INTRODUCTION TO THE MODEL THEORY OF DIFFERENTIAL FIELDS

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Contents

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Literature related to this course:

(1) The main reference for Differential Algebra is [Kolchi1973], but we only need a small fragment of this book in a special case.
(2) More suitable references for our course are [Kaplan1957], [Ritt1950], [MaMePi1996, Chapter II] and [Poizat2000, Sections 6.2].
(3) For (linear) differential Galois Theory, principal resources are [vdPSin2003], [CreHaj2011] and [Magid1994].
Other resources are [Poizat1983], [Pillay1998], [Pillay2004], [Pillay2009], [Kolchi1953], [Kolchi1985] [Cassid1972], and [Pommar1983].
(4) A principal resource for differential algebraic geometry is [Buium1994].
(5) For applications of derivations in classical algebraic geometry, see [Kunz1986].
(6) Background on derivations of non-commutative rings may be found in [Bourba1989, Chapter 1]

1. Differential Algebra

By a **ring** in this chapter we mean a unital, associative and commutative ring. Furthermore, ringhomomorphism are unital.

1.1. Commutative Differential Rings.

1.1.1. **Definition.** Let A be a ring. A map $d : A \longrightarrow A$ is called a **derivation** if it satisfies

$$d(a+b) = d(a) + d(b) \qquad (\textbf{Additivity}), \text{ and}$$
$$d(a\cdot b) = d(a)\cdot b + a\cdot d(b) \qquad (\textbf{Leibniz rule}),$$

for all $a, b \in A$. The pair (A, d) is then called a **differential ring**. We will also write a' instead of $d(a)$ and call it the **derivative** of a (for d). If $n \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ we write $d^n(a) = a^{(n)}$ for the n^{th} derivative of a, hence $d^0(a) = a$ and $d^{n+1}(a) = d(d^n(a))$. A **differential field** is a differential ring (A, d) where A is a field. Similarly differential domains, differential valuation rings etc. are defined.

1.1.2. **Definition.** Let (A, d) and (B, ∂) be differential rings. A **differential homomorphism** $(A, d) \longrightarrow (B, \partial)$ is a ring homomorphism $h : A \longrightarrow B$ (in particular $h(1) = 1$) that preserves the derivations; hence $h(d(a)) = \partial(h(a))$ for all $a \in A$. If $A \subseteq B$ and h is the inclusion map $A \hookrightarrow B$, then h is differential, just if d is the restriction of ∂ to A and in this case (A, d) is called a **differential subring** of (B, ∂).

Clearly, for a differential homomorphism $h : (A, d) \longrightarrow (B, \partial)$, the image $h(A)$ is closed under ∂ and so $h(A)$ is (or 'carries') a differential subring of (B, ∂).

1.1.3**.** Arbitrary intersections of differential subrings of a differential ring (A, d) are again differential and therefore every subset S of A is contained in a smallest differential subring. This ring is called the **differential subring generated by** S (in A) and we see from the Leibniz rule that it is equal to the subring of A generated by $S \cup d(S) \cup d^2(S) \cup \ldots$.

1.1.4. **Definition.** Let $A = (A, d)$ be a differential ring. An element $c \in A$ with $d(c) = 0$ is called a **constant**. Clearly the set C of all constants is a differential subring of A and $d : A \longrightarrow A$ is a C-module homomorphism.

1.1.5. *Examples.*

(i) For any ring A, the map $A \longrightarrow A$ that is constantly zero obviously is a derivation, called the **trivial derivation** of A.

(ii) If $U \subseteq \mathbb{R}$ is open then the ring $C^\infty(U)$ of all infinitely many times differentiable functions $U \longrightarrow \mathbb{R}$ is a differential ring with respect to the standard derivation. Notice that $C^\infty(U)$ is not a domain, e.g. the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(t) = e^{-\frac{1}{t}}$ for $t > 0$ and $f(t) = 0$ for $t \leq 0$ is in $C^\infty(\mathbb{R})$ with $f(t) \cdot f(-t) = 0$ and $f(t), f(-t) \neq 0$.

Similarly, if $U \subseteq \mathbb{C}$ is open then the ring $C^\omega(U)$ of all holomorphic (i.e., complex differentiable) functions $U \longrightarrow \mathbb{C}$ becomes a differential ring with respect to the standard derivation. If U is connected, then $C^\omega(U)$ is a domain by the identity theorem of complex analysis.

In both examples, the constants of the differential ring are the constant functions.

(iii) Let R be a ring. For a power series $f = r_0 + r_1 t + r_2 t^2 + r_3 t^3 + \ldots \in R[[t]]$ in one variable over R we define

$$\frac{\mathrm{d}}{\mathrm{d}t} f = r_1 + 2r_2 t + 3r_3 t^2 + \ldots .$$

Straightforward checking shows that $d(f) := \frac{\mathrm{d}}{\mathrm{d}t} f$ defines a derivation of $R[[t]]$ - sometimes referred to as the **standard derivation** - and we obtain a differential ring $(R[[t]], \frac{\mathrm{d}}{\mathrm{d}t})$.[1]

(iv) Let $R = (R, ')$ be a differential ring. For a power series $f = r_0 + r_1 t + r_2 t^2 + r_3 t^3 + \ldots \in R[[t]]$ in one variable over R we define

$$\partial(f) = r_0' + r_1' t + r_2' t^2 + r_3' t^3 + \ldots$$

Straightforward checking shows that ∂ defines a derivation of $R[[t]]$ and we obtain a differential ring $(R[[t]], \partial)$.

A situation where this derivation occurs "naturally" is when $R = (\mathbb{R}[[x]], \frac{\mathrm{d}}{\mathrm{d}x})$, so $R[[t]] = \mathbb{R}[[x, t]]$ and ∂ is defined by taking partial derivatives of powerseries $f(x, t)$ with respect to x.

(v) Restricting the derivations in (iii) and (iv) to polynomials, defines the differential subrings rings $(R[t], \frac{\mathrm{d}}{\mathrm{d}t})$ and $(R[t], \partial)$.

Hence the passage from (R, d) to $(R[t], \partial)$ introduces a new constant t. A natural example of such differential ring is given by partial derivatives similar to the example in (iv). Another example is given by $R = (\mathbb{Q}[x], \frac{\mathrm{d}}{\mathrm{d}x})$, where the role of t is taken by a transcendental number, say e: Here $(R[t], \partial)$ is isomorphic to $(\mathbb{Q}[x, e], \frac{\mathrm{d}}{\mathrm{d}x})$ and e is a new constant.

(vi) Another source of derivations comes from Lie-Algebra and in fact we will see in 3.1.7 that every (non-trivial) derivation $d : A \longrightarrow A$ is of the following form: There is a (necessarily non-commutative) ring Λ containing A as a subring, and some $\lambda \in \Lambda$ such that for all $a \in A$ we have $d(a) = \lambda \cdot a - a \cdot \lambda$.

[1]Recall that $\sum_n r_n t^n \cdot \sum_n s_n t^n = \sum_n (\sum_{k \leq n} r_k s_{n-k}) t^n$.

1.1.6**. Structure of derivations** Let A be a ring.

(i) If d,δ are derivations of A, then so are $d+\delta$ and $a{\cdot}d$ for any $a\in A$: Hence with these operations, the set of all derivations of A is an A-module.

(ii) Another important operation supported by derivations is the **Lie bracket**: If d,δ are derivations of A, then the Lie bracket $[d,\delta]= d\circ\delta-\delta\circ d$ is again a derivation of A.

Proof. (i) Additivity is clear in both cases; for the Leibniz rule we verify $(d+\delta)(x{\cdot}y)=d(x){\cdot}y+x{\cdot}d(y)+\delta(x){\cdot}y+x{\cdot}\delta(y)=(d+\delta)(x){\cdot}y+x{\cdot}(d+\delta)(y)$, and $(a{\cdot}d)(x{\cdot}y)=a{\cdot}d(x){\cdot}y+a{\cdot}x{\cdot}d(y)=a{\cdot}d(x){\cdot}y+x{\cdot}a{\cdot}d(y)=(a{\cdot}d)(x){\cdot}y+x{\cdot}(a{\cdot}d)(y)$ (using commutativity of A).

(ii) Clearly $d\circ\delta-\delta\circ d$ is additive. Take $x,y\in A$. Then

$$\begin{aligned}(d\circ\delta-\delta\circ d)(xy) &= d(\delta(x)y+x\delta(y))-\delta(d(x)y+xd(y))\\ &= d(\delta(x))y+\delta(x)d(y)+d(x)\delta(y)+xd\delta(y)\\ &\quad -\delta(d(x))y-d(x)\delta(y)-\delta(x)d(y)-x\delta(d(y))\\ &= d(\delta(x))y+xd\delta(y)-\delta(d(x))y-x\delta(d(y))\\ &= (d\circ\delta-\delta\circ d)(x)y+x(d\circ\delta-\delta\circ d)(y).\end{aligned}$$

□

1.1.7. **Proposition.** *Let (R,d_0) be a differential ring. If T is a set of indeterminates over R and $f_t\in R[T]$ for each $t\in T$, then there is a unique derivation $d:R[T]\longrightarrow R[T]$ that extends d_0 and that satisfies $d(t)=f_t$ for all $t\in T$. Explicitly, for $g\in R[T]$ we have*

$$d(g)=g^*+\sum_{t\in T}f_t{\cdot}\frac{\partial g}{\partial t},$$

where g^* is the polynomial obtained from g by replacing its coefficients with their derivatives for d_0. (Note that the formula really is the Leibniz rule with the standard chain rule where we think of the $t\in T$ as functions, e.g. $d(r{\cdot}t_1^2{\cdot}t_2^3)=d_0(r)t_1^2t_2^3+r2t_1d(t_1)t_2^3+3rt_1^2t_2^2{\cdot}d(t_2)$.)

`Remark:` *If R is a field, the proposition can be significantly strengthened, see 2.1.6.*

Proof. Uniqueness is clear, since $R[T]$ is generated as a ring by $R\cup T$. For existence, let $\frac{\mathrm{d}}{\mathrm{d}t}:R[T]\longrightarrow R[T]$ be the derivation from 1.1.5(v) for the coefficient ring $R[T\setminus\{t\}]$.[2] By 1.1.6, for any finite set $F\subseteq T$, the map $d_F=\sum_{t\in F}f_t{\cdot}\frac{\mathrm{d}}{\mathrm{d}t}$ is again a derivation of $R[T]$ and we define

$$\tilde d(g)=d_F(g)$$

for any finite $F\subseteq T$ with $g\in R[F]$. This is well defined because for all finite $E\subseteq F\subseteq T$ with $g\in R[E]$ we have $d_E(g)=d_F(g)$. For additivity and the Leibniz rule of d, take $g,h\in R[T]$ and let $F\subseteq T$ be finite with $g,h\in R[F]$. Then additivity and the Leibniz rule for d_F show that $\tilde d(f+g)=\tilde d(f)+\tilde d(g)$ and $\tilde d(fg)=\tilde d(f)g+f\tilde d(g)$.

Hence $\tilde d$ is a derivation of $R[T]$ that is trivial on R and satisfies $\tilde d(t)=f_t$. Finally, the map ∂ that sends g to g^* is also a derivation of $R[T]$ by iteration of the case of

[2]Normally one would write $\frac{\partial}{\partial t}$ instead of $\frac{\mathrm{d}}{\mathrm{d}t}$ and we will also do so in the future.

one transcendental element, see 1.1.5(v), or just by straightforward checking. Now $d = \partial + \tilde{d}$ has the required properties. □

To name a natural example of 1.1.7: Let $R = \mathbb{C}$ equipped with the trivial derivation. Let d be the derivation of $R[t_1, t_2]$ that is trivial on R with $d(t_1) = 1$ and $d(t_2) = t_2$. Then $(R[t_1, t_2], d)$ is isomorphic to $(\mathbb{C}[x, e^x], \frac{\mathrm{d}}{\mathrm{d}x})$.

1.1.8. **Definition.** Let $A = (A, d)$ be a differential ring. A subsets I of A is called a **differential ideal** if I is an ideal of A (in particular $0 \in A$) with $d(I) \subseteq I$. Clearly every subset Z of A is contained in a smallest differential ideal of A. We write $[Z]$ for this ideal and see from the Leibniz rule that it the ideal generated in A by $Z \cup d(Z) \cup d^2(Z) \cup \ldots$. If I is a differential ideal and a prime ideal, or a radical ideal of A, then I is called a **differential prime ideal**, or a **radical differential ideal**.

1.1.9. **Observation.** *Let $A = (A, d)$ be a differential ring.*

(i) If I is a differential ideal of A, then there is a unique derivation $\delta : A/I \longrightarrow A/I$ making the diagram

$$\begin{array}{ccc} A & \xrightarrow{d} & A \\ \downarrow & & \downarrow \\ A/I & \xrightarrow{\delta} & A/I \end{array}$$

commutative. Uniqueness is clear and as I is differential we may define δ by $\delta(a + I) = d(a) + I$. Additivity and the Leibniz rule readily transfer from d to δ.

(ii) It follows from (i) that the differential ideals of A are precisely the kernels of differential homomorphisms $A \longrightarrow B$ to differential rings.

(iii) Preimages of differential ideals under differential ringhomomorphisms are again differential.

For example, in the differential ring $A = (R[t], t\cdot\frac{\mathrm{d}}{\mathrm{d}t})$, the ideal (t^2) is differential and the ring of dual numbers $R[t]/(t^2)$ is a differential ring satisfying $(a + bt)' = b$.

1.1.10**. Localization** *Let $A = (A, d)$ be a differential ring and let $S \subseteq A$ be multiplicatively closed with $1 \in S$. Let $S^{-1}A$ be the localization of A at S and let $\iota_S : A \longrightarrow S^{-1}A$ be the natural map sending a to $\iota(a) = \frac{a}{1}$.*[3]

Then there is a unique derivation $\delta : S^{-1}A \longrightarrow S^{-1}A$ making the diagram

$$\begin{array}{ccc} A & \xrightarrow{d} & A \\ \downarrow{\iota_S} & & \downarrow{\iota_S} \\ S^{-1}A & \xrightarrow{\delta} & S^{-1}A \end{array}$$

commutative. Explicitly we have

$$\delta(\frac{a}{s}) = \frac{d(a)s - ad(s)}{s^2}.$$

[3]Recall from commutative algebra that $S^{-1}A$ is defined as $A \times S$ modulo the equivalence relation $(a, s) \sim (b, t) \iff \exists r \in S : r\cdot(at - bs) = 0$. Equivalence classes are denoted by $\frac{a}{s}$ and the ring operations are (well) defined by $\frac{a}{s} + \frac{b}{t} = \frac{at+bs}{st}$ and $\frac{a}{s}\cdot\frac{b}{t} = \frac{ab}{st}$. Also recall that $\iota_S : A \longrightarrow S^{-1}A$ is the unique homomorphism with the property that every homomorphism $h : A \longrightarrow B$ with $h(S) \subseteq B^\times$ (the units of B) factors through ι_S.

Proof. We first show that δ is well defined. So assume $r\cdot(at-bs)=0$ with $r,s,t\in S$. Taking derivatives gives $d(r)(at-bs)+r(d(a)t+ad(t)-d(b)s-bd(s))=0$. Multiplication with r gives $r^2(d(a)t+ad(t)-d(b)s-bd(s))=0$ and further multiplication with $s\cdot t$ shows

$$\begin{aligned}0&=r^2(sd(a)t^2+sad(t)t-d(b)s^2t-bd(s)st)\\&=r^2(sd(a)t^2-d(b)s^2t)+r(ratsd(t)-rbsd(s)t)\\&=r^2(sd(a)t^2-d(b)s^2t)+r(rbs^2d(t)-rad(s)t^2),\ \text{since } rat=rbs\\&=r^2\Bigg(t^2(d(a)s-ad(s))-s^2(d(b)t-bd(t))\Bigg).\end{aligned}$$

Consequently $\frac{d(a)s-ad(s)}{s^2}=\frac{d(b)t-bd(t)}{t^2}$ and so δ is well-defined.

It is straightforward to check that δ is a derivation. Uniqueness follows from the requirement $\delta(\frac{s}{1}\cdot\frac{1}{s})=\frac{d(1)}{1}=0$, which implies $\delta(\frac{1}{s})=\frac{-d(s)}{s^2}$. □

For example, if $A=(A,')$ is a differential domain, then there is a unique extension d of $'$ to a derivation of the fraction field of A. Prominent differential fields are: $K(t),K((t))$ with derivations extending those from $R[t],R[[t]]$ in examples 1.1.5, as well as the differential field $\mathscr{M}(U)$ of meromorphic functions $U\longrightarrow\mathbb{C}$ for some open connected set $U\subseteq\mathbb{C}$.

1.1.11**. Derivations on Laurent series** Let $R=(R,d_0)$ be a differential ring and let $R((t))$ be the **Laurent series** ring over R, which may be defined as the localization $R[[t]]_t$ of $R[[t]]$ at t, or directly as the ring of all Laurent series $\sum_{n\geq k}r_nt^n$, $k\in\mathbb{Z}$, $a_n\in R$ with the natural addition and multiplication. So we consider $R[[t]]$ as a subring of $R((t))$. Let $R[t,t^{-1}]:=R[t]_t$ be the subring of $R((t))$ of **Laurent polynomials**.

(i) By 1.1.10, the derivations ∂ and $\frac{\mathrm{d}}{\mathrm{d}t}$ of $R[[t]]$ from 1.1.5(iii),(iv) extend uniquely to derivations of $R((t))$ such that the natural map $R[[t]]\longrightarrow R((t))$ are differential. We write $\partial,\frac{\mathrm{d}}{\mathrm{d}t}$ again for this derivation of $R((t))$ and obviously

$$\partial(\sum_{n\geq k}r_nt^n)=\sum_{n\geq k}d_0(r_n)t^n,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\sum_{n\geq k}r_nt^n)=\sum_{n\geq k}nr_nt^{n-1}.$$[4]

(ii) By 1.1.6, for each $f\in R((t))$ the map $d:=\partial+f\cdot\frac{\mathrm{d}}{\mathrm{d}t}$ is a derivation of $R((t))$ extending d_0 with the property $d(t)=f$.

(iii) If $f\in R[t,t^{-1}]$, then there is a unique derivation δ of $R[t,t^{-1}]$ extending d_0 with the property $\delta(t)=f$. Uniqueness is clear and for existence we may take δ to be the restriction of $\partial+f\cdot\frac{\mathrm{d}}{\mathrm{d}t}$ to $R[t,t^{-1}]$.

Warning: The standard derivation $d=\frac{\mathrm{d}}{\mathrm{d}t}$ of $R[[t]]$ is **not** the unique derivation of $R[[t]]$ satisfying $d(t)=1$ and $d(r)=0$ for all $r\in R$.

We conclude this section by recording

1.1.12. **Higher Leibniz rule.** If $A = (A, d)$ is a differential ring, then for all $a_1, \ldots, a_n \in A$ and every $\nu \in \mathbb{N}_0$ we have

$$d^\nu(a_1 \cdot \ldots \cdot a_n) = \sum_{k_1+\ldots+k_n=\nu} \frac{\nu!}{k_1! \cdot \ldots \cdot k_n!} d^{k_1}(a_1) \cdot \ldots \cdot d^{k_n}(a_n).$$ [5]

When $n = 2$ the formula reads as

$$d^\nu(ab) = \sum_{k \leq \nu} \frac{\nu!}{k!(\nu-k)!} d^k(a) d^{\nu-k}(b).$$

Proof. This is straightforward by induction on ν. □

[5]Notice that this formula is also correct when the characteristic of the ring is $\neq 0$. The coefficients in the formula are integers, or merely the value of that integer after application of the natural map $\mathbb{Z} \longrightarrow A$.

1.2. **Principal differential ideals in differential polynomial rings.**

1.2.1**. Definition of the differential polynomial ring.** Let $A = (A,')$ be a differential ring and let T be a nonempty set. We define the **differential polynomial ring** over A in the differential variables T, denoted by $A\{T\}$, as follows. For each $t \in T$, choose a countable family $\{t(i) \mid i \in \mathbb{N}_0\}$ of indeterminates over A with $t(0) = t$, such that the family $(t(i) \mid t \in T, i \geq 0)$ is algebraically independent over A. Then as a ring, $A\{T\}$ is $A[t(i) \mid t \in T, i \in \mathbb{N}_0]$. By 1.1.7 there is a unique derivation d on $A\{T\}$ that extends the given derivation on A satisfying $d(t(i)) = t(i+1)$ for all $i \geq 0$ and every $t \in T$. Hence $A\{T\} = A[t, d(t), d^2(t), \ldots \mid t \in T]$. Then $(A\{T\}, d)$ is the differential polynomial ring over A in the differential variables from T. If T is finite, $T = \{t_1, \ldots, t_n\}$ we just write $A\{t_1, \ldots, t_n\}$.

1.2.2**. Explicit computation of derivatives in** $A\{T\}$**.** In the situation of 1.2.1, if $P \in A\{T\}$, then

$$d(P) = P^* + \sum_{i \geq 0, t \in T} d^{i+1}(t) \cdot \frac{\partial}{\partial t(i)} P,$$

where P^* is the polynomial obtained from P by taking derivatives of coefficients. Notice that the sum is finite, because only finitely many of the $t(i)$ occur in P. If $T = \{t\}$ is a singleton, then

$$d(P) = P^* + \sum_{i \geq 0} d^{i+1}(t) \cdot \frac{\partial}{\partial t(i)} P.$$

1.2.3. **Proposition.** *If B is another differential ring and $\varphi : A \longrightarrow B$ is a differential ring homomorphism, then for every choice of $b_t \in B$ ($t \in T$) there is a unique differential homomorphism $A\{T\} \longrightarrow B$ extending φ and mapping t to b_t for all $t \in T$.*

Hence $A\{T\}$ indeed is the free differential ring over A in the set T, as instigated by the term "differential polynomial ring".

If $T = \{t_1, \ldots, t_n\}$ and $P \in A\{t_1, \ldots, t_n\}$ we write $P(b_1, \ldots, b_n)$ instead of $\psi(P)$.

Proof. Uniqueness is clear. For existence let $\psi : A\{T\} \longrightarrow B$ be the ring homomorphism extending φ satisfying $\psi(d^n(t)) = b_t^{(n)}$ for $t \in T$, $n \geq 0$. It suffices to show that ψ is differential. Using 1.2.2, this says that for $P \in A\{T\}$, the derivative of $\psi(P)$ in B is

$$\psi(d(P)) = \psi(P^*) + \sum_{i \geq 0, t \in T} b_t^{(i+1)} \cdot \psi(\frac{\partial}{\partial t(i)} P).$$

This follows easily from the additivity and the higher Leibniz rule 1.1.12 for the derivation of B. □

We will now have a closer look at the structure of differential polynomials in one differential variable x. We will just write P' for the derivative of $P \in A\{x\}$.

1.2.4. **Definition.** Let A be a differential ring and let $P \in A\{x\} \setminus A$. Then there is a smallest $n \in \mathbb{N}_0$ with $P \in A[x, x', \ldots, x^{(n)}]$, which is called the **order** of P and we write $\operatorname{ord}(P) = n$. The degree $\deg(P)$ of P is the degree of P when viewed as a polynomial in $x^{(n)}$. The **separant** of P is defined as the derivative of P with respect to the variable $x^{(n)}$:

$$S(P) = \frac{\partial}{\partial x^{(n)}} P.$$

The **leader** of P is the coefficient of P at $(x^{(n)})^{\deg(P)}$, when P is considered as a polynomial in $x^{(n)}$. We write $L(P)$ for the leader of P and notice that $\operatorname{ord}(L(P)) < \operatorname{ord}(P)$. If $a \in A \setminus \{0\}$ then we define $\operatorname{ord}(a) = -1, \deg(a) = 0$ and $S(a) = 0$. Further we set $\operatorname{ord}(0) = \infty, \deg(0) = -\infty$ and $S(0) = 1$.

1.2.5**. Higher derivatives of polynomials.** Let $P \in A\{x\} \setminus A$ and let $n = \operatorname{ord}(P)$. Using 1.2.2 we see that for each $k \geq 1$, there is a unique polynomial $P_k \in A\{x\}$ with

$$P^{(k)} = S(P)x^{(n+k)} + P_k \text{ and } \operatorname{ord}(P_k) < n + k.$$

We see that $\operatorname{ord}(P^{(k)}) = \operatorname{ord}(P)+k$, $\deg(P^{(k)}) = 1$ and $S(P^{(k)}) = L(P^{(k)}) = S(P)$.

1.2.6**. Construction of the weakly reduced remainder.** Let A be a differential ring and let $P \in A\{x\}$. Given $F \in A\{x\}$ we now construct a polynomial $Q \in A\{x\}$ such that

$$S(P)^m \cdot F \equiv Q \operatorname{mod} [P] \text{ and } \operatorname{ord}(Q) \leq \operatorname{ord}(P)$$

for some $m \geq 0$. This polynomial is called the **weakly reduced remainder** of F for P.

If $P \in A$, then we take $Q = P$ and $m = 1$. Hence we may assume that $P \in A\{x\} \setminus A$ and $n = \operatorname{ord}(P) \geq 0$. If $F = 0$ or $\operatorname{ord}(F) \leq n$ then we take $Q = F$. Hence we may assume that $\operatorname{ord}(F) = n + k > n$. Then $F = \sum_{i=0}^{m} F_i \cdot (x^{(n+k)})^i$ for some $m \geq 1$ and some polynomials $F_i \in A[x, x', \ldots, x^{(n+k-1)}]$. By 1.2.5, $S(P) \cdot x^{(n+k)} \equiv -P_k \operatorname{mod} [P]$ and $\operatorname{ord}(P_k) < n + k$. Then

$$S(P)^m \cdot F = \sum_{i=0}^{m} F_i \cdot S(P)^{m-i} \cdot (S(P)x^{(n+k)})^i \text{ and with}$$

$$\tilde{F} = \sum_{i=0}^{m} F_i \cdot S(P)^{m-i} \cdot (-P_k)^i$$

We see that $S(P)^m \cdot F \equiv \tilde{F} \operatorname{mod} [P]$ and $\operatorname{ord}(\tilde{F}) < n + k$. We now may iterate the construction of $\tilde{F}$ from F until the result Q has order $\leq n$.

1.2.7. **Definition.** Let A be a differential ring and let $P \in A\{x\}$. We define

$$I(P) = \{F \in A\{x\} \mid S(P)^m \cdot F \in [P] \text{ for some } m \in \mathbb{N}\}.$$

It is easy to check that $I(P)$ is a differential ideal containing P. A different way of looking at $I(P)$ is the following: Let $A\{x\}_{S(P)}$ be the localization of $A\{x\}$ at the element $S(P)$ [6] and let $\iota : A\{x\} \longrightarrow A\{x\}_S(P)$ be the associated natural map. Then the ideal $[P]A\{x\}_{S(P)}$ generated by $\iota([P])$ in $A\{x\}_{S(P)}$ is differential (because $\iota([P])$ is closed under the derivation) and

$$I(P) = \iota^{-1}([P]A\{x\}_{S(P)}).$$

The weakly reduced remainder is an instrument to solve the membership problem for $I(P)$: It reduces membership $F \in I(P)$ to membership $Q \in [P]$ for polynomials Q of order $\leq \operatorname{ord}(P)$. The latter problem is dealt with next:

[6]hence in the terminology of 1.1.10, $A\{x\}_{S(P)}$ is $S^{-1}A\{x\}$ for $S = \{1, S(P), S(P)^2, \ldots\}$.

1.2.8. **Proposition.** *Let A be a differential ring and let $P \in A\{x\} \setminus A$, thus $n = \operatorname{ord}(P) \geq 0$. Then*

$$I(P) \cap A[x, \dots, x^{(n)}] = \{Q \in A[x, \dots, x^{(n)}] \mid S(P)^r \cdot Q \in (P) \text{ for some } r \in \mathbb{N}\}.$$

Proof. The inclusion $\supseteq$ is clear. For the other inclusion it suffices to show by induction on $k \geq 0$ the following:

If $m \in \mathbb{N}$, $Q \in I(P)$ with $\operatorname{ord}(Q) \leq n$ and $H_0, \dots, H_k \in A\{x\}$ with

$$S(P)^m \cdot Q = H_0 \cdot P + \dots + H_k \cdot P^{(k)},$$

then there are $r \geq 0$ and some $H \in A[x, \dots, x^{(n)}]$ with $S(P)^r \cdot Q = H \cdot P$.

If $k = 0$, then as $S(P), Q, P \in A[x, \dots, x^{(n)}]$, we may replace the variables $x^{(n+1)}, x^{(n+2)}, \dots$ in $S(P)^m \cdot Q = H_0 \cdot P$ by 0 and take $H = H(x, \dots, x^{(n)}, 0, 0, \dots)$.

$k - 1 \to k$. We have $P^{(k)} = S(P) \cdot x^{(n+k)} + P_k$ and $\operatorname{ord}(P_k) < n + k$. Thus

$$(*) \qquad S(P)^m \cdot Q = H_0 \cdot P + \dots + H_{k-1} \cdot P^{(k-1)} + H_k \cdot (S(P) \cdot x^{(n+k)} + P_k)$$

and by replacing the variables $x^{(n+k+1)}, x^{(n+k+2)}, \dots$ by 0 we may assume that $H_i \in A[x, \dots, x^{(n+k)}]$. Let $B := A[x, \dots, x^{(n+k-1)}]$ and notice that $P, S(P), Q \in B$. Let $\varepsilon : B[x^{(n+k)}] \longrightarrow B_{S(P)}$ be the B-algebra homomorphism sending $x^{(n+k)}$ to $-\frac{P_k}{S(P)}$. Applying ε to equation $(*)$ gives

$$(\dagger) \qquad \frac{S(P)^m \cdot Q}{1} = \varepsilon(H_0) \cdot \frac{P}{1} + \dots + \varepsilon(H_{k-1}) \cdot \frac{P^{(k-1)}}{1}$$

in the localization $B_{S(P)}$. Choose $\tilde{H}_0, \dots, \tilde{H}_{k-1} \in B$ and some $d \in \mathbb{N}$ with $\varepsilon(H_i) = \frac{\tilde{H}_i}{S(P)^d}$ for $i \in \{0, \dots, k-1\}$. Back in B, equation $(\dagger)$ says that for some $l \in \mathbb{N}$ we have

$$S(P)^l \cdot S(P)^{m+d} \cdot Q = S(P)^l \tilde{H}_0 \cdot P + \dots + S(P)^l \tilde{H}_{k-1} \cdot P^{(k-1)}.$$

But now we may apply induction and get the assertion. □

1.2.9. *Remark.* If A is a differential domain and $P \in A\{x\} \setminus A$ with $S(P) \neq 0$, then P does not divide $S(P)$, because the degree of $S(P)$ with respect to $x^{\operatorname{ord}(P)}$ is strictly less than the degree of P.

1.2.10. **Corollary.** *If A is a differential domain and $P \in A\{x\} \setminus A$ is prime*[7] *with $S(P) \neq 0$, then*

(i) $I(P)$ is a prime ideal of $A\{x\}$, and
(ii) $I(P) \cap A[x, \dots, x^{(n)}] = P \cdot A[x, \dots, x^{(n)}]$, where $n = \operatorname{ord}(P)$.

Proof. (ii) follows from 1.2.8 using 1.2.9.

(i). Take $F_1, F_2 \in A\{x\}$ with $F_1 \cdot F_2 \in I(P)$. Let Q_i be the weakly reduced remainder of F_i for P, hence $S(P)^m \cdot F_i \equiv Q_i \bmod [P]$ for some $m \geq 0$ and $\operatorname{ord}(Q_i) \leq n$. Then $S(P)^{2m} \cdot F_1 \cdot F_2 \equiv Q_1 \cdot Q_2 \bmod [P]$ and from $F_1 \cdot F_2 \in I(P)$, $[P] \subseteq I(P)$ we get $Q_1 \cdot Q_2 \in I(P)$. However, $\operatorname{ord}(Q_1 \cdot Q_2) \leq \operatorname{ord}(P)$ and so by 1.2.8 we see that P divides $S(P)^r \cdot Q_1 \cdot Q_2$ for some $r \geq 0$. Since P is prime and P does not divide $S(P)$ (cf. 1.2.9), we may assume that $P|Q_1$. But then $S(P)^m \cdot F_1 \equiv Q_1 \bmod [P]$ shows $F_1 \in I(P)$ as required. □

[7]Recall that an element r of a ring is prime if it satisfies $r|st \Rightarrow r|s$ or $r|t$.

1.2.11**. Construction of the reduced remainder.** Let A be a differential domain and let $P \in A\{x\}$. Given $F \in A\{x\}$ we now construct a polynomial $G \in A\{x\}$ such that

$$L(P)^k \cdot S(P)^m \cdot F \equiv G \bmod [P] \text{ for some } k, m \geq 0 \text{ and}$$

$$G = 0, \text{ or, } \operatorname{ord}(G) < \operatorname{ord}(P), \text{ or, } \operatorname{ord}(G) = \operatorname{ord}(P) \text{ and } \deg(G) < \deg(P).$$

This polynomial is called the **reduced remainder** of F for P.

Let Q be the weakly reduced remainder of F for P. If $Q = 0$ or $\operatorname{ord}(Q) < \operatorname{ord}(P)$, then we may take $G = Q$. Hence was may assume that $\operatorname{ord}(Q) = \operatorname{ord}(P)$. Let K be the fraction field of $A[x, x', \dots, x^{(n-1)}]$. We now apply division with remainder over domains and get some minimal $k \in \mathbb{N}_0$ and $H, R \in K[x^{(n)}]$ with $Q = H \cdot P + R$, $\deg(R) < \deg(P)$ and $L(P)^k \cdot H, L(P)^k \cdot R \in A[x, x', \dots, x^{(n-1)}]$. We see that the choice $G = L(P)^k \cdot R$ satisfies the requirements.

1.2.12. **Theorem.** *Let A be a differential $\mathbb{Q}$-algebra and a unique factorization domain. Then the differential prime ideals of $A\{x\}$ that intersect A in $\{0\}$ are exactly the ideals $I(P)$, where $P = 0$ or $P \in A\{x\} \setminus A$ is irreducible.*

Proof. If $P = 0$, then $I(P) = (0)$ is prime, because A is a domain. If $P \in A\{x\} \setminus A$ is irreducible, then it is prime, because $A\{x\}$ is a again a unique factorization domain (by the Lemma of Gauß). Since $\mathbb{Q} \subseteq A$ we have $S(P) \neq 0$. Hence by 1.2.10, $I(P)$ is a differential prime ideal.

Conversely, let $\mathfrak{p} \subseteq A\{x\}$ be prime with $\mathfrak{p} \cap A = (0)$ and $\mathfrak{p} \neq (0)$. We choose a polynomial $P \in \mathfrak{p}$ as follows:

(a) Let $n = \min\{\operatorname{ord}(Q) \mid Q \in \mathfrak{p} \setminus \{0\}\}$. Since $\mathfrak{p} \cap A = (0)$ we know that $n \in \mathbb{N}_0$.
(b) Let $d = \min\{\deg(Q) \mid Q \in \mathfrak{p} \setminus \{0\}, \operatorname{ord}(Q) = n\} \in \mathbb{N}$.
(c) Take $P_0 \in \mathfrak{p} \setminus \{0\}$ with $\operatorname{ord}(P_0) = n$ and $\deg(P_0) = d$. Using that $A\{x\}$ is a unique factorization domain and $\mathfrak{p}$ is a prime ideal, there is an irreducible factor P of P_0 with $P \in \mathfrak{p}$. Obviously $\operatorname{ord}(P) = n$ and $\deg(P) = d$ again.

It remains to show that $\mathfrak{p} = I(P)$.

$\supseteq$. If $Q \in I(P)$ then $S(P)^m \cdot Q \in [P] \subseteq \mathfrak{p}$ for some m and by choice of P we have $S(P) \notin \mathfrak{p}$ (using $S(P) \neq 0$).

$\subseteq$. Let $F \in \mathfrak{p}$ and let G be the reduced remainder of F for P. Then $L(P)^k \cdot S(P)^m \cdot F \equiv G \bmod [P]$ for some k, m and so $G \in \mathfrak{p}$. Now $G = 0$, or, $\operatorname{ord}(G) < \operatorname{ord}(P)$, or, $\operatorname{ord}(G) = \operatorname{ord}(P)$ and $\deg(G) < \deg(P)$. Since $\mathfrak{p} \cap A = (0)$, the choice of P entails $G = 0$. Hence $L(P)^k \cdot F \in I(P)$. But P is irreducible, so we know already that $I(P)$ is prime. Since $\operatorname{ord}(L(P)) < \operatorname{ord}(P)$, 1.2.10 implies $L(P) \notin I(P)$, thus $F \in I(P)$. □

1.2.13. *Remark.*

(i) When P is not prime, the ideal $I(P)$ can differ dramatically from the ideal $[P]$. For example $I(x'^2) = A\{x\}$ for any differential ring A, because $S(x'^2) = 2x'$.
(ii) When F is a field of characteristic $p > 0$, then in general there are differential prime ideals of $F\{x\}$ that are not of the form $I(P)$ for any P. For example if F is such that the polynomial $Q := x^{p^k} - a \in F[x]$ is irreducible. We equip F with the trivial derivation. Then $\mathfrak{p} = Q \cdot F\{x\}$ is a differential prime ideal of $F\{x\}$, but one checks easily that it cannot be of the form $I(P)$ for any $P \in F\{x\}$. Notice: If we choose P for $\mathfrak{p}$ as in the proof of 1.2.12, then $P = a \cdot Q$ for some $a \in F \setminus \{0\}$ and so $S(P) = 0$.

1.3. **The Taylor morphism.**

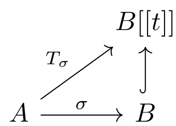
1.3.1. **Definition.** Let B be a ring and let $A = (A, d)$ be a differential ring containing $\mathbb{Q}$ as a subring. Let $\sigma : A \longrightarrow B$ be a ring homomorphism and let $a \in A$. We define

$$T_\sigma(a) := \sum_{\nu \geq 0} \frac{\sigma(d^\nu a)}{\nu!} t^\nu \in B[[t]].$$

Hence we obtain a map $T_\sigma \colon A \longrightarrow B[[t]]$, which is called the **Taylor morphism** of σ. If we need to specify the derivation d of A we also write $T_{d,\sigma}$.

1.3.2. **Proposition.** *In the situation of 1.3.1, T_σ is a differential ringhomomorphism $(A, d) \longrightarrow (B[[t]], \frac{\mathrm{d}}{\mathrm{d}t})$, where $\frac{\mathrm{d}}{\mathrm{d}t}$ is the standard derivativation of $B[[t]]$ as in 1.1.5(iii).*

Observe that the diagram



does in general **not** commute. For example, when $B = A$ and $\sigma = \mathrm{id}_A$. Then the diagram commutes if and only if d is the trivial derivation.

The case when σ is the identity function $\mathrm{id}_A : A \longrightarrow A$ also shows that every differential $\mathbb{Q}$-algebra is a differential subring of a power series ring with the standard derivation: $T_{\mathrm{id}_A} : (A, d) \longrightarrow (A[[t]], \frac{\mathrm{d}}{\mathrm{d}t})$ is an embedding of differential rings (it is injective: consider the first coefficient of $T_{\mathrm{id}_A}(a)$).

Proof. Since σ and d are additive, it is clear that T_σ is additive. Furthermore, the definition of T_σ immediately implies that $T_\sigma(d(a)) = \frac{\mathrm{d}}{\mathrm{d}t} T_\sigma(a)$. It remains to show that $T_\sigma(a \cdot b) = T_\sigma(a) \cdot T_\sigma(b)$: The coefficient of $T_\sigma(a) \cdot T_\sigma(b)$ at t^ν is

$$\sum_{k \leq \nu} \frac{\sigma(d^k(a))}{k!} \cdot \frac{\sigma(d^{\nu-k}(b))}{(\nu-k)!}.$$

The coefficient of $T_\sigma(a \cdot b)$ at t^ν is

$$\frac{\sigma(d^\nu(a \cdot b))}{\nu!} = \frac{\sigma(\sum_{k \leq \nu} \frac{\nu!}{k!(\nu-k)!} d^k(a) d^{\nu-k}(b))}{\nu!},$$

see 1.1.12. We see that these coefficients are equal, showing $T_\sigma(a \cdot b) = T_\sigma(a) \cdot T_\sigma(b)$. □

For $f \in A$, one should think of $\sigma : A \longrightarrow B$ as valuation at some abstract "point" $\sigma(f)$ and then $T_\sigma(f)$ is the Taylor expansion of f at that point. For example, in the case of C^∞-functions $U \longrightarrow \mathbb{R}$, if $p \in U$ and $\sigma : C^\infty(U) \longrightarrow \mathbb{R}$ is the evaluation map $\sigma(f) = f(p)$ at p, then $T_\sigma : C^\infty(U) \longrightarrow \mathbb{R}[[t]]$ and $T_\sigma(f)$ indeed is the Taylor expansion of f about p; hence

$$f(p + t) = T_\sigma(f)(t)$$

for small t, provided f is analytic (or a polynomial).

1.3.3. *Notation.* (Keigher) Let $A = (A, d)$ be a differential ring and let I be an ideal of A. From the higher Leibniz rule 1.1.12 we see that

$$I^{\#} := \{a \in A \mid d^n(a) \in I \text{ for all } n \geq 0\}.$$

is the largest differential ideal of A contained in I.

1.3.4. **Proposition.** *Let A be a differential $\mathbb{Q}$-algebra and let I be an ideal of A.*

(i) $I^{\#} = \ker(T_\pi)$, where $\pi : A \twoheadrightarrow A/I$ is the natural map onto the residue ring.

(ii) If I is radical[8], then $I^{\#}$ is radical.

(iii) If I is a prime ideal, then $I^{\#}$ is a prime ideal.

(iv) If I is a differential ideal, then

(a) Every prime ideal of A that is minimal with the property that it contains I, is differential.

(b) $\sqrt{I} = \bigcap\{\mathfrak{p} \mid \mathfrak{p} \subseteq A$ differential prime ideal with $I \subseteq \mathfrak{p}\}$.[9] This means the abstract Nullstellensatz of Krull holds for differential ideals.

(c) $\sqrt{I}$ is again a differential ideal. It follows that for any $S \subseteq A$, the smallest differentially radical ideal $\sqrt[d]{S}$ containing S is $\sqrt{[S]}$, where $[S]$ denotes the differential ideal generated by S.

(v) If $S \subseteq A$ is multiplicatively closed and I is maximal among differential ideals with the property that $I \cap S = \emptyset$, then I is prime.

(vi) If $S, T \subseteq A$, then $\sqrt[d]{S} \cdot \sqrt[d]{T} \subseteq \sqrt[d]{S \cdot T}$.

Proof. (i) is immediate from the definition of T_π.

(ii). The ideal I is radical if and only if the ring A/I is reduced[10]. But in that case also $A/I\,[[t]]$ is reduced (look at powers of the leading coefficient) and so by (i) we see that $I^{\#} = \ker(T_\pi)$ is radical.

(iii). The ideal I is prime if and only if the ring A/I is a domain. But in that case also $A/I\,[[t]]$ is a domain (look at products of leading coefficients) and so by (i) we see that $I^{\#} = \ker(T_\pi)$ is radical.

(iv) is a formal consequence of (iii):

(a). Let $\mathfrak{p}$ be a prime ideal of A that is minimal with the property that it contains I. By (iii), $\mathfrak{p}^{\#}$ is again prime. But $\mathfrak{p}^{\#}$ contains I because $I \subseteq \mathfrak{p}$ and $d(I) \subseteq I$. By minimality of $\mathfrak{p}$ we see that $\mathfrak{p}^{\#} = \mathfrak{p}$ as required.

(b) follows from (a) together with the classical abstract Nullstellensatz of Krull. Finally (c) is direct from (b).

(v). By Zorn's lemma there is an ideal J of A containing I with $J \cap S = \emptyset$ such that J is maximal with this property. By classical commutative algebra, J is a prime ideal of A. By (iii), also $J^{\#}$ is prime. Since I is differential we get $I \subseteq J^{\#}$. By maximality of I we see that $I = J^{\#}$ is prime.

(vi). By (iv) it suffices to show that every differential prime ideal $\mathfrak{p}$ containing $S \cdot T$ contains S or T. However, if $s \in S \setminus \mathfrak{p}$, then $s \cdot T \subseteq \mathfrak{p}$ implies $T \subseteq \mathfrak{p}$. ☐

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[8]meaning that $a^n \in I$ implies $a \in I$

[9]Recall that $\sqrt{I} = \{a \in A \mid a^n \in I \text{ for some } n \in \mathbb{N}\}$ is the smallest radical ideal containing I.

[10]Meaning that $x^n = 0 \Rightarrow x = 0$. This property of a ring is sometimes called *semiprime*

It should be noticed that when the characteristic of the ring A is not 0, then in general there are differential ideals I of A such that $\sqrt{I}$ is not differential. For example the ideal (t^p) is a differential ideal in $(\mathbb{Z}/p\mathbb{Z}\,[t], \frac{\mathrm{d}}{\mathrm{d}t})$, but if p is prime then $\sqrt{(t^p)} = (t)$ is not differential.

Now in the proof of 1.3.4, we have shown (iii) and then proceeded by proving implications (iii)⇒(iv)(a)⇒(iv)(b)⇒(iv)(c). Since (iv)(c) fails in general, all the other properties in this chain of implications also fail. Similarly (v) fails in general.

1.4. **The basis theorem of Ritt and Raudenbush.**

1.4.1. *Example.* If K is a differential field then in general not every differential ideal of $K\{x\}$ is finitely generated as a differential ideal. An example is

$$[x^2, x'^2, (x^{(2)})^2, (x^{(3)})^2, \ldots],$$

see [MaMePi1996, p. 43]

1.4.2. **Lemma.** *Let A be a differential ring and a $\mathbb{Q}$-algebra. Let $I \subseteq A\{x\}$ be a radical differential ideal with $I \neq J$, where J is the differential radical of $I \cap A$ in $A\{x\}$. Choose $P \in I \setminus J$ as follows:*

(a) Let $n = \min\{\mathrm{ord}(Q) \mid Q \in I \setminus J\}$. Since $I \neq J$ we know that $n \in \mathbb{N}_0$.
(b) Let $d = \min\{\deg(Q) \mid Q \in I \setminus J, \mathrm{ord}(Q) = n\} \in \mathbb{N}$.
(c) Take $P \in I \setminus J$ with $\mathrm{ord}(P) = n$ and $\deg(P) = d$. Write

$$P = L(P)\cdot(x^{(n)})^d + P_0,$$

where $\mathrm{ord}(P_0) \leq n$ and $\deg_{x^{(n)}}(P_0) < d$.

Then

(i) $L(P)\cdot S(P) \notin I$.
(ii) If I contains a prime ideal of A[11], then $L(P)\cdot S(P)\cdot I \subseteq \sqrt[d]{J, P}$.

Proof. We have

$$S(P) = \frac{\partial}{\partial x^{(n)}} P = d\cdot L(P)\cdot(x^{(n)})^{d-1} + \frac{\partial}{\partial x^{(n)}} P_0, \text{ and so}$$

$$L(P)\cdot S(P) = d\cdot L(P)^2\cdot(x^{(n)})^{d-1} + L(P)\cdot\frac{\partial}{\partial x^{(n)}} P_0.$$

Claim. $L(P)^2 \notin I$

Proof. Otherwise $L(P) \in I$ as I is radical. By choice of P and $\mathrm{ord}(L(P)) < n$ we get $L(P) \in J$. Further, $L(P) \in I$ implies $P_0 \in I$ and again by choice of P we get $P_0 \in J$. But then $P \in J$, a contradiction to the choice of P. $\diamond$

If $d = 1$, then

$$P = L(P)\cdot x^{(n)} + P_0,$$

where $\mathrm{ord}(P_0) < n$; but then $S(P) = L(P)$ and hence $L(P)\cdot S(P) \notin I$ by the claim.

Hence we may assume $d > 1$. Since $\mathrm{ord}(L(P)) < n$ we see that $\mathrm{ord}(L(P)\cdot\frac{\partial}{\partial x^{(n)}}P_0) \leq n$ and if $\mathrm{ord}(P_0) = n$, then $\deg_{x^{(n)}}(L(P)\cdot\frac{\partial}{\partial x^{(n)}}P_0) \leq \deg_{x^{(n)}}(\frac{\partial}{\partial x^{(n)}}P_0) < \deg_{x^{(n)}}(P_0) \leq d-1$.

Since $L(P)^2 \neq 0$ by the claim, and $d > 1$ we see that $\mathrm{ord}(L(P)\cdot S(P)) = n$ and $\deg(L(P)\cdot S(P)) < d$.

By choice of P we get $L(P)\cdot S(P) \notin I$, or, $L(P)\cdot S(P) \in J$. Assume that $L(P)\cdot S(P) \in J$. Since J is closed under $\frac{\partial}{\partial x^{(n)}}$ we get

$$J \ni \frac{\partial^{d-1}}{(\partial x^{(n)})^{d-1}}(L(P)\cdot S(P)) = d!\cdot L(P)^2.$$

But this contradicts the claim. Hence $L(P)\cdot S(P) \notin I$ showing (i).

(ii). Now assume that there is a prime ideal $\mathfrak{p}$ of A with $\mathfrak{p} \subseteq I$. By 1.3.4(iii) we know that $\mathfrak{p}^\# \subseteq \mathfrak{p}$ is again prime, hence we may assume that $\mathfrak{p}$ is a differential prime ideal. Let $\pi : A\{x\} \twoheadrightarrow (A/\mathfrak{p})\{x\}$ be the differential homomorphism extending $A \twoheadrightarrow A/\mathfrak{p}$

[11]For example if A is a domain, or if $I \cap A$ itself is prime

and sending x to x. Notice that π commutes with $\frac{\partial}{\partial x^{(k)}}$ for all $k \geq 0$, and $\ker(\pi) \subseteq J$ (using $\mathfrak{p} \subseteq I \cap A$). Since $L(P) \notin I$ by (i) we know that $\pi(L(P)) \neq 0$, which implies $\pi(L(P)) = L(\pi(P))$, $\operatorname{ord}(\pi(P)) = n$, $\deg(\pi(P)) = d$ and $\pi(S(P)) = S(\pi(P))$.

Now take $F \in I$ and let $H \in A/\mathfrak{p}\{x\}$ be the reduced remainder of $\pi(F)$ for $\pi(P)$. (Since $A/\mathfrak{p}$ is a domain, 1.2.11 is applicable.) Hence $H = 0$, or $\operatorname{ord}(H) < n$, or $\operatorname{ord}(H) = n$ and $\deg(H) < d$. Take a preimage $G \in A\{x\}$ of H under π with $G = 0$, or $\operatorname{ord}(G) < n$, or $\operatorname{ord}(G) = n$ and $\deg(G) < d$. Then for some $k, m \geq 0$ we see that

$$L(P)^k \cdot S(P)^m \cdot F \equiv G \bmod [P] + \ker(\pi).$$

Since P and $\ker(\pi)$ are in I we get $G \in I$. However, by the choice of P this is only possible if $G \in J$. It follows that $(L(P) \cdot S(P) \cdot F)^\alpha \in \sqrt[d]{\ker(\pi), P} \subseteq \sqrt[d]{J, P}$ for some $\alpha \geq 0$ and so $L(P) \cdot S(P) \cdot F \in \sqrt[d]{J, P}$. □

1.4.3. **Ritt-Raudenbush** *Let A be a differential $\mathbb{Q}$-algebra such that every differentially radical ideal is finitely generated as such.*[12] *Then also $A\{x\}$ has this property.*

Proof. Otherwise, use Zorn and take a maximal differentially radical ideal I that is not of the form $\sqrt[d]{E}$ for any finite $E \subseteq A\{x\}$. Then I is prime: If $a, b \notin I$, then by maximality of I we easily find a finite set $E \subseteq I$ with $\sqrt[d]{I, a} = \sqrt[d]{E, a}$ and $\sqrt[d]{I, b} = \sqrt[d]{E, b}$. Since I is differentially radical we get $I \subseteq \sqrt[d]{I \cdot I} \subseteq \sqrt[d]{\sqrt[d]{E, a} \cdot \sqrt[d]{E, b}} \subseteq \sqrt[d]{E, ab}$ using 1.3.4(vi). But then $ab \notin I$, otherwise $\sqrt[d]{E, ab} \subseteq I$ and so $I = \sqrt[d]{E, ab}$ in contradiction to the choice of I.

Let J be the differential radical ideal of $A\{x\}$ generated by $I \cap A$. By assumption on A there is a finite subset $E_0 \subseteq I \cap A$ such that $J = \sqrt[d]{E_0}$ (in $A\{x\}$). From $J \subseteq I$ we know that $I \cap A = J \cap A$. Take $P \in I \setminus J$ as in 1.4.2. Hence, as $I \cap A$ is prime we know that $L(P) \cdot S(P) \notin I$ and $L(P) \cdot S(P) \cdot I \subseteq \sqrt[d]{J, P}$. By maximality of I, there is a finite set $E \subseteq I$ such that $\sqrt[d]{I, L(P) \cdot S(P)} = \sqrt[d]{E, L(P) \cdot S(P)}$. Now

$$\begin{aligned}
I &\subseteq \sqrt[d]{I \cdot I} \subseteq \sqrt[d]{I \cdot \sqrt[d]{I, L(P) \cdot S(P)}} \\
&= \sqrt[d]{I \cdot \sqrt[d]{E, L(P) \cdot S(P)}} \text{ by choice of } E, \\
&\subseteq \sqrt[d]{I \cdot (E, L(P) \cdot S(P))} \text{ by 1.3.4}(vi) \\
&\subseteq \sqrt[d]{I \cdot E, I \cdot L(P) \cdot S(P)} \subseteq \sqrt[d]{E, I \cdot L(P) \cdot S(P)} \\
&\subseteq \sqrt[d]{E, J, P} \\
&\subseteq I.
\end{aligned}$$

Hence $I = \sqrt[d]{E, J, P} = \sqrt[d]{E, E_0, P}$ is the differential radical of a finite set. This contradicts the choice of I. □

[12] Equivalently: There is no infinite chain of strictly increasing differentially radical ideals $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \ldots$.

2. Differentially closed fields

We will now restrict ourselves to rings and fields containing $\mathbb{Q}$.

2.1. Quantifier elimination.

2.1.1. **Definition.** (Leonore Blum)
A **differentially closed field** (of characteristic 0, in one derivation) is a differential field M such that for all $P, Q \in M\{x\} \setminus \{0\}$ with $\mathrm{ord}(Q) < \mathrm{ord}(P)$ there is some $a \in M$ with $P(a) = 0$ & $Q(a) \neq 0$.

Observe that every DCF is algebraically closed (set $Q = 1$)

2.1.2. **Definition.** Let $\mathscr{L}_{\mathrm{ri}} = \{+, -, \cdot, 0, 1\}$ be the language of rings and let $\mathscr{L}_{\mathrm{ri}}(d)$ be the language obtained from $\mathscr{L}_{\mathrm{ri}}$ by adding a unary function symbol. Obviously, the class of differential fields of characteristic 0 is first order axiomatisable in $\mathscr{L}_{\mathrm{ri}}(d)$ and we denote its theory by DF. Furthermore, the class of differentially closed field is axiomatisable and we denote its theory by **DCF**.

2.1.3. **Theorem.** *The theory* DCF *is the modelcompletion of the theory* DF*. Furthermore,* DCF *has quantifier elimination and* DCF *is complete.*

Proof. Obviously, the ring $\mathbb{Z}$ with the trivial derivation is a common substructure of all differential fields. Hence completeness of DCF follows from quantifier elimination. Since DCF is an extension of DF we only need to show two things:

(a) Every differential field can be embedded into a differentially closed field, and
(b) DCF has quantifier elimination.

Proof of (a). Since DF is $\forall\exists$-axiomatised, every differential field is contained in an existentially closed differential field M. We show that M is a DCF:

Take $P, Q \in M\{x\}\setminus\{0\}$ with $\mathrm{ord}(Q) < \mathrm{ord}(P)$ and let P_1 be an irreducible factor of P of order $\mathrm{ord}(P)$. By 1.2.12, $I(P_1)$ is a differential prime ideal of $M\{x\}$ and by 1.2.10, $I(P_1)$ contains no polynomial of order $< \mathrm{ord}(P)$ and so $Q \notin I(P_1)$. It follows that the residue class of the variable x in $M\{x\}/I(P_1)$ is a differential solution of $P = 0$ & $Q \neq 0$. Since $M\{x\}/I(P_1)$ is a differential domain it is contained in a differential field. Hence the system $P = 0$ & $Q \neq 0$ has a differential solution in a differential field containing M. Since M is an existentially closed DF, there must also be such a solution in M. This shows that M is a DCF. $\diamond$

Proof of (b). We use the Shoenfield-Blum test, by which it suffices to solve the following embedding problem: Let $M, N \models$ DCF and suppose N is $\aleph_1$-saturated. Let A be a common differentially finitely generated subring of M, N. If $\alpha \in M$, then there is a differential embedding $A\{\alpha\} \longrightarrow N$ over A, where $A\{\alpha\}$ is the differential subring of M generated by $A \cup \{\alpha\}$. Firstly, the field generated by A in M and in N are isomorphic by a unique isomorphism and this isomorphism is differential. Hence we may replace A by its fraction field F and work over F. Let $\mathfrak{p} = \{Q \in F\{x\} \mid Q(\alpha) = 0\}$ be the differential ideal of vanishing of α. Clearly $\mathfrak{p}$ is a differential prime ideal of $F\{x\}$. By 1.2.12, either $\mathfrak{p} = (0)$, or there is an irreducible polynomial $P(x) \in F\{x\}$ with $\mathfrak{p} = I(P)$.

Let Γ be the following set of $\mathscr{L}_{\mathrm{ri}}(d)$-formulas with parameters in F:

$$\Gamma = \{P(x) = 0\} \cup \{Q(x) \neq 0 \mid \mathrm{ord}(Q) < \mathrm{ord}(P)\},$$

where we set $\mathrm{ord}(P)=\infty$ if $P=0$. Since N is a DCF, Γ is finitely realizable in N. Since F is countable and N is $\aleph_1$-saturated, there is a common realization β of all formulas in Γ. Let $\mathfrak{q}=\{Q\in F\{x\}\mid Q(\beta)=0\}$. If $\mathfrak{p}=(0)$, then also $\mathfrak{q}=(0)$. If $P\neq 0$, then take $Q\in F\{x\}$ irreducible with $\mathfrak{q}=I(Q)$. By definition of Γ we know that $\mathrm{ord}(Q)=\mathrm{ord}(P)$. Since $P\in\mathfrak{q}=I(Q)$ we know from 1.2.10 that Q divides P. Since P is irreducible this implies $P|Q$. As $F\{x\}$ is factorial, this is only possible if $P=a{\cdot}Q$ for some unit $a\in F$. Then $S(Q)=a{\cdot}S(P)$ as well and so $I(P)=I(Q)$, i.e. $\mathfrak{p}=\mathfrak{q}$.

So we know $\mathfrak{p}=\mathfrak{q}$ in either case. But then $F\{\alpha\}\cong_F F\{x\}/\mathfrak{p}=F\{x\}/\mathfrak{q}\cong_F F\{\beta\}$, solving our embedding problem. ⋄

This establishes the theorem. □

2.1.4**. Towards the construction of "concrete" DCFs** No very concrete DCF is known. However it is known that there are DCFs in the ring of germs of meromorphic functions over the complex numbers at 0 by applying Seidenberg's embedding theorem, cf. [Seiden1958a, Seiden1958b]; however this is an existence theorem only. Algebraically it is possible to construct a model of DCF, using a countable iteration of power series fields together with a twisted version of the Taylor morphism, cf. [LeSTre2020b, Examples 5.2].

Before exploiting 2.1.3 in the next sections we use it for a general extension property of derivations of fields, see 2.1.6 and 2.1.7.

2.1.5. **Observation.** *Let $L=(L,\delta)$ be a differential field and let $K\subseteq L$ be a (not necessarily differential) subfield of L. If $\beta\in L$ is algebraic over K with minimal polynomial $f(x)=x^k+a_{k-1}x^{k-1}+\ldots+a_0\in K[x]$, $k\geq 1$, then by applying δ to $f(\beta)=0$ we see that*

$$(*)\qquad \delta(\beta)=-\frac{\delta(a_{k-1})\beta^{k-1}+\ldots+\delta(a_1)\beta+\delta(a_0)}{\frac{\mathrm{d}f}{\mathrm{d}x}(\beta)}.$$

(Notice that the denominator here is nonzero by choice of f.) Consequently, if $\overline{K}$ denotes the algebraic closure of K in L, then:

- *(i) The restriction $\delta|_{\overline{K}}:\overline{K}\longrightarrow L$ of δ is uniquely determined by its restriction $\delta|_K:K\longrightarrow L$ to K.*
- *(ii) If K is a differential subfield of L, i.e. $\delta(K)\subseteq K$, then*
  - *(a) $\overline{K}$ is a differential subfield of L.*
  - *(b) The constant field of $\overline{K}$ is the algebraic closure of the constant field of K in L. This can be read off $(*)$ as follows: If all a_i are constants, then by $(*)$, also β is a constant; conversely, if $\delta(\beta)=0$, then by choice of f, using $\delta(K)\subseteq K$, we must have $\delta(a_i)=0$ for all i and so β is algebraic over the field of constants of K (with minimal polynomial f).*

2.1.6. **Proposition.** *Let $K\subseteq L$ be fields and let $d:K\longrightarrow K$ be a derivation. Let $T\subseteq L$ be a transcendence basis of L over K and for $t\in T$ let $b_t\in L$. Then there is a unique derivation δ of L extending d such that $\delta(t)=b_t$ for all $t\in T$.*

In particular, if L is algebraic over K, then there is a unique derivation of L extending d.

Proof. For uniqueness note first that δ is obviously uniquely determined on $K(T)$ by the requirements $\delta|_K=d$ and $\delta(t)=b_t$, $t\in T$. Since L is algebraic over $K(T)$

we may apply 2.1.5(i), which implies that there can then only be one derivation on L satisfying these requirements.

For existence we first do the case when $T = \emptyset$:

Claim. If L/K is algebraic, then d has a unique extension to a derivation on L.

Proof. Uniqueness follows from 2.1.5(i). For existence we apply 2.1.3 to find some DCF M extending K. Then M is algebraically closed and as L/K is algebraic we may assume that $L \subseteq M$ as a field. By 2.1.5(i)(a), the derivation of M restricts to a derivation of L. $\diamond$

Now we show the proposition. Let ∂ be the unique derivation of $K(T)$ extending d which satisfies $\partial(t) = 0$ for all $t \in T$. By the claim, ∂ can be extended to a derivation of L, which we denote by ∂ again. Similarly, for each $t \in T$, the derivation $\frac{\mathrm{d}}{\mathrm{d}t}$ of $K(T)$ has an extension d_t to L. We will now proceed as in the proof of 1.1.7. For $g \in L$ we define

$$\delta(g) = \partial(g) + \sum_{t \in T} b_t \cdot d_t(g).$$

We need to check that the sum is finite: Let $E \subseteq T$ be finite such that g is algebraic over $K(E)$. If $t \in T \setminus E$, then $d_t(g) = 0$, because d_t vanishes on $K(E)$ and so also vanishes on g by the claim. Hence $\delta(g)$ is well defined. It follows easily from 1.1.6 that δ is a derivation on L. Clearly δ extends d and satisfies $\delta(t) = b_t$ for all $t \in T$. $\square$

2.1.7. **Corollary.** *Let L be a field and let A be a subring of L. Let $d : A \longrightarrow L$ be a derivation (i.e. $d(a+b) = d(a) + d(b)$ and $d(ab) = ad(b) + d(a)b$ for all $a, b \in A$). Then there is a derivation δ of L extending d.*

Proof. Let $K \subseteq L$ be the fraction field of A. It is straightforward to check that d extends to a derivation $K \longrightarrow L$ by defining $d(\frac{a}{b}) = \frac{d(a)b - ad(b)}{b^2}$ (just follow the proof of 1.1.10 verbatim). We may thus assume that $A = K$ is a field.

Let T be a transcendence basis of K (over $\mathbb{Q}$). By 2.1.6 there is a derivation δ of L with $\delta(t) = d(t)$ for all $t \in T$. Then K is algebraic over $\mathbb{Q}(T)$ and one checks easily that the formula $(*)$ in 2.1.5 for δ is also valid for d instead of δ. Since δ extends d on $\mathbb{Q}(T)$ we then see from that formula that δ also extends d. $\square$

2.2. **Differential spectrum, Kolchin topology and differential Nullstellensatz.**

2.2.1. **The differential spectrum** Let A be a differential ring.

(i) The **differential spectrum of** A is the topological space with base set

$$\mathrm{Sped}(A)=\{\mathfrak{p}\subseteq A\mid \mathfrak{p}\text{ differential prime ideal}\}$$

equipped with the **Kolchin topology**, whose closed sets are the sets $V^{\delta}(Z)=\{\mathfrak{p}\in\mathrm{Sped}(A)\mid Z\subseteq\mathfrak{p}\}$, where Z ranges over subsets of A. The closed sets in this topology are called **Kolchin closed**.

Notice that $\bigcap_{i\in I}V^{\delta}(Z_i)=V^{\delta}(\bigcup_{i\in I}Z_i))$ and $V^{\delta}(Z_1)\cup V^{\delta}(Z_2)=V^{\delta}(Z_1\cdot Z_2)$, hence the $V^{\delta}(Z)$ indeed form the closed sets of a topology.

(ii) If $S\subseteq\mathrm{Sped}(A)$ we define $I^{\delta}(S)=\bigcap_{\mathfrak{p}\in S}\mathfrak{p}$ and call it the **differential vanishing ideal** of S.

(iii) If $\mathfrak{p},\mathfrak{q}\in\mathrm{Sped}(A)$, then $\mathfrak{p}\subseteq\mathfrak{q}$ if and only if $\mathfrak{q}$ is in the closure of the set $\{\mathfrak{p}\}$ (which is equal of $V^{\delta}(\mathfrak{p})$); we also say $\mathfrak{p}$ **specializes to** $\mathfrak{q}$.

(iv) As in the classical case, the assignments $Z\mapsto V^{\delta}(Z)$, $S\mapsto I^{\delta}(S)$ form a **Galois connection** $V^{\delta}:\mathcal{P}(A)\longrightarrow\mathcal{P}(\mathrm{Sped}(A))$, $I^{\delta}:\mathcal{P}(\mathrm{Sped}(A))\longrightarrow\mathcal{P}(A)$ between the powerset of A and the powerset of $\mathrm{Sped}(A)$. This means that both assignments are inclusion reversing and $S\subseteq V^{\delta}(I^{\delta}(S))$, $Z\subseteq I^{\delta}(V^{\delta}(Z))$. It follows that $V^{\delta}(I^{\delta}(S))$ is the Kolchin closure of S and $I^{\delta}(V^{\delta}(Z))=\sqrt[d]{Z}$ is the differentially radical ideal generated by Z. see 1.3.4(iv).

(v) The Galois connection restricts to an inclusion-reversing bijection between Kolchin closed subsets and differentially radical ideals.

(vi) The Galois connection restricts to a inclusion reversing bijection between differential prime ideals and nonempty, irreducible and closed subsets of $\mathrm{Sped}(A)$.

(vii) Since $V^{\delta}(S)=\mathrm{Sped}(A)\cap V(S)$, the inclusion map $\mathrm{Sped}(A)\hookrightarrow\mathrm{Spec}(A)$ is a homeomorphism onto its image.[13]

A first consequence of theorem 2.1.3 is:

2.2.2. **Corollary.** *If M is a differentially closed field and $K\subseteq M$ is a differential subfield, $x=(x_1,\dots,x_n)$, then the map*

$$\pi:S_n(M,K)\longrightarrow\mathrm{Sped}(K\{x\})$$
$$p\longmapsto\{f\in K\{x\}\mid f(x)=0\in p\}$$

is a bijection.[14] *The map π is continuous, but is not a homeomorphism, because $S_n(M,K)$ is Hausdorff and* $\mathrm{Sped}(K\{x\})$ *is not.*[15]

Proof. The map is injective, because for $p\neq q\in S_n(M,K)$ there is a quantifier free formula with parameters in K distinguishing them (use quantifier elimination in 2.1.3). Since all such formulas are Boolean combinations of formulas of the form

[13]For readers who know the terminology: $\mathrm{Sped}(A)$ is a spectral subspace of $\mathrm{Spec}(A)$, in particular $\mathrm{Sped}(A)$ is itself a spectral space. See [DiScTr2019, Section 1.1]

[14]Here, $S_n(M,K)$ is the set of all n-types of M over K, hence the Stone space of the Tarski-Lindenbaum algebra of all formulas $\varphi(x_1,\dots,x_n)$ in the language $\mathscr{L}_{\mathrm{ri}}(d)(K)$ of differential rings with parameters from K, modulo the theory of (M,K).

[15]However, π is a homeomorphism if we equip $\mathrm{Sped}(K\{x\})$ with the so-called constructible topology (aka patch topology), having the sets $V^{\delta}(f)\cap D^{\delta}(g)$, $f,g\in K\{x\}$ as a subbasis of open sets; here $D^{\delta}(g)=\mathrm{Sped}(K\{x\})\setminus V^{\delta}(g)$. See [DiScTr2019, Section 1.3]

$f(x) = 0$, $f \in K\{x\}$, there must be a formula of the form $f(x) = 0$ distinguishing them.

For surjectivity of π, take $\mathfrak{p} \in \mathrm{Sped}(K\{x\})$. Since every differential domain embeds into a DCF by 2.1.3, there is a differentially closed field N extending the differential domain $K\{x\}/\mathfrak{p}$. By quantifier elimination for DCF, M and N are elementary over K, in particular $S_n(M, K) = S_n(N, K)$. Let p be the type over K that is realized by $(x_1 + \mathfrak{p}, \dots, x_n + \mathfrak{p})$ in N^n. One checks easily that p is mapped to $\mathfrak{p}$ by π. □

2.2.3**. The Kolchin topology in the geometric context** Let K be a differential field, let $n \in \mathbb{N}$ and set $x = (x_1, \dots, x_n)$. For $a \in K^n$, the kernel $\iota(a)$ of the differential evaluation map $ev_a : K\{x\} \longrightarrow K$ at a is a maximal ideal of $K\{x\}$ and a differential ideal. We obtain a map $\iota : K^n \longrightarrow \mathrm{Sped}(K\{x\})$, which is obviously injective. If we identify K^n with the image of ι, the Kolchin topology restricts to a topology on K^n which is called the **Kolchin topology on** K^n. The closed sets are the common differential zero sets of sets of differential polynomials in n differential variables (because $V^\delta(Z) \cap K^n$ is the common zero set of all $f \in Z$).

A consequence of the Ritt-Raudenbush basis theorem 1.4.3 is

2.2.4. **Corollary.** *For any differential field K the Kolchin topology of* $\mathrm{Sped}(K\{x\})$*, $x = (x_1, \dots, x_n)$ is* **Noetherian***, i.e., there is no infinite strictly ascending chain of closed sets $A_1 \supsetneq A_2 \supsetneq A_3 \supsetneq \dots$.*

Since K^n is a subspace of $\mathrm{Sped}(K\{x\})$*, the Kolchin topology of K^n is Noetherian, too.*

Proof. Since there is no infinite strictly increasing chain of differentially radical ideals in $K\{x_1, \dots, x_n\}$ by Ritt-Raudenbush 1.4.3, the Galois connection in 2.2.1(vi) shows that the Kolchin topology of $\mathrm{Sped}(K\{x\})$ is Noetherian. □

2.2.5**. Differential Nullstellensatz**

Let M be a DCF *and let $I \subseteq M\{x\}$, $x = (x_1, \dots, x_n)$ be a differential ideal. Let $V_M(I) = \{a \in M^n \mid f(a) = 0$ for all $f \in I\}$ (in the setup of 2.2.3, $V_M(I) = V^\delta(I) \cap M^n$). Then*

$$\sqrt{I} = \{f \in M\{x\} \mid f \text{ vanishes on } V_M(I)\}.$$

Proof. Only the inclusion $\supseteq$ needs a proof. Take $f \in M\{x\} \setminus \sqrt{I}$. By 1.3.4(iv) we know that there is a differential prime ideal $\mathfrak{p}$ of $M\{x\}$ containing I with $f \notin \mathfrak{p}$. By the Ritt-Raudenbush theorem 1.4.3 there is a finite set $E \subseteq \mathfrak{p}$ such that $\mathfrak{p} = \sqrt[d]{E}$. Now in the differential domain $M\{x\}/\mathfrak{p}$, the n-tuple $(x_1 + \mathfrak{p}, \dots, x_n + \mathfrak{p})$ is a differential zero of all polynomials in $\mathfrak{p}$, which is not a zero of f. Since M is a DCF, it is existentially closed in $M\{x\}/\mathfrak{p}$ (use 2.1.3). Hence there is a differential zero $a \in M^n$ of all polynomials in E, which is not a zero of f. As $I \subseteq \mathfrak{p} = \sqrt[d]{E}$, all polynomials in I vanish at a, so $a \in V_M(I)$. As $f(a) \neq 0$ we get the assertion. □

As in the case of the classical Nullstellensatz, 2.2.5 implies that V_M is a bijection between the differentially radical ideals of $M\{x\}$ and the Kolchin closed subsets of M^n, or, in view of 2.2.1(vi), the closed subsets of $\mathrm{Sped}(M\{x\})$.

2.2.6. **Proposition.** *(Quantifier free stability of differential fields)*[16]
Let $K \subseteq L$ be differential fields (of arbitrary characetristic!) and let $X \subseteq L^n$ be a Boolean combination of Kolchin closed subsets of L^n. Then

(i) *$X \cap K^n$ a Boolean combination of Kolchin closed subsets of K^n.*
(ii) *If X is a Kolchin closed subset of L^n, then $X \cap K^n$ is a Kolchin closed subset of K^n.*

Notice that when K carries the trivial derivation, then the Kolchin topology of K^n is the Zariski topology of K^n. Hence the proposition also implies its field version, where "Kolchin" is replaced by "Zariski" everywhere.

Proof. Since Boolean operations commute with intersection with K^n it suffices to show that for all $P(x) \in L\{x\}$, $x = (x_1, \ldots, x_n)$, the intersection of K^n with the differential zero set of P in L^n is Kolchin closed in K^n. Take K-linearly independent elements $\beta_1, \ldots, \beta_m \in L$ and $P_1, \ldots, P_m \in K\{x\}$ with $P = P_1 \cdot \beta_1 + \ldots + P_m \cdot \beta_m$ (notice that any K-vector space basis of L is also a basis of the free $K\{x\}$-module $L\{x\}$). Then

$$\{a \in L^n \mid P(a) = 0\} = \{a \in L^n \mid P_1(a) \cdot \beta_1 + \ldots + P_m(a) \cdot \beta_m = 0\}$$

and therefore

$$\{a \in K^n \mid P(a) = 0\} = \{a \in K^n \mid P_1(a) = \ldots = P_m(a) = 0\}$$

is Kolchin closed in K^n. □

Another consequence of theorem 2.1.3 then is:

2.2.7. **Corollary.** *The constants of a differentially closed field are* **stably embedded***: Let M be a differentially closed field with constant field C. If $D \subseteq M^n$ is definable with parameters from M, then $D \cap C^n$ is definable in the field C.*

Proof. By quantifier elimination for DCF the set D is a Boolean combination of Kolchin closed subsets of M^n. Hence by 2.2.6, $D \cap C^n$ is a Boolean combination of Kolchin closed subsets of C^n (notice that C is a differential subfield of M). But C carries the trivial derivation, hence Kolchin closed sets in C^n are Zariski closed. □

[16]A theory is *stable* if and only if for all models $A \prec B$ and every subset $D \subseteq B^n$ that is definable in B with parameters from B, the intersection $D \cap A^n$ is definable in A (with parameters from A). The proposition, together with quantifier elimination for DCF implies stability of DCF, hence the naming. In fact DCF has the stronger property of being ω-stable, see 2.5.2.

2.3. DCF has elimination of imaginaries.

2.3.1. **Definition.** Let M be an $\mathscr{L}$-structure with two definable constants. We say that M has **elimination of imaginaries**[17] if for every 0-definable equivalence relation $E \subseteq M^n \times M^n$, there are some $d \in \mathbb{N}$ and a 0-definable function $f : M^n \longrightarrow M^d$ such that the equivalence classes of E are exactly the fibres of f; in symbols:

$$\{[\bar{a}]_E \mid \bar{a} \in M^n\} = \{f^{-1}(\bar{c}) \mid \bar{c} \in \text{ image of } f\}.$$

A complete theory has elimination of imaginaries if one (equivalently: all) of its models have elimination of imaginaries.

We will now show that DCF has elimination of imaginaries. This will follow the same route as the proof that ACF has elimination of imaginaries. We need three preparations, one from model theory (2.3.3), one from topology (2.3.4) and one from field theory (2.3.5).

2.3.2. *Warning.* Elimination of imaginaries looks like a weak form of definable Skolem functions. However this is not the case. For example the field of p-adic numbers has definable Skolem functions, but it does not have elimination of imaginaries (use the equivalence relation $v_p(a) = v_p(b)$ on $\mathbb{Q}_p$).

2.3.3. **Theorem.** *Let T be an $\mathscr{L}$-theory (not necessarily complete) that defines at least two constants. Then T has elimination of imaginaries if and only if for every $\mathscr{L}$-formula $\varepsilon(\bar{x}, \bar{y})$ that defines an equivalence relation on n-tupels in all models of T, the following condition holds.*

(∗) *For every model M of T and each $\bar{a} \in M^n$ there is a set $B \subseteq M$ (of some size, not necessarily finite) such that for all automorphisms σ of M we have*

$$\sigma(\varepsilon[M, \bar{a}]) = \varepsilon[M, \bar{a}] \iff \sigma|_B = \mathrm{id}_B\,.$$

If this is the case, then the set B above can also be chosen to be finite.

Proof. If we have a function f for ε as in 2.3.1 then we may take B as the finite set of coordinates of $f(\bar{a})$ (notice that $\sigma(\varepsilon[M, \bar{a}]) = \varepsilon[M, \sigma\bar{a}]$). Hence the condition is satisfied when T has elimination of imaginaries.

The converse essentially follows from standard definability tests in model theory, like Svenonius' theorem. For details see [Hodges1993, Section 4.4, Section 10.5 and Corollary 10.5.5].

Remark: The assumption that T defines 2 constants is undesirable in some contexts, for example for modules or groups. In that case, condition (∗) characterizes existence of functions as in 2.3.1, but only 'locally'. See [Hodges1993, Section 4.4] □

2.3.4. **Lemma.** *Let X be any topological space and let $C \subseteq X$ be a Boolean combination of open sets. Then the frontier $\overline{C} \setminus C$ of C is not dense in $\overline{C}$.*

Proof. We may replace X by $\overline{C}$ and assume that C is dense in X. Write $C = \bigcup_{i=1}^n A_i \cap O_i$ with $n \geq 1$, $O_i \subseteq X$ open and nonempty and $A_i \subseteq X$ closed. Let O be minimal among nonempty intersections of the $O_1, \ldots, O_n$. Then for each $i \in \{1, \ldots, n\}$ we have $O \subseteq O_i$ or $O_i \cap O = \emptyset$. Hence after a permutation of

[17] In [Hodges1993, Section 4.4] this property is called *uniform* elimination of imaginaries. However in the presence of two definable constants this is equivalent to elimination of imaginaries in the sense of [Hodges1993, Section 4.4].

$\{1,\dots,n\}$ we may assume that $C\cap O=\bigcup_{i=1}^{k}A_i\cap O$ for some $k\in\{1,\dots,n\}$. It follows that $C\cap O$ is closed in O. On the other hand C is dense in X and O is open in X, which implies that $C\cap O$ is dense in O. Consequently $C\cap O=O$, i.e. $O\subseteq C$. Since $O\neq\emptyset$, the set $X\setminus C$ is not dense in X. □

2.3.5. **Theorem.** *(André Weil)*
Let K be a field and let $\mathfrak{a}$ be an ideal of $K[T]$, where T is a (not necessarily finite) set of indeterminates. Then there is a smallest subfield $k\subseteq K$ such that $\mathfrak{a}$ is defined over k, i.e. $(\mathfrak{a}\cap k[T])\cdot K=\mathfrak{a}$. The field k is called the **field of definition** *of $\mathfrak{a}$ and has the following property: If σ is an automorphism of K, then*

$$\sigma(\mathfrak{a})=\mathfrak{a}\iff\sigma|_k=\mathrm{id}_k\,.$$

Proof. Reference: [Lang1972, Chap. III, Section 2, Theorem 7, p.62]. Here is how to find k: Let B be a set of monomials in the indeterminates T such that $b\neq b'\bmod\mathfrak{a}$ for all $b\neq b'\in B$ and such that $\{b\bmod\mathfrak{a}\mid b\in B\}$ is a basis of the K-vectorspace $K[T]/\mathfrak{a}$. For each monomial m in the indeterminates T, let $m_b\in K$ be the coefficient at $b\bmod\mathfrak{a}$, when $m\bmod\mathfrak{a}$ is written as a K-linear combination in that basis. Hence

$$m=\sum_{b\in B}m_b\cdot b\mod\mathfrak{a}.$$

The sum makes sense because $m_b=0$ for all but finitely many $b\in B$.

Let k be the field generated by all the m_b, $b\in B, m$ a monomial. Then one shows that k has the required properties. □

2.3.6. **Corollary.** *Let K be a differential field and let T be a set of differential indeterminates. Let $\mathfrak{a}$ be a differential ideal of $K\{T\}$. There is a smallest differential subfield $k\subseteq K$ such that $\mathfrak{a}$ is generated as a differential ideal by $\mathfrak{a}\cap k\{T\}$.*

The field k is the differential field generated by the field of definition of the ideal $\mathfrak{a}$ and is called the **differential field of definition** *of $\mathfrak{a}$. It has the following property: If σ is a differential automorphism of K then*

$$\sigma(\mathfrak{a})=\mathfrak{a}\iff\sigma|_k=\mathrm{id}_k\,.$$

Proof. Straightforward checking using 2.3.5. □

2.3.7. **Theorem.** *Differentially closed fields have elimination of imaginaries.*

Proof. We verify condition $(*)$ of 2.3.3 for $\mathscr{L}_{\mathrm{ri}}(d)$-formulas $\varepsilon(\bar{x},\bar{y})$, for which DCF defines an equivalence relation on n-tuples (in models). Take a differentially closed field M and some $\bar{a}\in M^n$. Let $E\subseteq M^n\times M^n$ be the set defined by $\varepsilon(\bar{x},\bar{y})$ in M and let $[\bar{a}]_E\subseteq M^n$ be the equivalence class of $\bar{a}$ for E. Hence $[\bar{a}]_E$ is defined by $\varepsilon(\bar{x},\bar{a})$ in M. Let $V\subseteq M^n$ be the closure of $[\bar{a}]_E$ for the Kolchin topology and let $I\subseteq M\{\bar{x}\}$ be the vanishing ideal of V, thus

$$I=\{P\in M\{\bar{x}\}\mid P|_V\equiv 0\}.$$

Obviously I is a differential ideal of $M\{\bar{x}\}$. Let k be the differential field of definition of I as in 2.3.6. We claim that k has the required property $(*)$ of 2.3.3 for $\bar{a}$, hence we claim for every differential automorphism σ of N that

$$(*)\qquad\sigma(\varepsilon[M,\bar{a}])=\varepsilon[M,\bar{a}]\iff\sigma|_k=\mathrm{id}_k\,.$$

By 2.3.6 we know that

$$\sigma(I)=I\iff\sigma|_k=\mathrm{id}_k,$$

which easily implies

(†) $$\sigma(V) = V \iff \sigma|_k = \mathrm{id}_k\,.$$

Since σ is a differential automorphism of M, its extension $M^n \longrightarrow M^n$ is a homeomorphism for the Kolchin topology. Hence if $\sigma(\varepsilon[M,\bar a]) = \varepsilon[M,\bar a]$, then also $\sigma(V) = V$ (recall that V is the Kolchin closure of $\varepsilon[M,\bar a]$), hence by (†) we get $\sigma|_k = \mathrm{id}_k$.

Conversely assume $\sigma|_k = \mathrm{id}_k$. By (†) we know $\sigma(V) = V$ and we need to show $\sigma(\varepsilon[M,\bar a]) = \varepsilon[M,\bar a]$. By quantifier elimination of DCF we know that $\varepsilon[M,\bar a]$ is a Boolean combination of Kolchin closed sets. Hence by 2.3.4, $V \setminus \varepsilon[M,\bar a]$ is not Kolchin dense in V. On the other hand, as σ is a homeomorphism, $\sigma(\varepsilon[M,\bar a])$ again has Kolchin closure V. Consequently, $\sigma(\varepsilon[M,\bar a])$ cannot be in $V \setminus \varepsilon[M,\bar a]$, which means that $\varepsilon[M,\bar a] \cap \sigma(\varepsilon[M,\bar a]) \neq \emptyset$. However, $\sigma(\varepsilon[M,\bar a]) = \varepsilon[M,\sigma(\bar a)]$ is also an equivalence class of E (using that E is 0-definable) and therefore $\varepsilon[M,\bar a] = \sigma(\varepsilon[M,\bar a])$, as required. □

2.3.8. *Examples.* Let M be a DCF. We exhibit two natural 0-definable equivalence relations. Let C be the constant field of M.

(i) The additive group M/C is coded by the derivation $\delta : M \longrightarrow M$, i.e. the diagram

$$\begin{array}{ccc} M & & \\ {\scriptstyle\delta}\uparrow & \nwarrow{\scriptstyle f} & \\ M & \longrightarrow & M/C \end{array}$$

where $f(a \bmod C) = a'$ is commutative and f is a bijection. f is even a group homomorphism. The compositional inverse of f maps $b \in M$ to $a \bmod C$, where a is any anti-derivative of b. In a sense, f^{-1} is like integration of M.

(ii) The multiplicative group $M^\times/C^\times$ is coded by the logarithmic derivation $\ell : M^\times \longrightarrow M$, $\ell(a) = \frac{a'}{a}$, i.e. the diagram

$$\begin{array}{ccc} M & & \\ {\scriptstyle\ell}\uparrow & \nwarrow{\scriptstyle g} & \\ M^\times & \longrightarrow & M^\times/C^\times \end{array}$$

where $g(a \bmod C^\times) = \frac{a'}{a}$ is commutative and g is a bijection. g is even a group homomorphism. The compositional inverse of g maps $b \in M$ to $a \bmod C^\times$, where $\frac{a'}{a} = b$.

2.4. **Cantor-Bendixson analysis of a topological space.**

2.4.1. **Definition.** Let ∞ be an element larger than every ordinal. For an arbitrary topological space X[18] we write δX for the set of all non-isolated (i.e., accumulation) points of X. The set δX is called the **Cantor–Bendixson derivative** of X. If $\delta X = X \neq \emptyset$, then X is called **perfect**. For an ordinal α we define subsets $\delta^\alpha X$ of X by transfinite recursion on α:

- $\delta^0 X = X$.
- $\delta^{\alpha+1} X = \delta\delta^\alpha X$.
- If α is a limit ordinal, then $\delta^\alpha X = \bigcap_{\beta<\alpha} \delta^\beta X$.

Finally we define $\delta^\infty X = \bigcap_\alpha \delta^\alpha X$. Clearly, $(\delta^\alpha X)_\alpha$ is a descending chain of closed subsets of X and $\delta^\infty X$ does not possess isolated points. For an element $x \in X$, the **Cantor–Bendixson rank** of x, denoted $\mathrm{CB}_X(x)$ or $\mathrm{CB}(x)$, is the largest ordinal α, or ∞, with the property that $x \in \delta^\alpha X$. (By the definition of $\delta^\alpha X$ this indeed makes sense.)

The **Cantor–Bendixson rank** $\mathrm{CB}(X)$ of a nonempty space X is the supremum of all α with $\delta^\alpha(X) \neq \emptyset$, or ∞ if this supremum does not exist. In other words

$$\mathrm{CB}(X) = \sup\{\mathrm{CB}_X(x) \mid x \in X\}.$$

We set $\mathrm{CB}(\emptyset) = -1$. If $\mathrm{CB}(X) = \alpha < \infty$, then the **Cantor-Bendixson degree** is defined as the cardinality of $\delta^\alpha(X)$:

$$\mathrm{CD}(X) = \mathrm{card}(\delta^\alpha(X)),\ \alpha = \mathrm{CB}(X).$$

2.4.2. *Examples.*

(i) If X is $\omega + 1$ with the interval topology (i.e., the one-point compactification of the discrete set ω), then $\mathrm{CB}_X(\omega) = 1$ and all points other than ω have Cantor–Bendixson rank 0. If X is a densely totally ordered set equipped with the interval topology, then no point of X is isolated, hence X is perfect and so $\mathrm{CB}_X(x) = \infty$ for all $x \in X$.

(ii) Let λ be an ordinal. We consider λ as a topological space having the **down-sets**[19] as open sets. Obviously, the unique isolated point of a nonempty subset Y is the minimum of Y. Consequently for any ordinal $\alpha \leq \lambda$, we have $\delta^\alpha(\lambda) = \{\beta \mid \alpha \leq \beta < \lambda\}$. Consequently $\mathrm{CB}(\lambda) = \lambda$, if λ is a limit ordinal and $\mathrm{CB}(\lambda) = \mu$, if λ is the successor of μ. Furthermore $\mathrm{CB}_\lambda(\alpha) = \alpha$ for all $\alpha \in \lambda$.

On the other hand, if we write λ_{opp} for λ with the topology that has the **up-sets**[20] as open sets, then $\mathrm{CB}(\lambda_{\mathrm{opp}}) = \infty$, unless λ is finite. The reason is that ω is a perfect subset.

2.4.3. **Generalities on the Cantor-Bendixson rank** Let $X = (X, \tau)$ be an arbitrary topological space, $X \neq \emptyset$.

(i) Clearly $\delta X = \emptyset$ if and only if X is discrete.

[18] In particular X does not need to be Hausdorff

[19] A down-set of a partially ordered set $(X, \leq)$ is a subset $Y \subseteq X$ with the property $x \leq y \in Y \Rightarrow x \in Y$.

[20] An up-set of a partially ordered set $(X, \leq)$ is a subset $Y \subseteq X$ with the property $x \geq y \in Y \Rightarrow x \in Y$.

(ii) If $Y \subseteq X$ is a subspace, then $\delta^\alpha Y \subseteq \delta^\alpha X$ as follows from $\delta Y \subseteq \delta X$ by straightforward transfinite induction on α. Consequently $\mathrm{CB}_Y(y) \leq \mathrm{CB}_X(y)$ for all $y \in Y$ and

$$\mathrm{CB}(Y) = \sup\{\mathrm{CB}_Y(y) \mid y \in Y\} \leq \sup\{\mathrm{CB}_X(y) \mid y \in Y\} \leq \mathrm{CB}(X).$$

(iii) $\delta^\infty X$ is nonempty if and only if X has a perfect subspace. Then $\delta^\infty X$ is the largest perfect subspace (by (ii)).

(iv) From (iii) we see that $\mathrm{CB}(X) < \infty$ if and only if X does not contain a perfect subset, which means that X is **scattered** (i.e., every nonempty subset S contains a point that is isolated in S).

(v) If $Y \subseteq X$ is open, then a straightforward induction on α implies $\delta^\alpha(Y) = \delta^\alpha(X) \cap Y$. Therefore $\mathrm{CB}_Y(y) = \mathrm{CB}_X(y)$ for all $y \in Y$ and

$$\mathrm{CB}(Y) = \sup\{\mathrm{CB}_X(y) \mid y \in Y\}.$$

(vi) For $x \in X$ and any basis of neighborhoods $\mathscr{N}$ of x we have

$$\mathrm{CB}_X(x) = \min\{\mathrm{CB}(Z) \mid x \in Z \in \mathscr{N}\}.$$

Proof. By (ii) we may assume that all sets in $\mathscr{N}$ are open.

$\leq$. Let $Z \in \mathscr{N}$. Since Z is open we know from (v) that $\mathrm{CB}_X(x) = \mathrm{CB}_Z(x)$, which is $\leq \mathrm{CB}(Z)$.

$\geq$: We may assume that $\alpha = \mathrm{CB}_X(x) < \infty$. Hence there is an open subset O of X with $O \cap \delta^\alpha X = \{x\}$ and we may assume that $O \in \mathscr{N}$.

By (ii) we see that $\mathrm{CB}(O) \leq \sup\{\mathrm{CB}_X(y) \mid y \in O\} = \alpha$. ◇

(vii) If τ' is another topology on X with $\tau \subseteq \tau'$, then a trivial induction on α shows that $\delta^\alpha(X, \tau') \subseteq \delta^\alpha(X, \tau)$ and therefore $\mathrm{CB}_{\tau'}(x) \leq \mathrm{CB}_\tau(x)$ for all $x \in X$.

(viii) If X is quasi-compact[21] then

$$\mathrm{CB}(X) = \max\{\mathrm{CB}_X(x) \mid x \in X\},$$

because all sets $\delta^\alpha X$ are closed. If $\alpha = \mathrm{CB}(X) < \infty$, the set $\delta^\alpha X$ is finite and so $\mathrm{CD}(X) \in \mathbb{N}$.

(ix) If $Y \subseteq X$ is open and quasi-compact, then by using (v) and (viii) we have

$$\mathrm{CB}(Y) = \max\{\mathrm{CB}_X(y) \mid y \in Y\}.$$

(x) If $\mathrm{CB}(X) \neq \infty$ then $\mathrm{card}(\mathrm{CB}(X)) \leq \mathrm{card}(X)$.

(xi) If X is a T_0-space then $X \setminus \delta X$ is contained in the set of minimal points (for specialization). The inclusion may be proper, for an example see 2.4.2(i).

(xii) Every indiscrete space with at least two points has Cantor–Bendixson rank ∞. If X is not a T_0-space then X has a perfect subspace, hence $\mathrm{CB}(X) = \infty$.

2.4.4. **Lemma.** *Let X be a compact Hausdorff space and let α be an ordinal. The following are equivalent.*

(i) $\mathrm{CB}(X) \geq \alpha + 1$
(ii) $\delta^\alpha(X)$ *is infinite.*
(iii) There are infinitely many open and pairwise disjoint subsets of X with Cantor-Bendixson rank $\geq \alpha$.

[21]A space is **quasi-compact** if every open cover has a finite subcover.

Proof. (i)⇒(iii). (here we only need the Hausdorff property, not compactness) By assumption there is some $p \in \delta^\alpha X$ that is not isolated in $\delta^\alpha X$. Choose $y_1 \in \delta^\alpha X$, $y_1 \neq p$ and disjoint open neighborhoods V_1 of p and U_1 of y_1 (using the Hausdorff property). Then $V_1 \cap \delta^\alpha X$ contains a point $y_2 \neq p$. Choose disjoint open neighborhoods V_2 of p and U_2 of y_2, contained in V_1. Now repeat this process with V_2 and iterate. We obtain open neighborhoods $V_1 \supseteq V_2 \supseteq \ldots$ of p, points $y_1, y_2, \ldots \in \delta^\alpha X$ and open neighborhoods U_i of y_i with $V_i \cap U_i = \emptyset$, $U_{i+1} \subseteq V_i$. It follows that $U_1, U_2, \ldots$ are infinitely many open and pairwise disjoint subsets of X with Cantor-Bendixson rank $\geq \alpha$.

(iii)⇒(ii) is obvious.

(ii)⇒(i). Since $\delta^\alpha X$ is a closed subset of the compact space X, it is itself compact. But $\delta^\alpha X$ is infinite and so it cannot be discrete. Consequently $\delta\delta^\alpha X \neq \emptyset$. □

2.4.5**. Cantor-Bendixson analysis in a Boolean space** Recall that a space X is **Boolean** if it is compact Hausdorff such that the set $\mathrm{Clop}(X)$ of **clopen** (=closed and at the same time open) subsets of X are a basis. The set $\mathrm{Clop}(X)$ is a Boolean algebra of subsets of X. By 2.4.3(vi) we know that the Cantor-Bendixson rank of points of a Boolean space may be computed from the Cantor-Bendixson ranks of clopen sets containing x:

$$(*) \qquad \mathrm{CB}_X(x) = \min\{\mathrm{CB}(Z) \mid x \in Z \in \mathrm{Clop}(X)\}.$$

Furthermore we may define $\mathrm{CB}(Z)$, $Z \in \mathrm{Clop}(X)$ in combinatorial terms within the Boolean algebra $\mathrm{Clop}(X)$. This goes as follows: If $Z \subseteq X$ is clopen then Z has the following properties:

(a) $\mathrm{CB}(Z) \geq -1$
(b) For every ordinal α, $\mathrm{CB}(Z) \geq \alpha + 1$ if and only if there are infinitely many nonempty clopen subsets $Z_1, Z_2, \ldots \subseteq Z$ with $Z_i \cap Z_j = \emptyset$ for all $i \neq j$ such that $\mathrm{CB}(Z_i) \geq \alpha$ for all i.
(c) If λ is a limit ordinal, then $\mathrm{CB}(Z) \geq \lambda$ if and only if $\mathrm{CB}(Z) \geq \alpha$ for all $\alpha < \lambda$.

Item (a) is clear, (b) holds by 2.4.4 using the assumption that $\mathrm{Clop}(X)$ is a basis, and (c) follows from 2.4.3(ix) using that Z is open and compact.

Hence in a Boolean space one may define Cantor-Bendixson rank by first using (a),(b) and (c) to define the expression $\mathrm{CB}(Z) \geq \alpha$ for $Z \in \mathrm{Clop}(X)$ by induction, and then to define $\mathrm{CB}(Z) = \max\{\alpha \mid \mathrm{CB}(Z) \geq \alpha\}$. Equation $(*)$ above extends the definition to points in x.

Remark. It is not difficult to show that a Boolean space X is scattered if and only if the Boolean algebra $\mathrm{Clop}(X)$ is **super atomic** (i.e., every homomorphic image has an atom); see *Handbook of Boolean Algebras, Vol. I, Chapter 17, p. 271f.*

The rank of a poset

2.4.6. **Proposition and Definition.** *Let $X = (X, \leq)$ be a poset. We equip X with the topology τ^U that has the up-sets as open sets and define the* **rank** *of $x \in X$ as*

$$\mathrm{rk}(x) = \mathrm{rk}_{(X,\leq)}(x) := \mathrm{CB}_{(X,\tau^U)}(x).$$

Hence $\mathrm{rk}(x)$ is an ordinal or ∞, which can be described directly in terms of the poset X as follows.

- $\mathrm{rk}(x) \geq 0$ *for all x.*

- *If α is an ordinal, then* $\mathrm{rk}(x) \geq \alpha + 1$ *if and only if there is some* $y \in X$ *with* $x < y$ *and* $\mathrm{rk}(y) \geq \alpha$.
- *If α is a limit ordinal, then* $\mathrm{rk}(x) \geq \alpha$ *if and only if* $\mathrm{rk}(x) \geq \beta$ *for all* $\beta < \alpha$.

We define $\mathrm{Rk}(X)$*, the* **rank of the poset***, to be the Cantor–Bendixson rank of* (X, τ^U)*. The* **rank of a space** (X, τ) *is the rank of the poset* $(X, \rightsquigarrow_\tau)$ *and is also denoted by* $\mathrm{Rk}(X)$.

In particular, for any ordinal α the set $\mathrm{Rk}^\alpha(X) = \{x \in X \mid \mathrm{rk}(x) \geq \alpha\}$ *is a down-set.*

Proof. By definition of the Cantor–Bendixson rank it suffices to show that for every ordinal α we have $\mathrm{CB}_{(X,\tau^U)}(x) \geq \alpha + 1$ if and only if there is some $y \in X$ with $x < y$ and $\mathrm{CB}_{(X,\tau^U)}(y) \geq \alpha$ (i.e., $y \in \delta^\alpha X$). Note that $\mathrm{CB}_{(X,\tau^U)}(x) \geq \alpha + 1$ means that $x \in \delta^\alpha X$ and x is not isolated in $\delta^\alpha X$ for τ^U. Since $x^\uparrow$ is the smallest open subset containing x, this is equivalent to saying that

$$x^\uparrow \cap \delta^\alpha X \supsetneq \{x\}.$$

But this condition says precisely that x is a non-maximal point of $\delta^\alpha X$ (i.e., there is some $y \in \delta^\alpha X$ with $x < y$). □

2.4.7. **Remarks and Examples** Let $(X, \leq)$ be a poset.

(i) Let $Y \subseteq X$ be a subset. Then $\mathrm{rk}_Y(y) \leq \mathrm{rk}_X(y)$ for each $y \in Y$.
(ii) An element $x \in X$ is maximal if and only if $\mathrm{rk}(x) = 0$. Thus, (X, τ^U) is perfect if and only if $(X, \leq)$ does not have any maximal elements.
(iii) If X has a bottom element then $\mathrm{rk}(\bot) = \mathrm{Rk}(X)$.
(iv) If $x \in X$ then $\mathrm{rk}(x) = \mathrm{Rk}(x^\uparrow)$, where $x^\uparrow = \{y \in X \mid x \leq y\}$.
(v) If $x < y$ then $\mathrm{rk}(y) < \mathrm{rk}(x)$ or $\mathrm{rk}(y) = \mathrm{rk}(x) = \infty$.
(vi) $\mathrm{rk}(x) = \infty$ if and only if there is an infinite chain $x = x_1 < x_2 < x_3 < \cdots$. (Use (v) for the implication $\Leftarrow$.)
(vii) If $\mathrm{rk}(x) = \alpha \neq \infty$ and $\beta < \alpha$, then there is some $x_\beta \in x^\uparrow$ with $\mathrm{rk}(x_\beta) = \beta$; in particular, $\mathrm{card}(\alpha) \leq \mathrm{card}(x^\uparrow)$.
Proof. Assume the claim is false. The description of the rank in 2.4.6 shows, together with (v), that there is some $x_1 \in x^\uparrow \setminus \{x\}$ with $\alpha > \mathrm{rk}(x_1) > \beta$. We repeat the construction with x_1 in place of x and obtain some $x_2 \in x_1^\uparrow$ with $\alpha > \mathrm{rk}(x_1) > \mathrm{rk}(x_2) > \beta$. Iteration yields a properly decreasing sequence $\mathrm{rk}(x_1) > \mathrm{rk}(x_2) > \mathrm{rk}(x_3) > \cdots$ of ordinals, a contradiction. ◇
(viii) Suppose $X \neq \emptyset$ is anti-well-ordered of order type λ_{opp} for an ordinal λ. Then a simple (transfinite) induction on $\alpha \in X$ shows that $\mathrm{rk}(\alpha) = \alpha$. Hence $\mathrm{Rk}(X) = \lambda$ if λ is a limit ordinal and $\mathrm{Rk}(X) = \beta$ if $\lambda = \beta + 1$. This should also be compared with 2.4.2(ii).

2.4.8. **Lemma.** *For a poset X the following conditions are equivalent:*

(i) $\mathrm{Rk}(X) < \infty$.
(ii) X has the ACC (i.e., every nonempty chain has a maximum).

Proof. (i)⇒(ii) Assume there is a chain in X without maximum. Then the chain contains a strictly increasing sequence $x_1 < x_2 < x_3 < \cdots$. It follows from 2.4.7(vi) that $\mathrm{rk}(x_1) = \infty$, which implies $\mathrm{Rk}(X) = \infty$ (by 2.4.3(ii) and 2.4.7(iv)).

(ii)⇒(i) Suppose $\mathrm{rk}(x) = \infty$ for some $x \in X$. Pick any ordinal α with $\mathrm{card}(\alpha) > \mathrm{card}(X)$. There is some $x_1 \in x^\uparrow \setminus \{x\}$ with $\mathrm{rk}(x_1) \geq \alpha$, 2.4.6. Now 2.4.3(x) implies $\mathrm{rk}(x_1) = \infty$ (since $\mathrm{card}(\mathrm{rk}(x_1)) = \mathrm{card}(\mathrm{Rk}(x_1^\uparrow)) > \mathrm{card}(x_1^\uparrow)$). Iterating this procedure we obtain a strictly increasing sequence $x < x_1 < x_2 < \cdots$, which is a chain without maximum. $\square$

2.4.9. **Proposition.** [22] *Let K be a differential subfield of a differentially closed field and let $p \in S_n(M,K)$. Let $\mathfrak{p} \in \operatorname{Sped}(K\{x\})$, $x = (x_1, \ldots, x_n)$ be the prime ideal corresponding to p in the bijection π of 2.2.2. Then the Cantor-Bendixson rank of p in $S_n(M,K)$ is less or equal to the rank of $\mathfrak{p}$ in* $\operatorname{Sped}(K\{x\})$.

Proof. The rank of $\pi(p)$ is the Cantor-Bendixson rank in $\operatorname{Sped}(K\{x\})$ for the topology that has the up-sets of $\operatorname{Sped}(K\{x\})$ (i.e. sets G satisfying $\mathfrak{q} \supseteq \mathfrak{p} \in G \Rightarrow \mathfrak{q} \in G$) as open sets. Since π is a bijection, the rank of $\pi(p)$ is the Cantor-Bendixson rank of p for the topology $\{\pi^{-1}(G) \mid G \text{ up-set of } \operatorname{Sped}(K\{x\})\}$ of $S_n(M,K)$. By 2.4.3(vii) it is therefore enough to show that for each up-set G of $\operatorname{Sped}(K\{x\})$, the set $\pi^{-1}(G)$ is open in $S_n(M,K)$ for the type space topology. So assume $\pi(p) \in G$. By Ritt-Raudenbush, 1.4.3, there is a finite subset $E \subseteq \pi(p)$ with $\pi(p) = \sqrt[d]{E}$. Then the set $\langle E = 0\rangle \subseteq S_n(M,K)$ is an open neighborhood of p, contained in $\pi^{-1}(G)$: If $q \in \langle E = 0\rangle$, then $\pi(p) = \sqrt[d]{E} \subseteq \pi(q)$. As G is an up-set we get $q \in \pi^{-1}(G)$. $\square$

2.4.10. *Remark.* The inequality in 2.4.9 is strict in general. We will see an example later. In the context of algebraically closed fields (of any characteristic), the inequality in 2.4.9 is indeed an equality, see [DiScTr2019, Theorem 12.4.12]. Hence in this context, the Krull-dimension of $K[x_1, \ldots, x_n]$ can be explained entirely with the type space topology.

2.4.11. **Theorem.** *If M is a DCF and $A \subseteq M$ is infinite, then $S_n(M,A)$ is scattered and of size* $\operatorname{card}(A)$.

Proof. We may assume that A is a differential field. By Ritt-Raudenbush, 1.4.3, differential prime ideals have the ACC. By 2.4.8, the rank of $\operatorname{Sped} K\{x_1, \ldots, x_n\}$ is an ordinal. Hence by 2.4.9, the Cantor-Bendixson rank of $S_n(M,A)$ is an ordinal as well and so this space is scattered. By 2.2.2, the size of $S_n(M,A)$ is $\operatorname{card}(\operatorname{Sped}(A\{x_1, \ldots, x_n\}))$. Since all differential prime ideals are finitely generated as differentially radical ideals we know that $\operatorname{card}(\operatorname{Sped}(A\{x_1, \ldots, x_n\}))$ is $\operatorname{card}(A)$. $\square$

Remark: There is a conceptually different proof of 2.4.11 that does not involve Ritt-Raudenbush, see 2.5.8.

[22]see [DiScTr2019, Theorem 8.1.25] for further topological explanations.

2.5. **Excursion: ω-stable theories.**

2.5.1. **Definition.** Let $\mathscr{L}$ be any language and let κ be a cardinal. An $\mathscr{L}$-theory T without finite models is called **κ-stable** if for all models M and any set $A \subseteq M$ of size κ we have $\operatorname{card}(S_1(M,A)) = \kappa$. T is called **stable** if T is stable for some cardinal $\kappa \geq \operatorname{card}(\mathscr{L})$. The theory T is called **totally transcendental theory** if for all $A \subseteq M \models T$ and each $n \in \mathbb{N}$ the type space $S_n(M,A)$ is scattered.

2.5.2**.** By 2.4.11 we know that DCF is ω-stable, totally transcendental and κ-stable for all infinite cardinals. One can prove 2.4.11 also without using Ritt-Raudenbush and this is interesting for general model theory. We give the arguments here, but only need a small fragment of them for DCF.

2.5.3. **Theorem.** *T is ω-stable $\Rightarrow$ T is totally transcendental $\Rightarrow$ T is κ-stable for all $\kappa \geq \operatorname{card}(\mathscr{L})$.*

Hence, if $\mathscr{L}$ is countable, then all three conditions are equivalent.

Proof. The first implication is done in 2.5.7 below, the second is done in 2.5.4. □

2.5.4. **Lemma.** *Let X be a scattered space and let $\mathscr{B}$ be a subbasis. Then $\operatorname{card}(X) \leq \operatorname{card}(\mathscr{B})$.*

Proof. Let $x \in X$ and $\alpha = \operatorname{rk}(x)$. Then x is isolated in $\delta^\alpha(X)$, hence there is some $U_x \in \mathscr{B}$ with $U_x \cap \delta^\alpha(X) = \{x\}$ and it suffices to show that the map $x \mapsto U_x$ is injective. So let $y \in X$ with $y \neq x$ and w.l.o.g. assume $\operatorname{rk}(y) \geq \alpha$. Then $y \in \delta^\alpha(X)$ and so $y \notin U_x$. Therefore $U_x \neq U_y$. □

2.5.5. **Lemma.** *Let $\mathscr{B}$ be a basis of a compact Hausdorff space X and suppose X is not scattered. Then there are $U_\sigma \in \mathscr{B}$, $\sigma \in 2^{<\omega}$ $(=\bigcup_{n<\omega} 2^n)$ with the following properties.*

(a) $U_\sigma \neq \emptyset$.
(b) If τ extends σ, then $U_\tau \subseteq U_\sigma$.
(c) If $\sigma, \tau \in 2^n$ with $\sigma \neq \tau$, then $\overline{U_\sigma} \cap \overline{U_\tau} = \emptyset$.

Proof. Since X is not scattered, there is a non-empty subset $S \subseteq X$ such that S has no isolated points (in S). We choose $U_\emptyset \in \mathscr{B}$ arbitrarily with $U_\emptyset \cap S \neq \emptyset$ and construct the U_σ with properties (b),(c) by induction on the length of σ, with the additional property that $U_\sigma \cap S \neq \emptyset$. Suppose we have already defined U_σ for all $\sigma \in 2^n$, $n \geq 0$ and fix $\sigma \in 2^n$. By induction we know $U_\sigma \cap S \neq \emptyset$. As U_σ is open, the set $U_\sigma \cap S$ has at least two different points x, y. In the compact Hausdorff space X, there are open neighborhoods $U_{\sigma,0}$ of x and $U_{\sigma,1}$ of y with disjoint closures. Since $\mathscr{B}$ is a basis of X we may shrink these sets if necessary and assume that $U_{\sigma,0}, U_{\sigma,1} \in \mathscr{B}$ as well as $U_{\sigma,0}, U_{\sigma,1} \subseteq U_\sigma$. By running through all $\sigma \in 2^n$, this gives a definition of U_τ for all $\tau \in 2^{n+1}$: Define $U_\tau = U_{\tau|_n, \tau(n)}$.

By construction, the U_σ satisfy (b) and (c) (as well as $U_\sigma \cap S \neq \emptyset$). □

2.5.6. **Proposition.** *Let $f : X \longrightarrow Y$ be a continuous map between topological spaces.*

(i) If Y and all fibers of f are scattered, then also X is scattered.

(ii) If f is a surjective and proper map[23]*, then for all α we have $\delta^\alpha(Y) \subseteq f(\delta^\alpha(X))$. Consequently, X is scattered if and only if Y and all fibres of f are scattered.*

Proof. (i). We claim that every nonempty subset S of X possesses a point that is isolated in S. Since Y is scattered, there is some $y \in f(S)$ and an open subset U of Y with $U \cap f(S) = \{y\}$. Since $f^{-1}(y)$ is scattered, there is some $x \in S \cap f^{-1}(y)$ and an open subset O of X with $O \cap S \cap f^{-1}(y) = \{x\}$. Then $O \cap f^{-1}(U) \cap S = \{x\}$, as required.

(ii). *Claim.* $\delta(Y) \subseteq f(\delta(X))$.

Proof of the claim. Take $y \in Y \setminus f(\delta(X))$. Then no x in the fiber of y is in $\delta(X)$ and so x is isolated. Consequently $f^{-1}(y)$ is open. Since f is closed, also $f(X \setminus f^{-1}(y))$ is closed. But f is surjective and so $f(X \setminus f^{-1}(y))$ is $Y \setminus \{y\}$. Hence y is isolated, i.e. $y \notin \delta(Y)$. $\diamond$

Now we do an induction by α, where $\alpha = 0$ holds by surjectivity of f. For the induction step we assume that $\delta^\alpha(Y) \subseteq f(\delta^\alpha(X))$. Then

$$\begin{aligned}\delta^{\alpha+1}(Y) &= \delta(\delta^\alpha(Y)) \subseteq \delta(f(\delta^\alpha(X))) \\ &\subseteq f(\delta(\delta^\alpha(X))), \text{ by the claim applied to the map} \\ &\qquad\qquad f|_{\delta^\alpha(X)} : \delta^\alpha(X) \longrightarrow f(\delta^\alpha(X)), \\ &\qquad\qquad \text{which is closed and surjective as well,} \\ &= f(\delta^{\alpha+1}(X)).\end{aligned}$$

If α is a limit ordinal or $\alpha = \infty$, then

$$\begin{aligned}\delta^\alpha(Y) &= \bigcap_{\beta<\alpha} \delta^\beta(Y) \subseteq \bigcap_{\beta<\alpha} f(\delta^\beta(X)), \text{ by induction,} \\ &= \{y \in Y \mid f^{-1}(y) \cap \delta^\beta(X) \neq \emptyset \text{ for all } \beta < \alpha\} \\ &= \{y \in Y \mid f^{-1}(y) \cap \delta^\alpha(X) \neq \emptyset\}, \text{ since } f^{-1}(y) \text{ is quasi-compact} \\ &= f(\delta^\alpha(X)).\end{aligned}$$

□

2.5.7. **Proposition.** *Let M be an $\aleph_1$-saturated $\mathscr{L}$-structure such that for all countable $A \subseteq M$ the set $S_1(M, A)$ is countable. Then for all $N \succ M$, every $B \subseteq N$ and each $n \in \mathbb{N}$ the space $S_n(N, B)$ is scattered and of size at most* $\mathrm{card}(\mathscr{L}(B))$.

Proof. By saturation of M we know that for all countable subsets A of any elementary extensions of N, the set $S_1(N, A)$ is countable as well.

Claim. For any $N \succ M$ and each set $B \subseteq N$, the space $S_1(N, B)$ is scattered.

Proof. Suppose otherwise. By 2.5.5 applied to the basis $\mathscr{B} = \{\langle\varphi(x)\rangle \mid \varphi \in \mathrm{Fml}(\mathscr{L}_1(B))\}$[24] of $S_1(N, B)$, there are $\varphi_\sigma \in \mathrm{Fml}(\mathscr{L}_1(B))$, where $\sigma \in 2^{<\omega}$ such that conditions (a),(b) and (c) of 2.5.5 hold. Let $A \subseteq N$ be countable such that $\varphi_\sigma \in \mathrm{Fml}(\mathscr{L}_1(A))$ for all $\sigma \in 2^{<\omega}$. Then for every $\gamma : \omega \longrightarrow \{0, 1\}$ the set $\Phi_\gamma = \{\varphi_{\gamma|_n} \mid n \in \mathbb{N}\}$ is finitely realizable in N (by (a) and (b)) and consequently there is some $p_\gamma \in S_1(N, B)$ containing Φ_γ. However, by (c), for $\gamma_1 \neq \gamma_2 : \omega \longrightarrow \{0, 1\}$, the sets Φ_{γ_1} and Φ_{γ_2} are inconsistent, which implies

[23]A continuous map is called proper if it is closed and all fibres are quasi-compact; for example continuous maps between compact Hausdorff spaces are proper.

[24]We write $\langle\varphi(x)\rangle$ for the set of types containing $\varphi(x)$.

$p_{\gamma_1} \neq p_{\gamma_2}$. Since there are uncountably many maps $\omega \longrightarrow \{0,1\}$ we see that $S_1(N, A)$ is uncountable, which contradicts the assumption. $\diamond$

Hence we know the proposition for $n = 1$. Suppose we know the proposition for n. We show that $S_{n+1}(N, B)$ is scattered by using 2.5.6(i) applied to the continuous map $\rho : S_{n+1}(N, B) \longrightarrow S_n(N, B)$, $\rho(p) = p \cap \mathrm{Fml}(\mathscr{L}_n(B))$. Since $S_n(N, B)$ is scattered by induction, we only need to show that each fibre $\rho^{-1}(p)$ with $p \in S_n(N, B)$, is scattered. Let $\bar{\alpha}$ be a realization of p in $N' \succ N$. Then the map

$$S_1(N', B \cup \{\alpha_1, \ldots, \alpha_n\}) \longrightarrow S_{n+1}(N', B) = S_{n+1}(N, B)$$

(induced by the map $\mathrm{Fml}(\mathscr{L}_{n+1}(B)) \longrightarrow \mathrm{Fml}(\mathscr{L}_1(B \cup \{\alpha_1, \ldots, \alpha_n\}))$ sending $\psi(x, y_1, \ldots, y_n)$ to $\psi(x, \alpha_1, \ldots, \alpha_n)$) is readily seen to be a homeomorphism onto $\rho^{-1}(p)$. By induction we see that all fibres of ρ are scattered. Hence by 2.5.6(i), $S_{n+1}(N, B)$ is scattered as well.

Finally we see that $S_n(N, B)$ has size at most $\mathrm{card}(\mathscr{L}(B))$ by 2.5.4. $\square$

2.5.8**.** Alternative proof of 2.4.11:
By 2.5.7 it suffices to show that for each countable differential field K, the set $S_1(M, K)$ is countable, where $M \supseteq K$ is a DCF. By 2.2.2 it suffices to show that $\mathrm{Sped}(K\{x\})$ is countable, where x is a single variable. But this is implied by 1.2.12, by which we only need to count irreducible polynomials of $K\{x\}$. Notice that this argument does not require Ritt-Raudenbush.

2.6. **Rank functions and stability.**

2.6.1. **Isolated types** If M is an $\mathscr{L}$-structure, then the isolated types in $S_n(M)$ are precisely the types that are realised in M. If M is a DCF, then under the bijection $\pi : S_n(M) \longrightarrow \mathrm{Sped}(M\{x\})$, $x = (x_1, \ldots, x_n)$, from 2.2.2, the isolated types correspond to those ideals that are maximal among proper differential ideals (and all these are maximal ideals of the ring $M\{x\}$).[25]

Proof. If $p \in S_n(M)$ is isolated by $\langle\varphi\rangle$, then p is realised by a realization $a \in M^n$ of φ. Hence $\pi(p)$ contains $x_i - a_i$ for all $i \in \{1, \ldots, n\}$ and therefore it is the kernel of the (differential) evaluation map $M\{x\} \longrightarrow M$ at a.

Conversely, if $\mathfrak{m}$ is maximal among differential (and proper) ideals of $M\{x\}$, then also $\sqrt[d]{\mathfrak{m}} = \sqrt{\mathfrak{m}}$ is a proper differential ideal and so $\mathfrak{m}$ is differentially radical. By the differential Nullstellensatz 2.2.5 there is a point $a \in V_M(\mathfrak{p})$. Then $\mathfrak{m}$ is contained in the kernel of the evaluation map $M\{x\} \longrightarrow M$ at a (which is even a maximal ideal of $M\{x\}$) and by maximality of $\mathfrak{m}$ we see that $\mathfrak{m}$ is that kernel. Hence $\mathfrak{m}$ is the differential prime ideal corresponding to the isolated type $\mathrm{tp}(a/M)$. $\diamond$

In the context of algebraically closed fields, the same correspondence holds true using the classical Nullstellensatz. In this context, the situation indeed extends to all fields, i.e., if K is a field and M is an ACF containing K, then the isolated types in $S_n(M, K)$ are in bijection with the maximal ideals of $\mathrm{Spec}(K[x])$. This is due to Zariski's version of the Nullstellensatz, see for example [DiScTr2019, Cor. 12.3.7].

In the differential context the correspondence does not descend to differential fields: Using Ritt-Raudenbush, 1.4.3, it is still true that every ideal $\mathfrak{m}$ that is maximal among proper differential ideals of $K\{x\}$ comes from an isolated type, a formula isolating that type is $\bigwedge\!\!\!\bigwedge_{P\in E} P(x) = 0$ for any finite $E \subseteq \mathfrak{m}$ with $\mathfrak{m} = \sqrt[d]{E}$. However, not for every isolated type $p \in S_1(M, K)$ is the prime ideal $\pi(p)$ maximal among proper differential ideals. An example is given in 2.6.4 below. This is a first indication that the Cantor-Bendixson rank does not properly describe the geometric situation and we will have to tweak it to obtain the *Morley rank*. In 2.6.4 we will see further evidence.

2.6.2. **Definition.** Let K be a differential field. If $p \in S_1(K)$, then by 1.2.12 and 2.2.2 we know that there is an irreducible polynomial $P \in K\{x\}$ such that $I(P) = \{Q \in K\{x\} \mid Q = 0 \in p\}$. Each such polynomial is called a (differential) **minimal polynomial** of p and of the prime ideal $\mathfrak{p} = I(P)$. It follows from 1.2.10(ii) that P is uniquely determined by this requirement, up to a non-zero scalar. We define the **dimension rank** of p and of $\mathfrak{p}$ to be the order of P:

$$\mathrm{RD}(p) = \mathrm{RD}(\mathfrak{p}) = \mathrm{ord}(P).$$

(If $P = 0$ then $I(p) = I((0)) = \mathrm{ord}(P) = \infty$.)

2.6.3. **Lemma.** *We have* $\mathrm{RD}(p) = \inf\{\mathrm{ord}(P) \mid P \in K\{x\},\ P(x) = 0 \in p\} \in \mathbb{N}_0 \cup \{\infty\}$ *and* $\mathrm{RD}(\mathfrak{p}) = \mathrm{tr.deg}_K \mathrm{Quot}(K\{x\}/\mathfrak{p})$.

Proof. This is clear if $P = 0$. So assume $\mathrm{ord}(P) = n \in \mathbb{N}_0$. Obviously $n = \mathrm{tr.deg}_K \mathrm{Quot}(K\{x\}/\mathfrak{p}) \in \mathbb{N}_0$. We write $a = x + \mathfrak{p} \in K\{x\}/\mathfrak{p}$. Since $P(a) = 0$

[25]Notice that not every maximal ideal of $M\{x\}$ occurs here, e.g. take a maximal ideal containing $\{x, x' - 1\}$. In this context it is worth mentioning that ideals of a differential ring that are maximal among proper differential ideals, are in general not maximal among all proper ideals. For example in the differential ring $(\mathbb{Q}[t], \frac{\mathrm{d}}{\mathrm{d}t})$, the only differential ideal is (0), because every nonzero polynomial has a higher derivation in $\mathbb{Q} \setminus \{0\}$.

it is clear that $a^{(n)}$ is algebraic over $K(a,a',\dots,a^{(n-1)})$. Since P is the minimal polynomial of $\mathfrak{p}$, the elements $a,a',\dots,a^{(n-1)}$ are algebraically independent. Using the formula $P^{(k)}=S(P)\cdot x^{(n+k)}+P_k$, $k>0$ from 1.2.5 and $S(P)(a)\neq 0$ we see that $a^{(n)}$ and all its derivatives are in $K(a,a',\dots,a^{(n)})$. This shows that $\mathrm{Quot}(K\{x\}/\mathfrak{p}$ has transcenedence degree n over K. □

2.6.4. *Example.* Let K be a differential field whose derivation is trivial.

(i) If $p\in S_1(K)$ with $\mathrm{RD}(p)=1$ and $x'\neq 0$ is in p, then p is isolated in $S_1(K)$. In particular $\mathrm{CB}(p)<\mathrm{RD}(p)$.

The type p has Cantor-Bendixson rank 0 in $S_1(K)$, yet p is not an **algebraic type**.[26] Notice that in the case of pure fields, isolation is equivalent to algebraicity.

(ii) The formula $x'=x\wedge x\neq 0$ isolates a type $p\in S_1(K)$ with $\mathrm{RD}(p)=1$ and the corresponding differential prime ideal $I(x'-x)=[x'-x]$ is not maximal among differential ideals of $K\{x\}$.

Proof. (i) Let P be a minimal polynomial of p. Since $\mathrm{RD}(p)=1$, there are $F(x)\in K\{x\}$, $F\neq 0$ and $h(x)\in K[x]$ with $P=F\cdot x'+h$. From $x'=0\notin p$ we get that $h\neq 0$. We claim that p is isolated by $\langle P(x)=0\wedge h(x)\neq 0\rangle$: To see this let q be a 1-type over K containing $P(x)=0$ and assume $q\neq p$. Using 1.2.10(ii) we see that the minimal polynomial of q cannot have order 1 and so $\mathrm{RD}(q)=0$. This implies that any realization a of q from some DCF containing K is algebraic over K. Since the derivation of K is trivial we get $a'=0$ (see 2.1.6) But then $P(a)=0$ implies $h(a)=0$. Hence q is not contained in $\langle P(x)=0\wedge h(x)\neq 0\rangle$ and so p is indeed isolated by this set.

(ii). By (i) applied to $F=1$ and $h=-x$ we see that p is isolated by $x'=x\neq 0$. Obviously $[x'-x]$ is properly contained in the differential ideal $[x]$ of $K\{x\}$. □

[26]A type $p\in S_n(K)$ is called algebraic if it has only finitely many realizations in any DCF containing K.

We now fix a complete theory T in an arbitrary language $\mathscr{L}$. We will start writing $S_n(A)$ for the type space $S_n(M,A)$, where $A \subseteq M \models T$. Notice: If A is a substructure of M and T has quantifier elimination, then $S_n(M,A)$ does not depend on M. In general one should keep in mind that A is taken from some model (which will be made explicit if this is important.)

2.6.5. **Definition.** Fix $n \in \mathbb{N}$. An n-rank is a map R defined on $\bigcup_{A\subseteq M\models T} S_n(M,A)$ with values in ordinals or ∞ satisfying the following properties.

R1. If $A \subseteq B \subseteq M \models T$ and $q \in S_n(M,B)$, then $R(q) \leq R(q|_A)$ (where $q|A = q \cap \mathrm{Fml}(\mathscr{L}_n(A))$).

R2. If $A \subseteq B \subseteq M \models T$ and $p \in S_n(M,A)$, then there is some extension q of p on B (i.e. $q \in S_n(M,B)$ and $q|A = p$) with $R(q) = R(p)$.

R3. If $f : M \longrightarrow N \models T$ is an isomorphism and $p \in S_n(M,M)$, then $R(f(p)) = R(p)$.

R4. If $p \in S_n(M,A)$ with $R(p) < \infty$, then there is a cardinal κ such that for all $N \succ M$, p has at most κ many extensions q on $S_n(N,N)$ of rank $R(p)$.

A rank for T is a map defined on $\bigcup_{n\in\mathbb{N},A\subseteq M\models T} S_n(M,A)$ with values in ordinals or ∞ such that for each n, the restriction of R to n-types is an n-rank.

An rank R is called **continuous** if for every set A, each $n \in \mathbb{N}$ and all α the set $R_n^\alpha(A) = \{p \in S_n(A) \mid R(p) \geq \alpha\}$ is a closed subset of $S_n(A)$. The rank R is called **Cantorian** if for all α, the Cantor-Bendixson derivative $\delta(R^\alpha(A))$ is contained in $R^{\alpha+1}(A)$ (this condition obviously implies that R is continuous).

2.6.6. *Remark.* Ranks are useful to understand *forking* in superstable (and totally transcendental) theories. We comment on this now but omit proofs as this is not needed later on.

(i) There is a smallest rank called the U-rank, or Lascar rank. U is defined as follows. By induction on α we define for types $p \in S_n(A)$ a property $U(p) \geq \alpha$ by
  (a) $U(p) \geq 0$
  (b) $U(p) \geq \alpha + 1 \iff$ for all cardinals κ there is some $B \supseteq A$ and at least κ many extensions q of p on B with $U(q) \geq \alpha$.
  (c) If α limit ordinal, then $U(p) \geq \alpha \iff U(p) \geq \beta$ for all $\beta < \alpha$.

  Finally define $U(p) = \max\{\alpha \mid U(p) \geq \alpha\}$, which is an ordinal or ∞.

(ii) If the U-rank has ordinal values (i.e. $U(p) < \infty$ for all types over all models), then T is called **superstable** (and T is indeed stable, see [Poizat2000, 17.5, p.335])

(iii) If T is superstable and R is any rank, $p \in S_n(A)$ with $R(p) < \infty$ and q is an extension of p on $B \supseteq A$, then $R(q) = R(p)$ if and only if q **does not fork** over p. We refer to the literature for the definition of forking. In the stable context one can use the following: $q \in S_n(B)$ does not fork over $p \in S_n(A)$ if and only if for all $M \prec N \models T$ with $A \subseteq M$, $B \subseteq N$ there is an extension p_1 of p on M and an *heir* q_1 of p_1 on N that extends q. Here q_1 is an heir of p_1 if for all formulas $\varphi(\bar{x},\bar{y})$ with parameters from M and every $\bar{b} \in N^{\bar{y}}$ with $\varphi(\bar{x},\bar{b}) \in q_1$ there is some $\bar{a} \in M^{\bar{y}}$ with $\varphi(\bar{x},\bar{a}) \in p_1$.

2.6.7. **Definition of Morley rank** Let $n \in \mathbb{N}$ and $\bar{x} = (x_1, \dots, x_n)$. By transfinite induction on ordinals α we define for every formula $\varphi = \varphi(\bar{x})$ with parameters from some model M of T the property $\mathrm{MR}(\varphi) \geq \alpha$ as follows:

MR1. $\mathrm{MR}(\varphi) \geq 0$
MR2. $\mathrm{MR}(\varphi) \geq \alpha + 1 \iff$ there are some $N \succ M$ and formulas $\psi_1(\bar{x}), \psi_2(\bar{x}), \dots$ with parameters in N such that $\psi_i[N] \cap \psi_j[N] = \emptyset$ for all $i \neq j$, and $\psi_i[N] \subseteq \varphi[N]$, $\mathrm{MR}(\psi_i) \geq \alpha$ for all i.
MR3. if α is a limit ordinal, then $\mathrm{MR}(\varphi) \geq \alpha \iff \mathrm{MR}(\varphi) \geq \beta$ for all $\beta < \alpha$.

We define $\mathrm{MR}(\varphi) = \max\{\alpha \mid \mathrm{MR}(\varphi) \geq \alpha\}$ (which is an ordinal, or ∞). Finally, for $p \in S_n(A)$ we define

$$\mathrm{MR}(p) = \min\{\mathrm{MR}(\varphi) \mid \varphi \in p\}.$$

$\mathrm{MR}(\varphi)$, $\mathrm{MR}(p)$ is called the **Morley rank** of φ, p respectively.

2.6.8. **Comparison of Morley rank and Cantor-Bendixson rank** Let T be any theory. Let $M \models T$ and let $A \subseteq M$. Let $\varphi(\bar{x}) \in \mathrm{Fml}(\mathscr{L}_n(A))$.

(i) By 2.4.5, we have $\mathrm{CB}(\langle\varphi\rangle_A) \leq \mathrm{MR}(\varphi)$, where $\langle\varphi\rangle_A$ is the subspace of $S_n(A)$ consisting of all types containing φ. Consequently by 2.4.5 we get $\mathrm{CB}(p) \leq \mathrm{MR}(p)$ for all $p \in S_n(A)$.

(ii) If $A = M$ and M is $\aleph_0$-saturated, then $\mathrm{CB}(\langle\varphi\rangle_M) = \mathrm{MR}(\varphi)$. Consequently by 2.4.5 we get $\mathrm{MR}(p) = \mathrm{CB}(p)$ for all $p \in S_n(M)$.
Proof. Assume $\mathrm{MR}(\varphi) \geq \alpha + 1$ and let $\psi_i(\bar{x})$ be formulas with parameters in some $N \succ M$ as in condition MR2 of 2.6.7. Let $B \subseteq N$ be countable such that $\psi_i \in \mathrm{Fml}(\mathscr{L}(B))$. Let A_0 be the finite set of parameters in φ. In the $\aleph_0$-saturated structure we may realize the type $\mathrm{tp}(B/A_0)$ by an infinite set $C \subseteq M$. Then condition MR2 in 2.6.7 is also satisfied for the formulas $\tilde{\psi}_i$ obtained from ψ_i by replacing parameters with the corresponding parameters from M. (Convince yourself that the property $\mathrm{MR}(\psi) \geq \alpha$ is preserved if we change from the base model to an elementary extension or if we apply an automorphism that fixes the parameters of ψ.) Now we may apply induction and 2.4.5. ⋄

(iii) For $p \in S_n(A)$ and $\tilde{M} \succ M$ $\aleph_0$-saturated we have

$$\begin{aligned}\mathrm{MR}(p) &= \max\{\mathrm{CB}(q) \mid q \text{ extension of } p \text{ on } \tilde{M}\}\\ &= \max\{\mathrm{CB}(q) \mid q \text{ extension of } p \text{ on some } N \succ M\}\\ &= \max\{\mathrm{MR}(q) \mid q \text{ extension of } p \text{ on some } N \succ M\}.\end{aligned}$$

Proof. Clearly $\max\{\mathrm{CB}(q) \mid q \text{ extension of } p \text{ on } \tilde{M}\} \leq \max\{\mathrm{CB}(q) \mid q \text{ extension of } p \text{ on some } N \succ M\}$.
By (i) we know $\max\{\mathrm{CB}(q) \mid q \text{ extension of } p \text{ on some } N \succ M\} \leq \max\{\mathrm{MR}(q) \mid q \text{ extension of } p \text{ on some } N \succ M\}$ and the definition of $\mathrm{MR}(p)$ easily implies $\max\{\mathrm{MR}(q) \mid q \text{ extension of } p \text{ on some } N \succ M\} \leq \mathrm{MR}(p)$. It remains to show that $\mathrm{MR}(p) \leq \max\{\mathrm{CB}(q) \mid q \text{ extension of } p \text{ on } \tilde{M}\}$. Write $\alpha = \mathrm{MR}(p)$, which could be ∞. Then by (ii) for all $\varphi \in p$ we know $\mathrm{CB}(\langle\varphi\rangle_{\tilde{M}}) = \mathrm{MR}(\varphi) = \alpha$ and therefore $\langle\varphi\rangle_{\tilde{M}} \cap \delta^\alpha(S_n(\tilde{M})) \overset{2.4.3(v)}{=} \delta^\alpha(\langle\varphi\rangle_{\tilde{M}}) \neq \emptyset$. By quasi-compactness, we get $\bigcap_{\varphi \in p} \langle\varphi\rangle_{\tilde{M}} \cap \delta^\alpha(S_n(\tilde{M})) \neq \emptyset$ and any type q in this intersection is an extension of p with $\mathrm{CB}(q) \geq \alpha$. ⋄

(iv) It follows from (ii) and (iii) that T is totally transcendental if and only if all formulas have Morley rank $< \infty$ if and only if all types of all models have Morley rank $< \infty$.

2.6.9. **Proposition.**

(i) $\mathrm{MR}(\varphi) \geq \alpha + 1 \iff$ *for all* $k \in \mathbb{N}$ *there are* $N \succ M$ *and formulas* $\psi_1(\bar{x}), \dots, \psi_k(\bar{x})$ *with parameters in* N *such that* $\psi_i[N] \cap \psi_j[N] = \emptyset$ *for all* $i \neq j$*,* $i, j \leq k$*, and* $\psi_i[N] \subseteq \varphi[N]$*,* $\mathrm{MR}(\psi_i) \geq \alpha$ *for all* $i \leq k$*.*

(ii) MR *(on types) is the smallest Cantorian rank.*

Proof. (i). The implication $\Rightarrow$ is clear. For $\Leftarrow$ assume $\mathrm{MR}(\varphi) = \alpha$. By 2.6.8(ii) we may assume that $\mathrm{CB}(\langle\varphi\rangle_M) = \mathrm{MR}(\varphi) = \alpha$. Since $\langle\varphi\rangle_M$ is quasi-compact we know from 2.4.3(viii) that the number of types of Cantor-Bendixson rank α in $\langle\varphi\rangle_M$ is finite, say of size k. But then using 2.6.8 we see that the property on the right hand side of (i) fails for $k + 1$.

(ii) We need to check the properties in 2.6.5. Properties R1, R2 and R3 are easily deduced from 2.6.8. For R4, assume $\mathrm{MR}(p) = \alpha < \infty$ and $A \subseteq B$. We claim that there are only finitely many extensions q of p on B with $\mathrm{MR}(q) = \alpha$. By R1 and R3 we may assume that $B = M$ is an $\aleph_0$-saturated model. But then CB = MR and there can only be finitely many extensions of q in $\delta^\alpha(S_n(M))$ (observe that the extensions of p on M are a closed set). This shows R4 for the Morley rank.

MR is Cantorian because if $p \in S_n(A)$ is a non-isolated point of $\{q \in S_n(A) \mid \mathrm{MR}(q) \geq \alpha\}$ and $M \supseteq A$ is an $\aleph_0$-saturated model, then a compactness argument and R2 show that there is an extension p' of p on M that is non-isolated in $\{q \in S_n(M) \mid \mathrm{MR}(q) \geq \alpha\}$. Thus $\mathrm{MR}(p) \geq \mathrm{MR}(p') = \mathrm{CB}(p') \geq \alpha + 1$.

Finally, if R is any Cantorian rank then using 2.6.8(iii) one shows without difficulty by induction on α that $\mathrm{MR}(p) \geq \alpha$ implies $R(p) \geq \alpha$. □

2.6.10. *Example.*

(i) Let K be a differential field and let $Q \in K\{X\}$ be irreducible with $\operatorname{ord} Q = 1$. If $XX'' - X' \in I(Q)$, then Q is associated to X'.

Consequently, the type $p \in S_1(K)$ with minimal polynomial $x''x - x'$ is the unique non-algebraic type in $\langle x''x - x', x' \neq 0\rangle$. Since K is arbitrary this implies that $\mathrm{MR}(p) = 1 < 2 = \mathrm{RD}(p)$.

Proof. Suppose X' does not divide Q. Let $f_0, \dots, f_n \in K[X]$ with $f_n \neq 0$, such that $Q = f_n{\cdot}(X')^n + \ldots + f_1{\cdot}X' + f_0$. As X' does not divide Q we have $f_0 \neq 0$ and we may assume that the leading coefficient of f_0 is 1.

Then $S(Q) = \sum_{i=1}^{n} i{\cdot}f_i{\cdot}(X')^{i-1}$ and with $Q^* := \sum_{i=0}^{n} f_i'{\cdot}(X')^i$ we have

$$Q' = S(Q){\cdot}X'' + Q^*$$

Since $XX'' - X' \in I(Q)$ we have $P := X{\cdot}Q' - S(Q){\cdot}(XX'' - X') \in I(Q)$. Furthermore

$$P = X{\cdot}(S(Q){\cdot}X'' + Q^*) - S(Q){\cdot}(XX'' - X') = X{\cdot}Q^* + X'{\cdot}S(Q)$$

It follows $\operatorname{ord} P \leq 1$ and the total degree of P is less or equal to 1+ the total degree of Q, so by 1.2.10(ii) there is some $H \in K\{X\}$ with $\operatorname{ord} H \leq 1$ and of total degree ≤ 1 such that $P = H{\cdot}Q$. We have $f_i' = S(f_i){\cdot}X' + f_i^*$, where f_i^*

denotes the polynomial which we get by differentiating the coefficients of f_i. From $Q^* = \sum_{i=0}^{n} f_i' \cdot (X')^i$ we get

$$Q^* = S(f_n) \cdot (X')^{n+1} + \sum_{i=1}^{n} (S(f_{i-1}) + f_i^*)(X')^i + f_0^*$$

From $P = X \cdot Q^* + X' \cdot S(Q)$ we get

$$P = X \cdot S(f_n) \cdot (X')^{n+1} + \sum_{i=1}^{n} [X \cdot S(f_{i-1}) + X \cdot f_i^* + i \cdot f_i] \cdot (X')^i + X \cdot f_0^*$$

Let $H = aX' + bX + c$ with $a, b, c \in K$. From $P = H \cdot Q$ we get $X \cdot f_0^* = (bX + c) \cdot f_0$. As the leading coefficient of f_0 is 1, the degree of f_0^* is less than the degree of f_0. Thus $b = 0$.

Suppose $c \neq 0$. Then X divides f_0. Let $f_0 = X^k \cdot g$ with $g \in K[X]$ and $X \nmid g$. Then $f_0^* = X^k \cdot g^*$ and $X^{k+1} \cdot g^* = X \cdot f_0^* = c \cdot f_0$ a contradiction. Hence c must be zero.

So $H = a \cdot X'$ and $f_0^* = 0$. By dividing the equation $P = H \cdot Q$ by X' we get

$$a \cdot Q = X \cdot S(f_n) \cdot (X')^n + \sum_{i=0}^{n-1} [X \cdot S(f_i) + X \cdot f_{i+1}^* + (i+1) \cdot f_{i+1}] \cdot (X')^i$$

hence

(a) $a \cdot f_n = X \cdot S(f_n)$ and

(b) $a \cdot f_i = X \cdot S(f_i) + X \cdot f_{i+1}^* + (i+1) \cdot f_{i+1}$ for all $i < n$.

By (a) X divides f_n. By (b) and induction we see that X divides all f_i. Hence X divides Q, a contradiction.

□

(ii) $\sqrt[d]{XX'' - X'}$ is prime.

Proof. Let $M \supseteq K$ be differentially closed. Since $(\sqrt[d]{XX'' - X'}) \cdot M$ lies over $\sqrt[d]{XX'' - X'}$ and is radical we may assume that $K = M$ is differentially closed. Let $Q \in K\{X\}$ be irreducible with $P \in I(Q)$. We have to show that $I(P) \subseteq I(Q)$. We have $\operatorname{ord} Q \leq \operatorname{ord} P = 2$, so if $\operatorname{ord} Q = 2$, then Q is associated to P and $I(P) = I(Q)$. So let $\operatorname{ord} Q \leq 1$ and let $F \in I(P)$. Take $m \in \mathbb{N}$ such that $X^m \cdot F \in [P]$. Then $X^m \cdot F \in I(Q)$. Since $X \notin I(Q)$ ($\operatorname{ord} X < 1$!) we have $F \in I(Q)$. Finally assume that $\operatorname{ord} Q = 0$ and $F \notin I(Q)$. Since K is algebraically closed, Q is of the form $x - a$ for some $a \in K$. Then $I(Q) = [Q] = (X - a, X' - a', X'' - a'', \ldots)$. Since $X \in I(Q)$ we must have $a = 0$. Since $F \notin I(Q)$ we have $F(0) \neq 0$. We write $F = F(X, X', \ldots, X^{(k)})$. Then $F(X, 0, \ldots, 0) \neq 0$. Take $a \in M$ with $a \cdot F(a, 0, \ldots, 0) \neq 0$ and $a' = 0$. Then $P(a) = 0$ and $F(a) \neq 0$ in contradiction to $X^m \cdot F \in [P]$. □

2.6.11. **Lemma.** *Let K be a perfect field with algebraic closure $\bar{K}$ and let $P \in K[x_1, \ldots, x_n]$ be irreducible. Write $P = \prod_{i=1}^{m} P_i^{k_i}$ with irreducible P_i such that P_i is not associated to P_j for all $i \neq j$. Then $k_i = 1$ for all i and for all i, j there is some $c \in \bar{K}$ and some σ in the Galois group $\operatorname{Gal}(\bar{K}/K)$ with $\sigma(P_i) = c \cdot P_j$.*

Proof. We may assume that one of the coefficients of P_1 is 1. Let L/K be a finite Galois extension containing all coefficients of P_1. Then the set $\{\sigma(P_1) \mid \sigma \in \operatorname{Gal}(\bar{K}/K)\}$ is finite and equal to $\{\sigma(P_1) \mid \sigma \in \operatorname{Gal}(L/K)\}$, say of size m. Enumerate the polynomials in this set as $P_1, \ldots, P_m$. Then $Q = P_1 \cdot \ldots \cdot P_m \in K[x_1, \ldots, x_n]$

divides P (since one of the coefficients of P_1 is 1, P_i is not associated to P_j for $i \neq j$) and therefore Q is associated to P. This shows the lemma. □

2.6.12. **Proposition.** *The dimension rank is a Cantorian 1-rank for DCF. If $K \subseteq L$ are differential fields and $P \in K\{x\}$ is irreducible, then the non-forking extensions of the type corresponding to $I(P, K)$ on $L \subseteq K$ are precisely the types of L corresponding to $I(Q, L)$, where Q runs through the irreducible factors of P in $L\{x\}$.*

Proof. R1 and R3 are clearly satisfied. R2 and R4 are easily deduced from the following

Claim. If $K \subseteq L$ are differential fields, $P \in K\{x\} \setminus \{0\}$ irreducible and $P_1 \in L\{x\}$ irreducible factor of P, then $I(P_1, L) \cap K\{x\} = I(P, K)$.

Proof. $\supseteq$: Take $F \in I(P, K)$, hence $S(P)^m \cdot F \in [P] \subseteq [P_1]$. Write $P = P_1 \cdot A, A \in L\{x\}$. Then $S(P) = A \cdot S(P_1) + P_1 \cdot \frac{\partial}{\partial x^{(n)}} A$, where $n = \mathrm{ord}(P)$. Hence

$$S(P_1)^m \cdot A^m \cdot F = (S(P) - P_1 \cdot \frac{\partial}{\partial x^{(n)}} A)^m \cdot F \in [P_1],$$

and so $A^m \cdot F \in I(P_1, L)$. But $P_1 \nmid A$ by 2.6.11 and so $F \in I(P_1, L)$.

$\subseteq$. Let $F \in I(P_1, l) \cap K\{x\}$ and let G be the reduced remainder of F for P. Suppose $G \neq 0$. For some $k, m \in \mathbb{N}$ we know that $L(P)^k \cdot S(P)^m \cdot F \equiv G \bmod [P]$ and so $L(P)^k \cdot S(P)^m \cdot F \equiv G \bmod [P_1]$ in $L\{x\}$.

Since $G \neq 0$ we have $\mathrm{ord}(G) \leq \mathrm{ord}(P) = \mathrm{ord}(P_1)$ (the latter equality from 2.6.11) and therefore $F \in I(P_1, l)$ entails $\mathrm{ord}(G) = \mathrm{ord}(P_1)$ and $P_1 \mid G$. By 2.6.11 then $P \mid G$, which is impossible by the choice of G. ◇

It remains to show that RD is Cantorian. This is deduced from the following two properties:

(a) $\mathrm{RD}(p) \leq \omega$ for all 1-types and the unique 1-type p with $\mathrm{RD}(p) = \omega$ is the one corresponding to the prime ideal (0).

(b) If $P \neq 0$ is a minimal polynomial of p with $n = \mathrm{ord}(P)$ then the formula $P(x) = 0$ isolates p in the set $\{q \in S_1(K) \mid \mathrm{RD}(q) \geq n\}$.

□

2.6.13**. Summary on ranks in differentially closed fields**

(1) $U \leq \mathrm{MR}$ and $\mathrm{CB} \leq \mathrm{MR}, \mathrm{rk}$ on all types, where $\mathrm{rk}(p)$ is the rank of the prime ideal $\pi(p)$ in the differential spectrum, see 2.4.9.

(2) On 1-types we have in addition $\mathrm{MR} \leq \mathrm{RD}$ by 2.6.12, but also $\mathrm{rk} \leq \mathrm{RD}$ as one checks without difficulty.

(3) On $\aleph_0$-saturated models we have $\mathrm{CB} = \mathrm{MR}$ and therefore $U \leq \mathrm{CB} = \mathrm{MR} \leq \mathrm{rk}$ for all types and

$$U \leq \mathrm{CB} = \mathrm{MR} \leq \mathrm{rk} \leq \mathrm{RD}$$

on 1-types.

2.7. **Prime models in totally transcendental theories and the differential closure.**

Our goal in this section is to show that every differential field K has a *differential closure* M (i.e., M is a DCF containing K such that M embeds over K into any other DCF containing K, compare with algebraic closures of fields) and to establish fundamental properties of M. The existence of such models follows solely from *atomicity* of DCF, which itself is implied by ω-stability. This is reviewed first, more can be found in [Poizat2000, Chapter 10].

Throughout we work with a complete $\mathscr{L}$-theory in some language $\mathscr{L}$.

2.7.1. **Definition.** A model of T is called **prime model** if for all $N \models T$ there is an elementary embedding $M \longrightarrow N$.

If $A \subseteq M \models T$, then M is called a **prime model over** A if (M, A) is a prime model of $\mathrm{Th}(M, A)$.[27]

The key feature of many theories with prime models (over all sets) is the presence of many isolated types.

2.7.2. **Definition.** If $A \subseteq B$ are sets from some T-model, then B is called **atomic over** A if for every $n \in \mathbb{N}$ and all $\bar{b} \in B^n$ the type $\mathrm{tp}(\bar{b}/A)$ is isolated in the space $S_n(A)$ (equivalently: there is a formula $\varphi(x_1, \ldots, x_n)$ with parameters from A such that $\mathrm{tp}(\bar{b}/A)$ is the unique type containing φ; we then say that φ **isolates the type**).

2.7.3**. Technical tool.** In the sequel we will many times tacitly apply the following characterization of tuples having the same types. Let A be a set and let $\bar{b}, \bar{c}$ be (not necessarily finite) tuples of the same length from some $M \models T$. Then $\mathrm{tp}(\bar{b}/A) = \mathrm{tp}(\bar{c}/A)$ if and only if there is some elementary extension N of M and an A-automorphism σ of N with $\sigma(\bar{b}) = \bar{c}$ (i.e. $\sigma(b_i) = c_i$ for all i).

2.7.4. **Lemma.** *Let $A \subseteq M \models T$.*

(i) Let $\bar{b}, \bar{c}$ be finite tuples from M. Then $\mathrm{tp}(\bar{b}\hat{\ }\bar{c}/A)$ is isolated if and only if $\mathrm{tp}(\bar{b}/A)$ and $\mathrm{tp}(\bar{c}/A \cup \bar{b})$ are isolated.

(ii) Atomicity is transitive, hence if $A \subseteq B \subseteq C \subseteq M$ and B is atomic over A, C is atomic over B, then C is atomic over A.[28]

Proof. We state the isolating formulas in all claims. For detailed verification use 2.7.3.

(i)$\Rightarrow$: If $\varphi(\bar{x}, \bar{y})$ isolates $\mathrm{tp}(\bar{b}\hat{\ }\bar{c}/A)$, then $\exists \bar{y} \varphi(\bar{x}, \bar{y})$ isolates $\mathrm{tp}(\bar{b}/A)$ and $\varphi(\bar{b}, \bar{y})$ isolates $\mathrm{tp}(\bar{c}/A \cup \bar{b})$.

(i)$\Leftarrow$: If $\psi(\bar{x})$ isolates $\mathrm{tp}(\bar{b}/A)$ and $\gamma(\bar{b}, \bar{y})$ isolates $\mathrm{tp}(\bar{c}/A \cup \bar{b})$, then $\psi(\bar{x}) \wedge \gamma(\bar{x}, \bar{y})$ isolates $\mathrm{tp}(\bar{b}\hat{\ }\bar{c}/A)$.

(ii). Let $\bar{c} \subseteq C$ be a finite tuple. Take an $\mathscr{L}(A)$-formula $\varphi(\bar{x}, \bar{y})$ and $\bar{b} \subseteq B$ such that $\mathrm{tp}(\bar{c}/B)$ is isolated by $\varphi(\bar{b}, \bar{y})$. Then take a formula $\psi(\bar{x})$ isolating $\mathrm{tp}(\bar{b}/A)$. Then $\mathrm{tp}(\bar{c}/A)$ is isolated by $\exists \bar{x}\, (\varphi(\bar{x}, \bar{y}) \wedge \psi(\bar{x}))$. □

[27]One should think of T having quantifier elimination (e.g. DCF) and A being some substructure of a model (e.g. a differential field); then M is a prime model over A just if A is a substructure of M and M embeds over A into any other model of T that has A as a substructure.

[28]Notice that the converse fails in general, i.e., if C is atomic over A then C is in general not atomic over B. For an example, see 2.7.13 below.

2.7.5. **Definition.** Let $A \subseteq B$. A **construction of** B **over** A is a surjective map $b : \lambda \longrightarrow B$ for some ordinal λ such that for all $\alpha < \lambda$ the type $\mathrm{tp}(b_\alpha/A \cup \{b_\beta \mid \beta < \alpha\})$ is isolated. We say that B is **constructed over** A if there is such a construction.

2.7.6. **Proposition.** *If $A \subseteq M \models T$ and M is constructed over A, then M is a prime model over A and M is atomic over A.*

Proof. We may assume that $A = \emptyset$ (otherwise replace T by $\mathrm{Th}(M, A)$). Let $b : \lambda \longrightarrow M$ be a construction of M over $\emptyset$ and let $B_\alpha = \{b_\beta \mid \beta < \alpha\}$.

Since $B_{\alpha+1}$ is atomic over B_α for all α we may use transitivity, cf. 2.7.4(ii), to do an induction on α, which shows that B_α is atomic over $\emptyset$. It follows that M is atomic over $\emptyset$.

Now let $N \models T$. Since T is complete we may assume that M and N are common elementary substructures of some $\Omega \models T$. All types are then to be understood with respect to Ω. We construct an elementary embedding $M \longrightarrow N$ as follows: Since $\mathrm{tp}(b_0/\emptyset)$ is isolated there is some $c_0 \in N$ such that $\mathrm{tp}(c_0/\emptyset) = \mathrm{tp}(b_0/\emptyset)$. Suppose by transfinite induction we have already found $c_\beta \in N$ with $\mathrm{tp}((b_\beta)_{\beta<\alpha}/\emptyset) = \mathrm{tp}((c_\beta)_{\beta<\alpha}/\emptyset)$.

Since $\mathrm{tp}(b_\alpha/B_\alpha)$ is isolated, we may use 2.7.3 to find $c_\alpha \in N$ such that $\mathrm{tp}((b_\beta)_{\beta\le\alpha}/\emptyset) = \mathrm{tp}((c_\beta)_{\beta\le\alpha}/\emptyset)$.

The map sending b_α to c_α is now an elementary embedding $M \longrightarrow N$. □

2.7.7. **Base example.** If $M \models T$ is countable and M is atomic (i.e. atomic over $\emptyset$), then M is a prime model of T.[29] The reason is that by 2.7.4(i), any enumeration $(a_n)_{n<\omega}$ of M is a constructions of M over $\emptyset$.

2.7.8. **Theorem.** *The following are equivalent.*

(i) *T is* **atomic**, *i.e. for all $A \subseteq M \models T$, the isolated points of $S_1(A)$ are dense in $S_1(A)$.*
(ii) *For every $A \subseteq M \models T$ there is a model $N \prec M$, $A \subseteq N$ such that N is constructed over A.*
(iii) *For every $A \subseteq M \models T$ there is an atomic prime model $N \prec M$ of A.*

If $\mathscr{L}$ is countable, then these conditions are equivalent to each of the the following conditions.

(iv) *For every countable $A \subseteq M \models T$ there is a prime model $N \prec M$ of A.*
(v) *For every countable $\Omega \models T$ and all $M, N \prec \Omega$ and any set $A \subseteq M \cap N$ there is $M' \prec M$ containing A and an elementary embedding $M' \longrightarrow N$ over A.*

Remark: *Notice that no stability theoretic assumption is made here. For example real closed fields have an atomic theory, but they are not stable.*

Proof. (ii)⇒(iii) holds by 2.7.6.

(iii)⇒(i). The isolated types of $S_n(A)$ are dense in $S_n(A)$, because every satisfiable formula is satisfied in a model M that is atomic over A. (We don't need that M is prime over A for this argument.)

(i)⇒(ii). Again we may assume that $A = \emptyset$. For an ordinal α we define elements $b_\alpha \in M$ as follows: Let $b_0 \in M$ be a realization of an isolated type in $S_1(\emptyset)$. If b_β

[29] If $\mathscr{L}$ is countable, then in fact by the *omitting types theorem*, every prime model of T is atomic.

for $\beta < \alpha$ have already been defined, then we write $B = \{b_\beta \mid \beta < \alpha\}$ and take

$$b_\alpha = \begin{cases} b_0, & \text{if every isolated type from } S_1(B) \text{ is realised in } B, \\ \text{any} & \text{element } b \in M \text{ realising an isolated type from} \\ & \quad S_1(B), \text{that is not realized in B, otherwise.} \end{cases}$$

Since M is a set there is a smallest $\alpha > 0$ with $b_\alpha = b_0$ and we show that $N = \{b_\beta \mid \beta < \alpha\}$ is an elementary substructure of M, by using the Tarski-Vaught test: Hence we need to show that for every formula $\varphi(x)$ in one free variable with parameters from B that is realized in M, there is some $b \in B$ with $M \models \varphi(b)$. Since φ is realized in M, we know from (i) that there is an isolated type $p \in S_1(B)$ with $\varphi(x) \in p$. But now by choice of α we see that p is already realized by some $b \in B$. Then $M \models \varphi(b)$ as required.

Hence B indeed is an elementary substructure of M and by definition, $(b_\beta)_{\beta<\alpha}$ is a construction of B over $\emptyset$. Thus $N = B$ has the required property for (ii).

This shows that (i),(ii) and (iii) are equivalent. Obviously (iii) implies (iv). Now assume $\mathscr{L}$ is countable and that there are prime models over every countable subset. We show (i) by following the proof from [MarTof2003, Theorem 6.4.16, p. 206].

Assume for a contradiction that there are $B \subseteq M \models T$ such that the isolated points of $S_1(B)$ are not dense in $S_1(B)$. Take $\varphi(x) \in \mathscr{L}_1(B)$ satisfiable in M and suppose there is no $\psi(x) \in \mathscr{L}_1(B)$ implying $\varphi(x)$ such that $\psi(x)$ isolates a type in $S_1(B)$. Let $A_0 \subseteq B$ be finite and containing the parameters of $\varphi(x)$. Then for each satisfiable formula $\psi(x) \in \mathscr{L}_1(A_0)$ with $\psi(x) \to \varphi(x)$ in M, there is formula $\vartheta_\psi(x) \in \mathscr{L}_1(B)$ such that $\vartheta_\psi(x) \to \varphi(x)$, $\neg\vartheta_\psi(x) \to \varphi(x)$ and both $\vartheta_\psi(x)$ and $\neg\vartheta_\psi(x)$ are satisfiable. Since $\mathscr{L}$ is countable, there is some countable $A_1 \subseteq B$ such that all the formulas $\vartheta_\psi(x)$ have parameters in A_1. We iterate this construction and obtain a countable chain $A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots$ such that for all $n < \omega$ and every satisfiable formula $\psi(x) \in \mathscr{L}_1(A_n)$ with $\psi(x) \to \varphi(x)$ in M, there is a formula $\vartheta_\psi(x) \in \mathscr{L}_1(A_{n+1})$ such that $\vartheta_\psi(x) \to \varphi(x)$, $\neg\vartheta_\psi(x) \to \varphi(x)$ and both $\vartheta_\psi(x)$ and $\neg\vartheta_\psi(x)$ are satisfiable.

Obviously then, the set $A = \bigcup_n A_n$ is countable and the formula $\varphi(x)$ does not contain an isolated 1-type of $S_1(A)$. Since $\mathscr{L}$ is countable, there is no prime model over A by the omitting types theorem. But this contradicts (iv).

Hence (i)-(iv) are equivalent and obviously (iv) implies (v). Finally, assume (v) holds. We show (iv) by showing that for every countable set A from some model Ω of T, the isolated 1-types of A are dense in $S_1(A)$ (this suffices, because then we may construct a countable atomic model of T inside Ω, cf. 2.7.7). Suppose this is not the case. Then there is a formula $\varphi(x)$ with parameters in A, such that $\langle\varphi(x)\rangle$ is nonempty but does not contain an isolated 1-type of $S_1(A)$. By passing to an elementary extension if necessary, we may assume that Ω is $\aleph_1$-saturated. Let $M \prec \Omega$ be countable with $A \subseteq M$. Then $X = \{p \in \langle\varphi(x)\rangle \mid p \text{ is realized in } M\}$ is countable without isolated points. By the omitting types theorem (and saturation of Ω), there is some countable $N \prec \Omega$ that omits all types from X. Obviously M and N are contained in a countable elementary substructure of Ω. Hence by (v) there are $M' \prec M$ with $A \subseteq M'$ and an elementary embedding $f : M' \longrightarrow N$ over A. Now take $b \in M'$ with $M' \models \varphi(b)$. Then $p = \mathrm{tp}(b/A) \in X$ and so p is realised in N by $f(b)$. But this contradicts the choice of N. □

We now turn to the subtle questions of uniqueness of prime models. We explain the matter but only hint at proofs.

2.7.9. **Theorem.** *[Ressayre] Any two constructed models of any complete theory in any language are isomorphic.*

Consequently, if T is an atomic and complete $\mathscr{L}$-theory, then by 2.7.8 any two constructed models over a set A of parameters are isomorphic over A.

Proof. The proof may be found in [Poizat2000, section 10.4] and uses a clever combinatorial generalization of the back and forth method, which delivers the claim quite easily when M, N are two countable models that have constructions $a : \omega \longrightarrow M$ and $b : \omega \longrightarrow N$. □

2.7.10. **Corollary.** *If M is constructed over A and $\bar{b}, \bar{c} \in M^n$ with $\mathrm{tp}(\bar{b}/A) = \mathrm{tp}(\bar{c}/A)$, then there is an A-automorphism σ of M with $\sigma(\bar{b}) = \bar{c}$.*

It follows that the definable closure of A in M is the set of all elements of M that are fixed under all A-automorphisms.

Proof. Let $b : \lambda \longrightarrow M$ be a construction of M over A. Using 2.7.4(i) we see that b is also a construction of M over $A \cup \bar{b}$ and of M over $A \cup \bar{c}$. By 2.7.9 there is an automorphism as required (first apply 2.7.3 to find an automorphism of some elementary extension of M that maps $\bar{b}$ to $\bar{c}$).

For the second assertion take $b \in M \setminus \mathrm{dcl}(A)$. Since $\mathrm{tp}(b/A)$ is isolated and $b \notin \mathrm{dcl}(A)$ it must have a realization different from b in M. By the first assertion, b is not fixed under all A-automorphisms of M. □

In [Poizat2000, end of section 18.1] there is an example of a constructed model and a prime model of a theory that are not isomorphic. However in the stable context we have

2.7.11. **Theorem.** *Let T be a stable and atomic theory. If T is superstable, or if the language is countable, then every prime model over any set A is constructed and consequently by 2.7.9, there is a unique prime model over A up to A-isomorphism.*

This applies to DCF by 2.4.11.

Proof. Let M be a prime model over A. Since T is atomic we know that $M \prec N$ for some $N \models T$ that is constructed over A.

The stability theoretic assumption now implies that constructibility on N over A descends to any set $B \subseteq N$ containing A. The proof of that is based on the combinatorial setup of Ressayre's theorem but also uses results about forking; it may be found in [Poizat2000, Proposition 18.1].

Consequently, M is also constructed over A and 2.7.9 applies. □

2.7.12. **Scholium.** *Let T be a totally transcendental theory in some language. For example DCF. Then for every set A of parameters there is a prime model M over A. M is constructed over A and unique up to an A-isomorphism.*

The prime model M is homogeneous over A in the sense that for all finite tuples $\bar{b}, \bar{c} \in M^n$ of the same type over A there is an A-automorphism of M mapping $\bar{b}$ to $\bar{c}$.

If K is a differential field, then the prime model over K is called the **differential closure** *of K and is sometimes denotes by $\hat{K}$.*

2.7.13**. Minimality of prime models** Before looking closer at the differential closure we will talk about minimality of prime models. We fix a totally transcendental theory T. For ACF, the prime model of a field K is its algebraic closure $\overline{K}$. This field is not only prime, but also *minimal*:

A prime model M of A is **minimal** if every elementary embedding $M \longrightarrow M$ over A is an automorphism of M. By 2.7.11 this is equivalent to saying that there are no proper elementary restrictions of M containing A. The following are equivalent for a prime model M over A:

(i) M is minimal over A.
(ii) M is, up to A-isomorphism, the unique model of T that is atomic over A.
(iii) There is no sequence $(a_n)_{n<\omega}$ in M with $a_n \neq a_m$ for $n \neq m$ such that for all $n < \omega$ the type $\mathrm{tp}(a_{n+1}/A \cup \{a_k \mid k \leq n\})$ extends the type $\mathrm{tp}(a_n/A \cup \{a_k \mid k < n\})$.

Furthermore, if the prime model of A is not minimal, then for each cardinal $\kappa \geq \mathrm{card}\,\mathscr{L}(A)$ there is an atomic model N over A of size κ.

For proofs see [Poizat2000, Theorem 18.7].

We now have a closer look at the differential closure.

2.7.14. **Theorem.** *The differential closure of any field K equipped with the trivial derivation is **not** minimal.*

Proof. This was shown independently by Kolchin, Rosenlicht and Shelah for $K = \mathbb{Q}$. Rosenlicht in [Rosenl1974] verifies that the set of solutions of $x' = x^3 - x^2$ in $\hat{K}$ is infinite and has a cofinite subset X that is algebraically independent over K. From X one can then construct easily a sequence with infinitely many terms as in 2.7.13(iii), witnessing non-minimality of $\hat{K}$. Rosenlicht proves the same for the equation $x'(x+1) - x = 0$. □

2.7.15. **Lemma.** *Let K be a differential field and let $p \in S_1(K)$ with minimal polynomial $P(x) \in K\{x\}$. Then the set*

$$\{\langle P(x) = 0 \wedge Q(x) \neq 0\rangle \mid Q \in K\{x\},\ \mathrm{ord}(Q) < \mathrm{ord}(P)\}$$

is a neighborhood basis of p in $S_1(K)$. Hence if p is isolated, then an isolating formula may be found in the form $P(x) = 0 \wedge Q(x) \neq 0$ with $\mathrm{ord}(Q) < \mathrm{ord}(P)$.

Proof. Let $\mathscr{U}$ be the set on the right hand side. Using 1.2.10(ii) we know that $\bigcap \mathscr{U} = \{p\}$. Now if $O \subseteq S_1(K)$ is any open neighborhood of p then $S_1(K) \setminus O \subseteq \bigcup_{U \in \mathscr{U}} S_1(K) \setminus U$ and by compactness there are finitely many $Q_1, \ldots, Q_n \in K\{x\}$ of order $< \mathrm{ord}(P)$ with $p \in \langle P(x) = 0 \wedge Q_1(x) \neq 0\rangle \cap \ldots \cap \langle P(x) = 0 \wedge Q_n(x) \neq 0\rangle \subseteq O$. If we set $Q = \prod_{i=1}^n Q_i$ we still have $\mathrm{ord}(Q) < \mathrm{ord}(P)$ and $p \in \langle P(x) = 0 \wedge Q(x) \neq 0\rangle \subseteq O$. □

2.7.16. **Corollary.** *Let K be a differential field.*

(i) The 1-type over K with minimal polynomial x' is not isolated.
(ii) If $L \supseteq K$ is a differential field extension of K and L is atomic over K, then the field of constants C_L of L is the algebraic closure (in the field theoretic sense) of C_K in L.

Proof. (i) For any polynomial $Q(x) \in K[x]$ there are infinitely many solutions of $x' = 0 \wedge Q(x) \neq 0$ in K: for example any rational number that is not a zero of Q.

(ii). By 2.1.5(ii)(b), the algebraic closure of C_K in L is contained in C_L. Conversely, if $a \in C_L$, then $a' = 0$ and as $\mathrm{tp}(a/K)$ is isolated we see from (i) that a is algebraic over K. By 2.1.5(ii)(b), a is even algebraic over C_K. □

2.7.17**. Algebraicity, differential algebraicity and isolation.** In the context of algebraically closed fields an element a is algebraic over a field K if and only if $\mathrm{tp}(a/K) \in S_1(K) = S_1(M, K)$ is isolated, where $K \subseteq M \models$ACF. The notion of an algebraic element also exists in general model theory: b is **algebraic over a set** A of parameters of a structure M just if there is a formula $\varphi(x)$ with parameters in A such that $M \models \varphi(b)$ and such that φ has only finitely many solutions in M. For ACF this coincides with field theoretic algebraicity. Similarly in DCF this notion agrees with field theoretic algebraicity over the differential field generated by $K \cup \{b\}$ (consider the minimal polynomial in the differential sense of $\mathrm{tp}(b/K)$).

If $K \subseteq L$ are differential fields and $a \in L$, then a is called **differentially algebraic** over K if it is the differential zero of a non-zero differential polynomial with coefficients in K. In stark contrast to the case of algebraically closed fields, differential algebraic elements over a differential field K are not all contained in $\hat{K}$. An example is given in 2.7.16(i). (Recall that $\hat{K}$ is atomic over K.)

On the other hand, every differential field L containing K, which is atomic and finitely generated as a differential field over K can be embedded over K into $\hat{K}$: realize the type of a set of finitely many generators of L in $\hat{K}$.

Hence isolation is the notion that corresponds more closely to the intuition that elements in $\hat{K}$ should be in some sense determined by K.

However, since $\hat{\mathbb{Q}}$ is not minimal, 2.7.13 tells us that not every differential field that is atomic over $\mathbb{Q}$ can be embedded into $\hat{K}$ and that there is no bound on the size of elements from a DCF, which have an isolated type over $\mathbb{Q}$.

3. Differential Galois Theory

3.1. **Non-commutative differential rings and differential modules.**

Reference: [vdPSin2003, Chapter 2], [Bourba1989].

Throughout, R is an associative unital ring, which is not necessarily commutative.

3.1.1. **Reminder.** *Let R be a ring.*

(i) Let M be a left module over R and let $\mathrm{End}_R(M)$ be the set of all R-endomorphism of M. Then $\mathrm{End}_R(M)$ is a ring via component wise addition and composition as multiplication. Then M is a left module over $\mathrm{End}_R(M)$, where scalar multiplication is defined as $\varphi{\cdot}m := \varphi(m)$.

(ii) Let M be a left R-module and let $\mathrm{End}_{\mathbb{Z}}(M)$ be the ring of additive maps $M \longrightarrow M$, where multiplication is given by composition. For $r \in R$, let ${}_r\Lambda : M \longrightarrow M$ be multiplication by r on the left, thus ${}_r\Lambda(m) = rm$. Then the map $\iota : R \longrightarrow \mathrm{End}_{\mathbb{Z}}(M)$ defined by $\iota(r) = {}_r\Lambda$ is a homomorphism of unital rings, because ${}_{r+s}\Lambda = {}_r\Lambda + {}_s\Lambda$ and ${}_{rs}\Lambda = {}_r\Lambda \circ {}_s\Lambda$ for all $r, s \in R$.[30] If M is not the null module, then ι is injective, because R is unital.

(iii) Now let $\varepsilon : R \longrightarrow S$ be a ring homomorphism. Then S is a left R-module with scalar multiplication $r{\cdot}s := \varepsilon(r)s$; cf. Bourbaki, bottom of page A II.2

3.1.2. **Definition.** A **derivation** of the ring R is an additive map $\mathrm{d} : R \longrightarrow R$ such that for all $r_1, r_2 \in R$ we have

$$\mathrm{d}(r_1 r_2) = \mathrm{d}(r_1)r_2 + r_1\mathrm{d}(r_2).$$

The pair (R, d) is then called a **differential ring**. We frequently write $'$ instead of d, thus $r' = \mathrm{d}(r)$ for $r \in R$. We write $\mathrm{Der}(R)$ for the set of derivations of R.

Given a differential ring $R = (R,')$ and a left R-module M, a map $\partial : M \longrightarrow M$ is a **derivation** of the R-module M if it is additive and for all $r \in R, m \in M$ we have

$$\partial(rm) = r'm + r\partial(m).$$

The pair (M, ∂) is called a **differential $(R,')$-module**. We will just say differential R-module if the derivation on R is clear. The set of all derivations of M over R for $'$ is denoted by $\mathrm{Der}_{(R,')}(M)$ or just $\mathrm{Der}_R(M)$ if $'$ is clear from the context.

Motivation. We will see shortly that every derivation ∂ of the module R^n is of the form

$$\partial\begin{pmatrix}\xi_1\\ \vdots\\ \xi_n\end{pmatrix} = \begin{pmatrix}\xi_1'\\ \vdots\\ \xi_n'\end{pmatrix} + B\begin{pmatrix}\xi_1\\ \vdots\\ \xi_n\end{pmatrix},$$

for some $B \in M_n(R)$. Hence the kernel of ∂ is the solution set of a linear system of ODEs. Conversely, we'll see that every such system comes from and defines a module derivation.

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[30] Observe that ${}_r\Lambda$ is not an R-module endomorphism in general.

3.1.3. *Remarks.*

(i) If (R, d) is a differential ring, then the constants $C = \{r \in R \mid \mathrm{d}(r) = 0\}$ is obviously a differential subring of R. Further, the center Z of R is a differential subring because for $z \in Z, r \in R$ we have $rz = zr$ and $\mathrm{d}(r)z = z\mathrm{d}(r)$ and so

$$\mathrm{d}(z)r = \mathrm{d}(zr) - z\mathrm{d}(r) = \mathrm{d}(rz) - \mathrm{d}(r)z = \mathrm{d}(r)z + r\mathrm{d}(z) - \mathrm{d}(r)z = r\mathrm{d}(z).$$

If $a \in Z$, then $a\mathrm{d}$ is again derivation of the ring R, because for $x, y \in R$ we have

$$(a\mathrm{d})(xy) = a\mathrm{d}(x)y + ax\mathrm{d}(y) = a\mathrm{d}(x)y + xa\mathrm{d}(y) = (a\mathrm{d})(x)y + x(a\mathrm{d})(y).$$

(ii) If ∂, δ are derivations of the ring R, then the Lie bracket $[\partial, \delta] = \partial \circ \delta - \delta \circ \partial$ is again a derivation of the ring R.[31]

Proof. Clearly $\partial \circ \delta - \delta \circ \partial$ is additive. Take $x, y \in R$. Then

$$\begin{aligned}(\partial \circ \delta - \delta \circ \partial)(xy) &= \partial(\delta(x)y + x\delta(y)) - \delta(\partial(x)y + x\partial(y)) \\ &= \partial(\delta(x))y + \delta(x)\partial(y) + \partial(x)\delta(y) + x\partial\delta(y) \\ &\quad - \delta(\partial(x))y - \partial(x)\delta(y) - \delta(x)\partial(y) - x\delta(\partial(y)) \\ &= \partial(\delta(x))y + x\partial\delta(y) - \delta(\partial(x))y - x\delta(\partial(y)) \\ &= (\partial \circ \delta - \delta \circ \partial)(x)y + x(\partial \circ \delta - \delta \circ \partial)(y).\end{aligned}$$

◇

(iii) If $x \in R$ is invertible and $\mathrm{d} : R \longrightarrow R$ is a derivation, then

$$\mathrm{d}(x^{-1}) = -x^{-1} \cdot \mathrm{d}(x) \cdot x^{-1},$$

because $0 = \mathrm{d}(1) = \mathrm{d}(x^{-1}x) = \mathrm{d}(x^{-1})x + x^{-1}\mathrm{d}(x)$.

(iv) If d is a derivation of the ring R, then obviously d is also a derivation of the left R-module R for d.

(v) If $'$ is a derivation on R and M is a left R-module, then a derivation ∂ of M is an R-module homomorphism if and only if $r' \cdot M = 0$ for all $r \in R$; this follows directly from $\partial(rm) = r'm + r\partial(m)$. In particular ∂ is a homomorphism for the restriction of the module M to the subring of constants of R.

(vi) By (v) the constant map 0 is in general not a derivation and so $\mathrm{Der}_{(R,')}(M)$ is not a group. In fact if ∂, δ are derivations of the R-module M for $'$, then in general $\partial + \delta$ is **not** a derivation and neither is $r\partial$ for $r \in R$.

3.1.4. *Examples.*

(i) Let $R = (R,')$ be a differential ring. If $(M_i, \mathrm{d}_i)_{i \in I}$ is a family of differential R-modules, then $\sum_{i \in I}(M_i, \mathrm{d}_i)$ is again a differential R-module with derivation $\mathrm{d}(\sum_i m_i) = \sum_i \mathrm{d}(m_i)$, which we refer to as the **natural derivation** of $\sum_i M_i$.

For any set Γ, let $R[\Gamma]$ be the direct sum of Γ copies of the left R-module R. Hence $R[\Gamma]$ is again a left differential R-module. We will write elements of $R[\Gamma]$ as $\sum_{\gamma \in \Gamma} r_\gamma \gamma$. Then the natural derivation of $R[\Gamma]$ reads as

$$(\sum_{\gamma \in \Gamma} r_\gamma \gamma)' := \sum_{\gamma \in \Gamma} r'_\gamma \gamma$$

and for any $r \in R$ we see that $\mathrm{d}(rm) = r'm + r\mathrm{d}(m)$, hence $\mathrm{d} \circ {}_r\Lambda = {}_{r'}\Lambda + {}_r\Lambda \circ \mathrm{d}$. The natural derivation on R^n reads as

$$(r_1, \dots, r_n)' = (r'_1, \dots, r'_n).$$

[31] This implies that $(\mathrm{Der}(R), +, [\ ,\ ])$ is a *Lie-ring*.

(ii) Now let $\varepsilon : R \longrightarrow S$ be a ring homomorphism. Then S is an R-module with scalar multiplication $r{\cdot}s := \varepsilon(r)s$. If $'$ is a derivation on R and if ∂ is a derivation on S such that ε is a homomorphism of differential rings, then ∂ is a derivation of the R-module S: For $r \in R$ and $s \in S$ we have

$$\begin{aligned}\partial(r{\cdot}s) &= \partial(\varepsilon(r)s) = \partial(\varepsilon(r))s + \varepsilon(r)\partial s\\ &= \varepsilon(r')s + \varepsilon(r)\partial s \text{ because } \varepsilon \text{ is a differential ring homomorphism}\\ &= r'{\cdot}s + r{\cdot}\partial s.\end{aligned}$$

3.1.5**. The adjoint derivation** Let R be any ring. For $x, y \in R$ define $[y, x] = yx - xy$. Then the map $\mathrm{ad}_y : R \longrightarrow R,\ x \mapsto [y, x]$ is a derivation of the ring R. This map is called the **adjoint map** of y (cf. [Bourba1989, Chapter 1,§1.2,Def. 2]). Further, for all $y, z \in R$ we have $\mathrm{ad}_{[y,z]} = [\mathrm{ad}_y, \mathrm{ad}_z]$ (in $\mathrm{Der}(R)$ by 3.1.3(ii))[32]; hence if $yz = zy$, then $\mathrm{ad}_y \circ \mathrm{ad}_z = \mathrm{ad}_z \circ \mathrm{ad}_y$.

Proof. We write $\mathrm{d} = \mathrm{ad}_y$, which clearly is additive. For the Leibniz rule:

$$\begin{aligned}\mathrm{d}(x_1)x_2 + x_1\mathrm{d}(x_2) &= (yx_1 - x_1y)x_2 + x_1(yx_2 - x_2y)\\ &= yx_1x_2 - x_1yx_2 + x_1yx_2 - x_1x_2y\\ &= yx_1x_2 - x_1x_2y\\ &= \mathrm{d}(x_1x_2).\end{aligned}$$

Hence ad_y is a derivation. If $x \in R$, then

$$\begin{aligned}[\mathrm{ad}_y, \mathrm{ad}_z](x) &= \mathrm{ad}_y(\mathrm{ad}_z(x)) - \mathrm{ad}_z(\mathrm{ad}_y(x))\\ &= \mathrm{ad}_y(zx - xz) - \mathrm{ad}_z(yx - xy)\\ &= y(zx - xz) - (zx - xz)y - z(yx - xy) + (yx - xy)z\\ &= yzx - yxz - zxy + xzy - zyx + zxy + yxz - xyz\\ &= yzx + xzy - zyx - xyz\\ &= (yz - zy)x - x(yz - zy)\\ &= \mathrm{ad}_{[y,z]}(x).\end{aligned}$$

□

3.1.6**. Derivations of Modules via Derivations of the endomorphism ring** Let M be a faithful left R-module (i.e. $r{\cdot}M = 0 \Rightarrow r = 0$). If $T \in \mathrm{End}_{\mathbb{Z}}(M)$ is arbitrary, then by 3.1.5, the map $\mathrm{ad}_T : \mathrm{End}_{\mathbb{Z}}(M) \longrightarrow \mathrm{End}_{\mathbb{Z}}(M)$ defined by $\mathrm{ad}_T(\varepsilon) = T \circ \varepsilon - \varepsilon \circ T$ is a derivation on the ring $\mathrm{End}_{\mathbb{Z}}(M)$. The following are equivalent (where $\iota : R \longrightarrow \mathrm{End}_Z(M)$ denotes the map from 3.1.1(ii))

(i) $\mathrm{ad}_T(\iota(R)) \subseteq \iota(R)$, thus ad_T induces a derivation on the image of ι.
(ii) There is a derivation $'$ on R such that T is a derivation of M for $'$.

If these conditions hold true, then

(iii) The derivation in (ii) is unique and the co-restriction of ι to its image is an isomorphism of differential rings $(R,') \longrightarrow (\iota(R), \mathrm{ad}_T|_{\iota(R)})$. Hence ad_T can be seen as an extension of a derivation on R.

[32]This statement says that ad is a Lie-ring homomorphism $(R, +, [\ ,\ ]) \longrightarrow (\mathrm{Der}(R), [\ ,\ ])$.

Proof. Since M is faithful, the map ι is injective.

(i)⇒(ii). By (i) and the injectivity of ι, for every $r \in R$ there is a unique $r' \in R$ with $\mathrm{ad}_T({}_r\Lambda) = {}_{r'}\Lambda$. Let $\mathrm{d} : R \longrightarrow R$ be the map defined by $\mathrm{d}(r) = r'$. Then the definition of r' says $\mathrm{ad}_T \circ \iota = \iota \circ \mathrm{d}$. But ad_T is a derivation on $\iota(R)$, hence d is a derivation on R and the corestriction of ι to its image is an isomorphism of differential rings $(R, d) \longrightarrow (\iota(R), \mathrm{ad}_T|_{\iota(R)})$.

To see that T is a derivation of the module M for d, take $r \in R$. Then $T \circ {}_r\Lambda - {}_r\Lambda \circ T = \mathrm{ad}_T({}_r\Lambda) = {}_{r'}\Lambda = {}_{d(r)}\Lambda$, which expresses $T(rm) - rT(m) = r'm$ for all $m \in M$, as required.

(ii)⇒(i). Let $r \in R$ and $m \in M$. Then

$$\begin{aligned} \mathrm{ad}_T({}_r\Lambda)(m) &= T({}_r\Lambda(m)) - {}_r\Lambda(T(m)) \\ &= T(rm) - rT(m) \\ &= r'm + rT(m) - rT(m) \text{ by (ii)}, \end{aligned}$$

which shows that $\mathrm{ad}_T({}_r\Lambda) = {}_{r'}\Lambda \in \iota(R)$. Since ι is injective, this computation also shows that the derivation $'$ on R is uniquely determined by the requirement that ad_T is a derivation of M for $'$.

This finishes the proof of the equivalence of (i) and (ii). Item (iii) has been verified on the way. □

3.1.7**.** If we apply 3.1.6 to the left R-module R and any derivation d of R, then item (iii) says that the embedding $\iota : R \longrightarrow \mathrm{End}_{\mathbb{Z}}(R)$ is a differential ring homomorphism, when R is equipped with d and $\mathrm{End}_{\mathbb{Z}}(R)$ is equipped with ad_d.

Hence every derivation of any (not necessarily commutative) ring is the restriction of an adjoint from a ring extension.

3.1.8. **Proposition.** *Let $R = (R,')$ be a differential ring and let $\partial : M \longrightarrow M$ be a derivation of the left R-module M.*

(i) If $\varphi : M \longrightarrow M$ is an R-module endomorphism, then $\partial + \varphi$ is again a derivation of M.
(ii) If $\delta : M \longrightarrow M$ is another derivation and $\varphi : M \longrightarrow M$ is an R-module endomorphism, then $\partial \circ \varphi - \varphi \circ \delta$ is again an R-module endomorphism.[33] *In particular*
  (a) $\partial - \delta$ is an R-module endomorphism (set $\varphi = \mathrm{id}_M$).
  (b) The derivation ad_∂ of the ring $\mathrm{End}_{\mathbb{Z}}(M)$ restricts to a derivation $\mathrm{End}(M) \longrightarrow \mathrm{End}(M)$ (set $\delta = \partial$). (Recall from 3.1.6 that ad_∂ also restricts to the given derivation of $R \subseteq \mathrm{End}_{\mathbb{Z}}(M)$ provided M is faithful.)

Hence $\mathrm{Der}_R(M)$ is the coset $\partial + \mathrm{End}_R(M)$ of $\mathrm{End}_{\mathbb{Z}}(M)$ for any derivation ∂ of M.

(iii) If $\psi : M \longrightarrow M$ is an R-module automorphism, then $\psi \circ \partial \circ \psi^{-1}$ is again a derivation. If $\delta : M \longrightarrow M$ is another derivation and $\varphi = \partial - \delta$, then

$$\psi^{-1} \circ \partial \circ \psi = \delta + \Big(\psi^{-1} \circ \mathrm{ad}_\delta(\psi) + \psi^{-1} \circ \varphi \circ \psi\Big).$$

This is called the **gauge transformation formula** *and will be explained further in 3.1.11 below.*

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[33]Notice that in general, $\partial \circ \varphi$ is neither a derivation (set $\varphi = 0$), nor an endomorphism (set $\varphi = \mathrm{id}_M$). Similarly for $\varphi \circ \delta$.

Proof. (i). Obviously $\partial+\varphi$ is additive. Let $r\in R$ and $m\in M$. Then

$$\begin{aligned}(\partial+\varphi)(rm)&=\partial(rm)+\varphi(rm)\\&=r'm+r\partial m+r\varphi(m)\\&=r'm+r(\partial+\varphi)(m),\end{aligned}$$

as required for a derivation.

(ii). It is clear that $\psi:=\partial\circ\varphi-\varphi\circ\delta$ is additive. Take $r\in R$ and $m\in M$. Then

$$\begin{aligned}\psi(rm)&=\partial(\varphi(rm))-\varphi(\delta(rm))\\&=\partial(r\varphi(m))-\varphi(r'm+r\delta(m))\\&=r'\varphi(m)+r\partial(\varphi(m))-(r'\varphi(m)+r\varphi(\delta(m)))\\&=r\partial(\varphi(m))-r\varphi(\delta(m))\\&=r\psi(m),\end{aligned}$$

as required for linearity.

(iii). By (ii), $\partial\circ\psi^{-1}-\psi^{-1}\circ\partial$ is an endomorphism. Hence $\psi\circ\partial\circ\psi^{-1}-\partial$ is an endomorphism as well. By (i), we may add ∂ and get the derivation $\psi\circ\partial\circ\psi^{-1}$.

Now let $\delta:M\longrightarrow M$ be another derivation and $\varphi=\partial-\delta$. For $m\in M$ we have

$$\begin{aligned}\partial\circ\psi(m)&=\delta(\psi(m))+\varphi(\psi(m))\text{ as }\partial=\delta+\varphi\\&=\mathrm{ad}_\delta(\psi)(m)+\psi(\delta(m))+\varphi(\psi(m)),\text{ because }\mathrm{ad}_\delta(\psi)=\delta\circ\psi-\psi\circ\delta,\end{aligned}$$

and so $\partial\circ\psi=\psi\circ\delta+\Big(\mathrm{ad}_\delta(\psi)+\varphi\circ\psi\Big)$. Multiplication from the left by ψ^{-1} shows the gauge transformation formula. □

For the rest of this section we work with a commutative differential ring R.

3.1.9. **Description of module derivations in coordinates** We describe derivations in coordinates and also make 3.1.8 more explicit. Let M be a differential R-module that is free and finite dimensional (as an R-module). Choose a basis $e_1,\dots,e_n$ of M. For $i\in\{1,\dots,n\}$ let $r_{ij}\in R$ with

$$\partial(e_i)=\sum_j r_{ij}e_j.$$

Then for $m\in M$, $m=\xi_1e_1+\ldots+\xi_ne_n$, $\xi_i\in R$ we have

$$\begin{aligned}\partial m&=\sum_{i=1}^n\xi_i'e_i+\sum_{i=1}^n\xi_i\sum_{j=1}^n r_{ij}e_j\\&=\sum_{i=1}^n\xi_i'e_i+\sum_{j=1}^n\xi_j\sum_{i=1}^n r_{ji}e_i\\&=\sum_{i=1}^n\xi_i'e_i+\sum_{i=1}^n\sum_{j=1}^n\xi_jr_{ji}e_i\\&=\sum_{i=1}^n\Big(\xi_i'+\sum_{j=1}^n\xi_jr_{ji}\Big)e_i.\end{aligned}$$

Hence when elements of M are written in coordinates with respect to $e_1, \dots, e_n$ we obtain

$$\partial \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} = \begin{pmatrix} \xi_1' \\ \vdots \\ \xi_n' \end{pmatrix} + A^T \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix},$$

where $A = (r_{ij})_{i,j=1}^n$. More precisely, this means that if δ is the natural derivation on R^n and $\varphi \in \mathrm{End}(R^n)$ is defined by $\varphi(\xi) = A^T\xi$, then the diagram

$$\begin{CD} M @>{\partial}>> M \\ @A{\gamma}AA @AA{\gamma}A \\ R^n @>>{\delta+\varphi}> R^n, \end{CD}$$

with $\gamma(\xi) = \sum_i \xi_i e_i$, commutes. Since $\gamma \circ (\delta + \varphi) = \partial \circ \gamma$ we see that $\gamma : (R^n, \delta + \varphi) \longrightarrow (M, \partial)$ is an isomorphism of differential modules.

We see that the kernel of ∂ is the solutions set of the system $y' = -A^T y$ of linear differential equations of order 1. If we identify such a system with the defining matrix, 3.1.8 says that every such system is given by a derivation of M (although solutions sets of these system do not uniquely define the derivation, nor the defining matrices).

3.1.10. **Extension of the derivation to $\mathrm{End}(R^n) = M_n(R)$.** Let ∂ be the natural derivation on $M = R^n$. By 3.1.8(ii)(b), ad_∂ restricts to a derivation of $\mathrm{End}_R(R^n)$. We write $\mathrm{End}_R(R^n) = M_n(R)$ and compute ad_∂ in terms of matrices:

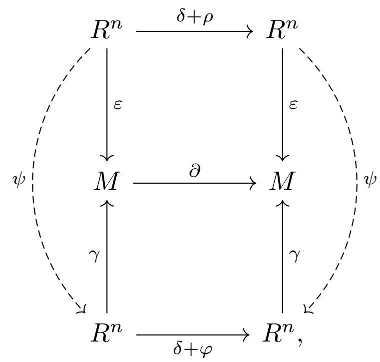
Let $\varphi \in \mathrm{End}(M)$ and $\varphi(x) = Ax$ with $A \in M_n(R)$. Then $\mathrm{ad}_\partial(\varphi)x = A'x$, where $A' \in M_n(R)$ is defined to be the matrix obtained from A by applying the derivation of R to all the entries of A. The reason is that $\mathrm{ad}_\partial(\varphi)x = \partial(Ax) - A\partial x$, which is readily seen to be equal to $A'x$.

Hence $\mathrm{ad}_\partial : M_n(R) \longrightarrow M_n(R)$ is the derivation defined by $A \mapsto A'$.

3.1.11. **Gauge transformation and change of basis** We continue the consideration on coordinates from 3.1.9. Let $f_1, \dots, f_n$ be another basis of M and let $\varepsilon : R^n \longrightarrow M$ be defined by $\varepsilon(\xi) = \sum_i \xi_i f_i$. As in 3.1.9, there is a linear map $\rho : R^n \longrightarrow R^n$ such that the following diagram to the right commutes (the diagram to the left is copied from 3.1.9):

$$\begin{CD} M @>{\partial}>> M \\ @A{\gamma}AA @AA{\gamma}A \\ R^n @>>{\delta+\varphi}> R^n, \end{CD} \qquad \begin{CD} M @>{\partial}>> M \\ @A{\varepsilon}AA @AA{\varepsilon}A \\ R^n @>>{\delta+\rho}> R^n. \end{CD}$$

We want to compute, ρ in terms of φ. If we flip and then glue the diagram on the right on top of the diagram on the left we get the following commutative diagram:



where $\psi := \gamma^{-1} \circ \varepsilon : R^n \longrightarrow R^n$. Then $\delta + \rho = \psi^{-1} \circ (\delta + \varphi) \circ \psi$. By 3.1.8(iii) we know

$$\psi^{-1} \circ (\delta + \varphi) \circ \psi = \delta + \psi^{-1} \circ \mathrm{ad}_\delta(\psi) + \psi^{-1} \circ \varphi \circ \psi$$

and so we obtain the base change formula

$$(\flat) \qquad \rho = \psi^{-1} \circ \mathrm{ad}_\delta(\psi) + \psi^{-1} \circ \varphi \circ \psi$$

Notice that for $A, B \in M_n(R)$ with $\psi(\xi) = B\xi$ and $\varphi(\xi) = A\xi$ (we have now replaced A by A^T compared to 3.1.9), then $(\flat)$ says that the matrix C with $\rho(\xi) = C\xi$ satisfies

$$(\dagger) \qquad C = B^{-1}B' + B^{-1}AB.$$

(See 3.1.10 for the computation of $\mathrm{ad}_\delta(\psi)$ as B'.) This formula is thus the **gauge transformation formula in matrix form** expressing the fact that the differential modules given by $\delta + A$ and $\delta + C$ on R^n are isomorphic as differential modules if and only if there is some $B \in GL_n(R)$ such that $(\dagger)$ holds.

3.2. **The ring of differential operators and classification of differential modules.**

3.2.1**. The ring of differential operators** We continue to work with a unital, but not necessarily commutative ring R. Let $'$ be a derivation on R. We write d for the natural derivation on the left R-module $R[\omega]$ as explained in 3.1.4, $\omega = \{0, 1, 2, \ldots\}$. We write elements in $R[\omega]$ as $\sum_{k\in\omega} r_k\tau^k$. Let $\Lambda : R[\omega] \longrightarrow R[\omega]$ be the R-module homomorphism obtained from the shift $\omega \longrightarrow \omega,\ k \mapsto k + 1$. Hence

$$\Lambda(\sum_{k\in\omega} r_k\tau^k) = \sum_{k\in\omega} r_k\tau^{k+1}.$$

One should think of Λ as multiplication by τ from the right. Now, $\partial := \mathrm{d} + \Lambda \in \mathrm{End}_{\mathbb{Z}}(R[\omega])$ is a module derivation of $R[\omega]$ (by 3.1.8(i)) and we define **the ring of differential operators over** $(R,')$ as the subring of $\mathrm{End}_{\mathbb{Z}}(R[\omega])$ generated by ∂ over the image of the embedding $\iota : R \longrightarrow \mathrm{End}_{\mathbb{Z}}(R[\omega]),\ r \mapsto {}_r\Lambda$ (cf. 3.1.1(ii)). We will identify R with its image under ι and thus obtain an overring $R[\partial]$ of R. We record some properties related to the construction of $R[\partial]$.

(i) Since $\mathrm{d} \circ \Lambda = \Lambda \circ \mathrm{d}$ we see that

$$\partial^n = (d + \Lambda)^n = \sum_{k=0}^{n} \binom{n}{k} \mathrm{d}^{n-k} \circ \Lambda^k,$$

where powers are taken in the ring $\mathrm{End}_{\mathbb{Z}}(R[\omega])$.

(ii) By 3.1.6, the ring homomorphism $\iota : R \longrightarrow R[\partial]$ (identified as being the inclusion) is differential, when R is equipped with $'$ and $R[\partial]$ is equipped with the derivation $\mathrm{d} = \mathrm{ad}_\partial$ of $R[\partial]$ from 3.1.5. Hence for $r \in R$ we have $\partial{\cdot}r = r' + r{\cdot}\partial$ in the ring $R[\partial]$.

(In 3.2.3 we will see see that $R[\partial]$ is the non-commutative ring $R\langle\partial\rangle/(\partial{\cdot}r = r' + r{\cdot}\partial \mid r \in R)$ and one could use this as the definition of $R[\partial]$ as well.)

(iii) Now let $\varepsilon : R \longrightarrow S$ be a ring homomorphism and let $s \in S$ be such that $\varepsilon : R \longrightarrow S$ is differential, when R is equipped with the derivation $'$ and S is equipped with the derivation ad_s from 3.1.5. Explicitly, this means that $s{\cdot}\varepsilon(r) = \varepsilon(r') + \varepsilon(r){\cdot}s$ for all $r \in R$. Then

(a) Left multiplication of S with s is a derivation of the left module S for $(R,')$, because for $f \in S$ and $r \in R$:

$$s{\cdot}(\varepsilon(r){\cdot}f) = (s{\cdot}\varepsilon(r)){\cdot}f = (\varepsilon(r') + \varepsilon(r){\cdot}s){\cdot}f = \varepsilon(r'){\cdot}f + \varepsilon(r){\cdot}(s{\cdot}f).$$

(b) $B := \{\sum_{k=0}^{n} \varepsilon(r_k)s^k \mid n \in \mathbb{N}_0, r_k \in R\}$ is the subring of S generated by $\varepsilon(R)$ and s, as follows by a straightforward induction on n showing that $s^n{\cdot}\varepsilon(r) \in B$.

If we apply (a) and (b) to $\varepsilon = \iota : R \longrightarrow R[\partial]$ we see, using (ii), that left multiplication of $R[\partial]$ by ∂ is a derivation of both left modules $\mathrm{End}_{\mathbb{Z}}(R[\omega])$ and $R[\partial]$ for $(R,')$ and

$$R[\partial] = \{\sum_{k=0}^{n} r_k\partial^k \mid n \in \mathbb{N}_0, r_k \in R\}.$$

We also see that

(c) for $f=\sum_{k=0}^n r_k\partial^k$,

$$\partial\cdot f=\sum_{k=0}^n r_k'\partial^k+f\partial=r_n\partial^{n+1}+(r_n'+r_{n-1})\partial^n+\ldots+(r_1'+r_0)\cdot\partial+r_0'.$$

(Notice that left multiplication of $R[\partial]$ by ∂ is not a derivation of the ring $R[\partial]$, because otherwise $\partial\cdot(1\cdot f)=(\partial\cdot 1)\cdot f+1\cdot(\partial\cdot f)$, implying $\partial\cdot r=0$.)

(iv) The kernel of d is $C[\partial]$, where C is the ring of constants of R. If C is commutative, then $C[\partial]$ is just the commutative polynomial ring in ∂ over C.

3.2.2. **Proposition.** *In the situation of 3.2.1, the evaluation map $ev_1:R[\partial]\longrightarrow R[\omega]$ at $1=\tau^0\in R[\omega]$ is a left R-module isomorphism. For $r_0,\ldots,r_n\in R$ we have*

$$(\sum_{k=0}^n r_k\partial^k)(1)=\sum_{k=0}^n r_k\tau^k.$$

In particular, the elements $1,\partial,\partial^2,\ldots$ are R-linearly independent.

Proof. First notice that by 3.2.1(iii), the set $R[\partial]$ indeed consist of left linear combinations of $1,\tau,\tau^2,\ldots$ over R. We have $\Lambda^k(1)=\tau^k$ and $\mathrm{d}^l(\tau^k)=0$ for all $k\geq 0$ and $l>0$. Hence by 3.2.1(i) we see that $\partial^k(1)=\tau^k$, which implies $(\sum_{k=0}^n r_k\partial^k)(1)=\sum_{k=0}^n r_k\tau^k$. It is now clear that ev_1 is an R-module isomorphism. □

3.2.3. **Theorem.** *Let $R=(R,')$ be a differential ring (not nec. commutative). Consider the ring homomorphism $\iota:R\longrightarrow R[\partial]$ together with the element $\partial\in R[\partial]$.*

(i) Then ι is a differential ring homomorphism $(R,')\longrightarrow(R[\partial],\mathrm{ad}_\partial)$ and left multiplication of $R[\partial]$ by ∂ is a derivation of the left R-module $R[\partial]$.

(ii) The pointed homomorphism $(\iota:R\longrightarrow R[\partial],\partial)$ is uniquely determined up to isomorphism of pointed homomorphisms in the following sense:

If $\varepsilon:R\longrightarrow S$ is a ring homomorphism and a differential map for $'$ and some $\mathrm{ad}_s=[s,\_]$, $s\in S$, then there is a unique ring homomorphism $R[\partial]\longrightarrow S$ extending ε mapping ∂ to s.

Proof. (i) has been shown in 3.2.1(iii) (the assumptions where verified in 3.2.1(ii)).
(ii). By 3.2.2, we may define an R-module endomorphism $\varphi:R[\partial]\longrightarrow S$ by $\varphi(\sum_{k=0}^n r_k\partial^k)=\sum_{k=0}^n\varepsilon(r_k)s^k$ and we claim that φ is a ring homomorphism.

Claim. For each n we have $s^n\varepsilon(r)=\sum_{k=0}^n\binom{n}{k}\varepsilon(r^{(n-k)})s^k$.
Proof. This is clear for $n=0,1$. For the induction step we have

$$\begin{aligned}s^{n+1}\varepsilon(r)&=s^n(\varepsilon(r')+\varepsilon(r)s)=s^n\varepsilon(r')+s^n\varepsilon(r)s\\&=\sum_{k=0}^n\binom{n}{k}\varepsilon(r^{(n-k+1)})s^k+(\sum_{k=0}^n\binom{n}{k}\varepsilon(r^{(n-k)})s^k)s\\&=\sum_{k=0}^n\binom{n}{k}\varepsilon(r^{(n-k+1)})s^k+\sum_{k=1}^{n+1}\binom{n}{k-1}\varepsilon(r^{(n-(k-1))})s^k\\&=\sum_{k=0}^{n+1}\binom{n+1}{k}\varepsilon(r^{(n+1-k)})s^k.\end{aligned}$$

◇

The claim also applies to $\partial \in R[\partial]$ (and the natural map $R \longrightarrow R[\partial]$) by 3.2.1(ii). Then for all $n, l \in \mathbb{N}_0$ and each $r \in R$ we see that

$$\begin{aligned}\varphi(\partial^n r \partial^l) &= \varphi(\sum_{k=0}^{n} \binom{n}{k} r^{(n-k)} \partial^{k+l}) \\ &= \sum_{k=0}^{n} \binom{n}{k} \varepsilon(r^{(n-k)}) s^{k+l} = (\sum_{k=0}^{n} \binom{n}{k} \varepsilon(r^{(n-k)}) s^k) s^l \\ &= s^n \varepsilon(r) s^l.\end{aligned}$$

It is now routine to check that φ is multiplicative. □

3.2.4. **Corollary.** *Let $\varphi : R \longrightarrow S$ be a homomorphism of differential rings. Then there is a unique ring homomorphism $\varphi[\partial] : R[\partial] \longrightarrow S[\partial]$ making the diagram*

$$\begin{array}{ccc} R[\partial] & \xrightarrow{\varphi[\partial]} & S[\partial] \\ \uparrow & & \uparrow \\ R & \xrightarrow[\varphi]{} & S \end{array}$$

commutative. Explicitly, $\varphi[\partial]$ maps $\sum r_i \partial^i$ to $\sum \varphi(r_i) \partial^i$.

Proof. By 3.2.3(i) applied to S implies that the composition ε of φ with the structure map $S \longrightarrow S[\partial]$ is a differential ring homomorphism $R \longrightarrow S[\partial]$, when the latter ring is equipped with the derivation ad_∂. By 3.2.3(ii) applied to ε and $\partial \in S[\partial]$ we get the corollary. □

3.2.5**. The formal adjoint of** $R[\partial]$**.** *Let $R = (R,')$ be a commutative differential ring and let $R[\partial]^{op}$ be the opposite ring of $R[\partial]$, hence $(R[\partial]^{op}, +) = (R[\partial], +)$ and multiplication in $R[\partial]^{op}$ is given by $s_1 * s_2 = s_2 \cdot s_1$, where $s_2 \cdot s_1$ is the product in $R[\partial]$. Let $\iota : R[\partial] \longrightarrow R[\partial]^{op}$ be defined by*

$$\iota(\sum r_i \partial^i) = \sum (-1)^i r_i * \partial^i.$$

Then ι is an isomorphism, called the **formal adjoint** *of $R[\partial]$. Notice that ι can also be seen as a map $\iota : R[\partial] \longrightarrow R[\partial]$, where $\iota(\sum r_i \partial^i) = \sum (-1)^i \partial^i \cdot r_i$.*

Proof. Let $\varepsilon : R \longrightarrow R[\partial]^{op}$ be defined by $\varepsilon(r) = r \cdot \partial^0$. Since R is commutative, ε is a ring homomorphism $R \longrightarrow R[\partial]^{op}$. Moreover, ε is differential when $R[\partial]^{op}$ is equipped with derivation $\mathrm{ad}_{-\partial}$, because for $r \in R$ we have

$$r' = \partial r - r\partial = r * \partial - \partial * r = (-\partial) * r - r * (-\partial).$$

By 3.2.3(ii) applied to ε we get the assertion. □

3.2.6. **Proposition.** *Let $R = (R,')$ be a differential ring. Then the category of left $R[\partial]$-modules is isomorphic to the category of left differential $(R,')$-modules. The isomorphism sends a left module M over $R[\partial]$ to $(M|_R, \vartheta)$, where $M|_R$ is the R-module obtained from M via scalar restriction, and $\vartheta : M \longrightarrow M$ is scalar multiplication of M by ∂.*

Proof. Let M be a left module over $R[\partial]$ and let $\vartheta : M \longrightarrow M$ be defined by (left) multiplication with ∂. Then for $r \in R$ and $m \in M$ we have

$$\begin{aligned}\vartheta(rm) &= \partial\cdot(rm) = (\partial\cdot r)\cdot m\\ &= (r' + r\cdot\partial)\cdot m,\ \text{by 3.2.1(ii)}\\ &= r'\cdot m + r\cdot\vartheta(m).\end{aligned}$$

This shows that ϑ is a derivation on M for $(R,')$.

Conversely, let $\vartheta : M \longrightarrow M$ be a derivation. Then M is an $R[\partial]$-module with $\partial\cdot m = \vartheta(m)$ in the following way. W.l.o.g. assume $M \neq 0$. Let S be the ring $\mathrm{End}_{\mathbb{Z}}(M)$ and consider S as an overring of R via the embedding ι from 3.1.1(ii). Since ϑ is a derivation on M, we have $\vartheta\cdot r = r' + r\cdot\vartheta$ (i.e. $\vartheta \circ {}_r\Lambda = {}_{r'}\Lambda + {}_r\Lambda \circ \vartheta$) for all $r \in R$. Hence by 3.2.3, there is an homomorphism $R[\partial] \longrightarrow S$ extending ι and mapping ∂ to ϑ. Since M is an S-module (via evaluation), the scalar restriction to $R[\partial]$ gives the required $R[\partial]$-module structure. □

3.2.7. **Example: The first Weyl Algebra over a ring.** Let k be a commutative ring and let $R = k[x]$ be the polynomial ring in one variable over k. Let $'$ be the standard derivative $\frac{\mathrm{d}}{\mathrm{d}x}$ of R. Then $R[\partial]$ is the first Weyl algebra over k: To see this, notice that the generators x, ∂ of the ring $R[\partial]$ over k satisfy $\partial\cdot x = x' + x\cdot\partial = 1 + x\cdot\partial$. Now the defining equation of the first Weyl algebra W is $XY - YX - 1 = 0$, i.e. $W := k\langle X, Y\rangle/(XY - YX - 1)$. Hence we have a k-algebra homomorphism $k\langle X, Y\rangle \longrightarrow R[\partial]$ mapping X to x and Y to ∂, which is obviously surjective. We obtain a map $k\langle X, Y\rangle/(XY - YX - 1) \longrightarrow R[\partial]$. To see that this map is injective we use the universal property of $R[\partial]$ from 3.2.3(ii) to produce a compositional inverse. The inclusion $R = k[x] \hookrightarrow W$ is a an R-algebra and the element $s = Y \bmod \langle XY - YX - 1\rangle$ satisfies $sr = r' + rs$ for all $r \in R$: Modulo the ideal $\langle XY - YX - 1\rangle$ we have $YX - XY = 1 = X'$ and by induction

$$\begin{aligned}YX^{n+1} - X^{n+1}Y &= (1 + XY)X^n - X^{n+1}Y = X^n + XYX^n - X^{n+1}Y\\ &= X^n + X(YX^n - X^nY)\\ &= X^n + X(X^n)',\ \text{by induction}\\ &= X^n + XnX^{n-1} = (n+1)X^n.\end{aligned}$$

Hence for $r = \sum_{k=0}^{n} a_k X^k$ we get (modulo $\langle XY - YX - 1\rangle$)

$$Yr - rY = \sum_{k=0}^{n} a_k(YX^k - X^kY) = \sum_{k=0}^{n} a_k(X^k)' = r'.$$

Thus, from 3.2.3(ii) we obtain an R-algebra homomorphism $R[\partial] \longrightarrow W$ mapping ∂ to s. This map is the compositional inverse we were looking for.

3.2.8. **Division with remainder in $R[\partial]$.** *(This is the proof of* [Lang2002, IV,§1,Theorem 1.1,p.173]*, where special attention to non-commutativity is given.)*

Let $R = (R,')$ be a differential ring. Let $f, g \in R[\partial]$ with $g \neq 0$ such that the leading coefficient of g is invertible in R. Then there are unique $q, r \in R[\partial]$ with $f = q\cdot g + r$ and $\deg(r) < \deg(g)$. (Here $\deg(0) = -\infty$.)

If R is commutative, then by 3.2.5 we see that there are unique $q, r \in R[\partial]$ with $f = g\cdot q + r$ and $\deg(r) < \deg(g)$.

Proof. First we do existence. If $\deg(f) < \deg(g)$, then take $q = 0$ and $r = f$. Hence we may assume that $n = \deg(f) \geq \deg(g) = d$. We do an induction on n. Since $g \neq 0$, also $f \neq 0$. We write

$$f = a_n\partial^n + \ldots + a_0,$$
$$g = b_d\partial^d + \ldots + b_0,$$

where $a_n, b_d \in R \setminus \{0\}$. By assumption b_d is invertible in R. If $n = 0$, then we may choose $q = a_0 \cdot b_0^{-1}$ and $r = 0$. Now assume $n > 0$. By 3.2.1(iii)(c) we see that $\partial \cdot g = b_d\partial^{d+1} + g^*$, where $g^* \in R[\partial]$ is of degree $< d + 1$. Consequently $\partial^{n-d} \cdot g = b_d\partial^n + g^*$, where $g^* \in R[\partial]$ is of degree $< n$. It follows that

$$f_1 = f - a_n b_d^{-1} \cdot \partial^{n-d} \cdot g$$

is of degree $< n$. By induction there are $q_1, r \in R[\partial]$ with $\deg(r) < \deg(g)$ such that $f_1 = q_1 \cdot g + r$. Consequently

$$f = q_1 \cdot g + a_n b_d^{-1} \cdot \partial^{n-d} \cdot g + r$$

and we may take $q = q_1 + a_n b_d^{-1} \cdot \partial^{n-d}$.

Now for uniqueness. Assume $f = q_i \cdot g + r_i$ with $\deg(r_i) < d$. Then $(q_1 - q_2) \cdot g = r_1 - r_2$. Since the leading coefficient of g is a unit and $\deg(r_1 - r_2) < \deg(g)$, this is only possible if $q_1 = q_2$, and consequently $r_1 = r_2$. □

3.2.9. **Corollary.** *Let $K = (K,')$ be a differential field. Then every left ideal of $K[\partial]$ is a principal left ideal and every right ideal of $K[\partial]$ is a principal right ideal.*

Proof. Let I be a left ideal of $K[\partial]$ and let $g \in I \setminus \{0\}$ be of minimal degree. Since K is a field we may assume that the leading coefficient of f is 1. Now if $f \in I$, then take $q, r \in K[\partial]$ with $f = q \cdot g + r$ and $\deg(r) < \deg(g)$. Since I is a left ideal of $K[\partial]$ we know that $q \cdot g \in I$. But then $r \in I$ as well. Since $\deg(r) < \deg(g)$ the choice of g implies $r = 0$. Thus $I = K[\partial] \cdot g$.

Since K is commutative, the assertion also holds for right ideals by 3.2.5 □

3.2.10. **Theorem.** [vdPSin2003, Proposition 2.9], *pdf If K is a differential field and M is a finitely generated $K[\partial]$-module then there is some $n \geq 0$ and some $f \in K[\partial]$ with*

$$M \cong K[\partial]^n \oplus K[\partial]/K[\partial]f$$

as $K[\partial]$-modules.

Hence if M is finite dimensional as a K-vector space, then $M \cong K[\partial]/K[\partial]f$ is cyclic, i.e. has a cyclic vector v, i.e. M is the differential submodule of M generated by v (take $v = \partial + K[\partial]f$).

3.2.11. **Corollary.** *If K is a differential field and $A \in M_n(K)$, then there is some $L(y) = y^n + a_{n-1}y^{(n-1)} + \ldots + a_0 y \in K\{y\}$ such that the matrix differential equation $Y' = A \cdot Y$ is equivalent to the matrix differential equation $Y' = A_L \cdot Y$, where*

$$A_L = \begin{pmatrix} 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ -a_0 & -a_1 & \ldots & -a_{n-2} & -a_{n-1} \end{pmatrix}.$$

Equivalence here means that the associated differential modules are isomorphic as differential modules. Alternatively, that A and A_L are obtained from a gauge transformation as in 3.1.11.

Proof. [vdPSin2003, Exercise 2.12.7], pdf One finds L as follows: Let M be the $K[\partial]$-module given by $d{+}A$. By 3.2.10 there is some $f = \partial^n + b_{n-1}\partial^{n-1} + \ldots + b_0\partial^0 \in K[\partial]$ with $M \cong K[\partial]/K[\partial]f$. Now take $a_i = (-1)^i \cdot b_i$. □

In [vdPSin2003, Section 2.2] one can find a wealth of properties of differential modules, which reflect natural questions about linear ODEs.

3.3. **Picard-Vessiot extensions.**

3.3.1**. Where to solve linear ODEs?** Let K be a differential field and let

$$P(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y \in K\{y\}.$$

Since P is irreducible with separant 1 we know from 1.2.12 that $I(P) = [P] = (P, P', P'', \ldots)$ is a differential prime ideal of $K\{y\}$. Hence the residue $\alpha = y + [P] \in K\{y\}/[P]$ is a solution of $P(y) = 0$ and the transcendence degree of $K\langle\alpha\rangle$ over K is n, see 2.6.3. Therefore, when $n > 0$:

(i) We can always find solutions of $P(y) = 0$ outside of K, in stark contrast to the classical case of polynomial equations in one variable (over algebraically closed fields).
(ii) By iteration we can find arbitrary many algebraically independent (over K) solutions of $P(y) = 0$ in some differential field containing K.

Hence if we want to study solutions of $P(y) = 0$ (and suppose there is only the trivial solution in K), we need to decide in which ambient differential field we want to look at solutions. The differential closure of K will provide a framework, suitable for Galois theory. Furthermore, the field $K\langle\alpha\rangle$ obtained from f by adding the generic solution α to K will in general **not** serve as splitting field of $f(y)$ in differential Galois theory.

3.3.2. **Proposition.** *Let $M = (M, \partial)$ be a differential module over the differential field $K = (K, ')$ and let C be the field of constants of K.*

(i) ∂ is a C-vector space homomorphism and therefore $\ker(\partial)$ *is a C-vector space.*
(ii) If $m_1, \ldots, m_n \in \mathrm{Ker}(\partial)$ are C-linearly independent, then they are K-linearly independent. In particular $\dim_C \ker(\partial) \le \dim_K(M)$.
(iii) If K is differentially closed and M is finite dimensional as a K-vector space, then $\ker(\partial)$ *generates M as a K-vector space and* $\dim_C \ker(\partial) = \dim_K(M)$.

Proof. (i) follows directly from the definition 3.1.2 of "module derivation".

(ii) Assume by induction that $m_1, \ldots, m_{n-1}$ are already K-linearly independent and that there are $r_1, \ldots, r_{n-1} \in K$ with $m_n = r_1 m_1 + \ldots + r_{n-1} m_{n-1}$. Applying ∂ gives

$$(*) \qquad \partial(m_n) = r_1' m_1 + \ldots + r_{n-1}' m_{n-1} + r_1 \partial(m_1) + \ldots + r_{n-1}\partial(m_{n-1}).$$

As all $m_i \in \mathrm{Ker}(\partial)$, this means

$$0 = r_1' m_1 + \ldots + r_{n-1}' m_{n-1}.$$

Since $m_1, \ldots, m_{n-1}$ are K-linearly independent we see that $r_1' = \ldots = r_{n-1}' = 0$, i.e. m_n is in the C-span of $m_1, \ldots, m_{n-1}$ by $(*)$.

(iii). We may assume that $M = K^n$ as vector space. By 3.1.9 we know that $\partial(x) = x' - A{\cdot}x$ $(x \in M)$ for some $A \in M_n(K)$. Choose generic solutions $Y_1, \ldots, Y_n$ of $\partial(x) = 0$, i.e., choose indeterminates $Y_{i,j}$ over K, $1 \le i, j \le n$ and let d be the derivation of the polynomial ring $K[Y_{i,j} \mid 1 \le i, j \le n]$ extending the derivation of K with

$$\begin{pmatrix} d(Y_{i,1}) \\ \vdots \\ d(Y_{i,n}) \end{pmatrix} = A \begin{pmatrix} Y_{i,1} \\ \vdots \\ Y_{i,n} \end{pmatrix}.$$

For existence of d use 1.1.7. Then the formula

$$\bigwedge_{i=1}^{n} y_i' = A{\cdot}y_i \ \&\ \det(y_1,\ldots,y_n)\neq 0$$

(in n^2 variables $y_{i,j}$) has a solution in the differential field extension $(K(Y_{i,j} \mid 1\leq i,j\leq n),d)$ of K given by $Y_{i,j}$; observe that $\det(Y_1,\ldots,Y_n)\neq 0$, since the $Y_{i,j}$ are algebraically independent. Since K is differentially closed there are $z_1,\ldots,z_n\in K^n$ with $z_i'=A{\cdot}z_i$ and $\det(z_1,\ldots,z_n)\neq 0$. It follows that $z_1,\ldots,z_n\in\ker(\partial)$ are K-linearly independent. Hence $\ker(\partial)$ generates M as a K-vector space and $\dim_C\ker(\partial)\geq\dim_K(z_1,\ldots,z_n)=\dim_K(M)$. By (ii) we get equality. □

3.3.3. **Definition of Picard-Vessiot extensions for matrix ODEs** Let $y'=Ay$ be a matrix ODE over the differential field $K=(K,')$, $A\in M_n(K)$.

(i) A **fundamental system of solutions** of $y'=Ay$ is a tuple $(y_1,\ldots,y_n)$, $y_i\in L^n$ of solutions $y_i'=Ay_i$ for all $i\in\{1,\ldots,n\}$ from some differential field extension L with $\det(y_1,\ldots,y_n)\neq 0$; equivalently: $y_1,\ldots,y_n$ are solutions of $y'=Ay$ and they are linearly independent over the constant field of L, see 3.3.2.)

(ii) A differential field $L\supseteq K$ is called a **Picard-Vessiot extension of** K **for** $y'=Ay$ if it is generated as a differential field by (the coordinates of) a fundamental system of solutions of $y'=Ay$ in L and if the constant field of L is the constant field of K. Notice that L is then already finitely generated as a field by $y_1,\ldots,y_n$, since $y_i'\in K(y_1,\ldots,y_n)$. A **Picard-Vessiot extension of** K is a Picard-Vessiot extension of K for some matrix ODE of K.

3.3.4. **Definition.** The **differential Galois group** of a differential field extension L/K is the group of differential K-automorphism of L. It is denoted by $\mathrm{Gal}(L/K)$. Notice that there is no conflict with the classical notation: If L/K is an algebraic extension, then every field automorphism of L over K is differential: deploy the formula $(*)$ in 2.1.5.

3.3.5. **Existence, uniqueness and normality of Picard-Vessiot extensions** *Let $K=(K,')$ be a differential field with algebraically closed constant field C. Let $y'=Ay$ be a matrix ODE over $K=(K,')$, where $A\in M_n(K)$.*

(i) If F is a differential field containing the differential closure $\hat{K}$ of K and the constant field of F is the constant field of $\hat{K}$[34]*, then the subfield L of F generated by all (entries of) solutions of $y'=Ay$ in F is the unique Picard-Vessiot extension of K for $y'=Ay$ inside F.*

(ii) Every differential K-automorphism of $\hat{K}$ restricts to a K-automorphism of L and the restriction map $\mathrm{Gal}(\hat{K}/K)\longrightarrow\mathrm{Gal}(L/K)$ is surjective.

(iii) All Picard-Vessiot extensions of K for $y'=Ay$ are differentially isomorphic over K.

Consequently: Up to differential K-isomorphism, there is a unique Picard-Vessiot extension L of K for $y'=Ay$, namely the field generated by all (entries of) solutions of $y'=Ay$ in the differential closure of K. By 3.3.2(iii), the solutions of $y'=Ay$ in L^n is a C-vector space of dimension n.

[34]The main case being $F=\hat{K}$. Notice that every differential field has an extension that does not change the constants: The quotient field of the differential polynomial ring.

Proof. (i). By 3.3.2(iii), there is a fundamental system $(y_1, \dots, y_n)$ of solutions of $y' = Ay$ in $\hat{K}$. Since $\hat{K}$ is atomic over K, we know from 2.7.16 that C is also the constant field of $\hat{K}$. Hence the field L generated by such a system also has constant field C and is therefore a Picard-Vessiot extension for $y' = Ay$.

Now let $\tilde{L}$ be another Picard-Vessiot extension for $y' = Ay$ inside F. and let $(z_1, \dots, z_n)$ be a fundamental system of solutions of $y' = Ay$ that generates $\tilde{L}$ as a field. Since the constant field of F is the constant field of $\hat{K}$ and this field is C we know that F and K have the same constant field. By 3.3.2(ii), the solutions of $y' = Ay$ in F is of C-dimension $\leq n$. However, both L as well as $\tilde{L}$ are generated by the entries of n elements of F^n that are linearly independent over C. Hence $\tilde{L} = L$.

(ii). By (i), every differential K-automorphism of $\hat{K}$ restricts to a K-automorphism of L. Since L is differentially finitely generated over K we may apply 2.7.10 and see that every differential K-automorphism extends to a differential K-automorphism of $\hat{K}$.

(iii). Let F be another Picard-Vessiot extensions of $y' = Ay$. Since F has constant field C, its differential closure $\hat{F}$ has again constant field C. Since $\hat{K}$ can be embedded into $\hat{F}$ over K, also L can be embedded into $\hat{F}$ over K. By (i) applied to $\hat{F}$ (which has constant field C), this embedding must be an isomorphism onto F. □

3.3.6. **Corollary.** *If L/K is a Picard-Vessiot extension with algebraically closed constant field, then the fixed field of* $\mathrm{Gal}(L/K)$ *is* K.

Proof. By 2.7.10 we know that the fixed field of $\mathrm{Gal}(\hat{K}/K)$ is K. Hence by 3.3.5(iii),(ii) we get the corollary. □

3.3.7. *Remark.* Without the assumption that the constants of K are algebraically closed, neither existence nor uniqueness of Picard-Vessiot extensions is true in general:

In [CreHaj2011, Exercises (25) for Chapter 5, page 137] an example for non-existence is given.

In [CreHaj2011, Exercises (26) for Chapter 5, page 137] an example of non-isomorphic Picard-Vessiot extensions for the linear ODE $y'' + y = 0$ over $\mathbb{R}$ (with the trivial derivation) is given.

3.3.8**. Characterization of Picard-Vessiot extensions** [Magid1994, Proposition 3.9]

Let $K \subseteq L$ be differential fields and assume K and L have the same constant field C and C is algebraically closed. Then L/K is Picard-Vessiot if and only if

(a) L is the differential field generated by a finite dimensional C-vectorspace V.
(b) There is a subgroup $G \subseteq \mathrm{Gal}(L/K)$ with fixed field K such that $\sigma(V) \subseteq V$ for all $\sigma \in G$.

If L/K is a Picard-Vessiot extension and $y_1, \dots, y_n$ is a C-basis of a C-vector space as in (a),(b), then L/K is a Picard-Vessiot extension for

$$\frac{\mathrm{wr}(Y, y_1, \dots, y_n)}{\mathrm{wr}(y_1, \dots, y_n)},$$

where wr is the **Wronskian**, See [Magid1994, Chapter 2] for definitions. In this course we are not talking about the Wronskian (due to time limitations), but the reader is strongly advised to connect to this important part of the theory.

3.4. The differential Galois group as definable group and as a linear algebraic group.

3.4.1. *Remark.* Let K be a differential field and let $A \in M_n(K)$. Suppose the matrix ODE $y' = Ay$ has a fundamental system of solutions $(y_1, \ldots, y_n) \in K^n$. We write $\mathcal{Y} = (y_1, \ldots, y_n)$ for the matrix from $M_n(K)$ with columns $y_1, \ldots, y_n$. By definition of "fundamental system of solutions" then $\mathcal{Y} \in \mathrm{GL}_n(K)$ and $\mathcal{Y}' = A \cdot \mathcal{Y}$ in $M_n(K)$, which can be seen as an equation in the differential ring $M_n(K)$, where derivations of matrices are taken entry wise, see 3.1.10. The matrix $\mathcal{Y}$ is called a **fundamental matrix for** $y' = Ay$. Let C be the field of constants of K. Then

(i) If $y \in K^n$ with $y' = Ay$, then $\mathcal{Y}^{-1} \cdot y \in C^n$ is the unique $c \in C^n$ with $y = c_1 y_1 + \ldots + c_n y_n$.
Proof. By 3.3.2, $y_1, \ldots, y_n$ is a C-basis of solutions of $y' = Ay$ in K^n. Hence there is a unique $c \in C^n$ with $y = c_1 y_1 + \ldots + c_n y_n$. Since $y_1, \ldots, y_n$ are the columns of $\mathcal{Y}$ this reads as $y = \mathcal{Y} \cdot c$ $(= \mathcal{Y} \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix})$, i.e. $\mathcal{Y}^{-1} \cdot y = c$. ⋄

(ii) The set of all fundamental matrices of $y' = Ay$ (from $\mathrm{GL}_n(K)$) is the coset $\mathcal{Y} \cdot \mathrm{GL}_n(C)$. Hence if $\mathcal{Z} \in \mathrm{GL}_n(K)$ is another fundamental matrix for $y' = Ay$, then
$$\mathcal{Y}^{-1} \cdot \mathcal{Z} \in \mathrm{GL}_n(C).$$
Proof. If $B \in \mathrm{GL}_n(C)$, then $(\mathcal{Y} \cdot B)' = \mathcal{Y}' \cdot B = A \cdot \mathcal{Y} \cdot B$, hence $\mathcal{Y} \cdot B$ is again a fundamental matrix for $y' = Ay$. For the other inclusion, if $\mathcal{Z} \in \mathrm{GL}_n(K)$ is another fundamental matrix for $y' = Ay$ with columns $z_1, \ldots, z_n$, then
$$\mathcal{Y}^{-1} \cdot \mathcal{Z} = (\mathcal{Y}^{-1} \cdot z_1, \ldots, \mathcal{Y}^{-1} \cdot z_n) \in M_n(C) \text{ by (i)}.$$
Hence $\mathcal{Y}^{-1} \cdot \mathcal{Z} \in \mathrm{GL}_n(C)$ and $\mathcal{Z} = \mathcal{Y} \cdot (\mathcal{Y}^{-1} \cdot \mathcal{Z}) \in \mathcal{Y} \cdot \mathrm{GL}_n(C)$.

⋄

3.4.2. **Proposition.** *Let K be a differential field with constant field C (which is not necessarily algebraically closed) and let L/K be a Picard-Vessiot extension for the matrix ODE $y' = Ay$, $A \in M_n(K)$. Fix a fundamental system of solutions $(y_1, \ldots, y_n) \in L^n$ of $y' = Ay$ and let $\mathcal{Y} = (y_1, \ldots, y_n) \in \mathrm{GL}_n(L)$. By 3.4.1(ii) we may define a map*
$$\Phi_{\mathcal{Y}} : \mathrm{Gal}(L/K) \longrightarrow \mathrm{GL}_n(C);\ \Phi_{\mathcal{Y}}(\sigma) = \mathcal{Y}^{-1} \cdot \sigma(\mathcal{Y}).$$
Then

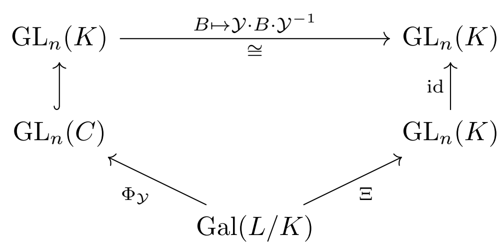
(i) $\Phi_{\mathcal{Y}}$ is an embedding of groups.
(ii) If $\mathcal{Z} \in \mathrm{GL}_n(K)$ is another fundamental matrix for $y' = Ay$, then by 3.4.1(ii) we know that $\mathcal{Z} = \mathcal{Y} \cdot B$ for $B = \mathcal{Y}^{-1} \cdot \mathcal{Z} \in \mathrm{GL}_n(C)$. Then
$$\Phi_{\mathcal{Z}}(\sigma) = \Phi_{\mathcal{Y} \cdot B}(\sigma) = B^{-1} \cdot \Phi_{\mathcal{Y}}(\sigma) \cdot B.$$
(iii) If $\sigma \in \mathrm{Gal}(L/K)$, then $\Phi_{\sigma(\mathcal{Y})} = \Phi_{\mathcal{Y}}$.
(iv) If $\sigma \in \mathrm{Gal}(L/K)$ and $y \in L^n$ with $y' = Ay$, then $\sigma(y) = \mathcal{Y} \cdot \Phi_{\mathcal{Y}}(\sigma) \cdot \mathcal{Y}^{-1} \cdot y$.

(v) The map

$$\Xi_{\mathcal{Y}} : \mathrm{Gal}(L/K) \longrightarrow \mathrm{GL}_n(K);\ \Xi_{\mathcal{Y}}(\sigma) = \sigma(\mathcal{Y}) \cdot \mathcal{Y}^{-1}.$$

is also an embedding of groups and for any other fundamental matrix $\mathcal{Z} \in \mathrm{GL}_n(K)$ *we have* $\Xi_{\mathcal{Y}} = \Xi_{\mathcal{Z}}$. *We may thus just write* Ξ *instead of* $\Xi_{\mathcal{Y}}$. *The image of* Ξ *(which is independent of the chosen* $\mathcal{Y}$ *but does not have values in the constants in general) is therefore called the* **intrinsic Galois group**, *whereas the image of* $\Phi_{\mathcal{Y}}$ *(which is dependent on* $\mathcal{Y}$ *but has values in the constants) is called the* **extrinsic Galois group**.

The automorphism $\mathrm{GL}_n(K) \longrightarrow \mathrm{GL}_n(K)$ *mapping* B *to* $\mathcal{Y} \cdot B \cdot \mathcal{Y}^{-1}$ *makes the diagram*



commutative. Hence $\Phi_{\mathcal{Y}} = \mathcal{Y}^{-1} \cdot \Xi \cdot \mathcal{Y}$.

Proof. (i). We write $\Phi = \Phi_{\mathcal{Y}}$. Let $\sigma, \tau \in \mathrm{Gal}(L/K)$. Then

$$\begin{aligned}
\Phi(\sigma \cdot \tau) &= \mathcal{Y}^{-1} \cdot (\sigma \circ \tau)(\mathcal{Y}) = \mathcal{Y}^{-1} \cdot \sigma(\tau(\mathcal{Y})) \\
&= \mathcal{Y}^{-1} \cdot \sigma(\mathcal{Y} \cdot \Phi(\tau)), \text{ since } \Phi(\tau) = \mathcal{Y}^{-1} \cdot \tau(\mathcal{Y}) \\
&= \mathcal{Y}^{-1} \cdot \sigma(\mathcal{Y}) \cdot \Phi(\tau) \cdot, \text{ because } \sigma|_K = \mathrm{id}_K \text{ and the constants of } L \text{ are in } K \\
&= \Phi(\sigma) \cdot \Phi(\tau).
\end{aligned}$$

Hence Φ is a group homomorphism. If $\Phi(\sigma) = I_n$, then $\mathcal{Y}^{-1} \cdot \sigma(\mathcal{Y}) = \Phi(\sigma) = I_n$, i.e. $\sigma(\mathcal{Y}) = \mathcal{Y}$ and since L is generated by $\mathcal{Y}$ over K we get $\sigma = \mathrm{id}_L$. Hence Φ is an embedding of groups.

(ii). Now take another fundamental matrix $\mathcal{Z} = \mathcal{Y} \cdot B$ for $y' = Ay$, $B \in \mathrm{GL}_n(C)$. For $\sigma \in \mathrm{Gal}(L/K)$ we have

$$\begin{aligned}
\Phi_{\mathcal{Z}}(\sigma) &= \mathcal{Z}^{-1} \cdot \sigma(\mathcal{Z}) = (\mathcal{Y} \cdot B)^{-1} \cdot \sigma(\mathcal{Y} \cdot B) \\
&= B^{-1} \cdot \mathcal{Y}^{-1} \cdot \sigma(\mathcal{Y}) \cdot B, \text{ as } \sigma|_K = \mathrm{id}_K \text{ and the constants of } L \text{ are in } K \\
&= B^{-1} \cdot \Phi_{\mathcal{Y}}(\sigma) \cdot B.
\end{aligned}$$

(iii). Now suppose $\mathcal{Z} = \sigma(\mathcal{Y})$. Then the matrix B in (ii) is

$$B = \mathcal{Y}^{-1} \cdot \mathcal{Z} = \mathcal{Y}^{-1} \cdot \sigma(\mathcal{Y}) = \Phi_{\mathcal{Y}}(\sigma)$$

and so by (ii) we see that $\Phi_{\sigma(\mathcal{Y})} = B^{-1} \cdot \Phi_{\mathcal{Y}}(\sigma) \cdot B = \Phi_{\mathcal{Y}}(\sigma)$.

(iv). We have

$$\begin{aligned}\mathcal{Y}\cdot\Phi(\sigma)\cdot\mathcal{Y}^{-1}\cdot y &= \mathcal{Y}\cdot\mathcal{Y}^{-1}\cdot\sigma(\mathcal{Y})\cdot\mathcal{Y}^{-1}\cdot y\\ &= \sigma(\mathcal{Y})\cdot\mathcal{Y}^{-1}\cdot y\\ &= \sigma(\mathcal{Y}\cdot\mathcal{Y}^{-1}\cdot y),\ \text{since } \mathcal{Y}^{-1}\cdot y\in C^n \text{ by 3.4.1(i), and because}\\ &\qquad\qquad \sigma|_K=\mathrm{id}_K \text{ and the constants of } L \text{ are in } K\\ &= \sigma(y).\end{aligned}$$

(v). We have

$$\begin{aligned}\Xi_{\mathcal{Y}}(\sigma\cdot\tau) &= \sigma(\tau(\mathcal{Y}))\cdot\mathcal{Y}^{-1}\\ &= \sigma(\mathcal{Y}\cdot\mathcal{Y}^{-1}\cdot\tau(\mathcal{Y}))\cdot\mathcal{Y}^{-1}\\ &= \sigma(\mathcal{Y})\cdot\mathcal{Y}^{-1}\cdot\tau(\mathcal{Y})\cdot\mathcal{Y}^{-1} \text{ because } \mathcal{Y}^{-1}\cdot\tau(\mathcal{Y})\in \mathrm{GL}_n(C)\\ &= \Xi_{\mathcal{Y}}(\sigma)\cdot\Xi_{\mathcal{Y}}(\tau).\end{aligned}$$

Further, write $\mathcal{Z}=\mathcal{Y}\cdot B$ with $B\in\mathrm{GL}_n(C)$ (using 3.4.1(ii)). Then

$$\begin{aligned}\Xi_{\mathcal{Z}}(\sigma) = \sigma(\mathcal{Z})\cdot\mathcal{Z}^{-1} &= \sigma(\mathcal{Y}\cdot B)\cdot(\mathcal{Y}\cdot B)^{-1}\\ &= \sigma(\mathcal{Y})\cdot B\cdot B^{-1}\cdot\mathcal{Y}^{-1}\\ &= \sigma(\mathcal{Y})\cdot\mathcal{Y}^{-1} = \Xi_{\mathcal{Y}}(\sigma).\end{aligned}$$

The diagram commutes because

$$\mathcal{Y}\cdot\Phi_{\mathcal{Y}}(\sigma)\cdot\mathcal{Y}^{-1} = \mathcal{Y}\cdot\mathcal{Y}^{-1}\cdot\sigma(\mathcal{Y})\cdot\mathcal{Y}^{-1} = \Xi(\sigma).$$

□

3.4.3. **Proposition.** *Let K be a differential field with algebraically closed constant field C and let $\hat{K}$ be the differential closure of K. Let $S\subseteq\hat{K}^n$ be the set of solutions of the matrix ODE $y'=Ay$, $A\in M_n(K)$, and let $L\subseteq\hat{K}$ be* the *Picard-Vessiot extension of K for $y'=Ay$ (cf. 3.3.5).*

Fix a fundamental matrix $\mathcal{Y}\in\mathrm{GL}_n(L)$ of $y'=Ay$ and let $H_{\mathcal{Y}}$ be the image of the embedding $\Phi:=\Phi_{\mathcal{Y}}:\mathrm{Gal}(L/K)\longrightarrow\mathrm{GL}_n(C)$ from 3.4.2.

Then the map $\Theta_{\mathcal{Y}}:H_{\mathcal{Y}}\times S\longrightarrow S$ defined by $\Theta_{\mathcal{Y}}(B,y)=\mathcal{Y}\cdot B\cdot\mathcal{Y}^{-1}\cdot y$ are K-definable in the differential field $\hat{K}$. (Hence also $H_{\mathcal{Y}}$ and S are K-definable.) The diagram

$$\begin{array}{ccc}\mathrm{Gal}(L/K)\times S & \xrightarrow{(\sigma,y)\mapsto\sigma(y)} & S\\ \Big\downarrow{\scriptstyle \Phi_{\mathcal{Y}}\times\mathrm{id}_S} & & \Big\downarrow{\scriptstyle \mathrm{id}_S}\\ H_{\mathcal{Y}}\times S & \xrightarrow[\Theta_{\mathcal{Y}}]{y\mapsto\mathcal{Y}\cdot B\cdot\mathcal{Y}^{-1}\cdot y} & S\end{array}$$

commutes, i.e. $\Theta_{\mathcal{Y}}(B,y)=\Phi_{\mathcal{Y}}^{-1}(B)(y)$.

We see that the action of $\mathrm{Gal}(L/K)$ on S is isomorphic to the K-definable (in $\hat{K}$) action $\Theta_{\mathcal{Y}}:H_{\mathcal{Y}}\times S\longrightarrow S$.

Proof. We write $\Phi=\Phi_{\mathcal{Y}},H=H_{\mathcal{Y}}$ and $\Theta=\Theta_{\mathcal{Y}}$ and keep in mind the dependency on $\mathcal{Y}$. The diagram commutes by 3.4.2(iv). Since $\hat{K}$ is atomic over K, the type $\mathrm{tp}(\mathcal{Y}/K)$ is isolated by some formula $\varphi(Y)$ (in the language of differential rings,

$Y = (y_{i,j})_{i,j=1}^n$), with parameters from K. We claim that the (graph of the) map $\Theta : H \times S \longrightarrow S$ is defined in $\hat{K}$ by the following formula $\vartheta(X, y, z)$ in the free variables $X = (x_{ij})_{i,j=1}^n$, y and z:

$$X' = 0 \;\&\; \det(X) \neq 0 \;\&\; y' = Ay \;\&\; \exists Y \Big(\varphi(Y) \;\&\; \varphi(Y{\cdot}X) \;\&\; z = Y{\cdot}X{\cdot}Y^{-1}{\cdot}y\Big).$$

(It follows that H is defined in $\hat{K}$ by the formula $X' = 0 \;\&\; \det(X) \neq 0 \;\&\; \exists Y(\varphi(Y) \;\&\; \varphi(Y{\cdot}X))$. Notice that ϑ indeed has parameters from K only.)

To see this, first suppose that $\Theta(B, y) = z$, i.e., $B \in H$, $y \in S$ and $\sigma(y) = z$ for the automorphism $\sigma \in \mathrm{Gal}(L/K)$ with $\Phi(\sigma) = H$, i.e. $\sigma(\mathcal{Y}) = \mathcal{Y}{\cdot}B$. Since $\hat{K} \models \varphi(\mathcal{Y})$ and φ has parameters in K we know $\hat{K} \models \varphi(\mathcal{Y}{\cdot}B)$. Since $z = \mathcal{Y}{\cdot}B{\cdot}\mathcal{Y}^{-1}{\cdot}y$ by definition of Θ, we see that $\hat{K} \models \vartheta(B)$.

Conversely take $B \in \mathrm{GL}_n(C)$, $y \in S$, $z \in \hat{K}^n$ and $\mathcal{Z} \in M_n(\hat{K})$ with $\hat{K} \models \vartheta(B, y, z)$. Then $B' = 0$ and $\det(B) \neq 0$ says $B \in \mathrm{GL}_n(C)$. Furthermore $y' = Ay$ and $\hat{K} \models \varphi(\mathcal{Z}) \;\&\; \varphi(\mathcal{Z}{\cdot}B) \;\&\; z = \mathcal{Z}{\cdot}B{\cdot}\mathcal{Z}^{-1}{\cdot}y$. Since $\hat{K} \models \varphi(\mathcal{Z})$, we know that $\mathrm{tp}(\mathcal{Y}/K) = \mathrm{tp}(\mathcal{Z}/K)$ and by 2.7.10 there is some $\hat{\tau} \in \mathrm{Gal}(\hat{K}/K)$ with $\hat{\tau}(\mathcal{Y}) = \mathcal{Z}$. By 3.3.5 we know that $\hat{\tau}$ restricts to an element $\tau \in \mathrm{Gal}(L/K)$. Then $\mathcal{Z}$ is a fundamental matrix for $y' = Ay$ and by 3.4.2(iii) we know that $\Phi_{\mathcal{Y}} = \Phi_{\mathcal{Z}}$. Similarly, from $\hat{K} \models \varphi(\mathcal{Z}{\cdot}B)$ we get some $\sigma \in \mathrm{Gal}(L/K)$ with $\sigma(\mathcal{Z}) = \mathcal{Z}{\cdot}B$. Consequently $\Phi_{\mathcal{Y}}(\sigma) = \Phi_{\mathcal{Z}}(\sigma) = B$ and it remains to show that $\sigma(y) = z$. Take $c \in C^n$ with $y = \mathcal{Z}{\cdot}c$. Then $\sigma(y) = \sigma(\mathcal{Z}){\cdot}c = \mathcal{Z}{\cdot}B{\cdot}c = \mathcal{Z}{\cdot}B{\cdot}\mathcal{Z}^{-1}{\cdot}y = z$ as required. □

3.4.4. **Proposition.** *The image $H_{\mathcal{Y}}$ of $\Phi_{\mathcal{Y}}$ in 3.4.3 is a Zariski closed subgroup of $\mathrm{GL}_n(C)$, hence it is a* **linear algebraic group**.

Proof. $H_{\mathcal{Y}}$ is of course a group. Let $Y = (y_{ij} \mid i, j \in \{1, \dots, n\})$ be indeterminates. Fix a fundamental matrix $\mathcal{Y} \in \mathrm{GL}_n(L)$ for $y' = Ay$ and let $\varepsilon : K[Y] \longrightarrow L$ be the evaluation map at $\mathcal{Y}$. We furnish $K[Y]$ with the derivation extending the one on K satisfying $y_i' = A{\cdot}y_i$, where $y_i = \begin{pmatrix} y_{i1} \\ \vdots \\ y_{in} \end{pmatrix}$, see 1.1.7. Then ε is a differential K-algebra homomorphism and its kernel $I = \{P(Y) \in K[Y],\ P(\mathcal{Y}) = 0\}$ is a maximal ideal of the ring $K[Y]$ (ε is surjective!) as well as a differential ideal of the differential ring $K[Y]$.

Claim. If $B \in \mathrm{GL}_n(C)$, then $B \in H_{\mathcal{Y}} \iff P(\mathcal{Y}{\cdot}B) = 0$ for all $P \in I$.

Proof. $\Rightarrow$. Since $B \in H_{\mathcal{Y}}$, there is some $\sigma \in \mathrm{Gal}(L/K)$ with $B = \Phi_{\mathcal{Y}}(\sigma)$. If $P \in I$, then $P(\mathcal{Y}{\cdot}B) = P(\mathcal{Y}{\cdot}\Phi_{\mathcal{Y}}(\sigma)) = P(\sigma(\mathcal{Y}))$. Since $\sigma|_K = \mathrm{id}_K$, we get $P(\sigma(\mathcal{Y})) = \sigma(P(\mathcal{Y})) = 0$, as required.

$\Leftarrow$. Let $\Sigma : K[Y] \longrightarrow K[Y]$ be the K-automorphism defined by $\Sigma(P(Y)) = P(Y{\cdot}B)$. Since B has constant entries, Σ is differential and the kernel J of $\varepsilon \circ \Sigma$ is

$$J = \{P(Y) \in K[Y] \mid P(\mathcal{Y}{\cdot}B) = 0\}.$$

By assumption, $I \subseteq J$. Since I is a maximal ideal we get $I = J$ and therefore $\varepsilon \circ \Sigma$ factors through a differential K-automorphism $\sigma : L \longrightarrow L$ so that the diagram

$$\begin{array}{ccc} K[Y] & \xrightarrow{\Sigma} & K[Y] \\ \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\varepsilon} \\ L & \xrightarrow{\sigma} & L \end{array}$$

commutes. We claim that $\Phi_{\mathcal{Y}}(\sigma) = B$ (showing $B \in H_{\mathcal{Y}}$):

$$\begin{aligned} \Phi_{\mathcal{Y}}(\sigma) &= \mathcal{Y}^{-1}\sigma(\mathcal{Y}) = \varepsilon(Y)^{-1}\cdot\sigma(\varepsilon(Y)) \\ &= \varepsilon(Y)^{-1}\cdot\varepsilon(\Sigma(Y)) \text{ by commutativity of the diagram} \\ &= \varepsilon(Y^{-1}\cdot\Sigma(Y)) = \varepsilon(Y^{-1}\cdot Y\cdot B) \\ &= B \text{ as } \varepsilon|_K = \mathrm{id}_K\,. \end{aligned}$$

◇

The claim says precisely that $H_{\mathcal{Y}}$ is the intersection of $\mathrm{GL}_n(C)$ with the Zariski closed subset of L^{n^2} defined by $\{P(\mathcal{Y}\cdot X) \in L[X] \mid P(Y) \in K[Y],\ P(\mathcal{Y}) = 0\}$, $X = (x_{ij})_{i,j=1}^n$. By 2.2.6(ii), the set $H_{\mathcal{Y}}$ is a Zariski closed subset of $\mathrm{GL}_n(C)$. □

3.4.5**. Fundamental Theorem of Differential Galois Theory for Picard-Vessiot extensions** *Let L/K be a Picard-Vessiot extension of differential fields for the matrix ODE $y' = Ay$, $A \in M_n(K)$. Suppose C has algebraically closed constant field. Let $\mathcal{Y} \in \mathrm{GL}_n(L)$ be a fundamental matrix for $y' = Ay$. Then the map*

(i)

$$\mathrm{Gal}(L/\_\_) : \left\{ L_1 \;\middle|\; \begin{array}{c} L_1 \textit{ differential field} \\ \textit{with } K \subseteq L_1 \subseteq L \end{array} \right\} \longrightarrow \left\{ G \;\middle|\; \begin{array}{c} G \subseteq \mathrm{Gal}(L/K) \\ \textit{linear algebraic group} \end{array} \right\}$$
$$L_1 \longmapsto \mathrm{Gal}(L/L_1)$$

is an inclusion reversing bijection. Its inverse sends G to the fixed field

$$L^G = \{a \in L \mid \sigma(a) = a \textit{ for all } \sigma \in G\}.$$

(ii) Under the bijection in (i) the differential fields L_1 that are normal (i.e. all $\sigma \in \mathrm{Gal}(L/K)$ restrict to automorphisms $L_1 \longrightarrow L_1$), are in bijection with the normal Zariski closed subgroups of $\mathrm{Gal}(L/K)$. Such a field L_1 is again a Picard-Vessiot extension of some matrix ODE of K.

Proof. [vdPSin2003, Proposition 1.34], [CreHaj2011, Proposition 6.3.1 and 6.3.2], [Magid1994, Theorem 6.5] □

3.4.6**. Classical Galois extensions** If K is a differential field with algebraically closed constant field, then every finite Galois extension L/K (in the classical sense) is a Picard-Vessiot extension, see [vdPSin2003, Exercises 1.24], or [CreHaj2011, Example 6.1.6].

Proof. Firstly, since L/K is algebraic and the constant field C of K is algebraically closed, 2.1.5(ii)(b) implies that L also has constant field C.

As we are in characteristic 0 the primitive element theorem tells us that $L = K[\alpha]$ for some $\alpha \in L$. Let d be the degree of L/K, i.e. the K-vector space dimension of L.

Then $\alpha, \alpha', \ldots, \alpha^{(d)}$ are linearly dependent over K and there is a maximal $k \leq d$ such that $\alpha, \alpha', \ldots, \alpha^{(k-1)}$ are linearly independent over K. Then there are $a_0, \ldots, a_{k-1} \in K$ such that α solves

$$y^{(k)} + a_{k-1}y^{(k-1)} + \ldots + a_1 y' + a_0 y = 0.$$

By 3.3.2(ii), the C-vector space $V \subseteq L$ of solutions of this equation is finite dimensional. As all field automorphisms of L/K preserve the derivation (by formula $(*)$ in 2.1.5), we see that conditions (a) and (b) of 3.3.8 are satisfied for L/K (take $G = \mathrm{Gal}(L/K)$), which implies the claim. □

3.4.7. *Examples.* [CreHaj2011, Examples 6.1.4 and 6.1.5] Let K be a differential field with (not necessarily algebraically closed) constant field C.

(i) (**Adjunction of an integral**).
Let $L = K\langle\alpha\rangle$, $\alpha' = a \in K$, $\alpha \notin K$.
If a has an integral $b \in K$, then $\frac{b}{\alpha}$ is a new constant. Now assume a has no integral in K. Then
- (a) α is transcendental over K
- (b) L/K is Picard-Vessiot for the equation $y'' - \frac{a'}{a}y = 0$
- (c) A fundamental system of solution for $y'' - \frac{a'}{a}y = 0$ is $\{1, \alpha\}$.
- (d) An element $\sigma \in \mathrm{Gal}(L/K)$ maps α to some $\alpha + c$, $c \in C$ and

$$\mathrm{Gal}(L/K) \cong \{\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \mid c \in C\} \cong (C, +).$$

(ii) (**Adjunction of an exponential of an integral**).
Let $L = K\langle\alpha\rangle$ with $\frac{\alpha'}{\alpha} = a \in K^{\times}$. Assume L has no new constants (e.g. $\alpha \in \hat{K}$). Then
- (a) α is a fundamental system of the linear ODE $y' - ay = 0$ and L/K is Picard-Vessiot for this linear ODE.
- (b) If α is algebraic over K, then $\alpha^n \in K$ for some K and $\mathrm{Gal}(L/K)$ is a finite cyclic group.
- (c) If α is transcendental over K, then $\mathrm{Gal}(L/K) \cong (C^{\times}, \cdot)$.

3.5. **Kolchin's strongly normal extensions.**

This is essentially an annotated write up of parts of [Poizat2000, Section 18.3], where definitions and statements are compared with the corresponding ones in Picard-Vessiot theory.

3.5.1. **Definition.** Let K be a differential field with algebraically closed constant field. A **Kolchin formula** of K is a formula $\varphi(x)$ in the language of differential rings in one free variable x, with parameters from K, that has the following property:

> There is a differentially closed field $M \supseteq K$ such that for every differentially closed field $L \supseteq M$ and each $\alpha \in L$ with $L \models \varphi(\alpha)$, the differential field $M\langle\alpha\rangle$ generated by α over M in L, is generated over M by constants of $M\langle\alpha\rangle$.

`In the case of linear ODEs:` Let $\varphi(x)$ be a homogeneous linear ODE of the form

$$(\mathcal{L}) \qquad x^{(n)} + a_{n-1}x^{(n-1)} + \ldots + a_1 x' + a_0 x = 0,$$

with $a_i \in K$. This formula is Kolchin, because by 3.3.2(iii), every differentially closed field contains a fundamental system of solutions of the matrix ODE $y' = A_{\mathcal{L}} y$ for

$$A_{\mathcal{L}} = \begin{pmatrix} 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ -a_0 & -a_1 & \ldots & -a_{n-2} & -a_{n-1} \end{pmatrix}$$

and by 3.3.2, for any differential field L containing a fundamental system, solutions of $y' = A_{\mathcal{L}} y$ are generated by this system over the constants. (Recall that solutions $y \in L^n$ of $y' = A_{\mathcal{L}} y$ are isomorphic to solutions of $\mathcal{L}$, i.e. realizations of $\varphi(x)$, by projecting onto the first coordinate of y.) An example of a Kolchin formula that is not linear is $x'^2 = x^3 + px + q$, where p and q are constants of the Weierstrass $\wp$-function.

3.5.2. **Lemma.** *Let $\varphi(x)$ be a Kolchin formula over K. Then there are 0-definable (for DCF) functions $f_i(\bar{y}, \bar{z})$, $i = \{1, \ldots, k\}$ such that for any differentially closed field M containing K there is a $\bar{y}$-tuple $\bar{a} \in M^{\bar{y}}$ with the following property:*

(†) *for all $\alpha \in N \succ M$ with $N \models \varphi(\alpha)$ there is a $\bar{z}$-tuple of constants of N with*

$$\alpha = f_1(\bar{a}, \bar{c}) \vee \ldots \vee \alpha = f_k(\bar{a}, \bar{c}).$$

A tuple $\bar{a}$ from a DCF $M \supseteq K$ is called a **fundamental system of parameters for** *$\varphi(x)$ if there are finitely many 0-definable functions f_i as above such that (†) holds. If this is the case, then every realization of the formula*

$$\forall x \Big(\varphi(x) \leftrightarrow \big(\bigvee_{i=1}^{k} \exists \bar{z}\ (d(\bar{z}) = 0\ \&\ x = f_{p_i}(\bar{y}, \bar{z})) \big) \Big).$$

(in the free variables $\bar{y}$) is a fundamental solutions of $\varphi(x)$ such that (†) holds for $f_1, \ldots, f_k$. ***`In the case of linear ODEs:`*** *$\bar{a}$ will be the first row of a fundamental matrix for $A_{\mathcal{L}}$, hence $\bar{a}$ is an n-tuple of solutions of $\mathscr{L}$ and we only need*

one function, namely

$$f(\bar{y},\bar{z}) = (y_1,\ldots,y_n)\cdot\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}.$$

Proof. Let $\tilde{M} \supseteq K$ be a DCF such that for every differentially closed field $L \supseteq \tilde{M}$ and each $\alpha \in L$ with $L \models \varphi(\alpha)$, the differential field $\tilde{M}\langle\alpha\rangle$ generated by α over $\tilde{M}$ in L, is generated over $\tilde{M}$ by constants of $\tilde{M}\langle\alpha\rangle$. Hence if $p \in S_1(\tilde{M})$ contains φ and α is a realization of p from some $L \succ M$. Then there is a 0-definable function $f_p(\bar{y},\bar{z})$ and some $\bar{a}_p \in \tilde{M}^{\bar{y}}$ such that p contains the $\tilde{M}$-formula $\varphi_p(x)$ defined as

$$\exists \bar{z}\ (d(\bar{z}) = 0\ \&\ x = f_p(\bar{a}_p,\bar{z})).$$

By compactness of $S_1(\tilde{M})$ there are $p_1,\ldots,p_k \in S_1(\tilde{M})$ containing $\varphi(x)$ such that

$$\tilde{M} \models \varphi(x) \leftrightarrow (\varphi_{p_1}(x) \vee \ldots \vee \varphi_{p_k}(x)).$$

Consequently

$$\tilde{M} \models \exists \bar{y}_1,\ldots,\bar{y}_k \forall x\Big(\varphi(x) \leftrightarrow (\bigvee_{i=1}^{k} \exists \bar{z}_i\ (d(\bar{z}_i) = 0\ \&\ x = f_{p_i}(\bar{y}_i,\bar{z}_i)))\Big).$$

By writing $\bar{y} = (\bar{y}_1,\ldots,\bar{y}_k)$, $\bar{z} = (\bar{z}_1,\ldots,\bar{z}_k)$ and modifying the f_{p_i} accordingly, we get

$$\tilde{M} \models \exists \bar{y} \forall x\Big(\varphi(x) \leftrightarrow (\bigvee_{i=1}^{k} \exists \bar{z}\ (d(\bar{z}) = 0\ \&\ x = f_{p_i}(\bar{y},\bar{z})))\Big).$$

This sentence has only parameters in K (only occurring in φ) and so by quantifier elimination for DCF, this sentence is true in every DCF containing K. This is what the lemma asserts. □

To proceed we need one model theoretic input:

3.5.3. **Definable sets in stable theories are large.** *Let M be a model of a stable theory and let $D \subseteq M^n$ be a set that is definable with parameters from M. Let A be the set of all entries of tuples from D. Then the natural restriction $S_n(M) \longrightarrow S_n(A)$ is injective on the set $\langle D\rangle$ of n-types of M containing (the formula defining) D.*

Proof. [Poizat2000, Theorem 12.30]. □

3.5.4. **Proposition.** *If $\varphi(x)$ is a Kolchin formula of the differential field K, then in any differentially closed field M containing K there is a fundamental system of parameters for $\varphi(x)$ in M consisting of realizations of $\varphi(x)$ in M.*

Proof. We can repeat the compactness argument of the proof of 3.5.2, once we have shown the following claim: If $N \succ M$ and $\alpha \in N$ with $N \models \varphi(\alpha)$, then there is some $\bar{a} \subseteq M$ whose coordinates are realizations of $\varphi(x)$ such that $\alpha \in \mathbb{Q}\langle \bar{a} \cup C_N \rangle$:

Let S be the set defined by φ in M and let $L = \mathbb{Q}\langle S \cup C_M \rangle$. Since $\varphi(x)$ is Kolchin, there is some tuple $\bar{c} \subseteq C_N$ with $\alpha \in M\langle \bar{c} \rangle$. Now, $\alpha \in S_N$ and the tuple $\bar{c}$ is in $C_N^{length(\bar{c})}$.

$$
\begin{array}{ccccc}
C_N & \hookrightarrow & N & & \\
\uparrow & & \uparrow & & \\
 & & M\langle \alpha \rangle & & \alpha = f(\bar{b}, \bar{c}),\ \bar{c} \in C_N \\
 & & \uparrow & & \\
C_M & \hookrightarrow & M & & \supseteq S \ni \bar{b} \\
 & & \uparrow & & \\
 & & L = \mathbb{Q}\langle S \cup C_M \rangle & &
\end{array}
$$

Since L contains S and C_M we may apply 3.5.3 and get that $\mathrm{tp}(\alpha, \bar{c}/L) \vdash \mathrm{tp}(\alpha, \bar{c}/M)$. Consequently $\mathrm{tp}(\alpha/L \cup \{\bar{c}\}) \vdash \mathrm{tp}(\alpha/M \cup \{\bar{c}\})$. Since α is in the definable closure of $M \cup \{\bar{c}\}$, it must also be in the definable closure of $L \cup \{\bar{c}\}$. But then there is some $\bar{a} \subseteq M$ whose coordinates are realizations of $\varphi(x)$ such that $\alpha \in \mathbb{Q}\langle \bar{a} \cup C_N \rangle$. □

3.5.5. **Corollary and Definition.** *If $\varphi(x)$ is a Kolchin formula over K and K has algebraically closed constants, then the* **Kolchin extension** *of K for $\varphi(x)$ is the differential field generated by K and the set defined by $\varphi(x)$ in the differential closure of $\hat{K}$. Kolchin calls these extensions* **strongly normal**. *Then*

- *(i) L is differentially finitely generated over K by any fundamental system of parameters of $\varphi(x)$ whose entries are realizations for φ.*
- *(ii) Every $\sigma \in \mathrm{Gal}(\hat{K}/K)$ restricts to an element of $\mathrm{Gal}(L/K)$ and the restriction map is surjective.*

Proof. (i) follows from 3.5.4.

(ii). It is clear that every $\sigma \in \mathrm{Gal}(\hat{K}/K)$ restricts to an automorphism of L. For the converse we use (i) and 2.7.10. □

3.5.6**. The Galois group as a definable group.** Let L/K be Kolchin for $\varphi(x)$ and fix a fundamental solution $\bar{a}$ for φ in L whose entries are realizations of $\varphi(x)$. Let $\psi(\bar{x})$ be a formula (with parameters from K) isolating $\mathrm{tp}(\bar{a}/K)$.

(i) The map $\Phi = \Phi_{\bar{a}} : \mathrm{Gal}(L/K) \longrightarrow \psi[\hat{K}]$ (realizations of ψ in $\hat{K}$) that sends σ to $\sigma(\bar{a})$ is bijective, because the action of σ is uniquely determined by its action on the generator $\bar{a}$ of L (note that K has alg. closed constant field by assumption). For surjectivity use the property of $\mathrm{tp}(\bar{a}/K)$ being isolated by ψ and apply 2.7.10 again.

(ii) We claim that the group law of $\mathrm{Gal}(L/K)$ is K-definable in $\hat{K}$ when passed through Φ.
Proof. Choose 0-definable functions $f_i(\bar{y}, \bar{z})$, $i = \{1, \ldots, k\}$ for $\bar{a}$ as in 3.5.2. Then for $\sigma_1, \sigma_2, \sigma_3 \in \mathrm{Gal}(L/K)$ we have

$$\sigma_2 \circ \sigma_1 = \sigma_3 \iff \forall \bar{c}\Big(d(\bar{c}) = 0 \to \bigwedge_{i=1}^{k} (\sigma_1(\bar{a}) = f_i(\bar{a}, \bar{c}) \to \sigma_3(\bar{a}) = f_i(\sigma_2(\bar{a}), \bar{c}))\Big).$$

Proof. $\Rightarrow$: if $\sigma_2 \circ \sigma_1 = \sigma_3$ and $\sigma_1(\bar{a}) = f_i(\bar{a}, \bar{c})$, then applying σ_2 gives $\sigma_3(\bar{a}) = f_i(\sigma_2(\bar{a}), \bar{c})$.
$\Leftarrow$: We know that for some i we have $\sigma_1(\bar{a}) = f_i(\bar{a}, \bar{c})$ and so $\sigma_3(\bar{a}) = f_i(\sigma_2(\bar{a}), \bar{c})$ by assumption. However we also have $\sigma_2(\sigma_1(\bar{a})) = f_i(\sigma_2(\bar{a}), \bar{c})$ and so $\sigma_3 = \sigma_2 \circ \sigma_1$. ◇
Hence the multiplication $\bar{y}_2 {\cdot} \bar{y}_1 = \bar{y}_3$ defined by

$$\forall \bar{c}\Big(d(\bar{c}) = 0 \to \bigwedge_{i=1}^{k} (\bar{y}_1 = f_i(\bar{a}, \bar{c}) \to \bar{y}_3 = f_i(\bar{y}_2, \bar{c})).$$

defines a group on $\psi(\hat{K})$ and Φ is an isomorphism of groups. ◇

(iii) The action of $\psi(\hat{K})$ on L is given as follows: Pick $\alpha \in L$, i.e. $\alpha = g(\bar{a})$ for some K-definable function $g(\bar{v})$, $\bar{v}$ an $\bar{a}$-tuple of variables. Then for $\sigma \in \mathrm{Gal}(L/K)$ and $\bar{y} = \sigma(\bar{a}) \in \psi(\hat{K})$ we have

$$\sigma(\alpha) = \sigma(g(\bar{a})) = g(\sigma(\bar{a})) = g(\bar{y}).$$

3.5.7. **Theorem.** [Poizat2000, Proposition 18.15] *If L/K is the Kolchin extension of the Kolchin formula $\varphi(x)$ and the constant field of K is algebraically closed, then for any fundamental system of parameters $\bar{a}$ for $\varphi(x)$ from the differential closure $\hat{K}$ of K, the map $\Phi = \Phi_{\bar{a}}$ from 3.5.6(i) induces a Galois correspondence*

$$\Phi\,\mathrm{Gal}(L/\_\_) : \left\{ L_1 \;\middle|\; \begin{array}{c} L_1 \textit{ differential field} \\ \textit{with } K \subseteq L_1 \subseteq L \end{array} \right\} \longrightarrow \left\{ G \;\middle|\; \begin{array}{c} G \subseteq \psi(\hat{K}) \textit{ subgroup,} \\ L\textit{-definable in } \hat{K} \end{array} \right\}$$

$$L_1 \longmapsto \Phi(\mathrm{Gal}(L/L_1)).$$

Hence $\Phi\,\mathrm{Gal}(L/\_\_)$ is an inclusion reversing bijection. Its inverse Fix *sends G to the fixed field*

$$\mathrm{Fix}(G) = \{a \in L \mid \sigma(a) = a \textit{ for all } \sigma \in \Phi^{-1}(G)\}.$$

Proof. If $\alpha \in L$, then $\mathrm{Gal}(L/K\langle\alpha\rangle)$ is the stabilizer of α in $\mathrm{Gal}(L/K)$. By 3.5.6(iii), the image $\Phi\,\mathrm{Gal}(L/K\langle\alpha\rangle)$ is the set of all $\bar{y} \in \psi(\hat{K})$ with $g(\bar{y}) = g(\bar{a})$. Hence $\Phi\,\mathrm{Gal}(L/K\langle\alpha\rangle)$ is defined by this formula.

Now if $K \subseteq L_1 \subseteq L$, then $\Phi\,\mathrm{Gal}(L/L_1) = \bigcap_{\alpha\in L_1} \Phi\,\mathrm{Gal}(L/K\langle\alpha\rangle)$. The latter group is a finite intersection, since DCF is totally transcendental, using the following

Claim. In a totally transcendental theory there is no strictly decreasing chain of definable subgroups $G_1 \supseteq G_2 \supseteq G_3 \supseteq \dots$ in any model.

Proof. Suppose there is such a chain. By passing to a subsequence we may assume that G_{i+1} has infinite index in G_i for all i. By definition of the Morley rank it follows by transfinite induction on ordinals α that $\mathrm{MR}(G_1) \geq \alpha$. But then $\mathrm{MR}(G_1) = \infty$ contradicting the assumption of total transcendence. $\diamond$

The claim implies that there are $\alpha_1, \dots, \alpha_k \in L$ with $\mathrm{Gal}(L/L_1) = \mathrm{Gal}(L/K\langle\alpha_1, \dots, \alpha_k\rangle)$. By 2.7.10 (and 3.5.6) we know that $K\langle\alpha_1, \dots, \alpha_k\rangle$ is the fixed field of $\mathrm{Gal}(L/K\langle\alpha_1, \dots, \alpha_k\rangle)$. Since L_1 is fixed by all automorphisms from $\mathrm{Gal}(L/K\langle\alpha_1, \dots, \alpha_k\rangle)$, we get that $L_1 \subseteq K\langle\alpha_1, \dots, \alpha_k\rangle$. But $\alpha_i \in L_1$ and so we see that that $L_1 = K\langle\alpha_1, \dots, \alpha_k\rangle$ is finitely generated and $\Phi(\mathrm{Gal}(L/L_1))$ is a definable subgroup of $\psi(\hat{K})$.

This shows that the map $\Phi\,\mathrm{Gal}(L/\_\_)$ is well defined and that the composition $\mathrm{Fix} \circ \Phi\,\mathrm{Gal}(L/\_\_)$ is the identity on the set of differential fields between K and L.

Conversely, let $G \subseteq \psi(\hat{K})$ be a subgroup that is definable in $\hat{K}$ by a formula $\gamma(\bar{v}, \bar{l})$, $\bar{l} \subseteq L$.

Claim. $\Phi^{-1}(G) = \{\sigma \in \mathrm{Gal}(L/K) \mid \sigma \text{ preserves } G \text{ setwise.}\}$

Proof. If σ preserves G, then from $\gamma(\bar{a}, \bar{l})$ (corresponding to $\mathrm{id} \in G$) we know $\gamma(\sigma(\bar{a}), \bar{l})$. But this means that $\sigma \in \Phi^{-1}(G)$, because $G = \{\tau(\bar{a}) \mid \tau \in \Phi^{-1}(G)\}$.

Conversely, if $\sigma \in \Phi^{-1}(G)$ and $\bar{z} = \tau(\bar{a}) \in G$ with $\tau \in \Phi^{-1}(G)$, then as $\sigma \circ \tau \in \Phi^{-1}(G)$ we have $\sigma\tau(\bar{a}) \in G$, meaning that $\models \gamma(\sigma(\bar{z}), \bar{l})$. $\diamond$

Now we invoke elimination of imaginaries of DCF, see 2.3.7. By 2.3.3 applied to the 0-definable equivalence relation $\varepsilon(\bar{u}, \bar{w})$ defined as $\forall\bar{v}(\gamma(\bar{v}, \bar{u}) \leftrightarrow \gamma(\bar{v}, \bar{w}))$, there is a finite set $B \subseteq \hat{K}$, such that the automorphisms of $\hat{K}$ that preserve $\gamma(\hat{K}, \bar{l})$ setwise (these are the same as the automorphisms preserving the ε-equivalence class of $\bar{l}$ setwise) are precisely the automorphisms that preserve B pointwise. Since $\gamma(\hat{K}, \bar{l})$ is defined over L it follows that $\mathrm{Gal}(\hat{K}/L) \subseteq \mathrm{Gal}(\hat{K}/\mathbb{Q}\langle B\rangle)$ and so by 2.7.10 we see that that $B \subseteq L$.

But now the the claim (together with 3.5.5(ii)) implies that $\Phi^{-1}(G)$ is the group of automorphisms of L that fix $L_1 = K\langle B\rangle$ pointwise, in other words $\mathrm{Fix}(G) = L_1$. This show that $\Phi\,\mathrm{Gal}(L/\_\_)\circ\mathrm{Fix}$ is the identity on the set of L-definable subgroups of $\Phi\,\mathrm{Gal}(L/K)$. $\square$

Further generalizations of the fundamental theorem of Galois theory, beyond (expansions of) fields, may be found in [Poizat1983].

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