STONE DUALITY FOR BOOLEAN ALGEBRAS

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Abstract. We give a detailed exposition of Stone duality for Boolean algebras, focussing on representation of Boolean algebras and Boolean spaces.

Contents

1. Introduction

Stone Duality is nowadays a term describing a tight relation between classes of algebraic structures and classes of topological spaces. In category theory terms it describes equivalences of certain categories of ordered algebraic structures and certain categories of topological spaces. The terminology is derived from Marshall Stone's work [Sto36] who established the duality for Boolean algebras and Boolean spaces. The note at hand explains this classical duality.

The backbone of Stone Duality for Boolean algebras is the following. Let A be a Boolean algebra (see 2.3.1 for the definition; for now it is enough to think of some algebraic structure, hence a set equipped with some operations). Then one can associate a topological space $\mathcal{U}(A)$ to A, called *the spectrum of* A, from which one can reconstruct the structure A. The process runs under the headline *Stone representation*, see 3.1.5 and the remark following it. The reconstruction process allows to analyze a Boolean algebra fully within its spectrum. This opens the possibility to think about the algebraic structure in terms of topological or

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even geometrical intuition. For example one can ask about the shape of the space locally (at each point) and what this means for the algebraic structure. As a matter of fact one can associate many spaces with the features above to almost every algebraic structure, but these spaces contain generally much more information than the original structure and they are therefore deemed to be too complicated to be analysable or of any help. This is not the case for the spectrum of a Boolean algebra: The reason is that they have an intrinsic topological description (as Boolean spaces) and the representation above also goes the other way. Hence for every Boolean space there is a unique Boolean algebra from which we can reconstruct the space. This is done in 3.1.6.

Although the main result 4.4 on Stone Duality of Boolean algebras is formulated in terms of categories and functors, the text is written so that the representation theorems 3.1.5 and 3.1.6 described above can be understood without any prior knowledge of category theory. Only in section 4 the reader is assumed to be familiar with the basic notions of category theory as for example exposed in [ML98, Chapter I, sections 1–4 and Chapter IV, section 4]

In section 2 the preliminary version 2.3.4 of the representation theorem 3.1.5 is presented, without reference to topology. This is developed in the more general context of *distributive lattices* (see, 2.1.1 for the definition). The rational here is twofold: On the one hand, the general case is not more complicated to prove and on the other hand, Stone Duality is also available in this more general context; the reader who wants to follow up this path will then have an adequate preparation.

Throughout the text some acquaintance with the basic notions of partially ordered sets is required, as may be found in [Fuc63]. In section 3, the reader is assumed to have basic knowledge of general (Hausdorff) topology as can be found in [Kel75; Eng89]. Section 3.2 gives an alternative description of the space $\mathcal{U}(A)$ for readers who are familiar with the prime spectrum of a ring; this section is not needed in the remainder of the text. Section 3.3 explains how the representation theorem 3.1.5 for Boolean algebras can be used to prove the completeness theorem for Propositional Logic.

In this text, the symbol $\mathbb{N}$ stands for the set of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$, whereas $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$.

2. Stone representation of distributive lattices

Summary In this section we describe the representation theorem for *distributive lattices* as lattices of sets due to Marshall Stone, cf. [Sto37]. Stone proved this first for the special case of Boolean algebras in [Sto36], but in fact this special case is not easier to show.

2.1. **Distributive lattices.**

We give a brief introduction to distributive lattices, suitable for our purposes. For more on the topic we refer to [Grä11, Chapter II].

2.1.1. **Definition.** A **distributive lattice** in this text[1] is a partially ordered set $L = (L, \leq)$ with the following properties:

[1]In the literature, condition **DL4** is not required and the objects that we are talking about are called *bounded* distributive lattices. However we will always work under assumption **DL4** and suppress the adjective *bounded*.

DL1 For all $a, b \in L$ the supremum of $\{a, b\}$ for the partial order $\leq$ exists. The supremum is denoted by $a \vee b$. It is also called the **join of** a **and** b.

DL2 For all $a, b \in L$ the infimum of $\{a, b\}$ for the partial order $\leq$ exists. The infimum is denoted by $a \wedge b$. It is also called the **meet of** a **and** b.

Hence we may view the operations $\wedge$, $\vee$ as functions $L \times L \longrightarrow L$. Notice that both operations are commutative and associative as follows immediately from their definitions; in particular, expressions of the form $a_1 \wedge \ldots \wedge a_n$ are unambiguous. However, the next requirement is not implied by the previous ones:

DL3 *Distributivity law for* $\wedge$ *and* $\vee$
For all $a, b, c \in L$ we have $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

DL4 There is a smallest element for $\leq$, which we denote by $\perp$, called **bottom**. There is a largest element for $\leq$, which we denote by $\top$, called **top**.

2.1.2. *Examples.*

(i) There is a smallest distributive lattice, consisting of exactly two elements $\perp < \top$. There is also a so called *terminal distributive lattice* consisting of exactly one element $\perp = \top$.

(ii) The most common example of a distributive lattice is the powerset $\mathfrak{P}(S)$ of a set S together with the partial order given by inclusion.

(iii) Generalising (ii), if L is a subset of $\mathfrak{P}(S)$ with $\emptyset, S \in L$ and if L is closed under taking finite intersections and finite unions, then $L = (L, \subseteq)$ is a distributive lattice. The operations and constants in definition 2.1.1 are given by

$$\perp = \emptyset, \ \top = S, \ a \wedge b = a \cap b, \ a \vee b = a \cup b.$$

Distributive lattices of this form are called **lattices of subsets** (of S).

(iv) The set of open subsets of a topological space is a distributive lattice. The set of closed subsets of a topological space is a distributive lattice. Both lattices are lattices of subsets of the space.

(v) A distributive lattice that is not by definition a lattice of subsets is the set of propositional sentences (expressions made up of letters $p, q, r, \ldots$ using the connectives $\neg, \vee, \wedge, \Rightarrow$) modulo the equivalence relation $s \sim t$ defined as "there is a proof of s=t" (think of "s,t have the same truth table"). The partial order $[s]_\sim \leq [t]_\sim$ is given by the property "$(s \Rightarrow t)$ is a tautology".

In section 2.3 below we will see another example of distributive lattices, namely *Boolean algebras.*

2.1.3. **Definition.** A map $\varphi : L \longrightarrow M$ between distributive lattices is called a **homomorphism (of lattice)** if it preserves $\perp, \top$, $\wedge$ and $\vee$. Explicitly, this means $\varphi(\perp_L) = \perp_M$, $\varphi(\top_L) = \top_M$, $\varphi(a \wedge_L b) = \varphi(a) \wedge_M \varphi(b)$ and $\varphi(a \vee_L b) = \varphi(a) \vee_M \varphi(b)$ for all $a, b \in L$. (For better readability we will drop the subscripts L, M of the operations when this is unambiguous.)

The homomorphism φ is called an **isomorphism (of lattices)** if it is bijective.

2.1.4. *Remark.* Let $\varphi : L \longrightarrow M$ be a homomorphism of lattices.

(i) The map φ preserves the partial orders given on L, M because $x \leq y$ is equivalent to $x = x \wedge y$ in every distributive lattice and this identity is preserved by φ.

(ii) If φ is an isomorphism, then its compositional inverse φ^{-1} is again a homomorphism: The proof is straightforward and follows tightly the lines of the proof that the compositional inverse of a bijective homomorphism of groups, is itself a homomorphism of groups.

2.2. **The representation of distributive lattices as lattices of subsets.**

2.2.1. We show that every distributive lattice is isomorphic to a lattice of subsets of some set S (cf. 2.2.12). The key issue is how to find S. In order to construct S we will need some preparations.

2.2.2. **Definition.** Let L be a distributive lattice. A **filter** of L is a subset F of L with the following properties.

F1 $F \neq \emptyset$.
F2 If $a, b \in F$, then $a \wedge b \in F$.
F3 If $a \in F$ and $a \leq b \in L$, then $b \in F$.[2]

Obviously $F = A$ is a filter of L. A filter is **proper** if it is different from L. In virtue of F3, this is equivalent to saying that $\perp \notin F$.

2.2.3. *Examples.* Let L be a distributive lattice.

(i) Clearly L is the largest filter of L and $\{\top\}$ is the smallest filter of L.
(ii) If $a \in L$, then the set $\mathfrak{f}_a := \{b \in L \mid a \leq b\}$ is obviously the smallest filter of L containing a, called the **principal filter** of a.
(iii) If L is a lattice of subsets of a set S (cf. 2.1.2(iii)) and $p \in S$, then the set $\{a \in L \mid p \in a\}$ is obviously a proper filter of L.
(iv) In the distributive lattice L of open subsets of a topological space X, the so called *neighborhood filter* $\mathcal{N}_p = \{O \in L \mid p \in O\}$ of a point $p \in X$ is a filter of L. This is a special case of (iii)

2.2.4. **Alternative description of filters.** *The following conditions are equivalent for every subset F of a distributive lattice L.*

(i) F is a filter.
(ii) $F \neq \emptyset$ and for all $a, b \in L$ we have

$$a \wedge b \in F \iff a \in F \text{ and } b \in F.$$

Proof. (i)⇒(ii). We know $F \neq \emptyset$ by condition **F1** in 2.2.2. The implication ⇐ of the equivalence holds by **F2** and the implication ⇒ follows from **F3** by noticing that $a \wedge b \leq a, b$.

(ii)⇒(i). Obviously conditions **F1** and **F2** follow from (ii). To see **F3**, assume If $F \ni a \leq b \in L$. Then $a \wedge b = a \in F$ and so implication ⇒ in (ii) implies $b \in F$. □

2.2.5. **Definition.** A filter F of a distributive lattice L is called a **prime filter** if

P1 F is proper, hence $F \neq L$.
P2 For all $a, b \in L$ with $a \vee b \in F$ we have $a \in F$ or $b \in F$.

The filters in example 2.2.3(iii) are clearly prime. On the other hand, principal filters may or may not be prime. For example the principal filter of $a \in \mathfrak{P}(S)$ in example 2.1.2(ii) is prime if and only if a has exactly one element.

[2]Hence by F1 we know $\top \in F$.

2.2.6. **Characterization of prime filters.** *The following conditions are equivalent for every subset F of a distributive lattice L.*

(i) F *is a prime filter.*

(ii) $F \neq \emptyset, L$ *and for all* $a, b \in L$ *we have*

$$\begin{aligned} a \wedge b \in F &\iff a \in F \textit{ and } b \in F \\ a \vee b \in F &\iff a \in F \textit{ or } b \in F. \end{aligned}$$

(iii) The map

$$\chi : L \longrightarrow \{\perp, \top\},\ a \longmapsto \begin{cases} \top & \textit{if } a \in F \\ \perp & \textit{if } a \notin F, \end{cases}$$

is a homomorphism of lattices.

(iv) The complement $I = L \setminus F$ *of* F *in* L *is a* ***prime ideal*** *of the distributive lattice* L*, i.e. the following conditions are satisfied:*

(a) $I \neq \emptyset$.

(b) For all $a \in I$ *and every* $b \in L$ *with* $a \leq b$ *we have* $b \in I$.

(c) For all $a, b \in I$ *we have* $a \vee b \in I$.[3]

(d) I *is proper, i.e.* $I \neq L$.

(e) For all $a, b \in L$*, if* $a \wedge b \in I$ *then* $a \in I$ *or* $b \in I$.

Proof. (i)⇒(ii). Since F is a proper filter we know that $F \neq L$ and by 2.2.4 we only need to show the second equivalence. The implication ⇒ holds by **P2** and the implication ⇐ follows from $a, b \leq a \vee b$ and **F3**.

(ii)⇒(i). By 2.2.4 we only need to show **P1** and **P2**. Since $F \neq L$ we know **P1**. The implication ⇒ in the second equivalence of (ii) is just **P2**.

Hence we know that (i) and (ii) are equivalent.

(ii)⇔(iii). The map χ preserves $\perp$ and $\top$ just if $\perp \notin F$ and $\top \in F$. Hence under both assumptions (ii) and (iii) we know $\perp \notin F$ and $\top \in F$. Furthermore, the equivalences in (ii) expressed in terms of the map χ translate into

$$\begin{aligned} \chi(a \wedge b) = \top &\iff \chi(a) = \top \text{ and } \chi(b) = \top \\ \chi(a \vee b) = \top &\iff \chi(a) = \top \text{ or } \chi(b) = \top. \end{aligned}$$

But this is just saying that χ preserves meet and join and consequently (ii) is equivalent to (iii).

(ii)⇔(iv). Property (ii) of F formulated in terms of $I = L \setminus F$ says $I \neq \emptyset, L$ and by considering contrapositives:

$$\begin{aligned} a \wedge b \in I &\iff a \in I \text{ or } b \in I \\ a \vee b \in I &\iff a \in I \text{ and } b \in I. \end{aligned}$$

Now the proof of (i)⇔(ii) above written out for I instead of F and using these new equivalences gives (ii)⇔(iv). □

2.2.7. *Notation.* Let L be a distributive lattice. We write

$$\mathrm{PrimF}(L) = \{P \subseteq L \mid P \text{ is a prime filter}\}$$

[3]A subset I of L satisfying conditions (a),(b) and (c) is called an *ideal* of the distributive lattice L.

for the set of prime filters of L. If $S \subseteq L$ we write

$$V(S) = \{P \in \mathrm{PrimF}(L) \mid S \subseteq P\}.$$

When $S = \{a\}$ with $a \in L$ we just write $V(a)$ instead of $V(\{a\})$, hence $V(a) = \{P \in \mathrm{PrimF}(L) \mid a \in P\}$. Finally we write

$$C(L) = \{V(a) \mid a \in L\}.$$

The set S promised in 2.2.1 is $\mathrm{PrimF}(L)$ and the lattice of subsets of this set, which is isomorphic to the given lattice L is supported by $C(L)$. All but one property of these statements are mere observations:

2.2.8. *Observation.* In the situation of 2.2.7 we observe the following properties.

(i) We have $V(\bot) = \emptyset$, because by **P1** no prime filter contains $\bot$. Furthermore $V(\top) = \mathrm{PrimF}(L)$ because every prime filter contains $\top$.

(ii) If $a, b \in L$ then by 2.2.6(i)⇒(ii) we know

$$V(a \wedge b) = V(a) \cap V(b), \text{ and}$$
$$V(a \vee b) = V(a) \cup V(b).$$

(iii) By (i) and (ii), the set $C(L)$ is a lattice of subsets of $\mathrm{PrimF}(L)$ and the map $\mathcal{V}_L : L \longrightarrow C(L)$ that sends $a \in L$ to $V(a)$ is a homomorphism of lattices.

Hence, once we know that the map $\mathcal{V}_L$ from 2.2.8(iii) is injective, then $\mathcal{V}_L$ is an isomorphism of distributive lattices and the goal laid out in 2.2.1 is achieved. However, injectivity requires some work; notice that at the moment we even do not know whether a given distributive lattice with at least two elements possesses a prime filter.

2.2.9. **Lemma.** *Let L be a distributive lattice and let $\emptyset \neq S \subseteq L$. Then there is a smallest filter of L containing S, namely*

$$\mathfrak{f}_S = \{a \in L \mid \exists n \in \mathbb{N},\, s_1, \ldots, s_n \in S : s_1 \wedge \ldots \wedge s_n \leq a\}.$$

*The filter $\mathfrak{f}_S$ is called the **filter generated by** S. Notice that $\mathfrak{f}_{\{a\}} = \mathfrak{f}_a$ for $a \in L$.*

Proof. Clearly $S \subseteq \mathfrak{f}_S$. We first show that $\mathfrak{f}_S$ is a filter: Since $S \neq \emptyset$ we have $\mathfrak{f}_S \neq 0$ and so **F1** of 2.2.2 holds. If $a \in \mathfrak{f}_S$ and $a \leq b \in L$ then clearly $b \in \mathfrak{f}_S$ and so **F3** holds. Now assume $a, b \in \mathfrak{f}_S$. Choose $k, n \in \mathbb{N}$ and $s_1, \ldots, s_k, t_1, \ldots, t_n \in S$ with $s_1 \wedge \ldots \wedge s_k \leq a$ and $t_1 \wedge \ldots \wedge t_n \leq b$. Then $s_1 \wedge \ldots \wedge s_k \wedge t_1 \wedge \ldots \wedge t_n \leq a \wedge b$, witnessing that $a \wedge b \in \mathfrak{f}_S$.

Hence indeed $\mathfrak{f}_S$ is a filter containing S and it remains to show that $\mathfrak{f}_S$ is contained in every filter F that contains S. Take $a \in \mathfrak{f}_S$. By definition of $\mathfrak{f}_S$ there are $n \in \mathbb{N}$ and $s_1, \ldots s_n \in S$ with $s_1 \wedge \ldots \wedge s_n \leq a$. As $S \subseteq F$, condition **F2** for F ensures $s_1 \wedge \ldots \wedge s_n \in F$. But now condition **F3** for F ensures that $a \in F$ as required. □

The next proposition contains the key argument in the proof of the representation theorem 2.2.12.

2.2.10. **Proposition.** *Let L be a distributive lattice, $a \in L$ and let F be a filter of L. Suppose $a \notin F$ such that for every filter G of L with $F \subseteq G$ and $a \notin G$ we have $F = G$ (hence F is maximal for inclusion among filters of L that do not contain a). Then F is a prime filter.*

Proof. Since $a \notin F$, the filter F is proper and thus satisfy condition **P1** of 2.2.5. We need to verify condition **P2**. So take $b, c \in L$ with $b \vee c \in F$. Assume by way of contradiction that $b, c \notin F$. Let G be the filter generated by $F \cup \{b\}$ and let H be the filter generated by $F \cup \{c\}$. By the maximality assumption on F in the proposition we know that $a \in G$ and $a \in H$. By 2.2.9 there are $s_1, \ldots, s_k, t_1, \ldots, t_n \in F$ with

$$s_1 \wedge \ldots \wedge s_k \wedge b \leq a \text{ and}$$
$$t_1 \wedge \ldots \wedge t_n \wedge c \leq a.$$

Then $z := s_1 \wedge \ldots \wedge s_k \wedge t_1 \wedge \ldots \wedge t_n \in F$ by **F2** of 2.2.2 and therefore $z \wedge b, z \wedge c \leq a$. But then $(z \wedge b) \vee (z \wedge c) \leq a$ and by the distributivity law **DL3** for distributive lattices we obtain

$$z \wedge (b \vee c) = (z \wedge b) \vee (z \wedge c) \leq a.$$

However, at the beginning of the proof we have assumed that $b \vee c \in F$. Then **F2** implies $z \wedge (b \vee c) \in F$ and consequently **F3** implies $a \in F$. This contradicts the assumption of the proposition. □

2.2.11. **Corollary.** *Let F be a filter of the distributive lattice L and let $a \in L \setminus F$. Then there is a prime filter P of L containing F with $a \notin P$.*

Proof. We apply the Lemma of Zorn (cf. [Cie97, Theorem 4.3.4, p. 53]) to the set

$$\mathcal{S} = \{G \subseteq L \mid G \text{ filter of } L \text{ with } F \subseteq G \text{ and } a \notin G\}$$

furnished with the partial order $\subseteq$. If $\mathcal{C} \subseteq \mathcal{S}$ is nonempty and totally ordered for inclusion, then routine checking shows that $\bigcup \mathcal{C}$ is again a filter of L and obviously $F \subseteq \bigcup \mathcal{C}$ (as $\mathcal{C} \neq \emptyset$) and $a \notin \bigcup \mathcal{C}$. Thus $\bigcup \mathcal{C}$ is an upper bound of $\mathcal{C}$ in the partially ordered set $(\mathcal{S}, \subseteq)$. Since $\mathcal{S}$ is nonempty (it contains F) we may apply Zorn's Lemma and see that $(\mathcal{S}, \subseteq)$ has a maximal element P. By 2.2.10 we know that P is a prime filter. Since $P \in \mathcal{S}$ we obtain $F \subseteq P$ and $a \notin P$, as required. □

2.2.12**. Representation theorem for distributive lattices as lattices of sets** (This was originally proved by Marshall Stone in [Sto37].) *Every distributive lattice L is isomorphic to the distributive lattice $C(L)$ of subsets of* $\mathrm{PrimF}(L)$ *(cf. 2.2.7). The isomorphism is given by the map $\mathcal{V}_L : L \longrightarrow C(L)$ that sends $a \in L$ to $V(a) = \{P \mid a \in P\}$.*

Proof. By 2.2.8, the only property that remains to be shown is injectivity of $\mathcal{V}_L$. So let $a, b \in L$ and without loss of generality assume that $b \not\leq a$. This means that a is not in the principal filter $\mathfrak{f}_b$ generated by b. By 2.2.11, there is a prime filter P of L with $\mathfrak{f}_b \subseteq P$ and $a \notin P$. Hence $P \in V(b)$ and $P \notin V(a)$ witnessing $V(a) \neq V(b)$. □

We conclude with one notion that becomes central in the rest of the text.

2.2.13. **Definition.** A filter F of a distributive lattice L is called a **ultrafilter** if it is a maximal proper filter, i.e.,

- **U1** F is proper, hence $F \neq L$.
- **U2** If G is a filter of L with $F \subseteq G$, then $G = F$ or $G = L$.

2.2.14. *Observation.* Ultrafilters are prime by 2.2.10 applied to $a = \perp$. Furthermore ultrafilters exist in any distributive lattice that satisfies $\perp \neq \top$: apply the proof of 2.2.11 to $F = \{\top\}$ and $a = \perp$.

2.3. **Boolean algebras as Boolean algebras of subsets.** We recall the definition of Boolean algebras in a way suitable for our purposes. As a general reference for Boolean algebras we mention [Kop89].

2.3.1. **Definition.** A **Boolean algebra** is a distributive lattice A that satisfies the following additional property:

BA *Existence of a complement*
For every $a \in A$ there is some $b \in A$ with $a \wedge b = \perp$ and $a \vee b = \top$.

If $b' \in A$ is another element satisfying **BA** for a, then $b' = b' \wedge \top = b' \wedge (a \vee b) = b' \wedge b$ by **DL3**; similarly $b = b' \wedge b$ showing that $b = b'$ is uniquely determined by a. We may therefore define $\neg a = b$ and called it the **complement of** a (in A).

A map $\varphi : A \longrightarrow B$ between Boolean algebras is called a **homomorphism (of Boolean Algebras)** if φ is a homomorphism of lattices. By uniqueness of complements, the properties defining the complement in **BA** readily imply that $\varphi(\neg a) = \neg\varphi(a)$ for all $a \in A$, thus φ preserves complements as well. An **isomorphism (of Boolean algebras)** is an isomorphism of distributive lattices between Boolean algebras.

2.3.2. *Example.* The prime example of a Boolean algebra is the powerset $\mathfrak{P}(S)$ of a set S, cf. 2.1.2(i). The distributive lattices in 2.1.2(iii),(iv) are in general not Boolean algebras. The distributive lattice in 2.1.2(v) is a Boolean algebra (called **Tarski-Lindenbaum algebra** of propositional calculus), because complements are given by $\neg[t]_\sim = [\neg t]_\sim$ for a propositional expression t.

If A is a nonempty subset of $\mathfrak{P}(S)$ that is closed under taking finite intersections and complements (in S), then $A = (A, \subseteq)$ is a Boolean algebra. The operations and constants in definitions 2.1.1 and 2.3.1 are given by

$$\perp = \emptyset,\ \top = S,\ a \wedge b = a \cap b,\ a \vee b = a \cup b\ \text{[4]},\ \text{and } \neg a = S \setminus a.$$

Boolean algebras of this form are called **Boolean algebras of subsets** (of S).

Obviously a Boolean algebras of subsets of S is the same as a lattice of subsets of S, which is at the same time a Boolean algebra. We therefore can write out the representation theorem 2.2.12 with the term “distributive lattice” replaced by “Boolean algebra”. However we can do slightly better because in Boolean algebras every prime filter is an ultrafilter:

2.3.3. **Characterization of ultrafilters in Boolean algebras.** *The following conditions are equivalent for every subset F of a Boolean algebra A.*

(i) *F is an ultrafilter.*
(ii) *F is a prime filter.*
(iii) *F is a proper filter and for all $a \in A$ we have $a \in F$ or $\neg a \in F$.*
(iv) *The map*

$$\chi : A \longrightarrow \{\perp, \top\},\ \longmapsto \begin{cases} \top & \text{if } a \in F \\ \perp & \text{if } a \notin F, \end{cases}$$

is a homomorphism of Boolean algebras.
(v) *$A \setminus F$ is a prime ideal of A.*

[4]The choice of $a \vee b$ here makes sense, because A is closed under finite intersections and complements; now apply DeMorgan’s law.

Proof. (i)⇒(ii) holds by 2.2.14 in any distributive lattice.

(ii)⇒(iii) The prime filter F is proper by **P1** of 2.2.5. If $a \in A$, then $a \vee \neg a = \top \in F$ and by **P2** we get $a \in F$ or $\neg a \in F$.

(iii)⇒(i). If G is a filter of A with $F \subsetneq G$, then take $a \in G \setminus F$. Since $a \notin F$ we know that $\neg a \in F$ by (iii). But $F \subseteq G$, hence $\neg a \in G$ and as $a \in G$ we obtain $\perp = a \wedge \neg a \in G$. Thus $G = A$ as required.

Hence we know that (i), (ii) and (iii) are equivalent. However, by 2.2.6 we already know that (ii), (iv) and (v) are equivalent (for (iv) recall that every homomorphism of lattices between Boolean algebras is a homomorphisms of Boolean algebras). □

Since ultrafilters are the same objects as prime filters for Boolean algebras by 2.3.3, theorem 2.2.12 entails

2.3.4**. Representation theorem for Boolean algebras as Boolean algebras of sets** ([Sto36]) *Every Boolean algebra A is isomorphic to the Boolean algebra $C(A)$ of subsets of the set of ultrafilters* $\mathrm{PrimF}(A)$ *of A.*

The isomorphism is given by the map $\mathcal{V}_A : A \longrightarrow C(A)$ that sends $a \in A$ to $V(a) = \{P \mid a \in P\}$. Consequently $V(\neg a) = \mathrm{PrimF}(A) \setminus V(a)$ for every $a \in A$. □

3. Topological representation theorems

We put the representation theorem 2.3.4 in a topological context, i.e., we define a topological space in which the Boolean algebra $C(A)$ can be described in purely topological terms of the space. This can also be done (in various formulations) for arbitrary distributive lattices, but requires more work on the topological side. For details we refer to [DST19, Section 3.2].

3.1. The Boolean space of ultrafilters.

3.1.1. **Definition.** A **Boolean space** is a topological space X that is compact Hausdorff and such that every open set is a union of **clopen** sets (clopen means "closed and open"); in other words, the clopen sets form a basis of X.

3.1.2. *Remark.* Let X be a any topological space.

(i) The set $\mathrm{Clop}(X)$ of clopen subsets of X is a Boolean algebra of subsets of X, because $\emptyset, X$ are clopen and clearly finite intersections and complements of clopen sets are again clopen.

(ii) The most prominent Boolean space is the Cantor ternary set. One can see this directly or by invoking the following characterization: A compact Hausdorff space is Boolean if and only if it is **totally disconnected**, i.e. the only nonempty connected subsets are singletons. This is an easy consequence of [Eng89, Theorem 6.1.23], which says that every connected component of any compact Hausdorff space is the intersection of its clopen supersets.

3.1.3. **Definition.** Let A be a Boolean algebra. We define a topological space $\mathcal{U}(A)$ associated to A, called the **spectrum of** A [5], as follows: The underlying set of $\mathcal{U}(A)$ is the set of ultrafilters of A; recall from 2.3.3 that this set is equal to the set of prime filters of A. The topology of $\mathcal{U}(A)$ is defined to be the smallest topology for which all sets of the form $V(a) = \{U \in \mathcal{U}(A) \mid a \in U\}$, $a \in A$, are closed.

[5]In the literature, $\mathcal{U}(A)$ is also called the **Stone space**, cf. [Joh86, II 4.2, bottom of p. 70] or **space of ultrafilters** of A.

3.1.4. **Proposition.**
The space $\mathcal{U}(A)$ *is a Boolean space and* $\mathrm{Clop}(\mathcal{U}(A)) = \{V(a) \mid a \in A\}$.

Proof. Recall that the set on right hand side was denoted by $C(A)$ in 2.2.7. By 2.3.4, the set $C(A)$ is a Boolean algebra of subsets of $\mathcal{U}(A)$ and the map $A \longrightarrow C(A)$ that sends a to $V(A)$ is an isomorphism of Boolean algebras. It follows that the set of all intersections of sets of the form $V(a)$ is the set of closed sets of a topology on $\mathcal{U}(A)$ and consequently this has to be the topology defined in 3.1.3. Consequently,

$(*)$ every closed set of $\mathcal{U}(A)$ is an intersection of sets of the form $V(a)$ with $a \in A$.

Claim 1. The space $\mathcal{U}(A)$ is compact.
Proof of claim 1. By virtue of property $(*)$, it suffices to show that every subset $\mathcal{S}$ of $C(A)$ with the property that every finite subset of $\mathcal{S}$ has nonempty intersection (this property of a set of subsets of a given set is referred to as **finite intersection property**), has nonempty intersection.

Let $F = \{a \in A \mid V(a) \in \mathcal{S}\}$. We first show that F is proper. Otherwise $\perp \in F$ and by 2.2.9 there are $a_1, \ldots, a_n \in S$ with $a_1 \wedge \ldots \wedge a_n = \perp$. But then $\emptyset = V(\perp) = V(a_1 \wedge \ldots \wedge a_n) = V(a_1) \cap \ldots \cap V(a_n)$; since all $V(a_i)$ are in $\mathcal{S}$, this contradicts the finite intersection property. Hence F indeed is a proper filter and by 2.2.11 there is a prime filter U of A containing F. By 2.3.3 we know $U \in \mathcal{U}(A)$ and we show that $U \in \bigcap \mathcal{S}$: Take $S \in \mathcal{S}$. Then $S = V(a)$ for some $a \in F$ by choice of F. Since $F \subseteq U$ we get $U \in V(a)$ as required. $\diamond$

Claim 2. $\mathrm{Clop}(\mathcal{U}(A)) = \{V(a) \mid a \in A\}$.
Proof of claim 2. $\supseteq$: Take $a \in A$. Since $V(\neg a) = \mathcal{U}(A) \setminus V(a)$, the set $V(a)$ is open. It is closed by definition of the topology, hence $V(A) \in \mathrm{Clop}(\mathcal{U}(A))$.

$\subseteq$. Let $\mathcal{K} \subseteq \mathcal{U}(A)$ be clopen. Since $\mathcal{K}$ is open we know from $(*)$ that the complement of $\mathcal{K}$ is an intersection of sets from $C(A)$. By taking complements and recalling that $V(a)$ has complement $V(\neg a)$ for $a \in A$, we see that $\mathcal{K}$ is a union of sets of the form $V(b)$ with $b \in A$. As $\mathcal{K}$ is also closed it is compact, using claim 1. It follows that $\mathcal{K}$ is a finite union of sets of the form $V(b)$ with $b \in A$. Hence there are $b_1, \ldots, b_n \in A$ with $\mathcal{K} = V(b_1) \cup \ldots \cup V(b_n)$. However, the latter set is equal to $V(b_1 \vee \ldots \vee b_n)$, which is in $C(A)$. $\diamond$

Claim 2 together with property $(*)$ also implies that every open set is a union of sets from $C(A)$ and so $\mathcal{U}(A)$ is Boolean. It remains to show that $\mathcal{U}(A)$ is Hausdorff. So take $U_1, U_2 \in \mathcal{U}(A)$ with $U_1 \neq U_2$. Without loss of generality we may assume that there is some $a \in U_1 \setminus U_2$. By 2.3.3 we know $\neg a \in U_2$. Hence $U_1 \in V(a)$, $U_2 \in V(\neg a)$ and $V(a) \cap V(\neg a) = V(a \wedge \neg a) = V(\perp) = \emptyset$. Since $V(a)$ and $V(\neg a)$ are open, this implies that $\mathcal{U}(A)$ is Hausdorff. $\square$

We can now improve 2.3.4 with the aid of 3.1.4, as follows:

3.1.5**. Representation theorem for Boolean algebras** (This was originally proved by Marshall Stone in [Sto36].) *Every Boolean algebra* A *is isomorphic to the Boolean algebra* $\mathrm{Clop}(\mathcal{U}(A))$ *of the Boolean space* $\mathcal{U}(A)$.

The isomorphism is given by the map $\mathcal{V}_A : A \longrightarrow \mathrm{Clop}(\mathcal{U}(A))$ *that sends* $a \in A$ *to* $V(a) = \{P \in \mathcal{U}(A) \mid a \in P\}$. $\square$

Theorem 3.1.5 says something remarkable: Given a Boolean algebra A we have constructed the topological space $\mathcal{U}(A)$. Now using 3.1.5 we see that we can reconstruct A (up to isomorphism) from this topological space. The mechanism also works in the opposite direction:

3.1.6**. Representation theorem for Boolean spaces**
Let X be a Boolean space. Then $\mathrm{Clop}(X)$ *is a Boolean algebra of subsets of X and the map $\Theta_X : X \longrightarrow \mathcal{U}(\mathrm{Clop}(X))$ defined by $\Theta_X(x) = \{K \in \mathrm{Clop}(X) \mid x \in K\}$ is a homeomorphism.*

The compositional inverse is given as follows: If $U \in \mathcal{U}(\mathrm{Clop}(X))$, then the intersection $\bigcap U$ has exactly one element and this element is $\Theta_X^{-1}(U)$.

Proof. Firstly we observe that the map Θ_X is indeed well defined, i.e., for $x \in X$ the set $\Theta_X(x)$ is an ultrafilter, also see example 2.2.3(iii). For the rest of the proof we suppress the index X from Θ_X and just write Θ. The essential part of the assertion is the following

Claim 1. For each $U \in \mathcal{U}(\mathrm{Clop}(X))$ there is some $x \in X$ with $\bigcap U = \{x\}$. We write $\Psi(U)$ for this element and obtain a map $\Psi : \mathcal{U}(\mathrm{Clop}(X)) \longrightarrow X$.
Proof of claim 1. As U is a proper filter it has the finite intersection property. Since all elements of U are closed sets and X is compact, we know that $\bigcap U \neq \emptyset$. We need to show that there is at most one point in $\bigcap U$. Suppose for way of contradiction that there are two points $x, y \in \bigcap U$. Since $x \neq y$ and X is Hausdorff, there are open and disjoint neighborhoods O, W of x, y respectively. Since X is Boolean there are clopen subsets K, L of X with $x \in K \subseteq O$ and $y \in L \subseteq W$. From $O \cap W = \emptyset$ we get $K \cap L = \emptyset$ and therefore $(X \setminus K) \cup (X \setminus L) = X$. Since U is a filter we know $X \in U$. However, $X \setminus K$ and $X \setminus L$ are in the Boolean algebra $\mathrm{Clop}(X)$ and so the ultrafilter property of U implies $X \setminus K \in U$ or $X \setminus L \in U$. By symmetry we may assume that $X \setminus L \in U$. But then $\bigcap U \subseteq X \setminus L$ and this contradicts $y \in L \cap \bigcap U$, establishing the claim. $\diamond$

We now proof that Ψ is the compositional inverse of Θ. For $x \in X$ we have $x \in \bigcap \Theta(x)$ by definition of $\Theta(x)$ and so by claim 1 this implies $\Psi(\Theta(x)) = x$. Thus $\Psi \circ \Theta = \mathrm{id}_X$. Further, if $U \in \mathcal{U}(\mathrm{Clop}(X))$ we have $\Theta(\Psi(U)) = \{K \in \mathrm{Clop}(X) \mid \Psi(U) \in K\} \supseteq U$ by definition of $\Psi(U)$; since both U and $\Theta(\Psi(U))$ are ultrafilters we get $\Theta(\Psi(U)) = U$. This shows $\Theta \circ \Psi = \mathrm{id}_{\mathcal{U}(\mathrm{Clop}(X))}$ and so indeed Ψ is the compositional inverse of Θ.

Finally we need to show that Θ is a homeomorphism. It is continuous because for $K \in \mathrm{Clop}(X)$ we have

$$\begin{aligned}
\Theta^{-1}(V(K)) &= \{x \in X \mid \Theta(x) \in V(K)\} \\
&= \{x \in X \mid K \in \Theta(x)\}, \text{ by definition of } V(K) \\
&= \{x \in X \mid x \in K\}, \text{ by definition of } \Theta(x) \\
&= K,
\end{aligned}$$

which is closed, and because every closed sets of $\mathcal{U}(\mathrm{Clop}(X))$ is an intersection of sets of the form $V(K)$ with $K \in \mathrm{Clop}(X)$ (see property $(*)$ in the proof of 3.1.4).

Hence we know that Θ is a continuous bijection between the compact Hausdorff spaces X and $\mathcal{U}(\mathrm{Clop}(X))$ (invoke 3.1.4) and every such map is a homeomorphism by general topology. $\square$

As indicated at the beginning of section 3, both representation theorems 3.1.5 and 3.1.6 can be generalised to all distributive lattices with the appropriate amendments on the topological side.

3.2. **The Stone space as prime spectrum.** This section gives an alternative description of the space of ultrafilters of a Boolean algebra for readers who are familiar with prime spectra of commutative rings. It is not needed in the remainder of the text and can be skipped.

3.2.1**.** Given a Boolean algebra A, we define new binary operations $+$ and $\cdot$ on A by

$$(*)\qquad a+b=(a\wedge\neg b)\vee(b\wedge\neg a)\ \text{and}\ \ a\cdot b=a\wedge b.$$

A straightforward calculation[6] shows that (the set underlying) A together with $+$ and $\cdot$ is a commutative unital ring. We denote this ring by $BR(A)$. Notice that 0 in $BR(A)$ is $\bot$ and 1 in $BR(A)$ is $\top$. Obviously the identity $x^2=x$ holds universally in $BR(A)$ and the Boolean operation of A are defined in terms of $+$ and $\cdot$ by

$$(+)\qquad a\vee b=a+b+a{\cdot}b,\ a\wedge b=a{\cdot}b,\ \neg a=1+a.$$

If B is another Boolean algebra and $\varphi:A\longrightarrow B$ is a map, then by $(*)$ and $(+)$ applied to A and B we see that φ is a Boolean algebra homomorphism if and only if φ is a unital ring homomorphism $BR(A)\longrightarrow BR(B)$.

3.2.2. **Definition.** A unital ring $R=(R,+,\cdot)$ is called a **Boolean ring** if it satisfies $r^2=r$ for all $r\in R$.

Hence in a Boolean algebra A, the operations $(*)$ of 3.2.1 define a Boolean ring. It is an exercise to show that every Boolean ring $R=(R,+,\cdot,1)$ is actually of this form: Define $x\le y$ as $x\cdot y=x$. Then one checks that $\le$ is a partial order with $\bot=0$, $\top=1$, $x\cdot y$ is the infimum of $\{x,y\}$ for $\le$, $x+y+x\cdot y$ is the supremum of $\{x,y\}$ for $\le$ and $1+x$ is the complement of x. We will get this for free from the next proposition.

3.2.3. **Proposition.** *Let R be a Boolean ring.*

(i) The ring R is commutative with $x+x=0$ for all $x\in R$.
(ii) Let $\operatorname{Spec}(R)$ be the prime spectrum of R. Then $\operatorname{Spec}(R)$ is a Boolean space and the map $\rho:R\longrightarrow\operatorname{Clop}(\operatorname{Spec}(R))$, $\rho(r)=\{\mathfrak{p}\in\operatorname{Spec}(R)\mid r\notin\mathfrak{p}\}$ is a ring isomorphism when $\operatorname{Clop}(\operatorname{Spec}(R))$ is considered as a Boolean ring.

Proof. (i). Since $2x=(2x)^2=4x^2=4x$ we get $2x=0$ for all $x\in R$. But then $x+y=(x+y)^2=x^2+xy+yx+y^2=x+y+xy+yx$ implies $xy=-yx=yx$.

(ii). Let A be the subset $\{\rho(r)\mid r\in R\}$ of the powerset of $\operatorname{Spec}(R)$.

Claim. For all $r,s\in R$ we have

(a) $\rho(r{\cdot}s)=\rho(r)\cap\rho(s)$,
(b) $\rho(r)\cup\rho(s)=\rho(r+s)\,\dot\cup\,\rho(r{\cdot}s)$, and
(c) $\rho(1+r)$ is the complement of $\rho(r)$ in $\operatorname{Spec}(R)$.

[6]Alternatively we may invoke 2.3.4 and assume that A is a Boolean algebra of subsets of some set. Then the required calculation consist of routine checking of identities of terms of sets involving the operations $\cap,\cup$ and complementation of sets.

Proof of the claim. (a) hold because for each prime ideal $\mathfrak{p}$ of the ring R we know that $r\cdot s \in \mathfrak{p} \iff r \in \mathfrak{p}$ or $s \in \mathfrak{p}$.

(b). The inclusion $\supseteq$ holds in any ring. For the inclusion $\subseteq$ we use $r = r\cdot(r+s)+(r\cdot s)$ (from (i)) to obtain $\rho(r) \subseteq \rho(r+s) \cup \rho(r\cdot s)$, and similar for $\rho(s)$. To see that $\rho(r+s)$ and $\rho(r\cdot s)$ are disjoint we use (a) and get $\rho(r+s) \cap \rho(r\cdot s) = \rho(r^2 s + srs)$. Since $r^2 s + srs = 0$ using (i), we see that the intersection is $\rho(0) = \emptyset$.

(c). We set $s = 1$ in (b) and get $\rho(r+1) \cup \rho(r) = \rho(r+1+r)$. As $1+r+r = 1$ by (i) and $\rho(1) = \mathrm{Spec}(R)$ we get (c). $\diamond$

Since $\rho(r)$ and $\rho(1+r)$ are open subsets of $\mathrm{Spec}(R)$ by definition of the prime spectrum, we see from (c) that $\rho(r)$ a clopen subset of $\mathrm{Spec}(R)$. Thus ρ is well defined. Since A is a basis of the topology of $\mathrm{Spec}(R)$ and $\mathrm{Spec}(R)$ is a compact T_0-space we obtain that $\mathrm{Spec}(R)$ is Boolean. Furthermore, (a) and (c) imply that A is a Boolean algebra of subsets of $\mathrm{Spec}(R)$. Since every clopen subset is compact and at the same time a union of sets from A we see that $A = \mathrm{Clop}(\mathrm{Spec}(R))$. It follows that ρ is surjective. It is also injective, because for $r \neq s$ we know that $r - s$ is not nilpotent (all powers are nonzero) and therefore there must be some $\mathfrak{p} \in \mathrm{Spec}(R)$ with $r + s = r - s \notin \mathfrak{p}$. Hence $\rho(r+s) \neq \emptyset$ and by (a),(b) this implies $\rho(r) \neq \rho(s)$.

Finally properties (b) and (a) imply that $\rho(r+s)$ is the symmetric difference of $\rho(r)$ and $\rho(s)$. This shows that ρ is a ring homomorphism when $\mathrm{Clop}(\mathrm{Spec}(R))$ is considered as a Boolean ring. $\square$

3.2.4. **Corollary.** *Every Boolean ring R is the Boolean ring of a Boolean algebra A. The operations of A are given by (+) in 3.2.1.*

If $I \subseteq R$, then I is a prime ideal of the ring R if and only if $A \setminus I$ is an ultrafilter. The map

$$\mathrm{Spec}(R) \longrightarrow \mathcal{U}(A),\ I \mapsto A \setminus I$$

is a homeomorphism.

Proof. By 3.2.3 we know that R is isomorphic to the Boolean ring of a Boolean algebra A of subsets of some set. Hence we may assume that R is this ring.

If $I \subseteq R$, then a straight forward calculation shows that I is a prime ideal of the ring R if and only if it is a prime ideal of the distributive lattice A (see 2.2.6(iv) for the definition), and this means that $A \setminus I$ is an ultrafilter of the Boolean algebra A, cf. 2.3.3.

It follows that the map

$$\mathrm{Spec}(R) \longrightarrow \mathcal{U}(A),\ I \mapsto A \setminus I$$

is a bijection. Obviously this bijection maps $\{I \in \mathrm{Spec}(R) \mid r \notin I\}$, $r \in R$, to $V(r)$ and therefore the map is a homeomorphism. $\square$

3.3. **Completeness of Propositional Logic.** The representation theorem 3.1.5 for Boolean algebras can be seen as an algebro-toplogical version of the completeness theorem of Propositional Logic. This is sketched here for the reader who is familiar with Propositional Logic. Let A be the Tarski-Lindenbaum algebra of Propositional Logic as explained in 2.1.2(v) and 2.3.2. The Completeness of Propositional Logic says that for all sentences $s_1,\ldots,s_n,t$ in Propositional Logic the following equivalence holds.

$$(\dagger)\qquad s_1,\ldots,s_n\vdash t \iff \text{for all valuations } w,\ \text{if } w(s_1)=\ldots=w(s_n)=\texttt{true},\ \text{then } w(t)=\texttt{true}.$$

This can be deduced from the representation theorem 3.1.5 as follows. The left hand side of (†) reads in the Tarski-Lindenbaum algebra as

$$(+)\qquad [s_1]_\sim\wedge\ldots\wedge[s_n]_\sim\leq[t]_\sim.$$

In order to interpret the right hand side of (†) we need the following

Claim If w is a valuation of Propositional Logic, then the set $\mathfrak{u}_w=\{[s]_\sim\mid w(s)=\texttt{true}\}$ is an ultrafilter of A. Conversely, if $\mathfrak{u}$ is an ultrafilter of A, then the assignment

$$w_{\mathfrak{u}}(s)=\begin{cases}\texttt{true} & \text{if } [s]_\sim\in\mathfrak{u},\\ \texttt{false} & \text{if } [s]_\sim\notin\mathfrak{u}\end{cases}$$

is a valuation of Propositional Logic. The proof of this claim is lengthy but straightforward and is left to the interested reader.

The claim implies that the assignment $\mathfrak{u}\longrightarrow w_{\mathfrak{u}}$ is a bijection between the set $\mathcal{U}(A)$ and the valuations of propositional calculus; its compositional inverse maps w to $\mathfrak{u}_w$.

Now we see that the right hand side of (†) just says that

$$V([s_1]_\sim)\cap\ldots\cap V([s_n]_\sim)\subseteq V([t]_\sim).$$

But this inclusion is equivalent to (+), because the map $\mathcal{V}_A$ in 3.1.5 is an isomorphism of Boolean algebras.

4. Anti-equivalence of the categories of Boolean algebras and Boolean spaces

The category of distributive lattice is anti-equivalent to a certain category of topological spaces, namely *spectral spaces*, cf. [DST19, Chapter 3]. This has been shown by Marshall Stone in [Sto37] (using different terminology). We focus here on Boolean algebras, which simplifies the matter to some extent, in particular at the topological side of the duality.

Concretely, we put the two representation theorems 3.1.5 and 3.1.6 into the context of category theory, which will tighten further the connection between Boolean algebras and Boolean spaces. The essential work has been done, we only need one further preparation.

4.1. **Lemma.**

(i) *If $\varphi:A\longrightarrow B$ is a homomorphism of Boolean Algebras, then the map*

$$\mathcal{U}(\varphi):\mathcal{U}(B)\longrightarrow\mathcal{U}(A),\ U\longmapsto\varphi^{-1}(U)$$

is continuous and satisfies $\mathcal{U}(\varphi)^{-1}(V(a))=V(\varphi(a))$ for all $a\in A$.

(ii) If $f : X \longrightarrow Y$ is a continuous map between Boolean spaces, then the map

$$\mathrm{Clop}(f) : \mathrm{Clop}(Y) \longrightarrow \mathrm{Clop}(X),\ K \longmapsto f^{-1}(K)$$

is a homomorphism of Boolean algebras.

Proof. (i). Firstly, $\mathcal{U}(\varphi)$ is well-defined: If $U \in \mathcal{U}(B)$, then the unique map $\chi : B \longrightarrow \{\bot, \top\}$ with $\chi^{-1}(\top) = U$ is a homomorphism of Boolean algebras by 2.3.3(i)⇒(iv). Then the composition $\varphi \circ \chi : A \longrightarrow \{\bot, \top\}$ is again a homomorphism of Boolean algebras and by 2.3.3(iv)⇒(i) the set $U_0 := (\varphi \circ \chi)^{-1}(\top)$ is an ultrafilter of A. But $U_0 = \varphi^{-1}(U)$, which shows that $\mathcal{U}(\varphi)$ is well-defined.

By definition of the topologies continuity of $\mathcal{U}(\varphi)$ is proved if we show that $\mathcal{U}(\varphi)^{-1}(V(a)) = V(\varphi(a))$ for all $a \in A$. So let $U \in (U)(B)$. Then

$$\begin{aligned} U \in \mathcal{U}(\varphi)^{-1}(V(a)) &\iff \mathcal{U}(\varphi)(U) \in V(a) \\ &\iff \varphi^{-1}(U) \in V(a) \\ &\iff a \in \varphi^{-1}(U) \\ &\iff \varphi(a) \in U \iff U \in V(\varphi(a)). \end{aligned}$$

(ii). Since f is continuous, $\mathrm{Clop}(f)$ is well-defined. Since the operation of taking preimages of a map commutes with all Boolean operations we see that $\mathrm{Clop}(f)$ is a homomorphism of Boolean algebras. □

4.2. **Definition.** Let `BoolAlg` be the category of Boolean algebras, which has Boolean algebras as objects and homomorphisms of Boolean algebras as morphisms. Let `BoolSp` be the category of Boolean spaces, which has Boolean spaces as objects and continuous maps as morphisms.

4.3. **Definition.** We define functors $\mathcal{U} : \mathtt{BoolAlg} \to \mathtt{BoolSp}$ and $\mathrm{Clop} : \mathtt{BoolSp} \to \mathtt{BoolAlg}$ as follows. The functor $\mathcal{U}$ acts on objects as in 3.1.3 and on morphisms as in 4.1(i). The functor Clop acts on objects as in 3.1.2(i) and on morphisms as in 4.1(ii). The verification that this defines functors is straightforward from the definitions and left to the reader.

4.4. **Stone Duality for Boolean algebras and Boolean spaces**
The category `BoolAlg` *of Boolean algebras and* `BoolSp` *of Boolean spaces are ant-equivalent. Explicitly:*

(i) The anti-equivalence is given by the functors $\mathcal{U} : \mathtt{BoolAlg} \to \mathtt{BoolSp}$ and its quasi-inverse $\mathrm{Clop} : \mathtt{BoolSp} \to \mathtt{BoolAlg}$.
(ii) The assignments given by $A \mapsto \mathcal{V}_A$ in 3.1.5 on Boolean Algebras and given by $X \mapsto \Theta_X$ in 3.1.6 on Boolean spaces define natural transformations $\mathcal{V} : \mathrm{id}_{\mathtt{BoolAlg}} \to \mathrm{Clop} \circ \mathcal{U}$ and $\Theta : \mathrm{id}_{\mathtt{BoolSp}} \to \mathcal{U} \circ \mathrm{Clop}$.
(iii) The natural transformations $\mathcal{V}$ and Θ are isomorphism of functors.

Proof. Item (i) is implied by (ii) and (iii).

(ii) In order to show that $\mathcal{V}$ and Θ are natural transformations we need verify that for every homomorphism $A \to B$ of Boolean algebras and each continuous function $f : X \to Y$ of Boolean spaces, the diagrams

$$\begin{tikzcd}
A \arrow[r, "\varphi"] \arrow[d, "\mathcal{V}_A"'] & B \arrow[d, "\mathcal{V}_B"] \\
\mathrm{Clop}(\mathcal{U}(A)) \arrow[r, "\mathrm{Clop}(\mathcal{U}(\varphi))"'] & \mathrm{Clop}(\mathcal{U}(A))
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\Theta_X"'] & Y \arrow[d, "\Theta_Y"] \\
\mathcal{U}(\mathrm{Clop}(X)) \arrow[r, "\mathcal{U}(\mathrm{Clop}(f))"'] & \mathcal{U}(\mathrm{Clop}(Y))
\end{tikzcd}$$

commute. For the diagram on the left, take $a \in A$. Then $\mathcal{V}_A$ maps a to $V(a)$ and $\mathrm{Clop}(\mathcal{U}(\varphi))$ maps $V(a)$ to $\mathcal{U}(\varphi)^{-1}(V(a))$. But this set is equal to $V(\varphi(a))$ by 4.1(i). Since $V(\varphi(a)) = (\mathcal{V}_B \circ \varphi)(a)$ we have confirmed the commutativity of the diagram on the left.

For the diagram on the right, take $x \in X$. Then $\Theta_X(x) = \{K \in \mathrm{Clop}(X) \mid x \in K\}$ and $\mathcal{U}(\mathrm{Clop}(f))$ maps this ultrafilter to

$$\begin{aligned}
\mathrm{Clop}(f)^{-1}(\Theta_X(x)) &= \{D \in \mathcal{U}(\mathrm{Clop}(Y)) \mid x \in \mathrm{Clop}(f)(D)\} \\
&= \{D \in \mathcal{U}(\mathrm{Clop}(Y)) \mid x \in f^{-1}(D)\} \\
&= \{D \in \mathcal{U}(\mathrm{Clop}(Y)) \mid f(x) \in D\} \\
&= \Theta_Y(f(x)),
\end{aligned}$$

as required.

(iii) is the key statement: The natural transformation $\mathcal{V}$ is an isomorphism by the representation theorem 3.1.5 for Boolean algebras. The natural transformation Θ is an isomorphism by the representation theorem 3.1.6 for Boolean spaces. $\square$

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