An introduction to Real Closed Rings

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1. Real Closed Fields

A field K is **real closed**, if $\{x^2 \mid x \in K\}$ is the positive cone of an ordering of K such that the intermediate value theorem holds for all $P(t) \in K[t]$. Examples: $\mathbb{R}$, $\mathbb{R}_{\text{alg}}$ (real algebraic numbers) and $\mathbb{R}((t^{\frac{1}{\infty}}))$ (Puiseux series).

Let K be a real closed field and let $X \subseteq K^n$. Then X is **semi-algebraic** (s.a.) if it is a Boolean combination of sets of the form

$$\{a \in K^n \mid P(a) \geq 0\}$$

for some $P \in K[t_1, \ldots, t_n]$.

A **semi-algebraic map** $X \longrightarrow K^m$ is a map with semi-algebraic graph.

For example $\sqrt{x} : [0, \infty) \longrightarrow \mathbb{R}$ is semi-algebraic. Its graph is defined by the *formula*

$$y^2 = x \;\&\; y \geq 0.$$

Tarski: Every set/function that is definable in first order predicate logic in the language $\{+, \cdot, \leq\}$ is semi-algebraic. (The talk will not depend in any way on familiarity with this theorem.)

2. Real Closed Rings: Examples

Examples of real closed rings are:

(a) Real closed fields, denoted by K.

$f_K :=$ the function defined by the formula that defines f

(b) Convex valuation rings V of a real closed field.

$f_V := f_K|_{V^n}$

(c) Rings $C(T)$ of continuous, $\mathbb{R}$-valued functions, where T is a top. space.

$f_{C(T)}(F_1, \dots, F_n) := f \circ (F_1, \dots, F_n)$

(d) For a real closed field K and a semi-algebraic set $X \subseteq K^n$ the rings

$$C_{\mathrm{sa}}(X) = \{F : X \longrightarrow K \mid F \text{ is continuous and semi-algebraic}\}$$

$f_{C_{\mathrm{sa}}(X)}(F_1, \dots, F_n) := f_K \circ (F_1, \dots, F_n)$

Common feature of these rings:
If A is any ring as above and $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous, s.a. and <u>defined over $\mathbb{Z}$</u>, then it makes sense to define a "scalar extension" $f_A : A^n \longrightarrow A$. In other words, it makes sense to apply f to n-tuples of A.

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3. Real Closed Rings: Definition

A **real closed structure** on a ring A is a family of functions

$$(f_A : A^n \longrightarrow A \mid n \in \mathbb{N}, f : \mathbb{R}^n \longrightarrow \mathbb{R} \text{ cont., s.a. and defined over } \mathbb{Z}),$$

satisfying the following conditions:

S1: Given $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $g_1, ..., g_n : \mathbb{R}^k \longrightarrow \mathbb{R}$ continuous, s.a., defined over $\mathbb{Z}$, then

$$(f \circ (g_1, ..., g_n))_A = f_A \circ ((g_1)_A, ..., (g_n)_A)$$

S2: If $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is addition/multiplication of $\mathbb{R}$, then f_A is addition/multiplication of A. Similarly, $0_A = 0$ (in A), $1_A = 1$ (in A) and p_A is the i^{th} projection $A^n \longrightarrow A$, when $p : \mathbb{R}^n \longrightarrow \mathbb{R}$ is the i^{th} projection.

3. Real Closed Rings: Definition

(N. Schwartz, 1981. Tampered with by me)

A ring is called **real closed** if there exists a real closed structure on it.

How to use the existence of a real closed structure on a ring A (teaser):

The ring A is reduced, because for $a \in A$ with $a^3 = 0$ we may use the identities $x = \sqrt[3]{x} \circ x^3$ and $\sqrt[3]{x}(0) = 0$ on $\mathbb{R}$ to get $a = 0$.

Here is a direct algebraic definition: A ring A is real closed if it satisfies all of the following three conditions.

(i) The set of squares of A is the set of nonnegative elements of a partial order $\leq$ on A such that $(A, \leq)$ is a reduced f-ring

(ii) Convexity condition: $0 \leq x \leq y \Longrightarrow x^2 \in y \cdot A$.

(iii) For every prime ideal $\mathfrak{p}$ of A, the ring $A/\mathfrak{p}$ is integrally closed and its field of fractions is a real closed field.

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4. Fundamental Theorem

Let A be a real closed ring.

(a) There is a unique real closed structure on A, we refer to it by $(f_A|\ldots)$.

(b) Every ring homomorphism $h : A \longrightarrow B$ between real closed rings, preserves the real closed structures, i.e., $h(f_A(a)) = f_B(h(a))$.

(c) For model theorists: Each f_A is definable in the ring A by some explicit formula in the ring language. In particular, the class of real closed rings is first order axiomatizable.

An important consequence: Every ring R has a **real closure**, i.e., there is a real closed ring $\rho(R)$ and a ring homomorphism $\rho_R : R \longrightarrow \rho(R)$ such that for every real closed ring A and all ring homomorphisms $\varphi : R \longrightarrow A$ there is a unique ring homomorphism $\psi : \rho(R) \longrightarrow A$ making the following diagram commutative:

$$\begin{array}{ccc} \rho(R) & \xrightarrow{\exists! \psi} & A \\ {\scriptstyle \rho_R}\uparrow & \nearrow{\scriptstyle \forall \varphi} & \\ R & & \end{array}.$$

Examples: **(a)** $\rho(\mathbb{C}) = 0$ **(b)** $\rho(\mathbb{Q}(\sqrt{2})) = \mathbb{R}_{\mathrm{alg}} \times \mathbb{R}_{\mathrm{alg}}$

(c) $\rho(\mathbb{R}[x_1, \ldots, x_n]) = C_{\mathrm{sa}}(\mathbb{R}^n)$.

5. Incentive for studying Real Closed Rings

My interest is framed in terms of the following question: Given a category of geometric/topological objects, is there an algebraic category dual to it?

In the example of rings of continuous functions such a duality is

$$\left\{ \begin{array}{c} \text{Real compact spaces} \\ \text{\& continuous functions} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Rings of the form } C(T) \\ \text{\& ring homomorphisms} \end{array} \right\}.$$

For example Lindelöf-spaces and particularly compact spaces are real compact. Hence the duality extends the (real) Gelfand-Naimark-Stone duality.

In algebraic geometry we have

$$\left\{ \begin{array}{c} \text{Varieties } \subseteq \mathbb{C}^n, n \in \mathbb{N} \\ \text{\& polynomial maps} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Affine } \mathbb{C}\text{-algebras} \\ \text{\& } \mathbb{C}\text{-algebra } \textit{homomorphisms} \end{array} \right\},$$

which extends to $\{\text{Affine Schemes}\} \longrightarrow \{\text{Commutative Rings}\}.$

↷

5. Incentive for studying Real Closed Rings

Spaces that are covered by real closed rings are called *affine real closed spaces* (adapting the notion of an affine scheme to the semi-algebraic context).

The full duality extends the one for rings of continuous functions above and restricts in the geometric situation as follows.

$$\left\{ \begin{array}{c} \text{Closed s.a. subsets of } \mathbb{R}^n, n \in \mathbb{N} \\ \text{\& s.a. continuous maps} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{f.g. real closed } \mathbb{R}\text{-algebras} \\ \text{\& } \mathbb{R}\text{-algebra homomorphisms} \end{array} \right\}$$

Here f.g. means: finitely generated in terms of the real closed structure. For example, if $V \subseteq \mathbb{R}^n$ is a variety with coordinate ring R, then $\rho(R) = C_{sa}(V)$ is in the category on the right hand side.

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6. Algebraic properties of real closed rings

Let A be a real closed ring.

(i) If $I \subseteq A$ is a radical ideal, then A/I is real closed.

(ii) If $S \subseteq A$ with $S \cdot S \subseteq S$, then the localization $S^{-1} \cdot A$ is real closed.

Hence if $\mathfrak{p}$ is a prime ideal of A, then the fraction field of $A/\mathfrak{p}$ is real closed.

(iii) Arbitrary products, direct limits and inverse limits of real closed rings, taken in the category of commutative unital rings are again real closed. If B, C are real closed A-algebras, then $\rho(B \otimes_A C)$ is the fibre sum of B, C over A in the category of real closed rings.

(iv) Local real closed rings are local henselian rings. (All real closed domains are local, but in general are not valuation rings.)

(v) Every convex subring of a real closed ring is again real closed.

(vi) There is a largest real closed ring B that contains A as a convex subring. B is called the convex closure of A. For example, the convex closure of $C(\beta\mathbb{R}^n)$ is $C(\mathbb{R}^n)$.

(vii) The Gelfand-Kolmogoroff theorem generalises to real closed rings. It says that for every real closed ring A and every convex subring B, the inclusion $B \hookrightarrow A$ induces a homeomorphism $\mathrm{Max}(A) \longrightarrow \mathrm{Max}(B)$.

7. The prime spectrum of a real closed ring

Reminder: The spectrum of a ring.

(i) For a ring R, let $\mathrm{Spec}(R)$ be the set of prime ideals of R endowed with the Zariski topology having the sets $D(f) = \{\mathfrak{p} \mid f \notin \mathfrak{p}\}$ as a basis of open sets. Recall that the closed sets are then given by the sets $V(I) = \{\mathfrak{p} \mid I \subseteq \mathfrak{p}\}$, where I runs through the ideals of R.

(ii) Specialization in $\mathrm{Spec}(R)$ is given by inclusion, i.e. for $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(R)$ we have $\mathfrak{p} \subseteq \mathfrak{q} \iff \mathfrak{q} \in \overline{\{\mathfrak{p}\}}$.

(iii) Any order theoretic statement made below is w.r.t. to the poset $(\mathrm{Spec}(R), \subseteq)$; for example: The maximal ideals are the closed points.

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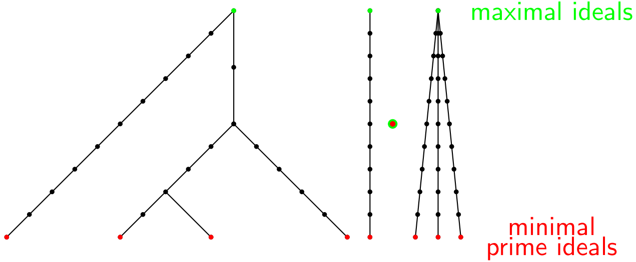
7. The prime spectrum of a real closed ring

Now let A be a real closed ring.

(i) $\operatorname{Spec}(A)$ is a root system, i.e. for each $\mathfrak{p} \in \operatorname{Spec}(A)$ the set $\{\mathfrak{q} \mid \mathfrak{p} \subseteq \mathfrak{q}\}$ is a chain. Example:



(ii) The root systems occurring in this way are precisely those that are jump dense and complete in chains (in particular all finite root systems occur).

(iii) If $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec}(A)$ with $1 \notin \mathfrak{p} + \mathfrak{q}$, then $\mathfrak{p} + \mathfrak{q}$ (sum of ideals) is again in $\operatorname{Spec}(A)$.

7. Interpretation of points and specializations in $\mathrm{Spec}(A)$

(a) Let $X \subseteq \mathbb{R}^n$ be semi-algebraic and locally compact (think of open sets or closed sets) and let $A = C_{sa}(X)$.

(b) Consider the lattice

$$L(X) = \{\text{closed semi-algebraic subsets of } X\}$$

and notice that every $C \in L(X)$ is the zero set $Z(f)$ for some $f \in A$: Take f to be the distance function of C.

(c) Now if $\mathfrak{p} \in \mathrm{Spec}(A)$, then the set $\mathfrak{f}(\mathfrak{p}) = \{Z(f) \mid f \in A\}$ is a prime filter of the lattice $L(X)$. It turns out that

$$\mathrm{Spec}(A) \longrightarrow \mathsf{PrimeFilters}\,(L(X)),\ \mathfrak{p} \mapsto \mathfrak{f}(\mathfrak{p})$$

is a homeomorphism. Here the right hand side is the Stone dual of the distributive lattice $L(X)$.

(d) The map $X \longrightarrow \mathrm{Spec}(A),\ x \mapsto \ker(\mathrm{evaluation}_x)$ is an embedding of topological spaces.

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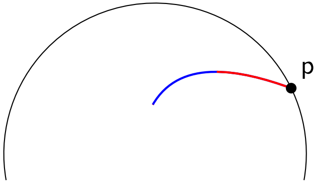
Example: Let X be the closed unit ball and $\gamma : (0,1) \longrightarrow X$ a s.a. curve with $\lim_{t\to 0} \gamma(t) = p$. Let $A = C_{sa}(X)$ and

$$\mathfrak{p} = \{f \in A \mid f \text{ vanishes on a small segment of } \gamma \text{ near } p\}.$$



Then the filter corresponding to $\mathfrak{p}$ is the prime filter of closed semi-algebraic subsets of X containing some red segment of γ.

The ideal $\mathfrak{p}$ specializes to the maximal ideal $\mathfrak{m}_p = \{f \in A \mid f(p) = 0\}$ of A and $A/\mathfrak{p}$ is a valuation ring.

For arbitrary X, the maximal length of chains in $\mathrm{Spec}(A)$ is the dimension of X (like in classical algebraic geometry).

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