# Non-Hausdorff Topology in Algebra and Logic
# – a short course –

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# Rough outline of the course

1. Stone duality for distributive lattices
2. Topology in logic and algebra
3. Not prepared yet. Could be
   (a) The domain theoretic view, or
   (b) model theory, or
   (c) a case study.

**Course notes (for session 1)**:

Chapters 1-3 of my book on *Spectral Spaces*. New Mathematical Monographs, vol. 35. Cambridge University Press. March 2019. Available at

https://personalpages.manchester.ac.uk/staff/Marcus.Tressl/tacl.pdf

# Preamble: Two perspectives of topology and lattices

There are two perspectives one can have when comparing topology and (distributive) lattice theory. The functors in the following diagram will be addressed in the course:

$$\mathtt{Top} \xrightarrow[\text{adjunction}]{X \mapsto \mathcal{O}(X)} \mathtt{Frm} \xhookrightarrow[\text{embedding}]{F \mapsto F} \mathtt{BDLat} \xrightarrow[\text{equivalence}]{L \mapsto \mathrm{Spec}(L)} \mathtt{Spec} \xhookrightarrow[\text{embedding}]{X \mapsto X} \mathtt{Top}$$

One may read these functors from the left to the right and consider them as generalizations, a point is made here in [Joh83]. In this course we often read the diagram from the right to the left. In applications one benefits most by taking both approaches on board. Note that both points of views are starting with the category of topological spaces.

# 1. Distributive lattices

Definition A **distributive lattice** in this course is a partially ordered set $L = (L, \leq)$ with the following properties:

DL1 For all $a, b \in L$ the supremum (aka **join**) of $\{a, b\}$ for the partial order $\leq$ exists and is denoted by $a \vee b$.

DL2 For all $a, b \in L$ the infimum (aka **meet**) of $\{a, b\}$ for the partial order $\leq$ exists and is denoted by $a \wedge b$.

We view the operations $\wedge$, $\vee$ as functions $L \times L \longrightarrow L$. Both operations are commutative and associative as follows immediately from their definitions.

DL3 *Distributivity law for $\wedge$ and $\vee$:*
For all $a, b, c \in L$ we have $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

DL4 There is a smallest element (aka **bottom**) for $\leq$, denoted by 0.
There is a largest element (aka **top**) for $\leq$, denote by 1.

Principal example: Any subset $L$ of the powerset of a set $S$ that is closed under finite unions and intersections and with $\emptyset, S \in L$. Such lattices are called **lattices of subsets** (of $S$).

# 1. Distributive lattices: Morphisms

Definition

A **morphism** of a distributive lattice $L$ is a map $\varphi : L \longrightarrow L$ preserving $\wedge, \vee$ and $0, 1$.

Since $x \leq y \iff x \wedge y = x$, each such map is then monotone.

# 1. Distributive lattices: Ideals

Definition An ideal of a distributive lattice $L$ is a nonempty **down-set** $I$ of $L$ satisfying $x, y \in I \Rightarrow x \vee y \in I$.

An ideal is called

- **proper**, if $I \neq L$, i.e.: $1 \notin I$.
- **principal**, if there is some $a \in L$ with $I = a^{\downarrow} := \{b \in I \mid b \leq a\}$.

Equivalent description For a nonempty subset $I$ of $L$, the following conditions are equivalent:

1. $I$ is an ideal.
2. $I$ satisfies $x \vee y \in I \iff x \in I$ and $y \in I$.
3. $I$ is the kernel of a lattice morphism $\varphi : L \longrightarrow M$ for some lattice $M$, hence $I = \mathrm{Ker}(\varphi) := \{a \in L \mid \varphi(a) = 0\}$.

Generation of ideals. If $S \subseteq L$ then the **ideal generated by** $S$ in $L$ is

$$\mathfrak{i}(S) = \{b \in L \mid b \leq \bigvee F \text{ for some finite set } F \subseteq S\}.$$

$\curvearrowright$

# 2. Prime ideals

Definition A **prime ideal** of $L$ is a **proper** ideal $P$ of $L$ satisfying

$$x \wedge y \in P \Rightarrow x \in P \text{ or } y \in P.$$

Equivalent description For a proper ideal $P$ of $L$ TFAE:

1. $P$ is prime.
2. $P$ satisfies $x \wedge y \in I \iff x \in I$ or $y \in I$.
3. $I$ is the kernel of a lattice morphism $\varphi : L \longrightarrow 2$, where 2 stands for the 2-element lattice $\{0, 1\}$ in the order $0 < 1$.

Important example. If $L$ is a lattice of subsets of $S$ and $s \in S$, then $P := \{U \in L \mid s \notin U\}$ is a prime ideal of $L$, because

- $P \neq L$ as $S = 1 \notin P$.
- $P$ satisfies $s \notin U \cup V \iff s \notin U$ and $s \notin V$.
- $P$ satisfies $s \notin U \cap V \iff s \notin U$ or $s \notin V$.

# 3. The spectrum

Let $L$ be a distributive lattice.

Definition The **spectrum** of $L$ is defined (as a set) by

$$\mathrm{Spec}(L) = \{P \subseteq L \mid P \text{ is a prime ideal}\}$$

For $a \in L$ we define $\quad D(a) = \{P \in \mathrm{Spec}(L) \mid a \notin P\}$.

Hence $D(a)$ is a subset of $\mathrm{Spec}(L)$, which should be thought of as an open 'neighborhood' of $P$.

In the example of a lattice of subsets $L$ of $S$ and the prime ideal $P = \{U \in L \mid s \notin U\}$ from above we have for any $U \in L$:

$$P \in D(U) \iff U \notin P \iff s \in U$$

Hence $P$ here corresponds to the point $s$ and $P \in D(U)$ resembles $U$ as a set containing $s$.

The setup looks odd as there is double negation involved. However the setup will pay off in many applications.

$\curvearrowright$

# 4. Reminder of session 1

Definition Hochster, [Hoc69]

A space $X$ is called **spectral** if it satisfies all of the following conditions.

S1 $X$ is quasi-compact and $T_0$.
Recall that $\mathrm{T}_0$ means: any two points are separated by an open set.

S2 $\mathring{\mathcal{K}}(X)$ $(:= \{U \subseteq X \mid U \text{ is quasi-compact open}\})$ is a basis of $X$.

S3 $\mathring{\mathcal{K}}(X)$ is closed under finite intersections.

S4 $X$ is sober (i.e. every nonempty closed and irreducible subset is the closure of a point).

Notation: For a subset $S$ of any space $X$ we write
$\mathrm{Gen(S)} = \{x \in X \mid \exists s \in S : x \rightsquigarrow s\}$ for the set of **generalizations** of $S$.

Exercise Every finite $\mathrm{T}_0$-space is spectral.

# 4. Reminder of session 1

Let $L$ be a distributive lattice (bounded always). We have seen that $\mathrm{Spec}(L)$ is a spectral space, more concretely:

(i) $\mathring{\mathcal{K}}(\mathrm{Spec}(L)) = \{D(a) \mid a \in L\}$, $D(a) := \{P \in \mathrm{Spec}(L) \mid a \notin P\}$.

(ii) The closed subsets of $\mathrm{Spec}(L)$ are precisely those of the form $V(I) := \{P \in \mathrm{Spec}(L) \mid I \subseteq P\}$, where $I$ is an ideal of $L$.

(iii) The nonempty closed and irreducible subsets of $\mathrm{Spec}(L)$ are precisely the sets of the form $\overline{\{P\}}$ $(= V(P))$ for some point $P \in \mathrm{Spec}(L)$.

Stone representation of a distributive lattice.
The map

$$\mathcal{D}_L : L \longrightarrow \mathring{\mathcal{K}}(\mathrm{Spec}(L)),\ \mathcal{D}_L(a) = \{P \mid a \notin P\}$$

is a poset isomorphism. Core ingredient: Krull's Ideal Theorem (done on the board, see [Tre26b, p. 2.2.10] for the version in the language of filters), saying that any ideal $I$ of $L$ that is maximal (for inclusion) among ideals not containing a given $a \in L$, is prime.

# 5. Stone duality

Representation theorem for spectral spaces

Let $X$ be a spectral space. Then $\mathring{\mathcal{K}}(X)$ is a distributive lattice and the map

$$\Theta_X : X \longrightarrow \mathrm{Spec}(\mathring{\mathcal{K}}(X)),\ x \mapsto \{U \in \mathring{\mathcal{K}}(X) \mid x \notin U\}$$

is a homeomorphism with $\Theta_X^{-1}(D(U)) = U$.

Proof: That $\mathring{\mathcal{K}}(X)$ is a distributive lattice follows directly from the axioms S1-S3 of spectral spaces. The assertion then is straightforward except for surjectivity of $\Theta_X$. To see this one proceeds as follows. Take a prime ideal $P$ of $\mathring{\mathcal{K}}(X)$. Then define $A = \bigcap\{X \setminus V \mid V \in \mathring{\mathcal{K}}(X), P \notin V\}$. One can then verify that $A$ is nonempty, closed and irreducible, hence $A = \overline{\{x\}}$ for some (unique by the $T_0$-assumption) $x \in A$. This $x$ satisfies $\Theta_X(x) = P$. $\square$

Example Consider the chain $\omega + 1 := \{0, 1, 2, 3, \ldots\} \cup \{\infty\})$ and let $\tau$ be the Alexandroff topology $\{U \subseteq \omega + 1 \mid U = U^{\downarrow}\}$ of $\omega + 1$.

Then $\omega + 1$ is spectral with $\mathring{\mathcal{K}}(\omega + 1) \cong \omega + 1$, [DST19, 1.6.6 and 3.6].

# 5. Stone duality - spectral maps

Let $\varphi : L \longrightarrow M$ be a homomorphism of lattices. Then the map

$$\mathrm{Spec}(\varphi) : \mathrm{Spec}(M) \longrightarrow \mathrm{Spec}(L),\ \ Q \longmapsto \varphi^{-1}(Q)$$

is continuous and satisfies $\mathrm{Spec}(\varphi)^{-1}(D(a)) = D(\varphi(a))$ for all $a \in L$.

This is verified straightforwardly. Crucially, this implies that **not** every continuous map between spectral spaces comes from a lattice morphism (For instance in the example above consider the map $\omega + 1 \longrightarrow 2 = \{0, 1\}$ that sends $\infty$ to 1 and everybody else to 0.)

Hence if we want to fully reflect distributive lattices in topology we need to choose the correct definition of a morphism between spectral spaces:

Definition A map $f : X \longrightarrow Y$ between spectral spaces is called a **spectral map** if for all $V \in \mathring{\mathcal{K}}(Y)$ we have $f^{-1}(V) \in \mathring{\mathcal{K}}(X)$. Observe that $f$ then is automatically continuous since $\mathring{\mathcal{K}}(Y)$ is a basis.

# 5. Stone duality

Let BDLat be the category of distributive lattices with morphisms as defined earlier. Let Spec be the category of spectral spaces with spectral maps as morphisms.

Then BDLat and Spec are anti-equivalent. Explicitly:

1. The anti-equivalence is given by the **contra-variant** functors $\mathrm{Spec} : \mathtt{BDLat} \to \mathtt{Spec}$ and $\mathring{\mathcal{K}} : \mathtt{Spec} \to \mathtt{BDLat}$. Here $\mathring{\mathcal{K}}$ acts on a morphism $f : X \longrightarrow Y$ via $\mathring{\mathcal{K}}(f)(V) = f^{-1}(V)$.
2. The assignments given by $L \mapsto \mathcal{D}_L$ in the representation theorem for distributive lattices and $X \mapsto \Theta_X$ in the representation theorem for spectral spaces define natural transformations $\mathcal{D} : \mathrm{id}_{\mathtt{BDLat}} \to \mathring{\mathcal{K}} \circ \mathrm{Spec}$ and $\Theta : \mathrm{id}_{\mathtt{Spec}} \to \mathrm{Spec} \circ \mathring{\mathcal{K}}$.
3. The natural transformations $\mathcal{D}$ and $\Theta$ are isomorphism of functors.

Proof: 1. and 2. are easily verified, in fact the setup is made so that they are true. The crucial statement really is 3. and we have seen this explicitly. $\square$

# 6. Spectra of (commutative, unital) rings

Let $R$ be a ring. We define

$$\mathrm{Spec}(R) = \{P \subseteq R \mid P \text{ is a prime ideal}\}$$

[A prime ideal is the kernel of a ring homomorphism $R \to$ some field.]

Let $D(f) = \{P \in \mathrm{Spec}(R) \mid f \notin P\}$. Then $\{D(f) \mid f \in R\}$ is the basis of a spectral topology on $\mathrm{Spec}(R)$ called the **Zariski spectrum** (or prime ideal spectrum). We again have $P \rightsquigarrow Q \iff P \subseteq Q$.
Proof: Follow the same strategy as for lattices. The algebraic setup changes (a bit) and Krull's Theorem then actually becomes Krull's original theorem (proof very similar).

Hochster's theorem Every spectral space is homeomorphic to the Zariski spectrum of a ring.

In fact Hochster shows that every spectral map comes form a spectral map induced by a ring homomorphism.

Red flag Be careful when studying spectral spaces using this theorem. The construction is not functorial (already for finite spaces).

$\curvearrowright$

# 6. Spectra of abelian $\ell$-groups (and MV-algebras)

For an introduction, see [Tre26a]

Let $G = (G, +, \leq)$ be an $\ell$-group. In this course this means

"An **abelian** group $(G, +)$ with a compatible **lattice order** $\leq$, i.e. $f \leq g \Rightarrow f + h \leq g + h$."

Prime example: $C(T) :=$ continuous functions $X \to \mathbb{R}$ for some space $T$ (typically Hausdorff), with pointwise $+, \leq$.

We define

$$\mathrm{Spec}(G) = \{P \subseteq G \mid P \text{ is a prime ideal}\}$$

[A prime ideal is the kernel of a morphism $G \to$ some totally ordered abelian group – including $\{0\}$.]

Let $D(f) = \{P \in \mathrm{Spec}(G) \mid f \notin P\}$. Then $\{D(f) \mid f \in R\}$ is the basis of a spectral topology on $\mathrm{Spec}(G)$ called the $\ell$-**spectrum**. We again have $P \rightsquigarrow Q \iff P \subseteq Q$.

Proof: Follow the same strategy as for lattices. The algebraic setup changes (a bit) and Krull's Ideal Theorem needs an $\ell$-group proof.

# 7. Why bother?

Classical - Logic (everybody's view), see [Tre26b, p. 3.3].

Classical - Algebraic (everybody's view): Representation theory of algebraic structures; done on the board.

My add on - Model Theory: What algebraic information of the structure (e.g. a ring or an $\ell$-group) is encoded in its spectrum? Done on the board.

$\curvearrowright$

# 8. The inverse and the patch topology

Let $X$ be a spectral space. The set

$$\overline{\mathcal{K}}(X) := \{X \setminus U \mid U \in \mathring{\mathcal{K}}(X)\}$$

is again a distributive lattice (isomorphic to the order dual of $\mathring{\mathcal{K}}(X)$).

We write $\mathcal{K}(X)$ for the Boolean algebra generated by $\mathring{\mathcal{K}}(X) \cup \overline{\mathcal{K}}(X)$, hence

$\mathcal{K}(X) = \{\,\text{finite unions of sets of the form } U \cap A \mid U \in \mathring{\mathcal{K}}(X), A \in \overline{\mathcal{K}}(X)\,\}.$

Definition The **inverse topology** on $X$ is the topology having $\overline{\mathcal{K}}(X)$ as a basis. The **inverse space** of $X$, denoted by $X_{\mathrm{inv}}$ is the set $X$ endowed with the inverse topology.

The **patch topology** (aka **constructible topology**) is the topology having $\mathcal{K}(X)$ as a basis. The **patch space** of $X$, denoted by $X_{\mathrm{con}}$ is the set $X$ endowed with the patch topology.

Topological constructions for these topologies will be annotated by superscripts. For example $\overline{S}^{\mathrm{con}}$ is the closure for the patch topology and $\overline{S}^{\mathrm{inv}}$ is the closure for the inverse topology.

Whenever we use topological terminology (closure, interior, etc.) this refers to the original, spectral topology of $X$.

# 9. The inverse and the patch topology

Let $X$ be a spectral space.

Theorem

$X_{\mathrm{inv}}$ and $X_{\mathrm{con}}$ are spectral spaces, $X_{\mathrm{con}}$ is even **Boolean**.

See [DST19, 1.3.14] for a purely topological proof (without reference to Stone duality).

It follows that

1. $\mathring{\mathcal{K}}(X_{\mathrm{inv}}) = \overline{\mathcal{K}}(X)$, $\overline{\mathcal{K}}(X_{\mathrm{inv}}) = \mathring{\mathcal{K}}(X)$ and $x \rightsquigarrow y$ in $X_{\mathrm{inv}}$ iff $y \rightsquigarrow x$ in $X$.
2. $\mathcal{K}(X_{\mathrm{inv}}) = \mathcal{K}(X) = \mathcal{K}(X_{\mathrm{con}}) = \mathring{\mathcal{K}}(X_{\mathrm{con}}) = \overline{\mathcal{K}}(X_{\mathrm{con}})$, and $X_{\mathrm{con}}$ only has trivial specializations.

# 10. Priestley spaces

A spectral space $X$ (like every $T_0$-space) carries a partial order

$$x \rightsquigarrow y \iff y \in \overline{\{x\}}.$$

The patch space $X_{\rm con}$ has forgotten this poset, because $X_{\rm con}$ is Boolean (i.e. compact with clopen sets as a basis). A very important point about spectral spaces is that the specialization order carries all information about $\mathring{\mathcal{K}}(X)$ - given the patch space:

Lemma A subset $U$ of a spectral space $X$ is in $\mathring{\mathcal{K}}(X)$ if and only if the following two conditions are satisfied:

(a) $U$ is compact open in the patch topology, and,

(b) $U$ is closed under generalization, i.e. $x \rightsquigarrow u \in U \Rightarrow x \in U$.

Proof: This is clear in one direction. If (a) and (b) holds, then suppose the complement of $U$ is not closed. Take $u \in U$ with $u \in \overline{X \setminus U}$. For each $V \in \mathring{\mathcal{K}}(X)$ with $u \in V$ we have $V \cap X \setminus U \neq \emptyset$. By compactness of $X_{\rm con}$ using (a) there is then some $x \in \bigcap\{X \setminus U \mid x \in V \in \mathring{\mathcal{K}}(X)\}$. But this contradicts (b). $\square$

# 10. Priestley spaces

A **Priestley space** is a pair $(Z, \rightsquigarrow)$ such that

PR1 $Z$ is a Boolean space, and,

PR2 $\rightsquigarrow$ is a partial order such that for all $z \not\rightsquigarrow y$ there is some clopen subset $U$ of $Z$ with $z \notin U \ni y$ such that $U$ is closed under generalisation of $\rightsquigarrow$ (i.e. $x \rightsquigarrow u \in U \Rightarrow x \in U$)

[You may think here that $\rightsquigarrow$ is $\leq$ or $\geq$.]

Recent book: [GG24].

Warning: A Priestley space is not a topological space, it is an **enriched** topological space.

Theorem H. Priestley [Pri70], also see [DST19, 1.5.11].
Let `Priestley` be the category of Priestley spaces together with continuous order preserving maps. Then the functor $\texttt{Spec} \longrightarrow \texttt{Priestley}$, $X \mapsto (X_{\mathrm{con}}, \rightsquigarrow)$ is an isomorphism (not just an equivalence).

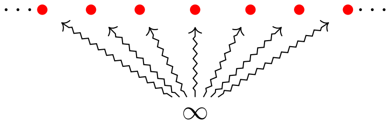
The inverse maps $(Z, \rightsquigarrow)$ to the space $X$ with base set $Z$ generated by the clopen and $\rightsquigarrow$-generalisation closed sets (and these sets form $\mathring{\mathcal{K}}(X)$.)

$\curvearrowright$

# 10. Priestley spaces - Examples

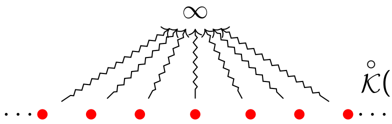
Let $S$ be an infinite set. We consider two spectral topologies with base set $S^* := S \dot\cup \{\infty\}$, see [DST19, 1.6D].

The first one is denoted by $S_\infty$ and has $\infty$ as a generic point:



$$\mathcal{O}(S_\infty) = \mathring{\mathcal{K}}(S_\infty)$$
$$\|$$
$$\{S^* \setminus F \mid F \subseteq S \text{ finite}\} \cup \{\emptyset\}$$

The second one is denoted by $S^\infty$ and has $\infty$ as a closed point:



$$\mathcal{O}(S^\infty) = \mathfrak{P}(S) \cup \{S^*\}$$
$$\mathring{\mathcal{K}}(S_\infty) = \{F \mid F \subseteq S \text{ finite}\} \cup \{S^*\}$$

# 11. Heyting algebras, Stone algebras and Frames in spectral spaces

Heyting algebras are special distributive lattices, namely they have a trace of a Boolean algebra in it: relative pseudocomplementation, corresponding to intuitionistic implication.

Other lattices of interest in algebraic logic are Stone algebras and frames. We have a look at how these conditions look on the spectral space side.

A general background here is pseudocomplementation, which we need to understand first:

# 12. Pseudo-complementation

In this section, $L$ is always a (bounded) distributive lattice and we may always assume that $L = \mathring{\mathcal{K}}(X)$ for some spectral space $X$.

Definition Given $a \in L$, an element $b \in L$ is the **pseudo-complement** of $a$ if $b$ is the largest element of $L$ satisfying $a \wedge b = 0$. We write $a^* = b$.

If all elements of $L$ have a pseudo-complement, then $L$ itself is called **pseudo-complemented**.

In topological terms: An element $U \in L = \mathring{\mathcal{K}}(X)$ has a pseudo-complement if and only if $U^* = X \setminus \overline{U}$ is in $L$.

Proposition The following are equivalent for every spectral space $X$.

(i) $\mathring{\mathcal{K}}(X)$ is pseudo-complemented - we say that $X$ is a **PC-space**.

(ii) For every $U \in \mathring{\mathcal{K}}(X)$ we have $\overline{U} \in \mathcal{K}(X)$ (=patch clopen subsets).

(iii) [DST19, 4.4.6, 4.4.16, 8.3.9] The following two conditions hold, where $X^{\min} := \{x \in X \mid \forall y \in X : y \rightsquigarrow x \Rightarrow y = x\}$

(a) $X^{\min}$ is compact $\iff$ $X^{\min}$ is patch-closed $\iff$ $X$ and $X_{\mathrm{inv}}$ induce the same topology on $X^{\min}$.

(b) For $U \in \mathring{\mathcal{K}}(X)$ the open regularization $\mathring{\overline{U}}$ belongs to $\mathring{\mathcal{K}}(X)$. This is a weakening of the *subfitness* condition "$U \in \mathring{\mathcal{K}}(X) \Rightarrow U$ regular open"; cf. [BMMTW24, 5.3]

# 12. Pseudo-complementation - Heyting Algebras

For every bounded distributive lattice $L = \mathring{\mathcal{K}}(X)$, TFAE:

(i) $L$ is a **Heyting algebra**, i.e. for all $a, b \in L$ there is a largest $c \in L$ with $a \wedge c \leq b$.

(ii) Every $A \in \overline{\mathcal{K}}(X)$ is a PC-space.

(iii) For all $C \in \mathcal{K}(X)$ we have $\overline{C} \in \mathcal{K}(X)$ (=patch clopen subsets).
`Terminology:` The Priestley space $(X_{\mathrm{con}}, \rightsquigarrow)$ is an **Esakia space**, [Esa19]. We call $X$ itself a **Heyting space** as a memory hook.

(iv) [BGJ13, 4.2] If $S \subseteq X$, then the closure of $S$ for the inverse topology is $\overline{\mathrm{Gen(S)}}^{\mathrm{con}}$. [Note that in an arbitrary spectral space the closure of a set $S$ for the inverse topology is $\mathrm{Gen}(\overline{S}^{\mathrm{con}})$.]

For metrizable Priestley spaces there is also a characterization in terms of avoiding three specific posets, see [BC19, 2.2].

Example If $L$ itself is a topology, then $L$ is Heyting: Given $U, V \in L$, the largest element $W \in L$ with $U \cap W \subseteq V$ is the interior of $U^c \cup V$.

A more general example is the lattice $L$ of open subsets of $\mathbb{R}^n$ that are **definable** in some fixed first-order structure expanding $(\mathbb{R}, \leq)$ because closures and interiors are definable operations).

# 12. Pseudo-complementation - Stone Lattices

Theorem For every bounded distributive lattice $L = \mathring{\mathcal{K}}(X)$, TFAE:

(i) $L$ is a **Stone lattice**, i.e. $L$ is pseudo-complemented and each $a \in L$ satisfies $a^* \vee a^{**} = 1$, i.e. $a^*$ has a complement.

(ii) For all $U \in \mathring{\mathcal{K}}(X)$ we have $\overline{U} \in \mathring{\mathcal{K}}(X)$ (=compact open subsets).

(iii) There is a (necessarily unique) continuous retraction of the embedding $\mathrm{GenericPoints}(X) \hookrightarrow X$. This retraction is then a spectral map.

This is an extension of the Grätzer-Schmidt theorem characterizing Stone lattices, as asked for by Marshall Stone.

Examples are open sets of an extremally disconnected set (closures of open sets are open), intervals of existentially closed $\ell$-groups and the lattice of positive divisors of $n \in \mathbb{N}$.

$\curvearrowright$

# 12. Pseudo-complementation - Overview

Recall: $X = \mathrm{PrimI}(L)$

$L \cong \mathring{\mathcal{K}}(X) = \{\text{quasi-compact open sets}\}$, $\overline{\mathcal{K}}(X) = \{X \setminus U \mid U \in \mathring{\mathcal{K}}(X)\}$
and $\mathcal{K}(X) = \{\text{patch clopen sets}\}$.

Heying algebra

$C \in \mathcal{K}(X)$
$\Downarrow$
$\overline{C} \in \overline{\mathcal{K}}(X)$

Boolean algebra

Boolean space

pseudo complemented

$U \in \mathring{\mathcal{K}}(X)$
$\Downarrow$
$\overline{U} \in \overline{\mathcal{K}}(X)$

$\Leftrightarrow$

$X^{\min}$ compact and

$U \in \mathring{\mathcal{K}}(X) \Rightarrow \mathring{\overline{U}} \in \mathring{\mathcal{K}}(X)$

Stone lattice

$U \in \mathring{\mathcal{K}}(X)$
$\Downarrow$
$\overline{U} \in \mathrm{Clop(X)} = \mathring{\mathcal{K}}(X) \cap \overline{\mathcal{K}}(X)$

# 13. Frames in spectral spaces

Let $X$ be a spectral space.

Theorem [DST19, 9.2.5]

$\mathring{\mathcal{K}}(X)$ is a frame if and only if the patch closure of **any** open set is open (and is therefore in $\mathring{\mathcal{K}}(X)$).

In other words, locales can be realized as actual topological spaces via Stone duality.

Theorem [DST19, 9.4.3]

If $\mathring{\mathcal{K}}(X)$ is a frame, then the co-frame of sublocales of $F$ (called the **assembly**) is poset isomorphic to the patch closed subsets $S$ of $X$ satisfying the following condition:

$$A \in \overline{\mathcal{K}}(S) \Rightarrow \text{the closure of } A \text{ in } X \text{ is in } \overline{\mathcal{K}}(X).$$

Remark. For the question on how spectral spaces may be seen as frames, i.e. the characterization of a space $X$ being spectral by saying that it is **coherent**, I refer to [Joh86, Chpater 2, section 3] and [DST19, 9.5.11].

# 14. Modal logic in Stone duality

**Rough outline:** In (propositional) modal logic we are dealing with formulas built from variables $p_1, p_2, \ldots$ and logical connectives $\{\wedge, \vee, \neg, 0, 1\} \cup \{\Box\}$, where $\Box$ is a new unary operator (for 'necessity'). The Tarski-Lindenbaum algebra here is the set of well formed formulas in that language, modulo the equivalence relation of provable equivalence for the proof system

> *"classical propositional logic + the new rule* $\vdash \varphi \Rightarrow \vdash \Box\varphi$ *+ the axiom* $\Box(\theta \to \varphi) \to (\Box\theta \to \Box\varphi)$*"*
> *This proof system is denotes by* $K$*.*

As in the classical case, the connectives preserve the equivalence classes, the new operator acts as $\Box[\varphi] := [\Box\varphi]$. So here the Tarski-Lindenbaum algebra is a Boolean algebra $A$ equipped with an operator $\Box : A \to A$. The axioms easily imply that this operator preserves $\wedge$.

One can then replace $\Box$ by the set under the graph of $\Box$, i.e. by the set $L = \{(a, b) \in A^2 \mid b \leq \Box a\}$. It turns out that $L$ is a bounded distributive lattice and there are very simple first order axioms describing the sublattices of $A^2$ occurring from such operators. Hence we may use Stone duality of distributive lattices to describe a topological representation of $(A, \Box)$.

Continued on the board...

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