$ via $\tau(F(C), C)$ gives rise to a natural transformation $W \circ F \longrightarrow \mathrm{id}_{\mathcal{C}}$,
     called the **counit of the adjunction**; F_C is called the **component** at C of
     that counit.

(iii) If (i)b holds, then for any choice of objects $W(D)$ and morphisms W_D as in
      (i)b, for $D \in \mathcal{D}$, the assignment $D \mapsto W(D)$ can be extended to a functor
      $W : \mathcal{D} \longrightarrow \mathcal{C}$ satisfying (i)a as follows: Take a morphism $f_0 : D \longrightarrow D'$ and
      set $f := W_{D'} \circ f_0$ and $C := W(D')$. Then define $W(f_0) : W(D) \longrightarrow W(D')$
      as the unique $\mathcal{C}$-morphism $W(D) \longrightarrow W(D')$ such that the diagram

$$
\begin{array}{ccc}
F(W(D)) & \xrightarrow{\;F(W(f_0))\;} & F(C) = F(W(D')) \\
{\scriptstyle W_D}\big\uparrow & \nearrow{\scriptstyle f} & \big\uparrow{\scriptstyle W_{D'}} \\
D & \xrightarrow[\;f_0\;]{} & D'
\end{array}
$$

      commutes, according to (†).
(iv) Any two functors that are left adjoint to F are isomorphic.
(v) If W is left adjoint to F, then W preserves all co-limits, cf. [ML98, p. 119,
    last paragraph]. For example W preserves direct limits and fibre sums (aka
    pushouts).

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[2] Recall that two functors are isomorphic if there is an invertible natural transformation between
them.

2.2. **Notation and setup.** We return to our setup of a ring A and an A-algebra B. Let
$$F : A\text{-}\mathtt{Alg} \longrightarrow B\text{-}\mathtt{Alg}$$
be the functor defined by $F(C) = C \otimes B$ and for $\varphi : C \longrightarrow C'$, $F(\varphi) = \varphi \otimes \mathrm{id}_B$. Here and below, tensor products are taken over A, unless stated otherwise.

We will from now on assume that B is free and finitely generated as an A-module of dimension ℓ over A. We also fix generators $b_1, \ldots, b_\ell$ of the A-module B. For $i \in \{1, ..., \ell\}$ let
$$\lambda_i : B \longrightarrow A, \ \lambda_i\left(\sum_{j=1}^{\ell} a_j b_j\right) = a_i$$
be the A-module homomorphism dual to $A \longrightarrow B$, $a \mapsto a \cdot b_i$. If C is an A-algebra we write $\lambda_i^C = \mathrm{id}_C \otimes \lambda_i : C \otimes B \longrightarrow C \otimes A = C$ for the base change of λ_i to C. Since $b = \sum_i \lambda_i(b) b_i$ for $b \in B$ we obtain
$$c \otimes b = c \otimes \sum_i \lambda_i(b) b_i = \sum_i (\lambda_i(b) c) \otimes b_i,$$
for $c \in C$. Hence $\lambda_i^C(c \otimes b) = \lambda_i(b) \cdot c$ is the coefficient of $c \otimes b$ at $1 \otimes b_i$ when it is written in the basis $1 \otimes b_1, \ldots, 1 \otimes b_\ell$ of the free C-module $C \otimes B$. This extends to all $f \in C \otimes B$, thus

$$(*) \qquad f = \sum_{i=1}^{\ell} \left(\lambda_i^C(f) \otimes b_i\right).$$

2.3. **Definition.** Let T be a set of indeterminates for A and B. We define an A-algebra $W(B[T]) = A[T]^{\otimes \ell} \ (= \underbrace{A[T] \otimes \ldots \otimes A[T]}_{\ell\text{-times}})$. For $i \in \{1, \ldots, \ell\}$ and $t \in T$ we write
$$t(i) := 1 \otimes ... \otimes 1 \otimes \underbrace{t}_{i\text{-th position}} \otimes 1 \otimes ... \otimes 1 \in A[T]^{\otimes \ell}.$$
Let $W_{B[T]}$ be the unique B-algebra homomorphism
$$W_{B[T]} : B[T] \longrightarrow F(W(B[T])) = A[T]^{\otimes \ell} \otimes B \text{ with}$$
$$W_{B[T]}(t) = \sum_{i=1}^{\ell} (t(i) \otimes b_i) \quad (t \in T).$$
Further, let $F_{A[T]}$ be the unique A-algebra homomorphism
$$F_{A[T]} : W(F(A[T])) = A[T]^{\otimes \ell} \longrightarrow A[T]$$
with the property $F_{A[T]}(t(i)) = \lambda_i(1) \cdot t$ for $t \in T$, $i \in \{1, \ldots, \ell\}$.

2.4. **Explicit description of the Weil descent of polynomial algebras.** The A-algebra $W(B[T])$ and the morphism $W_{B[T]}$ described above satisfy condition (†) of 2.1(i)b. Hence by 2.1(iii) we may choose $W(B[T])$ as the Weil descent of $B[T]$, and $W_{B[T]}$ as the unit of the adjunction at $B[T]$; these choices are then independent of the basis $b_1, \ldots, b_\ell$ up to a natural A-algebra isomorphism (see 2.1(iv)).

Explicitly, for every $C \in A\text{-}\mathtt{Alg}$, the map
$$\tau = \tau(B[T], C) : \ \mathrm{Hom}_{A\text{-}\mathtt{Alg}}(A[T]^{\otimes \ell}, C) \ \longrightarrow \ \mathrm{Hom}_{B\text{-}\mathtt{Alg}}(B[T], C \otimes B)$$
$$\varphi \ \longmapsto \ F(\varphi) \circ W_{B[T]} = (\varphi \otimes \mathrm{id}_B) \circ W_{B[T]}$$

is bijective, where $\varphi \otimes \mathrm{id}_B = F(\varphi) : F(W(B[T])) = A[T]^{\otimes d} \otimes B \longrightarrow C \otimes B$ is the base change of φ. For $t \in T$ we have

$$\tau(\varphi)(t) = \sum_{i=1}^{\ell} (\varphi(t(i)) \otimes b_i).$$

The compositional inverse of $\tau = \tau(B[T], C)$ is defined as follows. Let $\psi : B[T] \longrightarrow C \otimes B$ be a B-algebra homomorphism. We define an A-algebra homomorphism $\varphi : A[T]^{\otimes d} \longrightarrow C$ by

$$\varphi(t(i)) := \lambda_i^C(\psi(t)) \ (t \in T, \ i = 1, \ldots, \ell).$$

Since $A[T] \otimes B \cong_B B[T]$, ψ is uniquely determined by $\{\psi(t) \mid t \in T\}$ and we see that φ is the unique preimage of ψ under τ.

Further, one checks easily that $F_{A[T]}$ is the component of the counit of the adjunction at $A[T]$.

2.5. **Explicit description of the Weil descent of B-algebras.** Now let D be a B-algebra. Take a surjective B-algebra homomorphism $\pi_D : B[T] \twoheadrightarrow D$ for some set T of indeterminates. Let I_D be the ideal generated in $W(B[T]) = A[T]^{\otimes \ell}$ generated by all the $\lambda_i^{W(B[T])}(W_{B[T]}(f))$, where $i \in \{1, \ldots, \ell\}$ and $f \in \ker(\pi_D)$. We define

$$W(D) := W(B[T])/I_D$$

and write

$$W(\pi_D) : W(B[T]) \longrightarrow W(D)$$

for the residue map. Then the bijection $\tau(B[T], C)$ from 2.4 induces a bijection

$$\tau(D, C) : \mathrm{Hom}_{A\text{-}\mathtt{Alg}}(W(D), C) \longrightarrow \mathrm{Hom}_{B\text{-}\mathtt{Alg}}(D, F(C))$$

such that the diagram

$$
\begin{array}{ccc}
\mathrm{Hom}_{A\text{-}\mathtt{Alg}}(W(D), C) & \xdashrightarrow{\ \tau(D,C)\ } & \mathrm{Hom}_{B\text{-}\mathtt{Alg}}(D, F(C)) \\
{\scriptstyle \_\circ W(\pi_D)}\big\uparrow & & \big\downarrow{\scriptstyle \_\circ \pi_D} \\
\mathrm{Hom}_{A\text{-}\mathtt{Alg}}(W(B[T]), C) & \xrightarrow[\tau(B[T],C)]{\ \cong\ } & \mathrm{Hom}_{B\text{-}\mathtt{Alg}}(B[T], F(C))
\end{array}
$$

commutes.

The commutativity of the diagram above says that for $\varphi \in \mathrm{Hom}_{A\text{-}\mathtt{Alg}}(W(D), C)$ we have

$$
\begin{aligned}
(+) \quad \tau(D, C)(\varphi) \circ \pi_D &= \tau(B[T], C)(\varphi \circ W(\pi_D)) \\
&= ((\varphi \circ W(\pi_D)) \otimes \mathrm{id}_B) \circ W_{B[T]}.
\end{aligned}
$$

Finally, we display the map $W_D := \tau(D, W(D))(\mathrm{id}_{W(D)}) : D \longrightarrow F(W(D))$ explicitly and show that – together with $W(D)$ – it satisfies the mapping property of ($\dagger$) in 2.1(i)b. Take $t \in T$. Then by (+) with $C = W(D)$, $\varphi = \mathrm{id}_{W(D)}$ we see that

$$(\sharp) \qquad W_D(\pi_D(t)) = \sum_{i=1}^{\ell} W(\pi_D)(t(i)) \otimes b_i = \sum_{i=1}^{\ell} (t(i) \bmod I_D) \otimes b_i.$$

Pick an A-algebra C. Since $\tau(D, C)$ is bijective, the mapping property of (†) in 2.1(i)b follows after checking $\tau(D, C)(\varphi) = F(\varphi) \circ W_D$ for all $\varphi \in \mathrm{Hom}_{A\text{-}\mathtt{Alg}}(W(D), C)$. Using ($\sharp$) this is a straightforward computation.

Using 2.1(iii),(iv) we have justified our choice of $W(D)$ and W_D for the Weil descent. Finally, 2.1(iii) gives the definition of W on morphisms.

3. Differential Weil Descent

In this section we present a construction of a Weil descent functor in the category of differential algebras in arbitrary characteristic. We first recall some basic facts about differential algebras and their tensor products. We continue to assume that our rings and algebras are unital and commutative.

3.1. **Generalities about differential algebra.** The following are well known generalities on differential algebras whose proofs are straightforward. For a ring A we let $\mathrm{Der}(A)$ denote the family of derivations on A.

(i) Let A be a ring and let T be a not necessarily finite set of indeterminates over A. For each $t \in T$ let $f_t \in A[T]$. Let $d \in \mathrm{Der}(A)$. Then there is a unique derivation δ of $A[T]$ extending d with $\delta(t) = f_t$ for all $t \in T$.

For $d, \delta \in \mathrm{Der}(A)$ we write $[d, \delta] : A \longrightarrow A$ for the Lie-bracket of d and δ, defined by $[d, \delta](a) = d\delta(a) - \delta d(a)$. Notice that $[d, \delta]$ is again a derivation of A.

(ii) Let A be a ring and let $S \subseteq A$ be a set of generators of the ring A.
  (a) Let $d_1, d_2, \delta_1, \ldots, \delta_n \in \mathrm{Der}(A)$ and suppose there are $a_i \in A$ with $[d_1, d_2](s) = \sum_{i=1}^n a_i \delta_i(s)$ for all $s \in S$. Then $[d_1, d_2] = \sum_{i=1}^n a_i \delta_i$.
  (b) Let $\varphi : A \longrightarrow B$ be a ring homomorphism and let $d : A \longrightarrow A$, $\delta : B \longrightarrow B$ be derivations. If $\varphi(ds) = \delta(\varphi(s))$ for all $s \in S$, then φ is a differential homomorphism $(A, d) \longrightarrow (B, \delta)$.

(iii) Let $d \in \mathrm{Der}(A)$ and let $(B, \delta), (C, \partial)$ be differential (A, d)-algebras. Then there is a unique derivation $\delta \otimes \partial$ on $B \otimes_A C$ such that the natural maps $B \longrightarrow B \otimes_A C, C \longrightarrow B \otimes_A C$ are differential maps, cf. [Bui94, Chapter 2 (1.1), p. 21].

(iv) Now let $d_1, d_2 \in \mathrm{Der}(A)$, $\delta_1, \delta_2 \in \mathrm{Der}(B)$ and $\partial_1, \partial_2 \in \mathrm{Der}(C)$ such that $(B, \delta_i), (C, \partial_i)$ are differential (A, d_i)-algebras. Then, for $a_1, a_2 \in A$, straightforward checking shows that
  (a) $(a_1 \delta_1 + a_2 \delta_2) \otimes (a_1 \partial_1 + a_2 \partial_2) = a_1(\delta_1 \otimes \partial_1) + a_2(\delta_2 \otimes \partial_2)$.
  (b) $[\delta_1, \delta_2] \otimes [\partial_1, \partial_2] = [\delta_1 \otimes \partial_1, \delta_2 \otimes \partial_2]$.

As in Section 2 we work with a ring A and an A-algebra B that is free and finitely generated by $b_1, \ldots, b_\ell$ as an A-module. We fix a derivation d on A and a derivation δ on B such that (B, δ) is a differential (A, d)-algebra (meaning that the structure map $A \longrightarrow B$ is differential).

By 3.1(iii), for any differential (A, d)-algebra (C, ∂_C), there is a unique derivation $\partial_C \otimes \delta$ on $F(C) = C \otimes B$ such that the natural map $C \longrightarrow F(C)$ is a differential (B, δ)-algebra morphism.

3.2. **Theorem.** *Let (D, ∂_D) be a differential (B, δ)-algebra. Then there is a unique derivation ∂_D^W on $W(D)$ such that $(W(D), \partial_D^W)$ is a differential (A, d)-algebra and*

$$W_D : (D, \partial_D) \longrightarrow (F(W(D)), \partial_D^W \otimes \delta)$$

is a differential (B, δ)-algebra homomorphism, i.e., $W_D \circ \partial_D = (\partial_D^W \otimes \delta) \circ W_D$.

Furthermore, ∂_D^W only depends on ∂_D and not on δ.

Proof. Take any set T of differential indeterminates and a surjective (B, δ)-algebra homomorphism $\pi_D : (B\{T\}, \partial) \twoheadrightarrow (D, \partial_D)$. Here, the differential polynomial ring $B\{T\}$ is considered just as polynomial ring over B in the algebraic indeterminates t_θ, where $t \in T$ and $\theta \in \Theta := \{\partial^i : i \geq 0\}$. Further, $\partial = \partial_{B\{T\}} : B\{T\} \longrightarrow B\{T\}$ is the natural derivation, thus $\partial t_\theta = t_{\partial\theta}$.

We choose $W_{B\{T\}} : B\{T\} \longrightarrow F(W(B\{T\}))$ according to 2.3 for the set of indeterminates $\{t_\theta \mid t \in T, \ \theta \in \Theta\}$ and $W_D : D \longrightarrow F(W(D))$ according to 2.5. Also recall ($\sharp$) in 2.5, which says that $W_D(\pi_D(t_\theta)) = \sum_{i=1}^{\ell} W(\pi_D)(t_\theta(i)) \otimes b_i$

<u>Claim 1.</u> If $\varepsilon : W(D) \longrightarrow W(D)$ is a derivation such that $(W(D), \varepsilon)$ is a differential A-algebra, then for all $t \in T$ and any $\theta \in \Theta$ we have

$$((\varepsilon \otimes \delta) \circ W_D)(\pi_D(t_\theta)) = \sum_{i=1}^{\ell} \left(\varepsilon(W(\pi_D)(t_\theta(i))) + \sum_{j=1}^{\ell} \lambda_i(\delta b_j) \cdot W(\pi_D)(t_\theta(j)) \right) \otimes b_i.$$

See 3.1(iii) for the definition of $\varepsilon \otimes \delta$.

Proof. This is a straightforward calculation using $\delta b_i = \sum_{j=1}^{\ell} \lambda_j(\delta b_i) b_j$. $\diamond$

<u>Claim 2.</u> If $\varepsilon : W(D) \longrightarrow W(D)$ is a derivation such that $(W(D), \varepsilon)$ is a differential A-algebra, then $W_D \circ \partial = (\varepsilon \otimes \delta) \circ W_D$ if and only if for all $t_\theta(i)$ we have

$$(*) \qquad \varepsilon(W(\pi_D)(t_\theta(i))) = W(\pi_D)(t_{\partial\theta}(i) - \sum_{j=1}^{\ell} \lambda_i(\delta(b_j)) \cdot W(\pi_D)(t_\theta(j)).$$

Proof. By 3.1(ii)(b), $W_D \circ \partial = (\varepsilon \otimes \delta) \circ W_D$ if and only if $((\varepsilon \otimes \delta) \circ W_D)(\pi_D(t_\theta)) = (W_D \circ \partial)(\pi_D(t_\theta))$ for all t_θ. By Claim 1 this is equivalent to

$$\sum_{i=1}^{\ell} \left(\varepsilon(W(\pi_D)(t_\theta(i))) + \sum_{j=1}^{\ell} \lambda_i(\delta b_j) \cdot W(\pi_D)(t_\theta(j)) \right) \otimes b_i$$
$$= W_D(\partial(\pi_D(t_\theta)))$$
$$= W_D(\pi_D(\partial t_\theta)), \text{ since } \pi_D \text{ is a differential map}$$
$$= W_D(\pi_D(t_{\partial\theta}))$$
$$= \sum_{i=1}^{\ell} W(\pi_D)(t_{\partial\theta}(i)) \otimes b_i, \text{ by } (\sharp) \text{ in 2.5.}$$

Since $1 \otimes b_1, \ldots, 1 \otimes b_\ell$ is a basis of $F(W(D))$ over $W(D)$, the identity is equivalent to $(*)$ being true for all $i \in \{1, \ldots, \ell\}$. $\diamond$

Claim 2 implies the uniqueness statement of the theorem, because the set of all the $W(\pi_D)(t_\theta(i))$ generates $W(D)$. For existence, we first deal with $B\{T\}$ instead of D. In that case, Claim 2 says that we only need to find a derivation $\partial_{B\{T\}}^W$ on $W(B\{T\})$ such that $(W(B\{T\}), \partial_{B\{T\}}^W)$ is a differential (A, d)-algebra with the property

$$\partial_{B\{T\}}^W(t_\theta(i)) = t_{\partial\theta}(i) - \sum_{j=1}^{\ell} \lambda_i(\delta(b_j)) \cdot t_\theta(j).$$

By 3.1(i) applied to the polynomial ring $W(B\{T\})$ over A, such a derivation indeed exists.[3]

It remains to prove that there is a derivation ∂_D^W of $W(D)$ as required.

<u>Claim 3.</u> The ideal I_D of $W(B\{T\})$ (see 2.5) is a differential ideal for $\partial_{B\{T\}}^W$.

Proof. Let $f \in \ker(\pi_D)$. Then $W_{B\{T\}}(f) = \sum_{i=1}^{\ell} g_i \otimes b_i$, where $g_i = \lambda_i^{W(B\{T\})}(W_{B\{T\}}(f))$. By definition of I_D it suffices to show that $\partial_{B\{T\}}^W(g_i) \in I_D$ [4]. Now one checks that

$$W_{B\{T\}}(\partial_{B\{T\}}(f)) = \sum_{i=1}^{\ell} \left(\partial_{B\{T\}}^W(g_i) + \sum_{j=1}^{\ell} \lambda_i(b_j)g_j \right) \otimes b_i$$

Since $1 \otimes b_1, \ldots, 1 \otimes b_\ell$ is a basis of $F(W(B\{T\}))$ over $W(B\{T\})$ we see that

$$\lambda_i^{W(B\{T\})}(W_{B\{T\}}(\partial_{B\{T\}}(f))) = \partial_{B\{T\}}^W(g_i) + \sum_{j=1}^{\ell} \lambda_i(b_j)g_j.$$

The left hand side here is in I_D by definition of I_D and because $\ker(\pi)$ is differential for $\partial_{B\{T\}}$. As all $g_i \in I_D$ this entails $\partial_{B\{T\}}^W(g_i) \in I_D$. $\diamond$

By Claim 3, the derivation $\partial_{B\{T\}}$ induces a derivation δ_D^W of $W(D) = W(B\{T\})/I_D$ such that $(W(D), \partial_D^W)$ is a differential (A, d)-algebra. It remains to show that W_D is a differential (B, δ)-algebra homomorphism, i.e., $W_D \circ \partial_D = (\partial_D^W \otimes \delta) \circ W_D$. This can be seen by a diagram chase as follows. Consider the diagram of maps

The claim is that the back side of this cube is commutative. Now, all other sides of the cube are commutative squares, because

- Bottom and top of the cube are identical and commute as a property of the classical Weil descent.

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[3]Notice that $W(B\{T\})$ naturally is a differential polynomial ring over A, but $\partial_{B\{T\}}^W$ is in general not the natural derivation of $W(B\{T\})$.

[4]Notice that the module homomorphism $\lambda_i^{W(B(\{T\}))} : F(W(B\{T\})) \longrightarrow W(B\{T\})$ does not in general commute with the derivations.

- The front of the cube commutes as we know the theorem already for $(B\{T\}, \partial_{B\{T\}})$.
- The square on the left hand side commutes by choice of $(B\{T\}, \partial_{B\{T\}})$.
- The square on the right hand side commutes by applying base change to B to the definition of ∂_D^W.

Since π is surjective, we see that the back of the cube also commutes. This finishes the proof of existence and uniqueness of ∂_W. From Claim 2 we see that the definition of $\partial_{B\{T\}}^W$ only depends on $\partial_{B\{T\}}$ and not on δ, because the structure map $B \longrightarrow D$ is differential. But then by construction of ∂_D^W after Claim 3, ∂_D^W only depends on ∂_D and not on δ. $\qquad\square$

3.3. **Theorem.** *Let again B be an A-algebra that is finitely generated and free as an A-module and let D be a B-algebra.*

Let $\mathrm{Der}_B(D)$ be the set of all $\partial \in \mathrm{Der}(D)$ for which there are derivations d of A and δ of B such that the structure maps of B and D are differential maps $(A, d) \longrightarrow (B, \delta)$ and $(B, \delta) \longrightarrow (D, \partial)$, respectively.[5]

Then $\mathrm{Der}_B(D)$ is an A-submodule and a Lie subring of $\mathrm{Der}(D)$ and the map $\mathrm{Der}_B(D) \longrightarrow \mathrm{Der}(W(D))$ that sends ∂ to the derivation ∂^W defined in 3.2, is an A-module and a Lie ring homomorphism. Explicitly, given $\partial_1, \partial_2 \in \mathrm{Der}_B(D)$ we have

(i) $(a_1\partial_1 + a_2\partial_2)^W = a_1\partial_1^W + a_2\partial_2^W$ *for all $a_1, a_2 \in A$.*
(ii) $[\partial_1, \partial_2]^W = [\partial_1^W, \partial_2^W]$. *In particular, $\partial_1^W, \partial_2^W$ commute if ∂_1, ∂_2 commute.*

Proof. In each case, the derivation of $W(D)$ on the right hand side turns $W(D)$ into a differential A-algebra, when A is furnished with the derivation $a_1d_1 + a_2d_2$ and $[d_1, d_2]$ respectively. By uniqueness in 3.2 we thus only need to verify the defining equation of the left hand side for the right hand side.

(i) Using 3.1(iv)(a) we get $((a_1\partial_1^W + a_2\partial_2^W) \otimes (a_1\delta_1 + a_2\delta_2)) \circ W_D = a_1(\partial_1^W \otimes \delta_1) \circ W_D + a_2(\partial_2^W \otimes \delta_2) \circ W_D = a_1 W_D \circ \partial_1 + a_2 W_D \circ \partial_2 = W_D \circ (a_1\partial_1 + a_2\partial_2)$, since W_D is an A-algebra homomorphism.

(ii) Using 3.1(iv)(b) we get $([\partial_1^W, \partial_2^W] \otimes [\delta_1, \delta_2]) \circ W_D = [\partial_1^W \otimes \delta_1, \partial_2^W \otimes \delta_2] \circ W_D = (\partial_1^W \otimes \delta_1) \circ (\partial_2^W \otimes \delta_2) \circ W_D - (\partial_2^W \otimes \delta_2) \circ (\partial_1^W \otimes \delta_1) \circ W_D = (\partial_1^W \otimes \delta_1) \circ W_D \circ \partial_2 - (\partial_2^W \otimes \delta_2) \circ W_D \circ \partial_1 = W_D \circ \partial_1 \circ \partial_2 - W_D \circ \partial_2 \circ \partial_1 = W_D \circ [\partial_1, \partial_2]$. $\qquad\square$

Theorems 3.2 and 3.3 establish

3.4. **The differential Weil descent.** Let A be a ring and let $d_1, \ldots, d_m \in \mathrm{Der}(A)$. We write $d = (d_1, \ldots, d_m)$. A differential (A, d)-algebra is an A-algebra C together with $\eta_1, \ldots, \eta_m \in \mathrm{Der}(C)$, such that the structure map $A \longrightarrow C$ is a differential morphism $(A, d_i) \longrightarrow (C, \eta_i)$ for all $i \in \{1, \ldots, m\}$. Let (A, d)-$\mathtt{Alg}$ be the category of differential (A, d)-algebras whose morphisms are ring homomorphisms preserving the appropriate derivations.

We fix a differential (A, d)-algebra (B, δ) such that B is finitely generated and free as an A-module. Then

(i) The functor $F^{\mathrm{diff}} : (A, d)$-$\mathtt{Alg} \longrightarrow (B, \delta)$-$\mathtt{Alg}$ that sends (C, η) to $(C \otimes B, \eta_1 \otimes \delta_1, \ldots, \eta_m \otimes \delta_m)$ has a left adjoint $W^{\mathrm{diff}} : (B, \delta)$-$\mathtt{Alg} \longrightarrow (A, d)$-$\mathtt{Alg}$, which

---

[5]If the structure morphism of D as a B-algebra is injective and we think of the structure maps $A \longrightarrow B$ and $B \longrightarrow D$ as inclusions, then $\mathrm{Der}_B(D)$ is the set of all derivations of D that restrict to derivations on A and B.

we call the **differential Weil descent** (or differential Weil restriction) from (B, δ) to (A, d). It sends (D, ∂) to $(W(D), \partial^W)$ with $\partial^W = (\partial_1^W, \ldots, \partial_m^W)$, ∂_i^W as defined in 3.2, and a morphism f to $W(f)$.

(ii) Let $(C, \eta) \in (A, d)$-$\mathtt{Alg}$ and let $(D, \partial) \in (B, \delta)$-$\mathtt{Alg}$. Then the bijection

$$\tau(D, C) : \mathrm{Hom}_{A\text{-}\mathtt{Alg}}(W(D), C) \longrightarrow \mathrm{Hom}_{B\text{-}\mathtt{Alg}}(D, F(C)), \ \varphi \longmapsto F(\varphi) \circ W_D$$

from the classical Weil descent 2.5 restricts to a bijection

$$\mathrm{Hom}_{(A,d)\text{-}\mathtt{Alg}}(W^{\mathrm{diff}}(D, \partial), (C, \eta)) \longrightarrow \mathrm{Hom}_{(B,\delta)\text{-}\mathtt{Alg}}((D, \partial), F^{\mathrm{diff}}(C, \eta)).$$

(iii) If $(D, \partial) \in (B, \delta)$-$\mathtt{Alg}$ and the $\partial_1, \ldots, \partial_m$ are Lie commuting with structure coefficients $a_{ij}^k \in A$ ($1 \leq i, j, k \leq m$), i.e.,

$$[\partial_i, \partial_j] = \sum_{k=1}^{m} a_{ij}^k \partial_k \quad (1 \leq i, j \leq m),$$

then also the derivations $\partial_1^W, \ldots, \partial_m^W$ of $W(D)$ are Lie commuting with structure coefficients a_{ij}^k.

Proof. By Theorem 3.2, the map $W_D : D \to F(W(D))$ is differential. Hence, if a morphism $\varphi : W(D) \to C$ is differential, so is $F(\varphi) \circ W_D$. Thus the map $\tau(D, C)$ restricts to differential morphisms as claimed in (ii). Now recall from 2.5 (and 2.1(iii)) that $W(f)$ is the unique map that corresponds to the morphism $W_{D'} \circ f : D \to F(W(D'))$ under the bijection $\tau(D, W(D'))$. As the latter morphism is differential, $W(f)$ must be differential. This entails (i), see 2.1(i)a. Item (iii) follows immediately from 3.3. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

4. Differential Algebraic Setup

One of our applications of the differential Weil descent constructed in Section 3 is to show that every algebraic extension of a large field that is a model of UC_m is again a model of UC_m. This will be achieved in a similar manner to the fact that algebraic extensions of large fields are again large [Pop96, Proposition 2.1]. The latter uses classical Weil descent and, in particular, the fact that the Weil functor preserves smoothness [Oes84, Appendix 2]. Before presenting the applications we will review the theory UC_m and establish some useful characterizations. But first some preliminaries.

4.1. Some preliminaries and conventions. We fix a distinguished set of commuting derivations $\Delta = \{\delta_1, \ldots, \delta_m\}$. We assume that all our fields are of characteristic zero. We work inside a large (saturated) differentially closed field $(\mathcal{U}, \Delta)$, and K denotes a differential subfield of $\mathcal{U}$. A **Kolchin-closed** subset of $\mathcal{U}^n$ is the common zero set of a set of differential polynomials over $\mathcal{U}$ in n differential variables; such sets are also called **affine differential varieties**. If the definining polynomials can be chosen with coefficients in K we will say the set is **defined over** K. The Kolchin-closed sets (defined over K) are the closed sets of a topology, called the **Kolchin-topology** of $\mathcal{U}^n$ (over K).

By a **differential variety** V we mean a topological space which has as finite open cover $V_1, \ldots, V_s$ with each V_i homeomorphic to an affine differential variety (inside some power of $\mathcal{U}$) such that the transition maps are regular as differential morphisms; see [LS13, Chap. 1, section 7]. We will say that the differential variety is over K when all objects and morphisms can be defined over K. This definition

also applies to our use of algebraic varieties, replacing Kolchin-closed with Zariski-closed in powers of $\mathcal{U}$ (recall that $\mathcal{U}$ is algebraically closed and a universal domain for algebraic geometry in Weil's "foundations" sense).

4.2. *Remark.* Suppose L/K is a finite field extension. Recall that the derivations $\delta_1, \ldots, \delta_m$ extend uniquely from K to L.

  (i) Given a differential L-algebra D, by 3.4, there is a natural one-to-one correspondence between the differential L-points of D and the differential K-points of $W^{\mathrm{diff}}(D)$.
 (ii) In the case when D is the differential coordinate ring of an affine differential variety, say $D = L\{V\}$, and when L has a K-basis $b_1, \ldots, b_\ell$ of constants (meaning that $\delta_i(b_j) = 0$ for all i, j), then a construction of the differential Weil descent $W^{\mathrm{diff}}(L\{V\})$ appears in [LSM16, §5]. However, a basis of constants does not always exist, see Example 4.3 below.

4.3. *Example.* We work in the ordinary case $\Delta = \{\delta\}$. Let $K = \mathbb{Q}(t)$ with $\delta t = 1$ and consider the finite extension $L = K(b)$ where $b^2 = t$. Then the (unique) induced derivation on L is given by $\delta b = \frac{1}{2b} = \frac{b}{2t}$. Fix the basis $\{1, b\}$ of L as a K-module. Consider the differential variety V given by $\delta x = 0$ (i.e., V is simply the constants of $\mathcal{U}$) viewed as a differential variety over L. The differential Weil descent $W^{\mathrm{diff}}(V)$ is obtained as follows; write x as $x_1 + x_2 b$ and compute

$$\delta(x_1 + bx_2) = \delta x_1 + (\delta b)x_2 + b\delta x_2 = \delta x_1 + \frac{b}{2t}x_2 + b\delta x_2 = \delta x_1 + \left(\frac{x_2}{2t} + \delta x_2\right)b.$$

Thus, $W^{\mathrm{diff}}(V)$ is the affine differential variety over K given by the equations

$$\delta x_1 = 0 \quad \text{and} \quad \delta x_2 + \frac{x_2}{2t} = 0.$$

Note that this is not contained in a product of the constants, as one might expect. Of course, if $\delta(b)$ were zero we would instead obtain the equations $\delta x_1 = 0$ and $\delta x_2 = 0$ (which would occur if δ were trivial on K, for instance).

4.4. We fix integers $n > 0$ and $r \geq 0$, and set

$$\Gamma_n(r) = \{(\xi, i) \in \mathbb{N}^m \times \{1, \ldots, n\} \mid \sum_{i=1}^{m} \xi_i \leq r\}.$$

We make use of prolongation spaces and recall the definition and some properties. The **r-th nabla map** $\nabla_r : \mathcal{U}^n \to \mathcal{U}^{\alpha(n,r)}$ with $\alpha(n,r) := |\Gamma_n(r)| = n \cdot \binom{r+m}{m}$ is defined by

$$\nabla_r(x) = (\delta^\xi x_i : (\xi, i) \in \Gamma_n(r)),$$

where $x = (x_1, \ldots, x_n)$ and $\delta^\xi = \delta_1^{\xi_1} \cdots \delta_m^{\xi_m}$. We order the elements of the tuple $(\delta^\xi x_i : (\xi, i) \in \Gamma_n(r))$ according to the canonical orderly ranking of the indeterminates $\delta^\xi x_i$; that is,

(4.1) $\qquad \delta^\xi x_i < \delta^\zeta x_j \iff \left(\sum \xi_k, i, \xi_1, \ldots, \xi_m\right) < \left(\sum \zeta_k, j, \zeta_1, \ldots, \zeta_m\right)$

where the ordering on the right-hand-side is the lexicographic one.

Let $\mathcal{U}_r := \mathcal{U}[\epsilon_1, \ldots, \epsilon_m]/(\epsilon_1, \ldots, \epsilon_m)^{r+1}$ where the ϵ_i's are indeterminates, and let $e : \mathcal{U} \to \mathcal{U}_r$ denote the ring homomorphism

$$x \mapsto \sum_{\xi \in \Gamma_1(r)} \frac{1}{\xi_1! \cdots \xi_m!}\, \delta^\xi(x)\, \epsilon_1^{\xi_1} \cdots \epsilon_m^{\xi_m}.$$

We call e the exponential $\mathcal{U}$-algebra structure of $\mathcal{U}_r$. To distinguish between the standard and the exponential algebra structure on $\mathcal{U}_r$, we denote the latter by $\mathcal{U}_r^e$.

4.5. Definition. Given an algebraic variety X the r-th **prolongation** τX is the algebraic variety given by the taking the $\mathcal{U}$-rational points of the Weil descent of $X \times_{\mathcal{U}} \mathcal{U}_r^e$ from $\mathcal{U}_r$ to $\mathcal{U}$. Note that the base change $V \times_{\mathcal{U}} \mathcal{U}_r^e$ is with respect to the exponential structure while the Weil descent is with respect to the standard $\mathcal{U}$-algebra structure.

For details and properties of prolongation spaces we refer to [MPS08, §2]; for a more general presentation see [MS10]. In particular, it is pointed out there that the prolongation $\tau_r X$ always exist when X is quasi-projective (an assumption that we will adhere to later on). A characterising feature of the prolongation is that for each point $a \in X = X(\mathcal{U})$ we have $\nabla_r(a) \in \tau_r X$. Thus, the map $\nabla_r : X \to \tau_r X$ is a differential regular section of $\pi_r : \tau_r X \to X$ the canonical projection induced from the residue map $\mathcal{U}_r \to \mathcal{U}$. We note that if X is defined over the differential field K then $\tau_r X$ is defined over K as well.

In fact, τ_r as defined above is a functor from the category of algebraic varieties over K to itself, and the maps $\pi_r : \tau_r X \to X$ and $\nabla_r : X \to \tau_r X$ are natural. The latter means that for any morphism of algebraic varieties $f : X \to Y$ we get

$$(4.2) \qquad f \circ \pi_{r,X} = \pi_{r,Y} \circ \tau_r f \quad \text{and} \quad \tau_r f \circ \nabla_{r,X} = \nabla_{r,Y} \circ f.$$

If G is an algebraic group, then $\tau_r G$ also has the structure of an algebraic group. Indeed, since τ_r commutes with products, the group structure is given by

$$\tau_r(*) : \tau_r G \times \tau_r G \to \tau_r G$$

where $*$ denotes multiplication in G. Moreover, by the right-most equality in (4.2), the map $\nabla_r : G \to \tau_r G$ is an injective group homomorphism. Hence, $\nabla_r(G)$ is a differential algebraic subgroup of $\tau_r G$.

Assume that V is a differential variety which is given as a differential subvariety of an algebraic variety X. We define the r-th jet of V to be the Zariski-closure of the image of V under the r-th nabla map $\nabla_r : X \to \tau_r X$; that is,

$$\mathrm{Jet}_r V = \overline{\nabla_r(V)}^{\mathrm{Zar}} \subseteq \tau_r X.$$

The jet sequence of V is defined as $(\mathrm{Jet}_r V : r \geq 0)$. Note that this sequence determines V, indeed

$$V = \{a \in X : \nabla_r(a) \in \mathrm{Jet}_r V \text{ for all } r \geq 0\}.$$

In the case when V is affine, say a Kolchin-closed subset of $\mathcal{U}^n$, and defined by differential polynomials of order at most r, then

$$V = \{a \in \mathcal{U}^n : \nabla_r(a) \in \mathrm{Jet}_r V\}.$$

4.6. Assumption. From now on we assume, whenever necessary for the existence of jets, that our differential varieties are given as differential subvarieties of quasi-projective algebraic varieties. Of course, in the affine case this is always the case. It is worth noting that for connected differential algebraic groups this is also true. Indeed, by [Pil97, Corollary 4.2(ii)] every such group embeds into a connected algebraic group and the latter is quasi-projective by Chevalley's theorem.

4.7. Reminder on characteristic sets. We recall some of the theory of characteristic set of prime differential ideals of the differential polynomial ring $K\{x\}$ with $x = (x_1, \ldots, x_n)$. For a detailed reference we refer the reader to [Kol73, Chapters I and IV]. Let $f \in K\{x\}$ be nonconstant. The leader of f, denoted v_f, is the highest ranking algebraic indeterminate that appears in f (according to the canonical orderly ranking of the indeterminates $\delta^\xi x_i$, as in the equivalence (4.1)). The leading degree of f, denoted d_f, is the degree of v_f in f. The rank of f, denoted $\mathrm{rk}(f)$, is the pair (v_f, d_f). The set of ranks is ordered lexicographically. The separant of f, denoted S_f, is the formal partial derivative of f with respect to v_f. The initial of f, denoted I_f, is the leading coefficient of f when viewed as a polynomial in v_f. Note that both S_f and I_f have lower rank than f. Given a finite subset $\Lambda \subseteq K\{x\} \setminus K$, we set $H_\Lambda := \prod_{f \in \Lambda} I_f S_f$.

One says that $g \in K\{x\}$ is weakly reduced with respect to $f \in K\{x\}$ if no proper derivative of v_f appears in g; if in addition the degree of v_f in g is strictly less than d_f we say that g is reduced with respect to f. A set $\Lambda \subseteq R\{x\}$ is said to be autoreduced if for any two distinct elements of Λ one is reduced with respect to the other. Autoreduced sets are always finite, and we always write them in nondecreasing order by rank. The canonical orderly ranking on autoreduced sets is defined as follows: $\{g_1, \ldots, g_r\} < \{f_1, \ldots, f_s\}$ means that either there is $i \leq r, s$ such that $\mathrm{rk}(g_j) = \mathrm{rk}(f_j)$ for $j < i$ and $\mathrm{rk}(g_i) < \mathrm{rk}(f_i)$, or $r > s$ and $\mathrm{rk}(g_j) = \mathrm{rk}(f_j)$ for $j \leq s$.

While it is not generally the case that prime differential ideals of $K\{x\}$ are finitely generated as differential ideals (though they are finitely generated as radical differential ideals), something close is true; they are determined by certain autoreduced subsets called characteristic sets. More precisely, if $P \subseteq K\{x\}$ is a prime differential ideal then a **characteristic set** Λ of P is a minimal autoreduced subset of P with respect to the ranking defined above. These minimal sets always exist, and determine the ideal P in the sense that

$$(4.3) \qquad P = \{f \in K\{x\} : H_\Lambda^\ell f \in [\Lambda] \text{ for some } \ell \geq 0\}.$$

The differential ideal on the right-hand-side is commonly denoted by $[\Lambda] : H_\Lambda^\infty$, where $[\Lambda]$ is the differential ideal generated by Λ in $K\{x\}$.

4.8. Fact. [Tre05, Proposition 2.7] *Suppose Λ is a characteristic set of a prime differential ideal $P \subseteq K\{x\}$. If $f \neq 0$ is reduced with respect Λ, then f is not in P.*

Given $I \subseteq \mathcal{U}\{x\}$, let $\mathcal{V}(I)$ denote the zeroes (as differential solutions) of the elements of I in $\mathcal{U}^n$. For a characteristic set Λ of a prime differential ideal $P \subseteq K\{x\}$, the description (4.3) implies

$$\mathcal{V}(P) \setminus \mathcal{V}(H_\Lambda) = \mathcal{V}(\Lambda) \setminus \mathcal{V}(H_\Lambda)$$

A consequence of Fact 4.8 is that $H_\Lambda \notin P$, and hence the above equality says that $\mathcal{V}(P)$ and $\mathcal{V}(\Lambda)$ agree off a proper Kolchin-closed subset, namely $\mathcal{V}(H_\Lambda)$.

We will need a bit more notation. We let $K\{x\}_{\leq r}$ denote the set of differential polynomials over K of order at most r. On the other hand, letting $(x_i^\xi : (\xi, i) \in \mathbb{N}^m \times \{1, \ldots, n\})$ be a collection of new variables, we set

$$K\{x\}_{\leq r}^{\mathrm{pol}} = K[x_i^\xi : (\xi, i) \in \Gamma_n(r)].$$

More generally, if $\mathcal{S}$ is a set of differential polynomials in $K\{x\}_{\leq r}$, we set

$$\mathcal{S}^{\mathrm{pol}} = \{f^{\mathrm{pol}} \in K\{x\}_{\leq r}^{\mathrm{pol}} : f \in \mathcal{S}\}$$

where f^{pol} denotes the polynomial obtained by replacing the variables $\delta^\xi x_i$ in f for the algebraic variables x_i^ξ. We also let $\mathcal{V}_r(\mathcal{S})$ denote the (algebraic) zero set of $\mathcal{S}^{pol}$ in $\mathcal{U}^{\alpha(n,r)}$ where recall that $\alpha(n,r) = |\Gamma_n(r)|$.

4.9. *Remark.* If V is an affine differential variety defined by the radical differential ideal $I \subseteq K\{x\}$, then for each r the jet $\mathrm{Jet}_r V$ has defining ideal given by $(I \cap K\{x\}_{\leq r})^{\mathrm{pol}}$. In other words,

$$\mathrm{Jet}_r V = \mathcal{V}_r(I \cap K\{x\}_{\leq r}).$$

We can now recall the uniform companion theory UC_m of differential fields of characteristic zero with m commuting derivations. For any set $\mathcal{S} \subseteq K\{x\}$ we let $\mathcal{S}^{(r)}$ denote the set of all $\delta^\xi f$ of order at most r with $f \in \mathcal{S}$.

4.10. **Definition.** [Tre05] A differential field K is a model of UC_m if the following condition is satisfied: for every characteristic set Λ of a prime differential ideal of $K\{x\}$, if $\mathcal{V}_r(\Lambda^{(r)}) \setminus \mathcal{V}_r(H_\Lambda) \subseteq \mathcal{U}^{\alpha(n,r)}$ has a smooth K-point for some r with $\Lambda \subseteq K\{x\}_{\leq r}$, then the differential variety

$$\mathcal{V}(\Lambda) \setminus \mathcal{V}(H_\Lambda) \subseteq \mathcal{U}^n$$

has a (differential) K-point.

4.11. *Remark.* The fact that the class of differential fields that satisfy the above condition is first-order axiomatizable in the language of differential rings is the content of [Tre05, §4]. The proof there relies heavily on Rosenfeld's Lemma which gives an algebraic characterization of characteristic sets of prime differential ideals [Kol73, Chapter IV, §9]. In Section 8 below we present an alternative (algebraic-geometric) axiomatization.

Next we prove two properties of characteristic sets of prime differential ideals that seem to be well known but to our knowledge are not explicitly stated elsewhere.

4.12. **Lemma.** *Let Λ be a characteristic set of a prime differential ideal $P \subseteq K\{x\}$. If $\Lambda \subseteq K\{x\}_{\leq r}$, then*

$$P \cap K\{x\}_{\leq r} = (\Lambda^{(r)}) : H_\Lambda^\infty$$

where $(\Lambda^{(r)})$ denotes the ideal generated by $\Lambda^{(r)}$ in $K\{x\}_{\leq r}$.

Proof. Since $P = [\Lambda] : H_\Lambda^\infty$, the containment $\supseteq$ is clear. Now let $f \in P \cap K\{x\}_{\leq r}$. Then f is weakly reduced with respect to $\Lambda^{(r)}$. By the differential division algorithm, there is g reduced with respect to $\Lambda^{(r)}$ and ℓ such that

$$H_\Lambda^\ell\, f - g \in (\Lambda^{(r)}).$$

But then, as f is in P, we get that $g \in [\Lambda] : H_\Lambda^\infty$. So, as g is also reduced with respect to Λ, Fact 4.8 implies that $g = 0$, and hence $f \in (\Lambda^{(r)}) : H_\Lambda^\infty$. $\qquad\square$

4.13. *Remark.* Suppose that V is a K-irreducible affine differential variety with corresponding prime differential ideal $P \subseteq K\{x\}$. Putting together Remark 4.9 and Lemma 4.12, we obtain that if Λ is a characteristic set of P and $\Lambda \subseteq K\{x\}_{\leq r}$, then the defining ideal of $\mathrm{Jet}_r V$ in $K\{x\}_{\leq r}^{\mathrm{pol}}$ is $((\Lambda^{(r)}) : H_\Lambda^\infty)^{\mathrm{pol}}$. As a result,

$$\mathrm{Jet}_r V \setminus \mathcal{V}_r(H_\Lambda) = \mathcal{V}_r(\Lambda^{(r)}) \setminus \mathcal{V}_r(H_\Lambda).$$

Recall that a field F is **existentially closed** (e.c. for short) in a field extension L if every algebraic variety over F with an L-point contains a F-point. The notion of e.c. for differential fields is defined similarly (in the category differential varieties).

4.14. **Proposition.** *Suppose Λ is a characteristic set of a prime differential ideal $P \subseteq K\{x\}$ and assume that $\Lambda \subseteq K\{x\}_{\leq r}$. Let $S = K\{x\}/P$ and $h = H_\Lambda/P \in S$, and let*

$$R = K\{x\}_{\leq r}/(\Lambda^{(r)}) : H_\Lambda^\infty.$$

Then, S_h is polynomial algebra over R_h. Consequently, $\mathrm{Frac}(R)$ is e.c. in $\mathrm{Frac}(S)$ as fields.

Proof. Let $\Theta(x)^{>r}$ denote the set of derivatives $\delta^\xi x_i$ of order strictly larger than r. We thus have

$$\Theta(x)^{>r} = \Theta_1(x) \cup \Theta_2(x)$$

where $\Theta_1(x)$ are elements of $\Theta^{>r}(x)$ that are not derivatives of any leader v_f with $f \in \Lambda$, and $\Theta_2(x) = \Theta(x)^{>r} \setminus \Theta_1(x)$. We write $\bar\theta(x)$ for the coset of $\theta(x)$ in S. We claim that the elements of $\bar\Theta_1(x) \subseteq S$ are algebraically independent over R_h. Indeed, if there were $\theta_1(x), \ldots, \theta_s(x) \in \Theta_1(x)$ such that $f(\bar\theta_1(x), \ldots, \bar\theta_s(x)) = 0$ for some nonzero $f \in R_h[t_1, \ldots, t_s]$, then for some ℓ we would get $H_\Lambda^\ell(x) f(\theta_1(x), \ldots, \theta_s(x)) \in P$. By the differential division algorithm, we can find g reduced with respect to Λ and ℓ' such that

$$H_\Lambda^{\ell'} f(\theta_1(x), \ldots, \theta_s(x)) - g \in [\Lambda],$$

but since Λ has order $\leq r$ and the $\theta_i(x)$'s are of order $> r$ and not a derivative a leader of Λ, we get that $g \neq 0$. But then P would contain a nonzero element, namely g, that is reduced with respect to Λ, this contradicts Fact4.8.

We now prove that all the elements of $\bar\Theta_2(x)$ are in $R_h[\bar\Theta_1(x)]$. The result follows from this, as $S_h = R_h[\bar\Theta(x)]$. Let $\theta(x) \in \Theta_2(x)$. By the differential division algorithm, there is g reduced with respect to Λ and ℓ such that $H_\Lambda^\ell \theta(x) - g \in [\Lambda]$. But then, as $g \in K\{x\}_{\leq r}[\Theta_1(x)]$, we get $\bar\theta(x) \in R_h[\bar\Theta_1(x)]$. $\square$

The above properties of characteristic sets are at the core of the proof of the Structure Theorem for differential algebras from [Tre02]. In the next section we will use the following slightly different version of this theorem.

4.15. **Theorem.** *Let B be a differential K-algebra that is differentially finitely generated and a domain. Then $B_h \cong_K A_h \otimes_K P$ where A is a domain and a finitely generated K-algebra, $h \in A$, and P is a polynomial algebra over K.*

Proof. By the assumptions, B is of the form $K\{x\}/P$ for some tuple of differential indeterminates $x = (x_1, \ldots, x_n)$ and P a prime differential ideal of $K\{x\}$. Let S, R, h and $\bar\Theta_1$ be as in the proof of Proposition 4.14, with Λ a characteristic set of P, then if we set $A = R$ and $P = K[\bar\theta_1]$ the proposition yields that $S_h \cong_K A_h \otimes P$ with the desired properties. $\square$

5. Differentially Large Fields

In the two applications of the differential Weil descent (see Sections 6 and 7), we will use several characterizations of UC_m given in Theorems 5.2 and 5.5 below. We first recall the definition of a large field, introduced by F. Pop in [Pop96].

5.1. **Definition.** A field F is said to be **large** (or *ample* in [FJ08, Rem. 16.12.3]) if every irreducible affine algebraic variety over F with a smooth F-point has a Zariski-dense set of F-points (Zariski-density is equivalent to saying that F is existentially closed in $F(V)$ as fields).

Another equivalent formulation of largeness is that F is existentially closed in the formal Laurent series field $F((t))$. Examples of large fields are pseudo algebraically closed fields, pseudo real closed fields, pseudo p-adically closed fields and the fraction field of any Henselian local ring, [Pop10]. Below, following Theorem 5.2, we will propose a notion of **differentially large**.

From now on a differential field will always mean a differential field in m commuting derivations and of characteristic 0.

5.2. **Theorem.** *Let K be a differential field and assume K is large as a pure field. Then, the following are equivalent.*

  (i) K is a model of UC_m.
  (ii) If L/K is a differential field extension such that K is e.c. in L as fields, then K is e.c. in L as differential fields.
  (iii) If V is a K-irreducible differential variety such that for infinitely many $r \geq 0$ (equivalently: for all $r \geq 0$) $\mathrm{Jet}_r(V)$ has a smooth K-point, then the set of K-rational points of V is Kolchin dense over K in V; in other words, for every proper differential subvariety $W \subseteq V$ over K there is a differential K-point in $V \setminus W$.

Further equivalent conditions may be found in 5.5, 7.2 and 8.4.

Proof. (i) $\Rightarrow$ (ii) Assume V is a K-irreducible affine differential variety over K with an L-point. We must show that it has a K-point. Without loss of generality, we may assume that V has a Kolchin-generic L-point over K, call it a (this can be achieved by replacing V with the Kolchin-locus of the given L-point over K, if necessary). Let P be the prime differential ideal of $K\{x\}$ defining V, and Λ a characteristic set of P. Let r be such that $\Lambda \subseteq K\{x\}_{\leq r}$. As a is a Kolchin-generic point of V, $\nabla_r(a)$ is a Zariski-generic L-point of $\mathrm{Jet}_r V$. By Remark 4.13, we have

$$\mathrm{Jet}_r V \setminus \mathcal{V}_r(H_\Lambda) = \mathcal{V}_r(\Lambda^{(r)}) \setminus \mathcal{V}_r(H_\Lambda),$$

and so the fact that K is e.c. in L as fields yields that $\mathcal{V}_s(\Lambda^{(r)}) \setminus \mathcal{V}_s(H_\Lambda)$ has a smooth K-point. Thus, by definition of UC_m, we get that $\mathcal{V}(\Lambda) \setminus \mathcal{V}(H_\Lambda)$ has a differential K-point. This point is also in V, as $V \setminus \mathcal{V}(H_\Lambda) = \mathcal{V}(\Lambda) \setminus \mathcal{V}(H_\Lambda)$. Note that this implication does *not* use the largeness assumption on K.

(ii) $\Rightarrow$ (iii) It suffices to consider the affine case. Let L be the differential function field of V over K; that is, L is the fraction field of $K\{x\}/P$ where P is the prime differential ideal defining V. By Remark 4.9, for any r we have that $\mathrm{Jet}_r(V) = \mathcal{V}_r(P \cap K\{x\}_{\leq r})$. By assumption, for infinitely many values of r we have that the latter has a smooth K-point; and by largeness of K, this means that K is e.c. in the function field of $\mathcal{V}_r(P \cap K\{x\}_{\leq r})$ as fields. As r can be taken to be arbitrarily large, we get that K is e.c. in L as fields. Hence, it is also e.c. as differential fields. As $V \setminus W$ contains an L-point, it also contains a K-point.

(iii) $\Rightarrow$ (i) Let Λ be a characteristic set of a prime differential ideal $P \subseteq K\{x\}$ such that $\mathcal{V}_r(\Lambda^{(r)}) \setminus \mathcal{V}_r(H_\Lambda)$ has a smooth K-point for some r with $\Lambda \subseteq K\{x\}_{\leq r}$. Let V be the K-irreducible affine differential variety defined by the prime differential

ideal P. Let $W = H_\Lambda$. We must show that $V \setminus W$ has a K-point (as the latter equals $\mathcal{V}(\Lambda) \setminus \mathcal{V}(H_\Lambda)$). So, it suffices to show that for all s the jet $\mathrm{Jet}_s(V)$ has a smooth K-point.

Let L be the fraction field of $K\{x\}/P$. We will first show that K is e.c. in L as fields. Let F be the fraction field of

$$R = K\{x\}_{\leq r}/(\Lambda^{(r)}) : H^\infty.$$

Then, as $\mathcal{V}_r(\Lambda) \setminus \mathcal{V}_r(H_\Lambda)$ has a smooth K-point and K is large, we have that K is e.c. in F (as fields). By Proposition 4.14, $(K\{x\}/P)_h$ with $h = H_\Lambda/P$ is a polynomial algebra over R_h, and so F is e.c. in L. This shows that K is e.c. in L as fields. As L contains a Kolchin-generic point of V, namely $a := x/P$, for each $s \geq 0$ we have that L contains a Zariski-generic point of $\mathrm{Jet}_s V$, namely $\nabla_s(a)$. It follows that $\mathrm{Jet}_s V$ has a smooth K-point for all s. $\qquad\square$

We define differentially large fields as follows:

5.3. **Definition.** A differential field K is said to be **differentially large** if it is large as a field and satisfies any of the equivalent conditions of Theorem 5.2.

5.4. *Remark.* The class of differentially large fields is first-order axiomatizable in the language of differential rings (with m derivations). Indeed, the class of large is axiomatizable and by Remark 4.11 the axioms of UC_m are first-order.

Using Theorem 4.15 we get the following algebraic characterization of UC_m:

5.5. **Theorem.** *Assume that the differential field K is large as a field. The following are equivalent.*

(i) *K is differentially large.*

(ii) *For every differentially finitely generated K-algebra S the following condition holds:*

>   *if S is a domain and $S \cong_K A \otimes_K P$ with K-algebras A, P such that*
>   >   *(a) A is a finitely generated K-algebra and a domain,*
>   >   *(b) P is a polynomial algebra over K, and,*
>   >   *(c) there is a smooth K-rational point $A \longrightarrow K$,* [6]
>   *then there is a differential K-rational point $S \longrightarrow K$.*

Proof. (i)$\Rightarrow$(ii) Assume $K \models \mathrm{UC}_m$ and we know the if-condition of (ii). Because there is a K-rational point $A \to K$ and K is large, K is existentially closed in $\mathrm{Frac}(A)$. Moreover, as S is a polynomial algebra over A, $\mathrm{Frac}(A)$ is e.c. in $\mathrm{Frac}(S)$ as fields, and hence K is e.c. in $\mathrm{Frac}(S)$ as fields. By Theorem 5.2, K is e.c. in $\mathrm{Frac}(S)$ also as differential fields. Thus, there is a differential K-rational point $S \to K$.

(ii)$\Rightarrow$(i) Assume condition (ii). Let Λ be a characteristic set of a prime differential ideal $P \subseteq K\{x\}$ such that $\mathcal{V}_r(\Lambda^{(r)}) \setminus \mathcal{V}_r(H_\Lambda)$ has a smooth K-point for some r with $\Lambda \subseteq K\{x\}_{\leq r}$. Let $S = (K\{x\}/P)_h$ where $h = H_\Lambda/P$, and let

$$A = \left(K\{x\}_{\leq r}/(\Lambda^{(r)}) : H_\Lambda^\infty \right)_h.$$

---

[6]Since K is assumed to be large as a field, condition (c) is equivalent to saying that for all $h \in A \setminus \{0\}$ there is a K-rational point $A_h \longrightarrow K$.

By Proposition 4.14 and the proof of Theorem 4.15, $S \cong_K A \otimes_K P$ for a polynomial K-algebra P. The given smooth K-point of $\mathcal{V}_r(\Lambda^{(r)}) \setminus \mathcal{V}_r(H_\Lambda)$ is a smooth K-rational point of $A \to K$. We thus get all the conditions in the hypothesis of (ii). This yields a differential K-rational point $S \to K$. This point lives in $V(\Lambda) \setminus \mathcal{V}(H_\Lambda)$ as desired. Note that this implication does *not* use the largeness assumption.

$\square$

For existence of differentially large fields we refer to [Tre05, Theorem 6.2 (II)]:

5.6. Theorem. *Let K be a differential field that is large as a field. Then there is differential field extension L of K such that L is differentially large and such that L as a pure field is an elementary extension of the field K.* $\square$

6. Algebraic Extensions of Differentially Large Fields and Minimal Differential Closures

We now prove that algebraic extensions of differentially large fields are again differentially large. This implies that the algebraic closure of a differentially large field is a model of $\mathrm{DCF}_{0,m}$. This yields new examples of differential fields with minimal differential closures. We carry on the notation and conventions from the previous section.

6.1. Theorem. *If K is differentially large, then so is every algebraic extension (equipped with the induced derivations).*

Proof. Let L/K be an algebraic extension. We must show that L is differentially large. By Theorem 5.2(ii), we may assume that L/K is finite. Now let us assume the hypothesis and notation of part (ii) of Theorem 5.5 with L in place of K. We must show that there is a differential L-rational point $S \to L$. Consider the differential Weil descent $S' = W^{\mathrm{diff}}(S)$ from L to K. Then, as the classical Weil descent commutes with tensors (cf. 2.1(v)), $S' \cong W(A) \otimes W(P)$ as K-algebras. Because Weil descent preserves smooth points (cf. [Oes84, Appendix 2]), we get a smooth K-rational point $W(A) \to K$. Thus, as K is differentially large, Theorem 5.5 yields a differential K-rational point $W^{\mathrm{diff}}(S) \to K$. By 3.4, we get a differential L-rational point $S \to L$, as desired. $\square$

6.2. Corollary. *The algebraic closure of a differentially large field is differentially closed. In particular, if $K \models \mathrm{CODF}_m$, the theory of closed ordered differential fields in m commuting derivations, then $K(i) \models \mathrm{DCF}_{0,m}$.*

Previously known examples of differential fields with minimal differential closures are models of CODF (which we denote as CODF_1), see [Sin78a], and fixed fields of models of $\mathrm{DCF}_{0,m}$ A, the theory differentially closed fields with a generic differential automorphism, see [LS16]. The corollary delivers a vast variety of new differential fields with this property, namely all differentially large fields, see also 5.6. A further application will be given in the next section.

7. Kolchin-Denseness of Rational Points in Differential Algebraic Groups

We present an application to a rationality question in differential algebraic groups. Note that if F is a large field, then it follows that for any connected algebraic group G over F the set of F-rational points of G is Zariski-dense. Indeed, G is smooth

as an algebraic variety and the identity $e \in G$ is a K-rational point, so largeness implies the Zariski-density. In this section, our goal is to prove

7.1. Theorem. *Assume K is differentially large. If G is a connected differential algebraic group over K, then the set of K-rational points of G, denoted $G(K)$, is Kolchin-dense in $G = G(\mathcal{U})$.*

Theorem 7.1 will follow from the next proposition, which states a further characterization of differential largeness (compare with 5.2(iii)).

7.2. Proposition. *Let K be a differential field and assume K is large as a field. Then K is differentially large if and only if the following condition holds:*

If V is a K-irreducible differential variety such that for infinitely many values of r the jet $\mathrm{Jet}_r V$ has a smooth K-point, then the set of K-rational points of V is Kolchin-dense in V.

Proof. The condition obviously implies 5.2(iii). Conversely suppose K is differentially large. By Corollary 6.2, K^{alg} is a model of $\mathrm{DCF}_{0,m}$, and so it suffices to show that $V(K)$ is dense in V with respect to the Kolchin topology over K^{alg}.

<u>Claim.</u> V is geometrically irreducible (i.e., K^{alg}-irreducible) as a differential variety. *Proof.* Since V is K-irreducible as a differential variety, for all r the jet $\mathrm{Jet}_r V$ is K-irreducible[7]. By assumption, there are infinitely many values of r for which $\mathrm{Jet}_r V$ has a smooth K-point; and so, for all these r the jet $\mathrm{Jet}_r V$ is geometrically irreducible. Now the defining ideal of the jet $\mathrm{Jet}_r V$ over K^{alg} is

$$(I \cap K^{\mathrm{alg}}\{x\}_{\leq r})^{\mathrm{pol}}$$

where I is the defining differential ideal of V over K^{alg} (see Remark 4.9). This implies that the ideal I is prime, hence that V is K^{alg}-irreducible.   ◇

We now show that $V(K)$ is dense in V with respect to the Kolchin topology over K^{alg}. Let Y be a proper differential subvariety of V over K^{alg}. Let W be the Kolchin closure of Y over K. Then Y is a geometric component of W. Since V is geometrically irreducible by the claim, W is a proper differential subvariety of V. By 5.2(iii), there is a K-point in $V \setminus W \subseteq V \setminus Y$ as required.   □

We conclude with the proof of Theorem 7.1.

Proof of Theorem 7.1. By Proposition 7.2, it suffices to show that for infinitely values of r the jet $\mathrm{Jet}_r G$ has a smooth K-rational point. By [Pil97, Corollary 4.2(ii)], G embeds over K into a connected algebraic group H defined over K. As we saw in Section 4, for each r, $\nabla_r G$ is a differential algebraic subgroup of $\tau_r H$. As a result, $\mathrm{Jet}_r G$ is an algebraic subgroup of $\tau_r H$, and so $\mathrm{Jet}_r G$ is smooth. If e denotes the identity of G, which is a K-point, then, for each r, the K-point $\nabla_r(e)$ is a smooth point of $\mathrm{Jet}_r G$.   □

7.3. Remark. If G is a connected linear algebraic group over a field F of characteristic zero (with no largeness assumptions), the Unirationality Theorem implies that the F-rational points of G are Zariski-dense. It would be interesting to study the analogous question for linear *differential* algebraic groups. We have not pursued this in this note.

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[7]K-irreducibility in the Kolchin sense is equivalent to K-irreducibility in the Zariski by Kolchin's irreducibility theorem [Kol73, Ch. IV,§17,Prop. 10, p. 200].

8. Algebraic-Geometric Axiomatization of Large Differential Fields

In this last section we present algebraic-geometric axioms for differentially large fields in the spirit of Pierce-Pillay [PP98]. The presentation follows the recent algebraic-geometric axiomatization of $\mathrm{DCF}_{0,m}$ established in [LS18]. In particular, we will use the recently developed theory of differential kernels for fields with several commuting derivations from [GL16]. One significant difference with the arguments in [LS18] is that there one only requires the existence of regular realizations of differential kernels, while here we need the existence of principal realizations, see Remark 8.1 and Fact 8.2. We carry on the notation and conventions from previous sections.

We will use two different orders $\leq$ and $\trianglelefteq$ on $\mathbb{N}^m \times \{1, \ldots, n\}$. Given two elements (ξ, i) and (τ, j) of $\mathbb{N}^m \times \{1, \ldots, n\}$, we set $(\xi, i) \leq (\tau, j)$ if and only if $i = j$ and $\xi \leq \tau$ in the product order of $\mathbb{N}^m$. On the other hand, we set $(\xi, i) \trianglelefteq (\tau, j)$ if and only if

$$\left(\sum \xi_k, i, \xi_1, \ldots, \xi_m \right) \text{ is less than or equal to } \left(\sum \tau_k, j, \tau_1, \ldots, \tau_m \right)$$

in the (left-)lexicographic order. Note that if $x = (x_1, \ldots, x_n)$ are differential indeterminates and we identify (ξ, i) with $\delta^\xi x_i := \delta_1^{\xi_1} \cdots \delta_m^{\xi_m} x_i$, then $\leq$ induces an order on the set of algebraic indeterminates given by $\delta^\xi x_i \leq \delta^\tau x_j$ if and only if $\delta^\tau x_j$ is a derivative of $\delta^\xi x_i$ (in particular this implies that $i = j$). On the other hand, the ordering $\trianglelefteq$ induces the canonical orderly ranking on the set of algebraic indeterminates.

We will look at field extensions of K of the form

(8.1) $$L := K(a_i^\xi : (\xi, i) \in \Gamma_n(r))$$

for some fixed $r \geq 0$. Here we use a_i^ξ as a way to index the generators of L over K. The element $(\tau, j) \in \mathbb{N}^m \times \{1, \ldots, n\}$ is said to be a leader of L if there is $\eta \in \mathbb{N}^m$ with $\eta \leq \tau$ and $\sum \eta_k \leq r$ such that a_j^η is algebraic over $K(a_i^\xi : (\xi, i) \triangleleft (\eta, j))$, and a leader (τ, j) is a minimal leader of L if there is no leader (ξ, i) with $(\xi, i) < (\tau, j)$. We note that the notions of leader and minimal leader make sense even when we allow $r = \infty$.

A (differential) kernel of length r over K is a field extension of the form

$$L = K(a_i^\xi : (\xi, i) \in \Gamma_n(r))$$

such that there exist derivations

$$D_k : K(a_i^\xi : (\xi, i) \in \Gamma_n(r - 1)) \to L$$

for $k = 1, \ldots, m$ extending δ_k and $D_k a_i^\xi = a_i^{\xi + \mathbf{k}}$ for all $(\xi, i) \in \Gamma_n(r - 1)$, where $\mathbf{k}$ denotes the m-tuple whose k-th entry is one and zeroes elsewhere.

Given a kernel $(L, D_1, \ldots, D_k)$ of length r, we say that it has a prolongation of length $s \geq r$ if there is a kernel $(L', D_1', \ldots, D_k')$ of length s over K such that L' is a field extension of L and each D_k' extends D_k. We say that $(L, D_1, \ldots, D_k)$ has a regular realization if there is a differential field extension $(M, \Delta' = \{\delta_1', \ldots, \delta_m'\})$ of $(K, \Delta = \{\delta_1, \ldots, \delta_m\})$ such that M is a field extension of L and $\delta_k' a_i^\xi = a_i^{\xi + \mathbf{k}}$ for all $(\xi, i) \in \Gamma_n(r - 1)$ and $k = 1, \ldots, m$. In this case we say that $g := (a_1^\mathbf{0}, \ldots, a_n^\mathbf{0})$ is a regular realization of L. If in addition the minimal leaders of L and those of the differential field $K\langle g \rangle$ coincide we say that g is a principal realization of L.

8.1. *Remark.* Note that if g is a principal realization of the differential kernel L, then L is existentially closed in $K\langle g\rangle$ as fields. Indeed, since the minimal leaders of L and $K\langle g\rangle$ coincide, for every $(\xi, i) \in \mathbb{N}^m \times \{1, \ldots, n\}$ we have that either $\delta^\xi g_i$ is in L or it is algebraically independent from $K(\delta^\eta g_j : (\eta, j) \lhd (\xi, i))$. In other words, the differential ring generated by g over L, namely $L\{g\}$, is a polynomial ring over L. The claim follows.

In general, it is not the case that every kernel has a principal realization (not even regular). In [GL16], an upper bound $C_{r,m}^n$ was obtained for the length of a prolongation of a kernel that guarantees the existence of a principal realization. This bound depends only on the data (r, m, n) and is constructed recursively as follows:

$$C_{0,m}^1 = 0, \quad C_{r,m}^1 = A(m-1, C_{r-1,m}^1), \quad \text{and} \quad C_{r,m}^n = C_{C_{r,m}^{n-1}, m}^1,$$

where $A(x, y)$ is the Ackermann function. For example,

$$C_{r,1}^n = r, \quad C_{r,2}^n = 2^n r \quad \text{and} \quad C_{r,3}^1 = 3(2^r - 1).$$

In [GL16, Theorem 18], it is proved that

8.2. **Fact.** *If a differential kernel $L = K(a_i^\xi : (\xi, i) \in \Gamma_n(r))$ of length r has a prolongation of length $C_{r,m}^n$, then there is $r \le h \le C_{r,m}^n$ such that the differential kernel $K(a_i^\xi : (\xi, i) \in \Gamma_n(h))$ has a principal realization.*

8.3. *Remark.* Note that in the ordinary case $\Delta = \{\delta\}$ (i.e., $m = 1$), we have $C_{r,1}^n = r$ by definition, and so the fact above shows that in this case every differential kernel has a principal realization (this is a classical result of Lando [Lan70]).

The fact above is the key to our algebraic-geometric axiomatization of differential largeness. But first we need some additional notation. For a given positive integer n, we let

$$\alpha(n) = n \cdot \binom{C_{1,m}^n + m}{m} \quad \text{and} \quad \beta(n) = n \cdot \binom{C_{1,m}^n - 1 + m}{m}.$$

We write $\pi : \mathcal{U}^{\alpha(n)} \to \mathcal{U}^{\beta(n)}$ for the projection onto the first $\beta(n)$ coordinates; i.e., setting $(x_i^\xi)_{(\xi,i)\in\Gamma_n(C_{1,m}^n)}$ to be coordinates for $\mathcal{U}^{\alpha(n)}$ then π is the map

$$(x_i^\xi)_{(\xi,i)\in\Gamma_n(C_{1,m}^n)} \mapsto (x_i^\xi)_{(\xi,i)\in\Gamma(C_{1,m}^n-1)}.$$

It is worth noting here that $\alpha(n) = |\Gamma_n(C_{1,m}^n)|$ and $\beta(n) = |\Gamma_n(C_{1,m}^n - 1)|$. We also use the projection $\psi : \mathcal{U}^{\alpha(n)} \to \mathcal{U}^{n\cdot(m+1)}$ onto the first $n \cdot (m+1)$ coordinates, that is,

$$(x_i^\xi)_{(\xi,i)\in\Gamma_n(C_{1,m}^n)} \mapsto (x_i^\xi)_{(\xi,i)\in\Gamma_n(1)}.$$

Finally, we use the embedding $\varphi : \mathcal{U}^{\alpha(n)} \to \mathcal{U}^{\beta(n)\cdot(m+1)}$ given by

$$(x_i^\xi)_{(\xi,i)\in\Gamma_n(C_{1,m}^n)} \mapsto \left((x_i^\xi)_{(\xi,i)\in\Gamma_n(C_{1,m}^n-1)}, (x_i^{\xi+\mathbf{1}})_{(\xi,i)\in\Gamma_n(C_{1,m}^n-1)}, \cdots\right.$$

$$\left.\ldots, (x_i^{\xi+\mathbf{m}})_{(\xi,i)\in\Gamma_n(C_{1,m}^n-1)}\right).$$

Recall from Section 4 that, given a Zariski-constructible set X of $\mathcal{U}^n$, the first-prolongation of X is denoted by $\tau X = \tau_1 X \subseteq \mathcal{U}^{n(m+1)}$. For the first-prolongation

it is easy to give the defining equations: $\tau(X)$ is the Zariski-constructible set given by the conditions

$$x \in X, \quad \text{and} \quad \sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(x) \cdot y_{i,k} + f_j^{\delta_k}(x) = 0 \text{ for } 1 \le j \le s, \ 1 \le k \le m$$

where $f_1, \ldots, f_s$ are generators of the ideal of polynomials over $\mathcal{U}$ vanishing at X, and each $f_j^{\delta_k}$ is obtained by applying δ_k to the coefficients of f_j. Note that $(a, \delta_1 a, \ldots, \delta_m a) \in \tau X$ for all $a \in X$. Further, if X is defined over the differential field K then so is τX.

8.4. **Theorem.** *Assume K is a differential field that is large as a field. Then, K is differentially large if and only*

($\diamondsuit$) *for every K-irreducible Zariski-closed set W of $\mathcal{U}^{\alpha(n)}$ with a smooth K-point such that $\varphi(W) \subseteq \tau(\pi(W))$, there is $a \in K^n$ with $(a, \delta_1 a, \ldots, \delta_m a) \in \psi(W)$.*

Proof. We will use the fact that a large and differential field K is differentially large just if K is existentially closed in every differential field extension L in which K is existentially closed as a field (see Theorem 5.2). The proof follows the strategy of [LS18], but here regular realizations are replaced by principal realizations with the appropriate adaptations. As the set up is technically somewhat intricate we give details.

Assume K is differentially large. Let W be as in condition ($\diamondsuit$), we must find a point $a \in K^n$ such that $(a, \delta_1 a, \ldots, \delta_m a) \in \psi(W)$. Let $b = (b_i^\xi)_{(\xi,i) \in \Gamma_n(C_{1,m}^n)}$ be a Zariski-generic point of W over K. Then $(b_i^\xi)_{(\xi,i) \in \Gamma_n(C_{1,m}^n - 1)}$ is a Zariski-generic point of $\pi(W)$ over K, and

$$\varphi(b) = \left((b_i^\xi)_{(\xi,i) \in \Gamma_n(C_{1,m}^n - 1)}, (b_i^{\xi+\mathbf{1}})_{(\xi,i) \in \Gamma_n(C_{1,m}^n - 1)}, \ldots, (b_i^{\xi+\mathbf{m}})_{(\xi,i) \in \Gamma_n(C_{1,m}^n - 1)} \right)$$
$$\in \tau(\pi(W))$$

By the standard argument for extending derivations (see [Lan02, Chapter 7, Theorem 5.1], for instance), there are derivations

$$D_k' : K(b_i^\xi : (\xi,i) \in \Gamma_n(C_{1,m}^n - 1)) \to K(b_i^\xi : (\xi,i) \in \Gamma_n(C_{1,m}^n))$$

for $k = 1, \ldots, m$ extending δ_k and such that $D_k' b_i^\xi = b_i^{\xi+\mathbf{k}}$ for all $(\xi,i) \in \Gamma_n(C_{1,m}^n - 1)$. Thus, $L' = K(b_i^\xi : (\xi,i) \in \Gamma_n(C_{1,m}^n))$ is a differential kernel over K and, moreover, it is a prolongation of length $C_{1,m}^n$ of the differential kernel $L = K(b_i^\xi : (\xi,i) \in \Gamma_n(1))$ of length 1 with $D_k = D_k'|_L$. By Fact 8.2, there is $r \le h \le C_{1,m}^n$ such that $L'' = K(b_i^\xi : (\xi,i) \in \Gamma_n(h))$ has a principal realization; in particular, there is a differential field extension (M, Δ') of (K, Δ) containing L'' such that $\delta_k' b^{\mathbf{0}} = b^{\mathbf{k}}$, where $b^{\mathbf{0}} = (b_1^{\mathbf{0}}, \ldots, b_n^{\mathbf{0}})$ and similarly for $b^{\mathbf{k}}$. Then

(8.2) $$(b^{\mathbf{0}}, \delta_1' b^{\mathbf{0}}, \ldots, \delta_m' b^{\mathbf{0}}) \in \psi(W)$$

Now, since W has a smooth K-point and K is large, K is e.c. in L' as fields; in particular, K is e.c. in L'' as fields. By Remark 8.1, L'' is e.c. in the differential field $K\langle b^{\mathbf{0}}\rangle$ as fields, and so K is e.c. in $K\langle b^{\mathbf{0}}\rangle$ as fields. Since K is differentially large, the latter implies that K is e.c. in $K\langle b^{\mathbf{0}}\rangle$ as differential fields as well; and so, by (8.2), we can find the desired point a in K^n.

For the converse, assume K is e.c. as field in a differential field extension F. We must show that K is also e.c. in F as differential field. Let $\rho(x)$ be a quantifier-free

formula over K (in the language of differential rings with m derivations) in variables $x = (x_1, \ldots, x_t)$ with a realization c in F. We may write

$$\rho(x) = \gamma(\delta^\xi x_i : (\xi, i) \in \Gamma_t(r))$$

where $\gamma((x^\xi)_{(\xi,i)\in\Gamma_t(r)})$ is a quantifier-free formula in the language of rings over K for some r. If $r = 0$, then ρ is a formula in the language of rings, and so $\rho(x)$ has a realization in K since K is e.c. in F as a field. Now assume $r > 0$. Let $n := t \cdot \binom{r-1+m}{m}$, $d := (\delta^\xi c_i)_{(\xi,i)\in\Gamma_t(r-1)}$, and

$$W := \text{Zar-loc}_K(\delta^\xi d_i : (\xi, i) \in \Gamma_n(C_{1,m}^n)) \subseteq \mathcal{U}^{\alpha(n)}.$$

We have that $\varphi(W) \subseteq \tau(\pi(W))$. Moreover, since W has a smooth F-point (namely $(\delta^\xi d_i)_{(\xi,i)\in\Gamma_n(C_{1,m}^n)}$) and K is e.c. in F as fields, W has a smooth K-point. By $(\diamondsuit)$, there is $a = (a_i^\xi)_{(\xi,i)\in\Gamma_t(r-1)} \in K^n$ such that $(a, \delta_1 a, \ldots, \delta_m a) \in \psi(W)$. This implies that $a_i^\xi = \delta^\xi a_i^{\mathbf{0}}$ for all $(\xi, i) \in \Gamma_t(r-1)$. Thus,

$$(\delta^\xi a_i^{\mathbf{0}})_{(\xi,i)\in\Gamma_t(r)} \in \text{Zar-loc}_K((\delta^\xi c_i)_{(\xi,i)\in\Gamma_t(r)}) \subseteq \mathcal{U}^{t\cdot\binom{r+m}{m}},$$

and so, since $(\delta^\xi c_i)_{(\xi,i)\in\Gamma_t(r)}$ realizes γ, the point $(\delta^\xi a_i^{\mathbf{0}})_{(\xi,i)\in\Gamma_t(r)}$ also realizes γ. Consequently, $K \models \rho(a^{\mathbf{0}})$, as desired. $\qquad\square$

It is worth noting that in the ordinary case ($m = 1$) we get the values $\alpha(n) = 2n$ and $\beta(n) = n$. Also, in this case, $\pi : \mathcal{U}^{2n} \to \mathcal{U}^n$ is just the projection onto the first n coordinates, and $\psi, \varphi : \mathcal{U}^{2n} \to \mathcal{U}^{2n}$ are both the identity map. We thus get the following

8.5. **Corollary.** *Assume that (K, δ) is an ordinary differential field of characteristic zero which is large as a field. Then, (K, δ) is differentially large if and only*

$(\diamondsuit')$ *for every K-irreducible Zariski-closed set W of $\mathcal{U}^{2n}$ with a smooth K-point such that $W \subseteq \tau_\delta(\pi(W))$, there is $a \in K^n$ with $(a, \delta a) \in W$.*

8.6. *Remark.*

(i) If K is algebraically closed of characteristic 0, then Corollary 8.5 yields the classical algebraic-geometric axiomatization of DCF_0 given by Pierce and Pillay in [PP98].

(ii) If K has a model complete theory T in the language of fields and if K is large, then Corollary 8.5 yields a slight variation of the geometric axiomatization of T_D given by Brouette, Cousins, Pillay and Point in [BCPP17, Lemma 1.6].

(iii) For large and topological fields with a single derivation, an alternative description of differentially large fields with reference to the topology may be found in [GR06].

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