ADVANCED CLASS: INTRODUCTION TO NIP

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Abstract. We give a self contained introduction to theories and formulas with the independence property and prove a theorem of Shelah (following Pillay) on externally definable sets in theories with the NIP.

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1. Definitions and examples

Let T be an $\mathscr{L}$-theory and let $\varphi(\bar{x},\bar{y})$ be an $\mathscr{L}$-formula. We say that φ has the **independence property w.r.t.** $\bar{x}$,$\bar{y}$ if in some model M of T there are

$$\bar{b}_S \in M^{\bar{x}} \text{ and } \bar{a}_i \in M^{\bar{y}}\ (i\in\omega, S\subseteq\omega)$$

such that

$$M\models\varphi(\bar{b}_S,\bar{a}_i) \iff i\in S.$$

Whenever the partitioning of the variables is clear we shall simply say φ has the independence property. If φ does not have the independence property then we say φ has the **NIP**. A theory has NIP if all formulas have NIP. Observe that in this case, also all formulas with parameters in a model of T have NIP (if $\varphi(\bar{x},\bar{y},\bar{c}))$ has IP, then $\varphi(\bar{x},\bar{y},\bar{z}))$ has IP w.r.t. $\bar{x}$ and $(\bar{y},\bar{z})$).

By compactness, φ has the independence property if for every finite set F there are a model M of T and

$$\bar{b}_S \in M^{\bar{x}} \text{ and } \bar{a}_i \in M^{\bar{y}}\ (i\in F, S\subseteq F)$$

Date: January 4, 2025.
2000 *Mathematics Subject Classification.* Primary: XXXXX, Secondary: XXXXX.

such that
$$M \models \varphi(\bar{b}_S, \bar{a}_i) \iff i \in S.$$
Syntactically, this means that T is consistent with the sentence
$$\exists \bar{x}_S (S \subseteq F)\ \exists \bar{y}_i (i \in F)\ \big[\bigwedge_{i \in S \subseteq F} \varphi(\bar{x}_S, \bar{y}_i) \wedge \bigwedge_{S \subseteq F, i \in F \setminus S} \neg\varphi(\bar{x}_S, \bar{y}_i) \big]$$

If M is a model of T and $\varphi(\bar{x}, \bar{y})$ is a formula with parameters in M, then we say φ has the independence property, if φ has the independence property with respect to $Th(M, M)$. This is the case if and only if for every $k \in \mathbb{N}$ there is a sequence $\bar{a}_1, ..., \bar{a}_k \in M^n$ such that for every subset $S \subseteq \{1, ..., k\}$ the formula
$$\bigwedge_{i \in S} \varphi(\bar{x}, \bar{a}_i) \wedge \bigwedge_{i \in \{1,...,k\} \setminus S} \neg\varphi(x, \bar{a}_i)$$
is satisfiable im M.

Again by compactness, if $\varphi(\bar{x}, \bar{y})$ has the independence property, then for every set I there is a model M of T and
$$\bar{a}_i \in M^{\bar{y}}, \bar{b}_S \in M^{\bar{x}}\ (i \in I, S \subseteq I)$$
such that
$$M \models \varphi(\bar{b}_S, \bar{a}_i) \iff i \in S.$$

1.1. **Proposition.** *If φ has the independence property w.r.t. $\bar{x}$, $\bar{y}$ then φ also has the independence property w.r.t. $\bar{y}$, $\bar{x}$.*

Proof. Pick $k \in \mathbb{N}$. We apply the independence property of φ w.r.t. $\bar{x}, \bar{y}$ to the finite set 2^k of subsets of $\{1, ..., k\}$: For each $T \in 2^k$ and each $S \subseteq 2^k$ there are $\bar{a}_T \in M^{\bar{y}}$ and $\bar{b}_S \in M^{\bar{x}}$ such that
$$M \models \bigwedge_{T \in S \subseteq 2^k} \varphi(\bar{b}_S, \bar{a}_T) \wedge \bigwedge_{T \in 2^k \setminus S} \neg\varphi(\bar{b}_S, \bar{a}_T).$$

For each $i \in k$ let $S(i) = \{Z \subseteq \{1, ..., k\} \mid i \in Z\} \subseteq 2^k$ and take $\bar{c}_i = \bar{b}_{S(i)}$. Then
$$M \models \bigwedge_{T \in S(i),\ i \in \{1,...,k\}} \varphi(\bar{b}_{S(i)}, \bar{a}_T) \wedge \bigwedge_{T \in 2^k \setminus S(i),\ i \in \{1,...,k\}} \neg\varphi(\bar{b}_{S(i)}, \bar{a}_T),$$
since this formula is a subformula of the one above. Since $T \in S(i)$ means $i \in T$, this shows that φ also has the independence property w.r.t. $\bar{y}$,$\bar{x}$. □

1.2. *Example.* The binary relation "y divides x" on $\mathbb{N}$ has the independence property. To see this take $k \in \mathbb{N}$, let $a_1, ..., a_k$ be an enumeration of the first k prime numbers and let $b_S = \prod_{i \in S} a_i$ for each $S \subseteq \{1, ..., k\}$. Then $i \in S \iff a_i$ divides b_S.

1.3. *Example.* In every infinite boolean algebra A, the relation $x \geq y$ has the independence property. To see this take $k \in \mathbb{N}$ and let $a_1, ..., a_k \in A$ be different from $\bot$ with $a_i \wedge a_j = \bot\ (i \neq j)$. Then with $b_S = \bigvee_{i \in S} a_i$, $S \subseteq \{1, ..., k\}$ we have $i \in S \iff b_S \geq a_i$. Note that the first order theory of a fixed boolean algebra A is well behaved (e.g. saying that A has no atoms gives a complete $\aleph_0$-categorical theory which is decidable).

1.4. *Example.* Stable theories have NIP. This is clear depending how one defines "stable". For example the definition "T stable $\iff$ no formula has the order property w.r.t T" immediately implies that stable theories have the NIP. Recall that a formula $\varphi(\bar{x}, \bar{y})$ has the order property if there are $\bar{x}$-tuples $\bar{b}_i$, $\bar{y}$-tuples $\bar{a}_j$ from some model of T with $M \models \varphi(\bar{b}_i, \bar{a}_j) \iff i \leq j$ $(i, j \in \omega)$. Notice that $i \leq j$ is equivalent to $i \in \{j' \mid j' \leq j\}$, hence a formula with the independence property also has the order property.

1.5. *Example.* More examples of theories with the NIP: p-adically closed fields, alg. closed valued fields, more generally c-minimal theories have NIP. Simple theories which have the NIP are stable (Reference missing); e.g. pseudo finite fields do have the independence property (see also [Dur])

1.6. *Example.* O-minimal structures have NIP. This will be proved in 7.2 as an easy consequence of 7.1 below.

2. Ramsey's theorem

2.1. *Notation.* Let X be a set and let $n \in \mathbb{N} = \{1, 2, 3, ...\}$. We write

$$[X]^n = \{S \subseteq X \mid |S| = n\}.$$

2.2. **Theorem.** *If X is infinite and $[X]^n = A_0 \cup A_1$, then there is an infinite subset Y of X with $[Y]^n \subseteq A_0$ or $[Y]^n \subseteq A_1$*

Proof. By induction on n, where $n = 1$ is trivial. Assume we know the assertion for $n-1$. Given an infinite subset Z of X, some element $z \in Z$ and $\delta \in \{0, 1\}$ we define

$$B_\delta(Z, z) := \{S \subseteq Z \setminus \{z\} \mid |S| = n-1 \text{ and } S \cup \{z\} \in A_\delta\}.$$

As $[X]^n = A_0 \cup A_1$, we have

$$[Z \setminus \{z\}]^{n-1} = B_0(Z, z) \cup B_1(Z, z)$$

and the induction hypothesis gives an infinite subset $Y_{Z,z}$ of $Z \setminus \{z\}$ and some $\delta_{Z,z} \in \{0, 1\}$ with

$$[Y_{Z,z}]^{n-1} \subseteq B_{\delta_{Z,z}}(Z, z).$$

We now iterate: Define a sequence $(Z_k, z_k, \delta_k)_{k \in \mathbb{N}}$, as follows: Choose $Z_1 := X$ and $z_1 \in Z_1$ arbitrarily. If (Z_k, z_k) is already defined, then define

$$Z_{k+1} = Y_{Z_k,z_k},\ \delta_k = \delta_{Z_k,z_k} \text{ and take } z_{k+1} \in Z_{k+1} \text{ arbitrarily.}$$

Hence

$$\boxed{z_{k+1} \in Z_{k+1} \subseteq Z_k \setminus \{z_k\} \text{ and } [Z_{k+1}]^{n-1} \subseteq B_{\delta_k}(Z_k, z_k).}$$

By symmetry, we may assume that there are infinitely many $k \in \mathbb{N}$ with $\delta_k = 0$. Then $Y = \{z_k \mid \delta_k = 0\}$ satisfies $[Y]^n \subseteq A_0$: First notice that $z_i \neq z_j$ for $i \neq j$, since $z_{k+1} \in Z_{k+1} \subseteq Z_k \setminus \{z_k\}$ for all k. In particular Y is infinite.

Now take $S \in [Y]^n$ and let $k_1 < \ldots < k_n \in \mathbb{N}$ with $S = \{z_{k_1}, \ldots, z_{k_n}\}$. Then

$$\{z_{k_2}, ..., z_{k_n}\} \subseteq Z_{k_2} \cup \ldots \cup Z_{k_n} \subseteq Z_{k_1+1}, \text{ hence}$$

$$\{z_{k_2}, ..., z_{k_n}\} \in [Z_{k_1+1}]^{n-1} \subseteq B_{\delta_{k_1}}(Z_{k_1}, z_{k_1}) = B_0(Z_{k_1}, z_{k_1}).$$

But this means $S = \{z_{k_1}\} \cup \{z_{k_2}, ..., z_{k_n}\} \in A_0$.

Second proof. We show that for every map $f : [X]^n \longrightarrow \{1, \ldots, k\}$ there is an infinite subset $Y \subseteq X$ such that f is constant on $[Y]^n$.

By induction on n, where $n = 1$ is clear.

$n \to n+1$. Pick $x^* \in X$. We have a map $g^* : [X \setminus \{x^*\}]^n \longrightarrow \{1, \ldots, k\}$, $g(S) = f(S \cup \{x^*\})$ and by induction there is an infinite set $X^* \subseteq X$ such that g is constant on $[X^*]^n$; hence there is some $d \in \{1, \ldots, k\}$ such that for all $S \in [X^*]^n$ we have $f(S \cup \{x^*\}) = d$.

We define $X_0 = X^*, x_0 = x^*$ and by induction $x^*_{i+1} \in X^*_i$ (arbitrarily) and $X_{i+1} = (X_i)^*$ (where we use the function $f|_{[X_i]^{n+1}}$). By construction, for each $i \in \mathbb{N}$ there is some $d_i \in \{1, \ldots, k\}$ such that $f_i(S \cup \{x_i\}) = d_i$ for all $S \in [X_i]^n$,

Now take some $d \in \{1, \ldots, k\}$ such that $I = \{i \in \mathbb{N} \mid d_i = d\}$ is infinite. Then the set $Y = \{x_i \mid i \in I\}$ has the required property.

□

We don't need, but state the Erdös-Rado Theorem:

2.3. **Theorem.** *(cf. [Hodges1993], section 11.1)*
Let λ be an infinite cardinal and let $k \in \mathbb{N}$. Let $\beth_{k-1}(\lambda)$ (pronounce"beth") be the $(k-1)$-st iteration of the function $\kappa \mapsto 2^\kappa$ starting with $\kappa = \lambda$.

If I is a set of size $> \beth_{k-1}(\lambda)$ and $[I]^k = \bigcup_{\alpha<\lambda} A_\alpha$, then there is a subset J of I of size (at least) λ^+ and some $\alpha < \lambda$ with $[J]^k \subseteq A_\alpha$.

2.4. *Remark.* One might ask whether there is a general partition theorem similar to 2.3 for infinite cardinals k. The answer is mainly negative. Here a striking statement proved in [EHMR], Theorem 12.1:

For all $\omega \leq \kappa \leq \lambda$ there is a partition of $[\lambda]^\kappa$ into 2^κ sets such that for every $J \subseteq \lambda$ of size κ, the set $[J]^\kappa$ intersects each member of the partition.

3. Indiscernible sequences

3.1. **Definition.** Let $(I, <)$ be a totally ordered set, $n \in \mathbb{N}$ and let $(\bar{a}_i)_{i\in I} \subseteq M^n$, M an $\mathscr{L}$-structure. Let Γ be a subset of $\mathrm{Fml}\,\mathscr{L}(M)$. We say that $(\bar{a}_i)_{i\in I}$ is a **Γ-indiscernible sequence** if for all $i_1 < ... < i_k$ and all $j_1 < ... < j_k$ from I we have

$$M \models \gamma(\bar{a}_{i_1}, ..., \bar{a}_{i_k}) \iff M \models \gamma(\bar{a}_{j_1}, ..., \bar{a}_{j_k}) \quad (\gamma(\bar{x}_1, ..., \bar{x}_k) \in \Gamma)$$

If $\Gamma = \mathrm{Fml}\,\mathscr{L}(A)$ with $A \subseteq M$ then we say "indiscernible sequence over A". If $\Gamma = \mathrm{Fml}\,\mathscr{L}$ then we say "indiscernible sequence".

We have the following corollary to Ramsey's theorem 2.2:

3.2. **Corollary.** *Given a finite subset Γ of $\mathrm{Fml}\,\mathscr{L}(M)$, every infinite sequence $(\bar{a}_i)_{i\in I} \subseteq M^n$ contains a Γ-indiscernible subsequence.*

Proof. As Γ is finite we may by induction assume that Γ is a singleton, say $\Gamma = \{\gamma(\bar{x}_1, ..., \bar{x}_k)\}$. Let

$$A_0 = \{\{i_1, ..., i_k\} \in [I]^k \mid i_1 < ... < i_k \text{ and } M \models \gamma(\bar{a}_{i_1}, ..., \bar{a}_{i_k})\}$$

and

$$A_1 = \{\{i_1, ..., i_k\} \in [I]^k \mid i_1 < ... < i_k \text{ and } M \models \neg\gamma(\bar{a}_{i_1}, ..., \bar{a}_{i_k})\}.$$

Then $[I]^k = A_0 \cup A_1$ and by 2.2, there is some infinite $J \subseteq I$ such that $[J]^k \subseteq A_0$, say. Clearly $(\bar{a}_j)_{i\in J}$ is Γ-indiscernible. □

3.3. *Remark.* One might wonder whether there is a cardinal λ such that every sequence of length λ from some model of T contains an infinite indiscernible subsequence. This is unlikely in general by 2.4 applied to $\kappa = \omega$ there. On the other hand in [TenZie2012, top of page 116] (also see [TenZie2012, Lemma 7.2.12]) we find

> The existence of a Ramsey cardinal $\kappa > \sup_{n<\omega} |S_n(A)|$ (see p. 210) would directly imply that any sequence of order type κ contains a countable indiscernible subsequence (in fact even an indiscernible subsequence of size κ).

In general, a central tool to produce indiscernible sequences out of a given sequence is explained next.

3.4. **Definition.** Let $s = (\bar{b}_i)_{i\in I}$ be a sequence of n-tuples of some structure M indexed by a chain I and let $A \subseteq M$. Then the **Ehrenfeucht-Mostowski type** of s over A is defined as

$$\mathrm{EM}(s/A) = \{\varphi(\bar{x}_1, \ldots, \bar{x}_k) \in \mathrm{Fml}\,\mathscr{L}(A) \mid k < \omega, M \models \varphi(\bar{b}_{i_1}, \ldots, \bar{b}_{i_k}) \text{ for all } i_1 < \ldots < i_k \in I\},$$

where $\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots$ are distinct n-tuples of variables. If $A = \emptyset$ we just write $\mathrm{EM}(s)$. Observe that every $\mathscr{L}(A)$-formula $\varphi(\bar{x})$ with $M \models \varphi(\bar{a}_i/A)$ $(i \in I)$ is in $\mathrm{EM}(s/A)$.

3.5. **Ehrenfeucht-Mostowski Theorem** *Let $s = (\bar{b}_i)_{i\in I}$ be an infinite sequence of n-tuples of some structure M indexed by a chain I and let $A \subseteq M$. Then for every infinite chain J there is an A-indiscernible sequence t indexed by J in some elementary extension of M with* $\mathrm{EM}(s/A) \subseteq \mathrm{EM}(t/A)$.

Proof. We may assume that $A = \emptyset$. Pick new n-tuples of constants $\bar{c}_j$ for $j \in J$ and consider the following set of $\mathscr{L}(\bar{c}_j \mid j \in J)$-sentences:

$$\Phi = \{\varphi(\bar{c}_{j_1}, \ldots, \bar{c}_{j_k}) \mid j_1 < \ldots < j_k \in J \text{ and } \varphi(\bar{x}_1, \ldots, \bar{x}_k) \in \mathrm{EM}(s)\}$$

$$\Psi = \{\psi(\bar{c}_{j_1}, \ldots, \bar{c}_{j_k}) \leftrightarrow \psi(\bar{c}_{n_1}, \ldots, \bar{c}_{n_k}) \mid j_1 < \ldots < j_k, n_1 < \ldots < n_k \in J \text{ and } \varphi(\bar{x}_1, \ldots, \bar{x}_k) \in \mathrm{Fml}(\mathscr{L})\}.$$

By compactness it suffices to show that $\Phi \cup \Psi$ is finitely satisfiable in M. Let $\Phi_0 \subseteq \Phi$, $\Psi_0 \subseteq \Psi$ be finite sets and choose $\Gamma \subseteq \mathrm{Fml}(\mathscr{L})$ finite such that each sentence in Ψ_0 is of the form $\psi(\bar{c}_{j_1}, \ldots, \bar{c}_{j_k}) \leftrightarrow \psi(\bar{c}_{n_1}, \ldots, \bar{c}_{n_k})$ for some $\psi(\bar{x}_1, \ldots, \bar{x}_k) \in \Gamma$ and some constants $\bar{c}_{j_i}, \bar{c}_{n_i}$. By 3.2 there is a subsequence of s_0 that is Γ_0-indiscernible. Since $\mathrm{EM}(s) \subseteq \mathrm{EM}(s_0)$ it is now clear that a long enough initial subsequence of s_0 realizes $\Phi_0 \cup \Psi_0$. □

3.6. **Definition.** Let $s := (\bar{a}_i)_{i\in\lambda} \subseteq M^n$, where λ is an ordinal and let $\varphi(\bar{x}, \bar{y})$ be an $\mathscr{L}$-formula. We say that s **is split by** $\varphi(\bar{x}, \bar{y})$ if in some elementary extension N of M there is some $\bar{b} \in N^{\bar{x}}$ such that

$$\{i \in I \mid \models \varphi(\bar{b}, \bar{a}_i)\} \text{ and } \{i \in I \mid \models \neg\varphi(\bar{b}, \bar{a}_i)\} \text{ are cofinal in } I$$

If s is split by some $\mathscr{L}$-formula, then we say that s is **splittable**.

3.7. *Remark.* $s := (\bar{a}_i)_{i\in\lambda} \subseteq M^n$ is unsplittable, then

$$\vec{\mathrm{tp}}(s/M) := \{\psi(\bar{x}, \bar{b}) \mid \psi(\bar{x}, \bar{y}) \in \mathrm{Fml}\,\mathscr{L}, \bar{b} \in M^{\bar{y}} \text{ and } M \models \psi(\bar{a}_i, \bar{b}) \text{ for } i \to \infty\}$$

is a complete n-type of M, called the **average type of** s.
Hence s is unsplittable if and only if s converges in $S_n(N)$ for every $N \succ M$.

3.8. **Proposition.** *The following are equivalent for every formula $\varphi(\bar{x}, \bar{y})$.*

(i) *$\varphi(\bar{x}, \bar{y})$ has the independence property*

(ii) *For every cardinal λ, there is an indiscernible sequence $(\bar{a}_i)_{i\in\lambda}$ of some model of T which is split by φ.*

(iii) *There is an indiscernible sequence $(\bar{a}_i)_{i\in\omega}$ of some model of T which is split by φ.*

(iv) *For every $k \in \mathbb{N}$ there is a k-indiscernible sequence $(\bar{a}_i)_{i\in\omega}$ of some model of T which is split by φ. Here,"k-indiscernible" means that for all $i_1 < ... < i_k$ and all $j_1 < ... < j_k$ from I we have*

$$M \models \gamma(\bar{a}_{i_1}, ..., \bar{a}_{i_k}) \iff M \models \gamma(\bar{a}_{j_1}, ..., \bar{a}_{j_k}) \quad (\gamma(\bar{x}_1, ..., \bar{x}_k) \in \Gamma).$$

Hence we do not demand that

$$M \models \gamma(\bar{a}_{i_1}, ..., \bar{a}_{i_l}) \iff M \models \gamma(\bar{a}_{j_1}, ..., \bar{a}_{j_l}) \quad (\gamma(\bar{x}_1, ..., \bar{x}_l) \in \Gamma).$$

when $l > k$.

Proof. (i)⇒(ii). By compactness it is enough to show that for every finite subset Γ of $\mathrm{Fml}\,\mathscr{L}$, there are a model M, some Γ-indiscernible sequence $\bar{a}_1 < \bar{a}_2 < ... \in M^n$ and some $\bar{b} \in M^{\bar{x}}$ such that

$$M \models \varphi(\bar{b}, \bar{a}_{2i}) \wedge \neg\varphi(\bar{b}, \bar{a}_{2i+1}) \ (i < \omega). \tag{$*$}$$

Since $\varphi(\bar{x}, \bar{y})$ has the independence property, there are $\bar{a}_0, \bar{a}_1, ...$ and $\bar{b}_S$ $(S \subseteq \omega)$ from some model such that

$$M \models \varphi(\bar{b}_S, \bar{a}_i) \iff i \in S. \tag{†}$$

By 3.2 there is an infinite Γ-indiscernible subsequence $(\bar{a}_j)_{j \in J}$ of $(\bar{a}_i)_{i\in\omega}$ (hence $J \subseteq \omega$). By replacing ω with J, property (†) remains true. Hence we may assume that $(\bar{a}_i)_{i\in\omega}$ itself is Γ-indiscernible. It remains to find $\bar{b} \in M^{\bar{x}}$ satisfying $(*)$. We pick $S = \{2i \mid i \in \omega\}$ and $\bar{b} = \bar{b}_S$. Then $(*)$ is an instance of (†).

(ii)⇒(iii) and (iii)⇒(iv) are weakenings.

(iv)⇒(i). By compactness it is enough to find for every $k \in \mathbb{N}$, n-tuples $\bar{a}_1, ..., \bar{a}_k$ in some model of T such that for every subset $S \subseteq \{1, ..., k\}$ the formula

$$\bigwedge_{i\in S} \varphi(\bar{x}, \bar{a}_i) \wedge \bigwedge_{i\in\{1,...,k\}\setminus S} \neg\varphi(\bar{x}, \bar{a}_i)$$

is satisfiable im M.

Fix $k \in \mathbb{N}$ and take a k-indiscernible sequence $(\bar{a}_i)_{i\in\omega}$ of some model M of T which is split by φ. By switching to a subsequence we may assume that for some $\bar{b} \in M^{\bar{x}}$ we have

$$M \models \varphi(\bar{b}, \bar{a}_{2i}) \wedge \neg\varphi(\bar{b}, \bar{a}_{2i+1}) \ (i < \omega). \tag{$*$}$$

We show that for every subset $S \subseteq \{1, ..., k\}$ the formula

$$\bigwedge_{i\in S} \varphi(\bar{x}, \bar{a}_i) \wedge \bigwedge_{i\in\{1,...,k\}\setminus S} \neg\varphi(\bar{x}, \bar{a}_i)$$

is satisfiable im M. In other words, $(\bar{a}_1, ..., \bar{a}_k)$ realizes

$$\psi(\bar{y}_1, ..., \bar{y}_k) := \exists\bar{x}\ \big[\bigwedge_{i\in S} \varphi(\bar{x}, \bar{y}_i) \wedge \bigwedge_{i\in\{1,...,k\}\setminus S} \neg\varphi(\bar{x}, \bar{y}_i)\big].$$

By $(*)$, the tuple $\bar{b}$ satisfies

$$\bigwedge_{i\in S} \varphi(\bar{x}, \bar{a}_{2i}) \wedge \bigwedge_{i\in\{1,...,k\}\setminus S} \neg\varphi(\bar{x}, \bar{a}_{2i+1}).$$

Hence with $j_i = \begin{cases} 2i & \text{if } i \in S \\ 2i+1 & \text{if } i \notin S \end{cases}$ we have a sequence $j_1 < ... < j_k$ such that $(\bar{a}_{j_1}, ..., \bar{a}_{j_k})$ realizes $\psi(\bar{y}_1, ..., \bar{y}_k)$. Since $(\bar{a}_i)_{i\in\omega}$ is k-indiscernible, also $(\bar{a}_1, ..., \bar{a}_k)$ realizes $\psi(\bar{y}_1, ..., \bar{y}_k)$, as desired. □

The interest in condition (iv) of 3.8 lies in the Erdös-Rado Theorem 2.3, which can be used to show that there is a cardinal λ so that every sequence of length λ has an infinite k-indiscernible subsequence (the proof is very similar to the proof of 3.2). We do not use this later on.

3.9. **Theorem.** *If T has the independence property, then there is a formula $\varphi(x,\bar{y})$ (where x is a single variable), which has the independence property.*

Proof. (cf. [Kud]).

We start with a formula $\varphi(\bar{x},\bar{y})$, which has the independence property. By 3.8(i)⇒(ii) applied to $\lambda = (\operatorname{card} \mathscr{L})^+$ there is an indiscernible sequence $(\bar{a}_i)_{i\in\lambda}$ of some model M of T which is split by φ. By switching to a subsequence we may assume that there is some $\bar{b} \in M^{\bar{x}}$ with

$$M \models \varphi(\bar{b},\bar{a}_{2i}) \wedge \neg\varphi(\bar{b},\bar{a}_{2i+1}). \qquad (*)$$

for all $i < \lambda$. Recall that $2i$ is the ordinal $\alpha + 2n$ where $i = \alpha + n$, α is a limit ordinal and $n < \omega$. Let $(\lambda_j)_{j<\lambda}$ be the strictly increasing enumeration of the limit ordinals in λ. We write $\bar{x} = (\bar{u}, v)$ and $\bar{b} = (\bar{c}, d)$. Suppose $\varphi(\bar{u}, v, \bar{y})$ does not have the independence property with respect to $\bar{u}$ and $(v, \bar{y})$.

Fix $j < \lambda$. Since $(*)$ holds for all indices $i \in \lambda$ between λ_j and λ_{j+1}, and $\varphi(\bar{u}, v, \bar{y})$ does not have the independence property, 3.8(iii)⇒(i) says that the sequence $(d, \bar{a}_i)_{\lambda_j \le i \le \lambda_{j+1}}$ is not indiscernible. This means that for some $\mathscr{L}$-formula $\gamma_j(v, \bar{y}_1, ..., \bar{y}_{l_j})$ there are indices $\lambda_j \le i_j(1) < ... < i_j(l_j) \le \lambda_{j+1}$ and $\lambda_j \le k_j(1) < ... < k_j(l_j) \le \lambda_{j+1}$ such that

$$\models \gamma_j(d, \bar{a}_{i_j(1)}, ..., \bar{a}_{i_j(l_j)}) \wedge \neg\gamma_j(d, \bar{a}_{k_j(1)}, ..., \bar{a}_{k_j(l_j)}). \qquad (+)$$

Since there are only card($\mathscr{L}$)-many formulas, there must be infinitely many j such that γ_j (and l_j) is independent of j. We may assume that this happens for all $j < \omega$. We write $\gamma(v, \bar{y}_1, ..., \bar{y}_l)$ instead of $\gamma_j(v, \bar{y}_1, ..., \bar{y}_{l_j})$ and claim that $\gamma(v, \bar{y}_1, ..., \bar{y}_l)$ has the independence property with respect to v and $(v, \bar{y}_1, ..., \bar{y}_l)$:

To see this, we use 3.8(iii)⇒(i). Define

$$\bar{c}_{2j} = (\bar{a}_{i_{2j}(1)}, ..., \bar{a}_{i_{2j}(l)}) \text{ and } \bar{c}_{2j+1} = (\bar{a}_{k_{2j+1}(1)}, ..., \bar{a}_{k_{2j+1}(l)}) \ (j < \omega).$$

Since $(\bar{a}_i)_{i\in\lambda}$ is indiscernible it is clear that also $(\bar{c}_j)_{j<\omega}$ is indiscernible and by (+), $\gamma(v, \bar{y}_1, ..., \bar{y}_l)$ splits this sequence. □

4. Indiscernible sequences from coheirs

4.1. **Definition.** Let $M \prec N$ and let $M \subseteq A \subseteq N$. An n-type $q \in S_n(A)$ of A is called a **coheir over** M, if every $\varphi(\bar{x}) \in q$ is satisfiable in M. In this case we say that q is a coheir of $q \restriction M$.

Observe that every n-type q over $A \supseteq M$ which is a coheir over M has an extension r on any $B \supseteq A$, which is a coheir over M: Any r containing

$$q \cup \{\neg\varphi(\bar{x}) \in \operatorname{Fml} \mathscr{L}_n(B) \mid \varphi(\bar{x}) \text{ is not satisfiable in } M\}$$

is such a coheir.

4.2. **Lemma.** *Let $M \prec N$, let $q \in S_n(N)$ be a coheir over M and let $I = (I, <)$ be a totally ordered set. For $i \in I$ let $\bar{a}_i \in N^n$ be such that $\bar{a}_i \models q \restriction (M \cup \{\bar{a}_j \mid j < i\})$ for all $i \in I$. Then $s = (\bar{a}_i)_{i\in I}$ is an indiscernible sequence over M.*

Proof. Firstly, notice that our assumption implies

$$\bar{a}_k \models q \restriction (M \cup \{\bar{a}_j \mid j < i\}) \text{ for all } i \le k \in I. \qquad (*)$$

Let $i_1 < ... < i_k$ and $j_1 < ... < j_k$ be finite sequences from I. We have to show that

$$\operatorname{tp}(\bar{a}_{i_1}, ..., \bar{a}_{i_k}/M) = \operatorname{tp}(\bar{a}_{j_1}, ..., \bar{a}_{j_k}/M). \qquad (+)$$

Claim. (+) holds true if $i_1 = j_1, ..., i_{k-1} = j_{k-1}$.
The claim holds true, since by (∗),

$$\operatorname{tp}(\bar{a}_{i_k}/M\bar{a}_{i_1}, ..., \bar{a}_{i_{k-1}}) = \operatorname{tp}(\bar{a}_{j_k}/M\bar{a}_{j_1}, ..., \bar{a}_{j_{k-1}}),$$

which gives (+) in the case $i_1 = j_1, ..., i_{k-1} = j_{k-1}$.

In order to show (+) we do an induction on k, where the case $k = 1$ holds true by (∗). Assume we know (+) for $k - 1$. We may first apply the claim and enlarge i_k, j_k such that $i_k = j_k$. We write $i := i_k = j_k$ Suppose (+) fails. Then there is some $\gamma(\bar{x}_1, ..., \bar{x}_k) \in \operatorname{Fml}\mathscr{L}(M)$ such that

$$N \models \gamma(\bar{a}_{i_1}, ..., \bar{a}_{i_{k-1}}, \bar{a}_i) \wedge \neg\gamma(\bar{a}_{j_1}, ..., \bar{a}_{j_{k-1}}, \bar{a}_i).$$

Since $i > i_1, ..., i_{k-1}, j_1, ..., j_{k-1}$ we get

$$\gamma(\bar{a}_{i_1}, ..., \bar{a}_{i_{k-1}}, \bar{x}_k) \wedge \neg\gamma(\bar{a}_{j_1}, ..., \bar{a}_{j_{k-1}}, \bar{x}_k) \in \operatorname{tp}(\bar{a}_i/M \cup \{\bar{a}_j \mid j < i\}).$$

By definition, q extends $\operatorname{tp}(\bar{a}_i/M \cup \{\bar{a}_j \mid j < i\})$. Since q is a coheir over M, there is some $\bar{m} \subseteq M$ with

$$N \models \gamma(\bar{a}_{i_1}, ..., \bar{a}_{i_{k-1}}, \bar{m}) \wedge \neg\gamma(\bar{a}_{j_1}, ..., \bar{a}_{j_{k-1}}, \bar{m}).$$

Consequently $\operatorname{tp}(\bar{a}_{i_1}, ..., \bar{a}_{i_{k-1}}/M) \neq \operatorname{tp}(\bar{a}_{j_1}, ..., \bar{a}_{j_{k-1}}/M)$, which contradicts (+) in the case $k - 1$. □

Here is an example how 4.2 can be used:

4.3. **Proposition.** [Poizat2000, Lemma 12.36] *Let $\varphi(\bar{x}, \bar{y})$ be an $\mathscr{L}$-formula, $n = |\bar{x}| = |\bar{y}|$, and suppose there is a sequence $(\bar{a}_i)_{i<\omega}$ in M^n such that*

$$i < j \iff \varphi(\bar{a}_i, \bar{a}_j) \text{ for all } i, j \in \omega \text{ with } i \neq j.$$

Then in some elementary extension N of M there is an indiscernible sequence $(\bar{b}_i)_{i<\omega}$ over M such that

$$i > j \iff \varphi(\bar{b}_i, \bar{b}_j) \text{ for all } i, j \in \omega \text{ with } i \neq j.$$

Proof. Let N be an $|M|^+$-saturated elementary extension of M. Consider the map $\iota : \omega \longrightarrow N^n, \iota(i) = \bar{a}_i$. This induces a map of Boolean algebras

$$\mathscr{L}_n(N) \hookrightarrow \mathcal{P}(N^n) \xrightarrow{T \mapsto \iota^{-1}(T)} \mathcal{P}(\omega),$$

which induces a continuous function $f : \beta(\mathbb{N}) \longrightarrow S_n(N)$. Let $\mathscr{U} \in \beta(\mathbb{N})$ be a non-principal ultrafilter of ω and let $q = f(\mathscr{U})$. Hence

$$(\dagger) \qquad q = \{\psi(\bar{x}, \bar{b}) \in \mathscr{L}_n(N) \mid \{k \in \omega \mid \ \models \psi(\bar{a}_k, \bar{b})\} \in \mathscr{U}\}.$$

Obviously q is a coheir over M. Since N is $|M|^+$-saturated there are $\bar{b}_k \in N^n$ such that $p = q|_M = \operatorname{tp}(\bar{b}_0/M)$ and $\bar{b}_{k+1} \models q|_{M\bar{b}_0...\bar{b}_k}$ for all $k < \omega$. By 4.2, $(\bar{b}_k)$ is an indiscernible sequence over M.

Claim 1. If $k < l$, then $\models \neg\varphi(\bar{b}_k, \bar{b}_l)$.
Proof. Otherwise $\varphi(\bar{b}_k, \bar{x}) \in \operatorname{tp}(\bar{b}_l/M\bar{b}_k) \subseteq q$ and so by (†) there is $j < \omega$ with $\models \varphi(\bar{b}_k, \bar{a}_j)$. But then $\varphi(\bar{x}, \bar{a}_j) \in \operatorname{tp}(\bar{b}_k/M) \subseteq q$ and as $\mathscr{U}$ is not principal, (†) implies that $\models \varphi(\bar{a}_i, \bar{a}_j)$ for infinitely many i. This contradicts our assumption that $\models \varphi(\bar{a}_i, \bar{a}_j)$ is equivalent to $i < j$ for $i \neq j$. ⋄

Claim 2. If $k > l$, then $\models \varphi(\bar{b}_k, \bar{b}_l)$.
Proof. Otherwise $\neg\varphi(\bar{x}, \bar{b}_l) \in \operatorname{tp}(\bar{b}_k/M\bar{b}_l) \subseteq q$ and so by (†) there is $i < \omega$ with $\models \neg\varphi(\bar{a}_i, \bar{b}_l)$. But then $\neg\varphi(\bar{a}_i, \bar{x}) \in \operatorname{tp}(\bar{b}_l/M) \subseteq q$ and as $\mathscr{U}$ is not principal, (†)

implies that $\models \neg\varphi(\bar{a}_i, \bar{a}_j)$ for infinitely many j. This contradicts our assumption that $\models \neg\varphi(\bar{a}_i, \bar{a}_j)$ is equivalent to $i > j$ for $i \neq j$. $\diamond$

Hence by the claims, the proposition is established. $\square$

5. Shelah's trace theorem

We work with models of the $\mathscr{L}$-theory T and we may assume that T is complete with quantifier elimination. An **externally definable subset of** M^k (or a **trace set**) is a set of the form $Y \cap M^k$, where Y is a subset of N^k for some $N \succ M$ such that Y is definable in N *with parameters from* N.

Fix a model M of T and let $\mathscr{L}^*$ be the language extending $\mathscr{L}$ which contains a predicate for every externally definable subset of M^k for every $k \in \mathbb{N}$.

Let M^* be the natural expansion of M to an $\mathscr{L}^*$-structure.

5.1. **Theorem.** *(Shelah)*
If $Th(M)$ has NIP, then the $\mathscr{L}^*$-theory $Th(M^*)$ has quantifier elimination.

The proof below is due to A. Pillay (cf. [Pi]). A more geometric version of 5.1 says: the projection of an externally definable subset of $M^n \times M$ to M^n is again externally definable.

Given $M_1 \succ M$ and $\varphi(\bar{x}) \in \mathrm{Fml}\,\mathscr{L}(M_1)$, we write $R_{\varphi(\bar{x})}$ for the predicate naming $\varphi[M_1] \cap M^{\bar{x}}$.

Let $M \prec M_1$ such that all externally definable subsets of any M^n are traces of M_1-definable sets. Let M^* be the natural expansion of M to an $\mathscr{L}^*$-structure, namely

$$(R_{\varphi(\bar{x})})^{M*} = \varphi[M_1] \cap M^{\bar{x}}.$$

5.2. **Observation.** *Every quantifier free M^*-definable subset of M^n is defined by some R_φ. In other words, every quantifier free $\mathscr{L}^*$-formula is modulo $Th(M^*)$ equivalent to some R_φ.*

Let (N_1, N) be an elementary and $(\mathrm{card}\, M)^+$-saturated elementary extension of the pair (M_1, M) of $\mathscr{L}$-structures. Let N^* be the extension of N to an $\mathscr{L}^*$-structure via

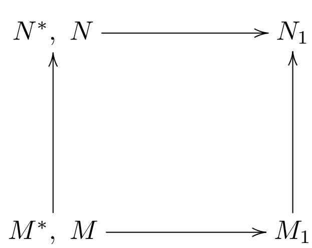
$$(R_{\varphi(\bar{x},\bar{c})})^{N^*} = \varphi[N_1, \bar{c}] \cap N^{\bar{x}},$$

where $\varphi(\bar{x}, \bar{y}) \in \mathrm{Fml}\,\mathscr{L}$ and $\bar{c} \in M_1^{\bar{y}}$. Notice that this is well defined, since for $\psi(\bar{x}, \bar{z}) \in \mathrm{Fml}\,\mathscr{L}$ and $\bar{d} \in M_1^{\bar{z}}$ with $R_{\varphi(\bar{x},\bar{c})} = R_{\psi(\bar{x},\bar{d})}$ we have $\varphi[M_1, \bar{c}] \cap M^{\bar{x}} = \psi[M_1, \bar{d}] \cap M^{\bar{x}}$ in other words

$$(M_1, M) \models \forall \bar{x}\ \bar{x} \subseteq M_1 \to (\varphi(\bar{x}, \bar{c}) \leftrightarrow \psi(\bar{x}, \bar{d})).$$

Since $(N_1, N) \succ (M_1, M)$ we get $(N_1, N) \models \forall \bar{x}\ \bar{x} \subseteq N_1 \to (\varphi(\bar{x}, \bar{c}) \leftrightarrow \psi(\bar{x}, \bar{d}))$, which shows that $\varphi[N_1, \bar{c}] \cap N^{\bar{x}} = \psi[N_1, \bar{d}] \cap N^{\bar{x}}$.

Here a diagram illustrating the involved structures:



Clearly N^* is a $(\operatorname{card} M)^+$-saturated elementary extension of M^*: Note that the structure M^* is definable in the $\mathscr{L}$-pair (M_1, M) and N^* is obtained from the pair (N_1, N) via the the same definition.

Proof of the trace theorem 5.1.

Suppose $Th(M^*)$ does not have quantifier elimination. Then by the general test for quantifier elimination, there is some $p^*(\bar{x}) \in S_n^{\mathrm{qf}}(M^*)$ (the quantifier free n-types of M^*, which are equal to the quantifier free n-types of $Th(M^*)$) and some $R(\bar{x}, \bar{y}) \in \mathscr{L}^*$ such that $p^*(\bar{x}) \cup \{\exists \bar{y}\ R(\bar{x}, \bar{y})\}$ and $p^*(\bar{x}) \cup \{\neg\exists \bar{y}\ R(\bar{x}, \bar{y})\}$ are consistent with $Th(M^*, M^*)$.

We shall construct a type $q(\bar{x}, \bar{y})$ of N such that

(a) $q(\bar{x}, \bar{y})$ is a coheir over M

(b) Both $q(\bar{x}, \bar{y}) \cup \{R(\bar{x}, \bar{y})\}$ and $q(\bar{x}, \bar{y}) \cup \{\neg R(\bar{x}, \bar{y})\}$ are finitely satisfiable in N^*.

With this type we can show that T does not have the NIP as follows: Choose

$$\begin{array}{lll} \text{a realization } (\bar{a}_0, \bar{b}_0) \subseteq N^* & \text{of} & (q \upharpoonright M) \cup \{R(\bar{x}, \bar{y})\} \\ \text{a realization } (\bar{a}_1, \bar{b}_1) \subseteq N^* & \text{of} & (q \upharpoonright M\bar{a}_0\bar{b}_0) \cup \{\neg R(\bar{x}, \bar{y})\} \\ \text{a realization } (\bar{a}_2, \bar{b}_2) \subseteq N^* & \text{of} & (q \upharpoonright M\bar{a}_0\bar{b}_0\bar{a}_1\bar{b}_1) \cup \{R(\bar{x}, \bar{y})\} \\ \text{a realization } (\bar{a}_3, \bar{b}_3) \subseteq N^* & \text{of} & (q \upharpoonright M\bar{a}_0\bar{b}_0\bar{a}_1\bar{b}_1\bar{a}_2\bar{b}_2) \cup \{\neg R(\bar{x}, \bar{y})\} \\ & \vdots & \end{array}$$

Then $(\bar{a}_i, \bar{b}_i)_{i<\omega} \subseteq N^*$ is a coheir sequence of $q(\bar{x}, \bar{y})$ (by (a)), hence by 4.2, $(\bar{a}_i, \bar{b}_i)_{i<\omega}$ is indiscernible over M. By construction we have

$$N^* \models R(\bar{a}_i, \bar{b}_i) \iff i \text{ is even.} \qquad (*)$$

Choose an $\mathscr{L}$-formula $\varphi(\bar{x}, \bar{y}, \bar{z})$ and some $\bar{z}$-tuple $\bar{c}$ from M_1 such that $R = R_{\varphi(\bar{x}, \bar{y}, \bar{c})}$. Then by choice of N^* we have

$$N^* \models R(\bar{a}_i, \bar{b}_i) \iff N_1 \models \varphi(\bar{a}_i, \bar{b}_i, \bar{c}).$$

Now $(*)$ shows that $\varphi(\bar{x}, \bar{y}, \bar{c})$ splits the indiscernible sequence $(\bar{a}_i, \bar{b}_i)_{i<\omega}$. Hence by 3.8, $\varphi(\bar{x}, \bar{y}, \bar{z})$ has the IP w.r.t. $(\bar{x}, \bar{y})$ and $\bar{z}$. So T does not have the NIP.

Construction of $q(\bar{x}, \bar{y})$ and proof of (a) and (b)

Let $p_+^*(\bar{x}, \bar{y})$ be a complete type of M^* containing $p^*(\bar{x}) \cup \{R(\bar{x}, \bar{y})\}$ and let $p_-^*(\bar{x}, \bar{y})$ be a complete type of M^* containing $p^*(\bar{x}) \cup \{\neg\exists \bar{y}\ R(\bar{x}, \bar{y})\}$

Let $q_\pm^*(\bar{x}, \bar{y})$ be a coheir of $p_\pm^*(\bar{x}, \bar{y})$ on N^*.

We choose $q(\bar{x}, \bar{y})$ as the (unique) type of N contained in $q_+^*(\bar{x}, \bar{y})$. Clearly q is a coheir over M and $q(\bar{x}, \bar{y}) \cup \{R(\bar{x}, \bar{y})\} \subseteq q_+^*(\bar{x}, \bar{y})$ is finitely satisfiable in N^*. It remains to show that $q(\bar{x}, \bar{y}) \cup \{\neg R(\bar{x}, \bar{y})\}$ is finitely satisfiable in N^*.

Claim. $q_+^*(\bar{x}, \bar{y})$ and $q_-^*(\bar{x}, \bar{y})$ induce the same $\mathscr{L}$-type in the variables $\bar{x}$ over N.

Proof of the claim. Suppose not. Take an $\mathscr{L}$-formula $\psi(\bar{x}, \bar{u})$ and some $\bar{u}$-tuple $\bar{b}$ from N with $\psi(\bar{x}, \bar{b}) \in q_+^*(\bar{x}, \bar{y})$ and $\neg\psi(\bar{x}, \bar{b}) \in q_-^*(\bar{x}, \bar{y})$.

Let $S(\bar{x})$ be the predicate naming the set $\psi[N, \bar{b}] \cap M^{\bar{x}}$.

If $S(\bar{x}) \in p^*(\bar{x})$, then as $p^*(\bar{x}) \subseteq q_-^*(\bar{x}, \bar{y})$ and $q_-^*(\bar{x}, \bar{y})$ is a coheir over M^*, there is some $\bar{a} \subseteq M$ with $N^* \models S(\bar{a}) \wedge \neg\psi(\bar{a}, \bar{b})$, contradicting the choice of S. Similarly, if $\neg S(\bar{x}) \in p^*(\bar{x})$, then as $p^*(\bar{x}) \subseteq q_+^*(\bar{x}, \bar{y})$ and $q_+^*(\bar{x}, \bar{y})$ is a coheir over M^*, there is some $\bar{a} \subseteq M$ with $N^* \models \neg S(\bar{a}) \wedge \psi(\bar{a}, \bar{b})$, contradicting the choice of S, too. □

Let $(\bar{\alpha}_\pm, \bar{\beta}_\pm)$ be a realization of $q^*_\pm(\bar{x}, \bar{y})$ in some elementary extension of N^*. By the claim, $\mathrm{tp}(\bar{\alpha}_+/N) = \mathrm{tp}(\bar{\alpha}_-/N)$, hence there is an $\mathscr{L}$-automorphism σ over N with $\sigma(\bar{\alpha}_+) = \bar{\alpha}_-$. Since $(\bar{\alpha}_+, \bar{\beta}_+) \models q(\bar{x}, \bar{y})$ $(\subseteq q^*_+(\bar{x}, \bar{y}))$, also

$$(\bar{\alpha}_-, \sigma\bar{\beta}_+) = (\sigma\bar{\alpha}_+, \sigma\bar{\beta}_+) \models q(\bar{x}, \bar{y})$$

As $(\bar{\alpha}_-, \bar{\beta}_-) \models q^*_-(\bar{x}, \bar{y}) \ni \neg\exists\bar{y}\ R(\bar{x}, \bar{y})$ we have $\models \neg\exists\bar{y}\ R(\bar{\alpha}_i, \bar{y})$. It follows that $\models \neg R(\bar{\alpha}_-, \sigma\bar{\beta}_+)$, which identifies $(\bar{\alpha}_-, \sigma\bar{\beta}_+)$ as a realization of $q(\bar{x}, \bar{y}) \cup \{\neg R(\bar{x}, \bar{y})\}$ in some elementary extension of N^*.

This finishes the proof of 5.1 □

6. Invariant extensions

6.1. **Definition.** Let $M \prec N$ and let $q \in S_I(N)$. q is called **invariant** over M, or **special** over M if for every $\varphi(\bar{x}, \bar{y}) \in \mathrm{Fml}\,\mathscr{L}$ and all $\bar{c}, \bar{d} \in N^{\bar{y}}$ with $\mathrm{tp}(\bar{c}/M) = \mathrm{tp}(\bar{d}/M)$ we have

$$\varphi(\bar{x}, \bar{c}) \in q \iff \varphi(\bar{x}, \bar{d}) \in q.$$

Warning: Poizat calls q special, only when q has an extension q' on some $N' \succ N$ which realizes all n-types over M, such that q' is special over M (in our sense).

Hence membership of $\varphi(\bar{x}, \bar{c})$ in q only depends on the type of $\bar{c}$ over M. Note that if $\varphi(\bar{x}, \bar{y}) \in \mathrm{Fml}\,\mathscr{L}(M)$ and $\bar{c}, \bar{d} \in N^{\bar{y}}$ with $\mathrm{tp}(\bar{c}/M) = \mathrm{tp}(\bar{d}/M)$ then we certainly also have $\varphi(\bar{x}, \bar{c}) \in q \iff \varphi(\bar{x}, \bar{d}) \in q$.

Also note that every $q \in S_n(N)$, invariant over M is fixed under each M-automorphism of N. If N is sufficiently saturated, this characterizes all types of N that are invariant over M.

6.2. *Example.* Coheirs are invariant.

Proof. Let $q \in S_n(N)$ and $M \prec N$. Let $\varphi(\bar{x}, \bar{y}) \in \mathrm{Fml}\,\mathscr{L}$ and $\bar{c}, \bar{d} \in N^{\bar{y}}$ with $\mathrm{tp}(\bar{c}/M) = \mathrm{tp}(\bar{d}/M)$ and suppose $\varphi(\bar{x}, \bar{c}), \neg\varphi(\bar{x}, \bar{d}) \in q$.

As q is a coheir over M, there is some $\bar{m} \in M^{\bar{x}}$ with $N \models \varphi(\bar{m}, \bar{c}) \wedge \neg\varphi(\bar{m}, \bar{d})$, in contradiction to $\mathrm{tp}(\bar{c}/M) = \mathrm{tp}(\bar{d}/M)$. □

6.3. **Lemma.** *Let $M \prec N \prec N'$ and let $q \in S_n(N)$ be invariant over M. If N realizes every type from $S_k(M)$ for every $k \in \mathbb{N}$, then there is a unique extension q' of q on N' which is invariant over M.*

Proof. We must define q' as

$$\{\varphi(\bar{x}, \bar{c}') \in \mathrm{Fml}\,\mathscr{L}_n(N') \mid \varphi(\bar{x}, \bar{c}) \in q \text{ for some } \bar{c} \in N^n \text{ with } \mathrm{tp}(\bar{c}/M) = \mathrm{tp}(\bar{c}'/M)\}.$$

Since N realizes every type from $S_k(M)$ for every $k \in \mathbb{N}$, q' is a type of N'. □

The following lemma produces indiscernible sequences from invariant extensions as we have obtained indiscernible sequences from coheirs in 4.2. The proof is identical to the proof of 4.2, in fact 4.2 is a corollary of 6.4, since coheirs are invariant. As this section was not a topic durng the lecture we repeat the proof.

6.4. **Lemma.** *Let $M \prec N$ and let $q \in S_n(N)$ be invariant over M and let $I = (I, <)$ be a totally ordered set. For $i \in I$ let $\bar{a}_i \in N^n$ such that $\bar{a}_i \models q \restriction (M \cup \{\bar{a}_j \mid j < i\})$ for all $i \in I$. Then $s = (\bar{a}_i)_{i \in I}$ is an indiscernible sequence over M.*

Proof. Firstly, notice that our assumption implies

$$\bar{a}_k \models q \upharpoonright (M \cup \{\bar{a}_j \mid j < i\}) \text{ for all } i \leq k <\in I. \qquad (*)$$

Let $i_1 < ... < i_k$ and $j_1 < ... < j_k$ be finite sequences from I. We have to show that

$$\operatorname{tp}(\bar{a}_{i_1}, ..., \bar{a}_{i_k}/M) = \operatorname{tp}(\bar{a}_{j_1}, ..., \bar{a}_{j_k}/M). \qquad (+)$$

Claim. (+) holds true if $i_1 = j_1, ..., i_{k-1} = j_{k-1}$.
The claim holds true, since by $(*)$,

$$\operatorname{tp}(\bar{a}_{i_k}/M\bar{a}_{i_1}, ..., \bar{a}_{i_{k-1}}) = \operatorname{tp}(\bar{a}_{j_k}/M\bar{a}_{j_1}, ..., \bar{a}_{j_{k-1}}),$$

which gives (+) in the case $i_1 = j_1, ..., i_{k-1} = j_{k-1}$.

In order to show (+) we do an induction on k, where the case $k = 1$ holds true by $(*)$. Assume we know (+) for $k-1$. We may first apply the claim and enlarge i_k, j_k such that $i_k = j_k$. We write $i := i_k = j_k$ Suppose (+) fails. Then there is some $\gamma(\bar{x}_1, ..., \bar{x}_k) \in \operatorname{Fml} \mathscr{L}(M)$ such that

$$N \models \gamma(\bar{a}_{i_1}, ..., \bar{a}_{i_{k-1}}, \bar{a}_i) \wedge \neg\gamma(\bar{a}_{j_1}, ..., \bar{a}_{j_{k-1}}, \bar{a}_i).$$

Since $i > i_1, ..., i_{k-1}, j_1, ..., j_{k-1}$ we get that

$$\gamma(\bar{a}_{i_1}, ..., \bar{a}_{i_{k-1}}, \bar{x}_k) \wedge \neg\gamma(\bar{a}_{j_1}, ..., \bar{a}_{j_{k-1}}, \bar{x}_k) \in \operatorname{tp}(\bar{a}_i/M \cup \{\bar{a}_j \mid j < i\}).$$

By definition, q extends $\operatorname{tp}(\bar{a}_i/M\bar{a}_0, ..., \bar{a}_{i-1})$. Since q is invariant over M it follows $\operatorname{tp}(\bar{a}_{i_1}, ..., \bar{a}_{i_{k-1}}/M) \neq \operatorname{tp}(\bar{a}_{j_1}, ..., \bar{a}_{j_{k-1}}/M)$, which contradicts (+) in the case $k-1$. □

6.5. **Lemma.** *Let $I = (I, <)$ be a totally ordered set and let $(\bar{a}_i)_{i\in I}$, $(\bar{a}_i)_{i\in I}$ be sequence of n-tuples from $M \models T$ with $\operatorname{tp}((\bar{a}_i)_{i\in I}) = \operatorname{tp}((\bar{b}_i)_{i\in I})$. Suppose $(\bar{a}_i)_{i\in I}$ is indiscernible. Then*

(i) $(\bar{b}_i)_{i\in I}$ is indiscernible.
(ii) If $\varphi(\bar{x}, \bar{y}) \in \operatorname{Fml} \mathscr{L}$ and if there is no $\bar{c}$ in any elementary extension of M such that $\varphi(\bar{x}, \bar{c})$ splits $(\bar{a}_i)_{i\in I}$, then there is no $\bar{c}$ in any elementary extension of M such that $\varphi(\bar{x}, \bar{c})$ splits $(\bar{b}_i)_{i\in I}$.

Proof. (i) is obvious and (ii) follows with compactness: there is some $N \in \mathbb{N}$ such that for all $i \geq N$, either for all $\bar{c}$, $\varphi(\bar{a}_i, \bar{c})$ is true or for all $\bar{c}$, $\neg\varphi(\bar{a}_i, \bar{c})$ is true. Note that we use the indiscernability of the sequence here. □

6.6. **Lemma.** *Let $M \prec N$ such that N realizes every type from $S_k(M)$ for every $k \in \mathbb{N}$. Let $q, r \in S_n(N)$ be invariant over M. Let $\bar{a}_i, \bar{b}_i \in N^n$ such that $\bar{a}_k \models q \upharpoonright (M \cup \{\bar{a}_0, ..., \bar{a}_{k-1}\})$ and $\bar{b}_k \models r \upharpoonright (M \cup \{\bar{b}_0, ..., \bar{b}_{k-1}\})$ for all $k < \omega$.*

If $(\bar{a}_i)_{i\in\omega}$ is unsplittable and $\operatorname{tp}(\bar{a}_0, \bar{a}_1, .../M) = \operatorname{tp}(\bar{b}_0, \bar{b}_1, .../M)$, then $q = r$.

Proof. (cf. [Poi], 12.26).

We construct a sequence $\bar{c}_0, \bar{c}_1, ...$ by induction as follows: If i is odd, then we take $\bar{c}_{i+1}$ to be a realization of the unique (by 6.3) extension of q on $N \cup \{\bar{c}_0, ..., \bar{c}_i\}$ which is invariant over M. If i is even, then we take $\bar{c}_{i+1}$ to be a realization of the unique extension of r on $N \cup \{\bar{c}_0, ..., \bar{c}_i\}$ which is invariant over M.

Claim. $\operatorname{tp}(\bar{a}_0, \bar{a}_1, .../M) = \operatorname{tp}(\bar{c}_0, \bar{c}_1, .../M) = \operatorname{tp}(\bar{b}_0, \bar{b}_1, .../M)$.

We prove by induction on i that

$$\operatorname{tp}(\bar{a}_0, ..., \bar{a}_i/M) = \operatorname{tp}(\bar{c}_0, ..., \bar{c}_i/M) = \operatorname{tp}(\bar{b}_0, ..., \bar{b}_i/M). \qquad (+)$$

If $i=0$ this holds true, since q and r extend $\operatorname{tp}(\bar{a}_0/M)=\operatorname{tp}(\bar{b}_0/M)$. Suppose we know (+) already for i. By symmetry we may assume that i is odd. Since $\bar{c}_{i+1}$ is a realization of an extension of q, $\bar{c}_{i+1}$ also realizes $q\restriction(M\cup\{\bar{a}_0,...,\bar{a}_i\})$. Hence by choice of $\bar{a}_{i+1}$ we have

$$\operatorname{tp}(\bar{c}_{i+1}/M\cup\{\bar{a}_0,...,\bar{a}_i\})=\operatorname{tp}(\bar{a}_{i+1}/M\cup\{\bar{a}_0,...,\bar{a}_i\}).$$

Pick $\psi(\bar{x}_0,...,\bar{x}_i,\bar{x}_{i+1})\in\operatorname{Fml}\mathscr{L}(M)$ with $N\models\psi(\bar{a}_0,...,\bar{a}_i,\bar{a}_{i+1})$. Then

$$\psi(\bar{a}_0,...,\bar{a}_i,\bar{x}_{i+1})\in\operatorname{tp}(\bar{a}_{i+1}/M\cup\{\bar{a}_0,...,\bar{a}_i\}),$$

thus

$$\psi(\bar{a}_0,...,\bar{a}_i,\bar{x}_{i+1})\in\operatorname{tp}(\bar{c}_{i+1}/M\cup\{\bar{a}_0,...,\bar{a}_i\})\subseteq\operatorname{tp}(\bar{c}_{i+1}/N\cup\{\bar{c}_0,...,\bar{c}_i\}).$$

Since the latter type is invariant over M by choice of $\bar{c}_{i+1}$ and by the induction hypothesis we have $\operatorname{tp}(\bar{a}_0,...,\bar{a}_i/M)=\operatorname{tp}(\bar{c}_0,...,\bar{c}_i/M)$, we get $\psi(\bar{c}_0,...,\bar{c}_i,\bar{x}_{i+1})\in\operatorname{tp}(\bar{c}_{i+1}/N\cup\{\bar{c}_0,...,\bar{c}_i\})$. This shows $\models\psi(\bar{c}_0,...,\bar{c}_i,\bar{c}_{i+1})$ and finishes the proof of $\operatorname{tp}(\bar{a}_0,...,\bar{a}_{i+1}/M)=\operatorname{tp}(\bar{c}_0,...,\bar{c}_{i+1}/M)$.

By assumption, we have $\operatorname{tp}(\bar{a}_0,...,\bar{a}_{i+1}/M)=\operatorname{tp}(\bar{b}_0,...,\bar{b}_{i+1}/M)$, hence we get the claim.

By 6.4, $(\bar{a}_i)_{i\in\omega}$ is indiscernible. Hence by 6.5, $(\bar{c}_i)_{i\in\omega}$ is unsplittable. Since the $\bar{c}_i$ are alternating between realizations of q and r, this is only possible if $q=r$. □

6.7. **Corollary.** *Let $M\prec N$ be models of T such that N realizes every type from $S_k(M)$ for every $k\in\mathbb{N}$. If T has NIP, then the number of n-types of N which are invariant over M is bounded by* $\operatorname{card}S_\omega(M)$.

Proof. By 3.8, every indiscernible sequence is unsplittable. Hence the corollary follows from 6.6 and 6.4. □

7. NIP via counting coheirs

7.1. **Theorem.** *T has the independence property if and only if there is a 1-type p over some model M of T with $\operatorname{card}M\geq\operatorname{card}\mathscr{L}$ and some $N\succ M$ such that p has $2^{2^{\operatorname{card}M}}$ coheirs on N.*

Proof. First suppose T has the independence property. By 3.9 (and 1.1) there is a formula $\varphi(\bar{x},y)$ which has the independence property. Take $\lambda>\operatorname{card}\mathscr{L}$, a model N of T and

$$a_i\in N,\bar{b}_S\in N^{\bar{x}}\ (i\in\lambda,S\subseteq\lambda)$$

such that

$$N\models\varphi(\bar{b}_S,a_i)\iff i\in S.$$

Since $\lambda\geq\operatorname{card}\mathscr{L}$, there is an elementary substructure $M\prec N$ of size λ containing each a_i.

Let $\mathfrak{u}$ be an ultrafilter of subsets of λ and define $p_{\mathfrak{u}}\in S_1(N)$ via

$$p_{\mathfrak{u}}=\{\psi(y,\bar{c})\in\operatorname{Fml}\mathscr{L}_1(N)\mid\{i\in\lambda\mid N\models\psi(a_i,\bar{c})\}\in\mathfrak{u}\}.$$

Straightforward checking shows that $p_{\mathfrak{u}}$ is indeed a 1-type of N.

$\mathfrak{p}_u$ is a coheir over M, since every formula of $p_{\mathfrak{u}}$ by definition is satisfiable in M. Since $N\models\varphi(\bar{b}_S,a_i)\iff i\in S$ for all $i\in S\subseteq\lambda$ we have $p_{\mathfrak{u}}\neq p_{\mathfrak{v}}$ whenever $\mathfrak{v}\neq\mathfrak{u}$ are ultrafilters of subsets of λ.

This shows that there are at least as many coheirs of 1-types of M on N as there are ultrafilters of subsets of λ. On the other hand, there are at most 2^λ 1-types of

M and there are 2^{2^λ} ultrafilters of subsets of λ (cf. A.5 in the appendix below). Since 2^λ is strictly less than the cofinality of 2^{2^λ} (cf. A.4 in the appendix below) there must be some $p \in S_1(M)$ which has at least 2^{2^λ} coheirs on N.

Conversely suppose T has the NIP. By 6.7, the number of invariant extensions of a given type $p \in S_n(M)$ on some $N \succ M$ is bounded by $\operatorname{card} S_\omega(M)$. Since $\operatorname{card} S_\omega(M) \leq 2^{\operatorname{card} M}$ and every coheir is invariant (cf. 6.2), this gives the assertion. □

As an application:

7.2. **Corollary.** *If T is a weakly o-minimal theory, then T has the NIP. In particular every o-minimal structure has NIP (recall that for every o-minimal structure, the theory of this structure is o-minimal).*

Proof. Recall that T is a theory in a language containing $<$ and weak o-minimality says that every parametrically definable subset of each T-model M is a finite union of convex sets. It is an exercise to show directly from this condition that

(a) every Dedekind cut ξ of a model M of T is induced by at most two 1-types p of M, i.e. there are at most two 1-types p with the property $a < \xi \iff a < x \in p$ $(a \in M)$.
(b) Every 1-type p of a model of M has at most two coheirs q on any $N \succ M$, namely the cut determined by q on N has to be the least or the largest extension of the cut determined by p on M.

Hence by 7.1, T has NIP. □

8. Vapnik-Chervonenkis dimension

8.1. **Definition.** Let X be a set and let $\mathcal{S}$ be a collection of subsets of X. We say that $\mathcal{S}$ **shatters** a subset $B \subseteq X$ if every subset of B is of the form $B \cap S$ for some $S \in \mathcal{S}$.

If there is some $d \in \mathbb{N}$ such that $\mathcal{S}$ does not shatter any subset of size d of X, then the smallest such d is called the **VC-dimension**, or **VC-index**, of $\mathcal{S}$. 'VC' stands for Vapnik-Chervonenkis. In this case $\mathcal{S}$ is called a **VC-class**.

If there is no such d, then $\mathrm{VC}(\mathcal{S}) := \infty$.

Let $\mathcal{S}$ be a collection of subsets of a set X. For $B \subseteq X$, let $B \cap \mathcal{S} = \{B \cap S \mid S \in \mathcal{S}\}$. For $n \in \mathbb{N}$ let

$$f_{\mathcal{S}}(n) = \max\{|B \cap \mathcal{S}| \mid B \subseteq X \text{ and } |B| = n\}.$$

Thus $f_{\mathcal{S}}(n) = 2^n$ if and only if $\mathcal{S}$ shatters a set of size n. Surprisingly, $f_{\mathcal{S}}(n)$ is polynomially bounded for large n, if $\mathcal{S}$ has finite VC-dimension:

8.2. **Theorem.** *Suppose $\mathcal{S}$ does not shatter any subset of X of size d. Then for all $n \geq d$, $f_{\mathcal{S}}(n)$ is at most the number of subsets of an n-element set of size $< d$, given by*

$$p_d(n) = \sum_{i<d} \binom{n}{i}.$$

Observe that $p_d(n)$ is a polynomial of degree $d - 1$.

Proof. First note (by counting subsets of size $< d$ of an n-element set) that $p_d(n) = p_{d-1}(n-1) + p_d(n-1)$.

We proceed by induction on n. If $n = d$ then $f_{\mathcal{S}}(n) < 2^n = p_n(n) - 1$. Now let $n > d$ and let $B \subseteq X$ be of size n. We must show that $|B \cap \mathcal{S}| \leq p_d(n)$ and of course we may replace $\mathcal{S}$ by $B \cap \mathcal{S}$. Fix $x \in B$ and define

$\mathcal{S}_0 = \{S \in \mathcal{S} \mid x \notin S \text{ and } S \cup \{x\} \in \mathcal{S}\}$
$\mathcal{S}_1 = \{S \in \mathcal{S} \mid x \in S \text{ or } S \cup \{x\} \notin \mathcal{S}\}$

Since $\mathcal{S}$ does not shatter any subset of X of size d, $\mathcal{S}_0$ does not shatter any subset of $X \setminus \{x\}$ of size $d-1$.

Hence the induction hypothesis says $|(B \setminus \{x\}) \cap \mathcal{S}_0| \leq p_{d-1}(n-1)$. As $x \notin S$ for any $S \in \mathcal{S}_0$, $(B \setminus \{x\}) \cap \mathcal{S}_0 = \mathcal{S}_0$ and $|\mathcal{S}_0| \leq p_{d-1}(n-1)$.

On the other hand $|\mathcal{S}_1| \leq |(B \setminus \{x\}) \cap \mathcal{S}_1|$ since the map $\mathcal{S}_1 \longrightarrow (B \setminus \{x\}) \cap \mathcal{S}_1$ which removes x is injective (by definition of $\mathcal{S}_1$ and since all $S \in \mathcal{S}_1$ are assumed to be a subset of B).

By the induction hypothesis we have $|\mathcal{S}_1| \leq p_d(n-1)$. Thus $|\mathcal{S}| = |\mathcal{S}_0| + |\mathcal{S}_1| \leq p_{d-1}(n-1) + p_d(n-1) = p_d(n)$. □

8.3. **Corollary.** *If M is a structure that has NIP and $S \subseteq M^n \times M^k$ is definable, then there is some $d \in \mathbb{N}$ such that for all sufficiently large $n \in \mathbb{N}$ and every subset $X \subseteq M^k$ of size n, there are at most n^d sets of the form $X \cap S_a$ where a varies in M^n.*

Proof. Since M has NIP, the collection $\{S_a \mid a \in M^n\}$ has finite VC-dimension d. Now apply 8.2 and notice that $p_d(n)$ is a polynomial of degree $d-1$. □

Appendix A. Theorems of Hausdorff and König

A.1. **Theorem.** *(Theorem of König)*
Let I be an index set and for each $i \in I$ let A_i, B_i be sets with $\operatorname{card} A_i < \operatorname{card} B_i$. *Then*

$$\operatorname{card} \sum_i A_i < \operatorname{card} \prod_i B_i.$$

Proof. We may assume that the A_i are disjoint. Let $s : \bigcup A_i \longrightarrow \prod_i B_i$ be a map. We construct an element not in the image of s: For $i \in I$ let $\pi_i : \prod_i B_i \longrightarrow B_i$ be the projection. By assumption $\pi_i(s(A_i)) \neq B_i$ for each i. Pick $b_i \in B_i \setminus \pi_i(s(A_i))$. We claim that $(b_i)_i$ is not in the image of s. Otherwise there is some $k \in I$ and some $a \in A_k$ with $s(a) = (b_i)_i$. But then $b_k = \pi_k(s(a)) \in \pi_k(s(A_k))$ in contradiction to the choice of b_k. □

A.2. **Corollary.** *For every infinite cardinal κ we have $\kappa < \kappa^{\mathrm{cf}\kappa}$.*

Proof. For $i < \mathrm{cf}\kappa$ let $\lambda_i < \kappa$ such that $\kappa = \sup_i \lambda_i$. Then $\kappa \leq \sum_i \lambda_i < \prod_i \kappa = \kappa^{\mathrm{cf}\kappa}$ by A.1. □

A.3. **Corollary.** *For all cardinals κ, λ with $\kappa \geq 2$ and $\lambda \geq \omega$ we have* $\mathrm{cf}(\kappa^\lambda) > \lambda$.

Proof. Otherwise $(\kappa^\lambda)^{\mathrm{cf}(\kappa^\lambda)} \leq (\kappa^\lambda)^\lambda = \kappa^{\lambda\cdot\lambda} = \kappa^\lambda$ in contradiction to A.2. □

A.4. *Example.* For every infinite cardinal κ we have $\mathrm{cf}(2^\kappa) > \kappa$.

A.5. **Theorem.** *(Theorem of Hausdorff)*
If κ is an infinite cardinal, then there are 2^{2^κ} ultrafilters of subsets of κ.

Proof. It is enough to construct an independent subset $\{S_i \mid i < 2^\kappa\}$ of subsets of κ, i.e for all distinct $i_1, ..., i_n < 2^\kappa$ and all $\varepsilon_1, ..., \varepsilon_n \in \{0, 1\}$ we have $S_{i_1}^{\varepsilon_1} \cap ... \cap S_{i_n}^{\varepsilon_n} \neq \emptyset$, where $S^0 = S$ and $S^1 = \kappa \setminus S$; then for every subset T of 2^κ the set $\{S_i \mid i \in T\} \cup \{\kappa \setminus S_i \mid i \in 2^\kappa \setminus T\}$ is a basis of a proper filter of subsets of κ and different T's can not be contained in the same ultrafilter.

Now the construction:

Let $\mathcal{F}$ be the set of all finite sequences $(F, F_1, ..., F_n)$, where $F \subseteq \kappa$ is finite and $F_1, ..., F_n \subseteq F$. Then $\operatorname{card} \mathcal{F} = \kappa$. We define a subset $\{S' \mid S \subseteq 2^\kappa\}$ of subsets of $\mathcal{F}$ as follows: Pick $S \subseteq 2^\kappa$ and define

$$S' = \{(F, F_1, ..., F_n) \in \mathcal{F} \mid S \cap F \in \{F_1, ..., F_n\}\}.$$

Now take distinct $S_1, ..., S_n \subseteq \kappa$ and $\varepsilon_1, ..., \varepsilon_n \in \{0, 1\}$. Take $F \subseteq \kappa$ finite such that the map $\{S_1, ..., S_n\} \longrightarrow \mathcal{P}(F)$, $S \mapsto F \cap S$ is injective. Let $i_1 < ... < i_k$ be an enumeration of those indices $i \in \{1, ..., n\}$ with $\varepsilon_i = 0$ and let $F_1 = F \cap S_{i_1}, ..., F_k = F \cap S_{i_k}$. Then for each $i \in \{i_1, ..., i_k\}$, S'_i contains the point $(F, F_1, ..., F_k)$. Whereas, if $i \in \{1, ..., n\} \setminus \{i_1, ..., i_k\}$ then $S_i \cap F$ is not among the $F_1, ..., F_k$, so $(F, F_1, ..., F_k) \in \mathcal{F} \setminus S'_i$.

This shows that $\{S' \mid S \subseteq 2^\kappa\}$ i an independent set of subsets of $\mathcal{F}$. Observe that $S_1 \neq S_2$ implies $S'_1 \neq S'_2$. As $\operatorname{card} \mathcal{F} = \kappa$, this finishes the proof. □

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