INTRODUCTION TO MODEL THEORY

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Abstract. These notes are based on my model theory course taught in semester 1 of 2010/2011. There are several additions and improvements compared to the original version.

URL of these notes::
http://personalpages.manchester.ac.uk/staff/Marcus.Tressl/papers/ModelTheory.pdf

Date: December 6, 2024.

Contents

1. <u>Revision of Predicate Logic</u>

1.1. **Languages and formulas.**

In this section we shall define what is a first order language, usually denoted by $\mathscr{L}$. $\mathscr{L}$ will consist of an <u>alphabet</u> and a set of finite sequences (strings) of elements of that alphabet, built according to certain rules; these string will be called <u>formulas</u>.

The alphabet of a language

1.1. **Definition.** (Alphabet)
The **alphabet** of a language $\mathscr{L}$ consists of the following data:

(I) A set of **logical symbols**, which are present in every language:
- $\neg$ ('not'), $\rightarrow$ ('implies'), $\forall$ ('for all')
- The equality symbol: $\doteq$
- Brackets:) (
- Comma: ,
- Symbols to denote **variables**: $v_0, v_1, v_2, ...$ Notice that each v_i is considered as a single symbol (and not as a concatenation of two symbols).

(II)
- Three mutually disjoint sets $\mathscr{R}$ (called the set of **relation symbols** or **predicate symbols**), $\mathscr{F}$ (called the set of **function symbols**) and $\mathscr{C}$ (called the set of **constant symbols**). Further, none of these sets contains a logical symbol.
- Maps

  $\lambda : \mathscr{R} \longrightarrow \mathbb{N}$ called the "**arity of relation symbols**"

  $\mu : \mathscr{F} \longrightarrow \mathbb{N}$ called the "**arity of function symbols**"

For $R \in \mathscr{R}$ and $F \in \mathscr{F}$, the numbers $\lambda(R)$ and $\mu(F)$ are called the **arity** of R, F respectively. We say that R, F is n**-ary**, if $\lambda(R) = n$, $\mu(F) = n$, respectively.

Every logical symbol and every element from $\mathscr{R} \cup \mathscr{F} \cup \mathscr{C}$ is called an $\mathscr{L}$-symbol or simply a **symbol** whenever $\mathscr{L}$ is clear from the context. We shall also use the term **($\mathscr{L}$-)letter** instead of ($\mathscr{L}$)-symbol.

We define the **set of variables** as

$$\mathrm{Vbl} := \{v_n \mid n \in \mathbb{N}_0\}.$$

The alphabet of a language $\mathscr{L}$ is called **finite** if $\mathscr{R}$, $\mathscr{F}$ and $\mathscr{C}$ are finite. Otherwise the alphabet of $\mathscr{L}$ is called **infinite**

The alphabet of a language $\mathscr{L}$ is called **countable** if $\mathscr{R}$, $\mathscr{F}$ and $\mathscr{C}$ are countable or finite. Otherwise the alphabet of $\mathscr{L}$ is called **uncountable**. .

In general, the **cardinality of an alphabet of a language** $\mathscr{L}$ is the cardinality of $\mathscr{R} \cup \mathscr{F} \cup \mathscr{C}$.

Notation. Obviously, the alphabet of a language is uniquely determined by the data in item II of definition 1.1. These data are called the **similarity type** of $\mathscr{L}$. Hence the similarity type of $\mathscr{L}$ is given by

$$(\lambda : \mathscr{R} \longrightarrow \mathbb{N}, \mu : \mathscr{F} \longrightarrow \mathbb{N}, \mathscr{C})$$

1.2. *Examples.*

(i) The empty similarity type. Here $\mathscr{R} = \mathscr{F} = \mathscr{C} = \emptyset$.
(ii) The similarity type of a composition (or of an operation): $(\emptyset, \mu : \{\circ\} \longrightarrow \{2\}, \emptyset)$. This means: $\mathscr{R} = \mathscr{C} = \emptyset$ and $\mathscr{F}$ consist of a single element $\circ$ of arity 2: $\mu(\circ) = 2$.
(iii) The similarity type of groups: $(\emptyset, \mu : \{\circ, ^{-1}\} \longrightarrow \mathbb{N}, \{e\})$ where $\mu(\circ) = 2$ and $\mu(^{-1}) = 1$; hence $\circ$ is a binary function symbol (i.e. of arity 2), $^{-1}$ is a function symbol of arity 1 and e is a constant symbol.
(iv) The similarity type of unital rings: $(\emptyset, \mu : \{+, -, \cdot\} \longrightarrow \mathbb{N}, \{0, 1\})$, where $\mu(+) = \mu(\cdot) = 2$ and $\mu(-) = 1$. Hence $-$ is a unary (i.e. 1-ary) and $+, \cdot$ are binary function symbols. 0 and 1 are constants.
(v) The similarity type of set theory: $(\lambda : \{\in\} \longrightarrow \{2\}, \emptyset, \emptyset)$. Here $\in$ is a binary predicate symbol. Sometimes this similarity type also contains a constant symbol (denoting the empty set).
(vi) The similarity type of partially ordered sets: $(\lambda : \{\leq\} \longrightarrow \{2\}, \emptyset, \emptyset)$. Here $\leq$ is a binary predicate symbol.
(vii) The similarity type of ordered groups: $(\lambda : \{\leq\} \longrightarrow \{2\}, \mu : \{\circ, ^{-1}\} \longrightarrow \mathbb{N}, \{e\})$. Here $\leq$ is a binary relation symbol.

Terms

1.3. **Definition.** ($\mathscr{L}$-term)
Given the similarity type $(\lambda : \mathscr{R} \longrightarrow \mathbb{N}, \mu : \mathscr{F} \longrightarrow \mathbb{N}, \mathscr{C})$ of $\mathscr{L}$, we define subsets $\mathrm{tm}_k \mathscr{L}$ of strings (i.e. of finite sequences) of the alphabet of $\mathscr{L}$ by induction on $k \in \mathbb{N}_0$ as follows:

$$\begin{aligned} \mathrm{tm}_0 \mathscr{L} &= \mathrm{Vbl} \cup \mathscr{C} \text{ and} \\ \mathrm{tm}_{k+1} \mathscr{L} &= \mathrm{tm}_k \mathscr{L} \cup \{F(t_1, t_2, ..., t_n) \mid n \in \mathbb{N}, F \in \mathscr{F}, \mu(F) = n,\ t_1, ..., t_n \in \mathrm{tm}_k \mathscr{L}\}. \end{aligned}$$

The set of **$\mathscr{L}$-terms** is defined as

$$\mathrm{tm}\, \mathscr{L} := \bigcup_{k \in \mathbb{N}_0} \mathrm{tm}_k \mathscr{L}.$$

The elements of $\mathrm{tm}\, \mathscr{L}$ are called **$\mathscr{L}$-terms** or simply 'terms' if $\mathscr{L}$ is clear from the context.

The **complexity of an $\mathscr{L}$-term** t - denoted by $c(t)$ - is the least $k \in \mathbb{N}_0$ such that $t \in \mathrm{tm}_k \mathscr{L}$. Notice that for $t \in \mathrm{tm}\, \mathscr{L}$ and $k \in \mathbb{N}_0$ we have by definition $c(t) \leq k \iff t \in \mathrm{tm}_k \mathscr{L}$.

1.4. **Theorem.** *(Unique readability theorem for terms) If t is an $\mathscr{L}$-term, then either t is a variable or t is a constant symbol or there are uniquely determined $n \in \mathbb{N}$, $F \in \mathscr{F}$ of arity n and $t_1, ..., t_n \in \mathrm{tm}\, \mathscr{L}$ such that*

$$t = F(t_1, t_2, ..., t_n).$$

1.5. **Corollary.** *For all $n \in \mathbb{N}$, all $\mathscr{L}$-terms $t_1, ..., t_n$ and each n-ary function symbol F of $\mathscr{L}$ we have*

$$c(F(t_1, ..., t_n)) = 1 + \max\{c(t_1), ..., c(t_n)\}.$$

Formulas

1.6. **Definition.** (formulas)
Given a similarity type $(\lambda : \mathscr{R} \longrightarrow \mathbb{N}, \mu : \mathscr{F} \longrightarrow \mathbb{N}, \mathscr{C})$ of a language $\mathscr{L}$, an **atomic $\mathscr{L}$-formula** is a string of the alphabet of $\mathscr{L}$ of the form

$$t_1 \doteq t_2,$$

where t_1, t_2 are $\mathscr{L}$-terms or

$$R(t_1, ..., t_n),$$

where R is a relation symbol of arity $n \in \mathbb{N}$ and $t_1, ..., t_n$ are $\mathscr{L}$-terms. The set of atomic $\mathscr{L}$-formulas is denoted by at-Fml$(\mathscr{L})$.

We define

$$\begin{aligned} \mathrm{Fml}_0(\mathscr{L}) &= \text{at-Fml}(\mathscr{L}) \text{ and inductively for each } k \in \mathbb{N}_0 : \\ \mathrm{Fml}_{k+1}(\mathscr{L}) &= \mathrm{Fml}_k(\mathscr{L}) \cup \{(\neg\varphi), (\varphi \to \psi), (\forall x\varphi) \mid \varphi, \psi \in \mathrm{Fml}_k \mathscr{L},\ x \in \mathrm{Vbl}\}. \end{aligned}$$

The set of **$\mathscr{L}$-formulas** is defined as

$$\mathrm{Fml}(\mathscr{L}) := \bigcup_{k \in \mathbb{N}_0} \mathrm{Fml}_k(\mathscr{L}).$$

If the letter $\forall$ does not occur in the $\mathscr{L}$-formula φ, then φ is called **quantifier-free**.

Warning. Not every formula that has (obvious) meaning in mathematics is a formula in our sense. This is in particular important after we have proved significant theorems involving formulas. Here an example:

$$\forall n \in \mathbb{N}\ \exists r, q \in \mathbb{N}_0\ n = q \cdot m + r \wedge r < m.$$

There is no language (according to our definition) such that the above is a formula in that language.

Notice that the quantifier introduced in the definition of $\mathrm{Fml}_{k+1} \mathscr{L}$ (cf. 1.6) is always applied in a nonrestricted way, e.g.

$$\forall n \exists r, q\ n \doteq q \cdot m + r \wedge r < m$$

will be a formula in the language of rings after we have introduced the appropriate abbreviations (concerning the symbols $\exists$ and $\wedge$)in 1.9.

The **language** or **signature** $\mathscr{L}$ is the triple consisting of the alphabet of $\mathscr{L}$, the set of $\mathscr{L}$-terms and the set of $\mathscr{L}$-formulas. Obviously, tm $\mathscr{L}$ and Fml $\mathscr{L}$ are uniquely determined by the similarity type of $\mathscr{L}$ and we shall simply communicate languages by their similarity type.

Hence the expression 'let $\mathscr{L} = (\lambda : \mathscr{R} \longrightarrow \mathbb{N}, \mu : \mathscr{F} \longrightarrow \mathbb{N}, \mathscr{C})$ be a language' stands for 'let $\mathscr{L}$ be the language with similarity type $(\lambda : \mathscr{R} \longrightarrow \mathbb{N}, \mu : \mathscr{F} \longrightarrow \mathbb{N}, \mathscr{C})$'.

We say that a language is **finite**, **infinite**, **countable** or **uncountable** if the alphabet of that language has this property. In general, the cardinality of a language $\mathscr{L}$, denoted by card$(\mathscr{L})$, is the cardinality of the alphabet of that language.

1.7. **Lemma.** *The cardinality of* Fml $\mathscr{L}$ *is the maximum of* $\aleph_0$ *and the cardinality of* $\mathscr{L}$. *If* $\mathscr{L}$ *is countable, then the sets* tm $\mathscr{L}$ *and* Fml $\mathscr{L}$ *are countable and infinite.*

As for terms we have a unique readability theorem:

1.8. **Theorem.** *(Unique readability theorem for formulas)*
Let $\mathscr{L} = (\lambda : \mathscr{R} \longrightarrow \mathbb{N}, \mu : \mathscr{F} \longrightarrow \mathbb{N}, \mathscr{C})$ *be a language and let* φ *be an* $\mathscr{L}$*-formula. Then exactly one of the following holds true:*

(i) φ *is atomic and there are uniquely determined* $t_1, t_2 \in \operatorname{tm} \mathscr{L}$ *such that* φ *is* $t_1 \doteq t_2$*, or*
(ii) φ *is atomic and there are a unique* $n \in \mathbb{N}$*, a unique* $R \in \mathscr{R}$ *and uniquely determined* $\mathscr{L}$*-terms* $t_1, ..., t_n$ *such that* φ *is* $R(t_1, ..., t_n)$*, or*
(iii) φ *is equal to a string of the form* $(\neg\psi)$ *for a uniquely determined* $\psi \in \operatorname{Fml} \mathscr{L}$*, or*
(iv) φ *is equal to a string of the form* $(\varphi_1 \rightarrow \varphi_2)$ *for uniquely determined* $\varphi_1, \varphi_2 \in \operatorname{Fml} \mathscr{L}$*, or*
(v) φ *is equal to a string of the form* $(\forall x \psi)$ *for uniquely determined* $\psi \in \operatorname{Fml} \mathscr{L}$ *and* $x \in \mathrm{Vbl}$.

1.9. **Domestication of the notation**

- We will omit brackets if this does not lead to ambiguity.
- We use the following abbreviation for $\mathscr{L}$-formulas φ, ψ: $\varphi \vee \psi := (\neg\varphi) \rightarrow \psi$, $\varphi \wedge \psi := \neg(\varphi \rightarrow (\neg\psi))$, $\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ and $\exists x \varphi := \neg\forall x(\neg\varphi)$ where x is a variable.
- We write

  $$\forall x_1, ..., x_n\ \varphi \text{ instead of } \forall x_1 ... \forall x_n \varphi \text{ and } \exists x_1, ..., x_n\ \varphi \text{ instead of } \exists x_1 ... \exists x_n \varphi$$

  where each x_i is a variable.

  The strings $\forall x$ and $\exists x$ are called **quantifiers**. A string of quantifiers is a string of the form $Q_1 x_1 ... Q_n x_n$, where each Q_i is either $\forall$ or $\exists$ and each x_i is a variable.
- We write

  $$\bigwedge_{i=1}^{n} \varphi_i \text{ instead of } \overbrace{(...(}^{n\text{-times}}\ \varphi_1 \wedge \varphi_2) \wedge \varphi_3) ... \wedge \varphi_n) \text{ and}$$

  $$\bigvee_{i=1}^{n} \varphi_i \text{ instead of } \overbrace{(...(}^{n\text{-times}}\ \varphi_1 \vee \varphi_2) \vee \varphi_3) ... \vee \varphi_n).$$
- We write $t_1 \not\doteq t_2$ instead of $(\neg t_1 \doteq t_2)$.
- If R is a binary relation, we write $t_1 R t_2$ instead of $R(t_1, t_2)$.

Complexity and subformulas

The unique readability theorems 1.4 and 1.8 allow us to define new objects from formulas, and to prove statements about formulas. This will be done via induction on the construction depth (or the 'complexity') of terms and formulas:

1.10. **Definition.** The **complexity of an** $\mathscr{L}$**-formula** φ - denoted by $c(\varphi)$ - is the least $k \in \mathbb{N}_0$ such that $\varphi \in \operatorname{Fml}_k \mathscr{L}$.

Notice that this is not in conflict with the definition of the complexity of $\mathscr{L}$-terms (cf. 1.3), since the set of $\mathscr{L}$-terms is disjoint from the set of $\mathscr{L}$-formulas. Notice also that for any given terms $t_1, t_2, ..., t_n$ and each n-ary relation symbol R of $\mathscr{L}$, $c(R(t_1, ..., t_n)) = 0$. Similarly $c(t_1 \doteq t_2) = 0$.

By definition, for every $\mathscr{L}$-formula φ and each $k \in \mathbb{N}_0$ we have

$$c(\varphi) \leq k \iff \varphi \in \mathrm{Fml}_k \mathscr{L}.$$

1.11. **Lemma.** *For all $\mathscr{L}$-formulas φ, ψ we have*

$$c(\neg\varphi) = 1 + c(\varphi),\ c(\varphi \to \psi) = 1 + \max\{c(\varphi), c(\psi)\} \text{ and } c(\forall x\varphi) = 1 + c(\varphi).$$

Proof. The proof is left as an exercise: Proceed as in 1.5 but use 1.8 instead of 1.4. □

1.12. **Definition.** (subformula)
We define a binary relation between $\mathscr{L}$-formulas φ and ψ

- called "... **is a subformula of** ..." -

inductively, w.r.t. the complexity of ψ:

(i) If $c(\psi) = 0$ (equivalently: ψ is atomic), then $\varphi = \psi$.
(ii) If $c(\psi) = k + 1$, then
  (a) If $\psi = (\forall x\vartheta)$ or $\psi = (\neg\vartheta)$, then φ is a subformula of ϑ or $\varphi = \psi$
  (b) If $\psi = (\psi_1 \to \psi_2)$, then φ is a subformula of ψ_1 or φ is a subformula of ψ_2 or $\varphi = \psi$.

Notice that this definition is correct by 1.11.

Of course, every subformula of φ occurs in φ at some position. Also note that by a straightforward induction on the complexity, we see that the subformula relation is transitive and a formula φ is a subformula of ψ is an only if φ occurs as a substring of ψ.

Free and bound occurrences of variables

Let $\mathscr{L} = (\lambda : \mathscr{R} \longrightarrow \mathbb{N}, \mu : \mathscr{F} \longrightarrow \mathbb{N}, \mathscr{C})$ be a language.

1.13. **Definition.** (scope of a quantifier)
The **scope** of a quantifier $\forall x$ in an $\mathscr{L}$-formula φ is the set of all positions of letters in φ, which are captured in a subformula of the form $(\forall x\psi)$ of φ.

More formally: the scope of $\forall x$ in φ is the set of all $k \in \mathbb{N}$ such that there is a subformula of the form $(\forall x\psi)$ of φ, of length $l \in \mathbb{N}$ which occurs at a position p in φ with $p \leq k < p + l$.

1.14. *Example.* For example, look at the formula of the language of ordered groups (cf. 1.2):

$$\varphi = (\forall v_2((\forall v_1\ \circ(e, v_1) \doteq v_5) \to\ ^{-1}(v_1) \leq e)).$$

Here the scope of the quantifier $\forall v_1$ in φ:

$$\varphi = (\forall v_2(\overbrace{(\forall v_1\ \circ(e, v_1) \doteq v_5)}^{\text{scope of } \forall v_1} \to\ ^{-1}(v_1) \leq e)).$$

1.15. **Definition.** (free and bound occurrence of variables)
Let φ be an $\mathscr{L}$-formula and let x be a variable.

(i) If x occurs in φ at position $k \in \mathbb{N}$ and if k is <u>not</u> in the scope of the quantifier $\forall x$ in φ, then we say x occurs **free** in φ at position k.
(ii) If x occurs in φ at position $k \in \mathbb{N}$ and if k is in the scope of the quantifier $\forall x$ in φ, then we say x occurs **bound** in φ at position k.
(iii) x is a **free variable** of φ if there is some $k \in \mathbb{N}$ such that x occurs free in φ at position k.

The set of free variables of φ is denoted by $\mathrm{Fr}(\varphi)$. If $\mathrm{Fr}(\varphi)=\emptyset$, then φ is called an ($\mathscr{L}$-)**sentence** and the set of all $\mathscr{L}$-sentences is denoted by $\mathrm{Sen}(\mathscr{L})$.

It is convenient to extend the notation to terms:

(iv) If t is an $\mathscr{L}$-term, then we define $\mathrm{Fr}(t)$ to be the set of all variables occurring in t and we will also say that x is free in t instead of $x \in \mathrm{Fr}(t)$. Notice that there are no variables which are possibly bound in t. If $\mathrm{Fr}(t)=\emptyset$, then t is called a **closed term** or a **constant term**.

Different occurrences of a given variable in a formula may be free or bound, depending on where they are. In example 1.14 above, v_1 occurs bound at two positions in φ and free at one position.

$$\varphi = (\forall v_2((\forall \overbrace{v_1}^{\text{bound occurrence}} \circ(e, \overbrace{v_1}^{\text{bound occurrence}}) \doteq v_5) \to {}^{-1}(\overbrace{v_1}^{\text{free occurrence}}) \leq e))).$$

We have $\mathrm{Fr}(\varphi)=\{v_1,v_5\}$.

1.16. **Lemma.** *Let φ,ψ be $\mathscr{L}$-formulas.*

(i) If φ is quantifier-free then $\mathrm{Fr}(\varphi)$ *is the set of variables occurring in φ.*
(ii) $\mathrm{Fr}(\neg\varphi)=\mathrm{Fr}(\varphi)$
(iii) $\mathrm{Fr}((\varphi\to\psi))=\mathrm{Fr}(\varphi)\cup\mathrm{Fr}(\psi)$.
(iv) $\mathrm{Fr}(\forall x\varphi)=\mathrm{Fr}(\varphi)\setminus\{x\}$ *for all* $x\in \mathrm{Vbl}$.

1.17. *Notation.*

- The expressions '$t(x_1,...,x_n)\in \mathrm{tm}\,\mathscr{L}$' or 'let $t(x_1,...,x_n)$ be an $\mathscr{L}$-term' are shorthand for

  "$t\in \mathrm{tm}\,\mathscr{L}, x_1,...,x_n\in \mathrm{Vbl}$ with $x_i\neq x_j$ $(i\neq j)$ and $\mathrm{Fr}(t)\subseteq\{x_1,...,x_n\}$".

  This is common practice in mathematics, for example a polynomial in two variables is also considered as a polynomial in three variables.
- The expressions '$\varphi(x_1,...,x_n)\in \mathrm{Fml}\,\mathscr{L}$' or 'let $\varphi(x_1,...,x_n)$ be an $\mathscr{L}$-formula' are shorthand for

  "$\varphi\in \mathrm{Fml}\,\mathscr{L}, x_1,...,x_n\in \mathrm{Vbl}$ with $x_i\neq x_j$ $(i\neq j)$ and $\mathrm{Fr}(\varphi)\subseteq\{x_1,...,x_n\}$".

1.18. **Definition.** Let φ be an $\mathscr{L}$-formula.

(i) Let x,y be variables. We define

x **is free in** φ **for** y or y **is substitutable for** x **in** φ

if no position of φ at which x occurs free in φ, is in the scope of the quantifier $\forall y$ in φ.

(ii) Let t be an $\mathscr{L}$-term. We define

x **is free in** φ **for** t or t **is substitutable for** x **in** φ

if x is free in φ for every variable which occurs in t.

So by definition, each variable x is free for x in φ and each variable which does not occur in φ is free in φ for every term.

In example 1.14, i.e.

$$\varphi = (\forall v_2((\forall v_1 \ \circ(e,v_1)\doteq v_5)\to \ {}^{-1}(v_1)\leq e)),$$

v_1 is free for v_5 but not free for v_2 in φ; v_5 is not free for the term $\circ(v_2,v_5)$.

1.19. **Definition.** Let $\varphi\in \mathrm{Fml}\,\mathscr{L}$, $t_1,...,t_n,t\in \mathrm{tm}\,\mathscr{L}$ and let $x_1,...,x_n$ be n distinct variables.

(i) The expression $t(x_1/t_1, ..., x_n/t_n)$ denotes the string obtained from t by replacing every occurrence of x_i in t with the string t_i $(1 \leq i \leq n)$.

(ii) If for each $i \in \{1, ..., n\}$ the variable x_i is free in φ for t_i then the expression $\varphi(x_1/t_1, ..., x_n/t_n)$ denotes the string obtained from φ by simultaneously replacing every free occurrence of x_i in φ with the string t_i $(1 \leq i \leq n)$. We call $\varphi(x_1/t_1, ..., x_n/t_n)$ the **substitution** of $x_1, ..., x_n$ by $t_1, ..., t_n$ in φ.

Warning. Notice that we replace the variables x_i by the terms t_i simultaneously and not consecutively: For example if φ is $(\forall x_2\ x_1 \doteq x_2) \to x_2 \doteq x_3$, then $\varphi(x_1/t_1, x_2/t_2)$ is $(\forall x_2\ t_1 \doteq x_2) \to t_2 \doteq x_3$.

However, in general $\varphi(x_1/t_1, x_2/t_2)$ is NOT the same as $\varphi(x_1/t_1)(x_2/t_2)$. Why?

1.20. **Lemma.** *Let $\varphi \in \mathrm{Fml}\,\mathscr{L}$, $t_1, ..., t_n, t \in \mathrm{tm}\,\mathscr{L}$ and let $x_1, ..., x_n$ be n distinct variables.*

(i) $t(x_1/t_1, ..., x_n/t_n)$ is an $\mathscr{L}$-Term and if $\mathrm{Fr}(t) \subseteq \{x_1, ..., x_n\}$, then

$$\mathrm{Fr}(t(x_1/t_1, ..., x_n/t_n)) \subseteq \mathrm{Fr}(t_1) \cup ... \cup \mathrm{Fr}(t_n).$$

(ii) If for each $i \in \{1, ..., n\}$ the variable x_i is free in φ for t_i then the string $\varphi(x_1/t_1, ..., x_n/t_n)$ is an $\mathscr{L}$-formula and in the case $\mathrm{Fr}(\varphi) \subseteq \{x_1, ..., x_n\}$ we have

$$\mathrm{Fr}(\varphi) \subseteq \mathrm{Fr}(t_1) \cup ... \cup \mathrm{Fr}(t_n).$$

Proof. (i) is a straightforward induction on the complexity of t. (ii) is a straightforward induction on the complexity of φ. □

1.2. **Structures and Tarski's definition of truth.**

Throughout, $\mathscr{L} = (\lambda : \mathscr{R} \longrightarrow \mathbb{N}, \mu : \mathscr{F} \longrightarrow \mathbb{N}, \mathscr{C})$ denotes a formal language.

1.21. **Definition.** An **$\mathscr{L}$-structure** is a tuple

$$\mathscr{M} = \Big(M \ , \ (R^{\mathscr{M}} \mid R \in \mathscr{R}) \ , \ (F^{\mathscr{M}} \mid F \in \mathscr{F}) \ , \ (c^{\mathscr{M}}, c \in \mathscr{C})\Big)$$

consisting of

(S1) A nonempty set M, called the **universe** or the **domain** or the **carrier** of $\mathscr{M}$. We shall also write $|\mathscr{M}|$ instead of M.

(S2) A family $(R^{\mathscr{M}} \mid R \in \mathscr{R})$ of relations of M such that for $R \in \mathscr{R}$, $R^{\mathscr{M}} \subseteq M^{\lambda(R)}$. Hence $R^{\mathscr{M}}$ is a $\lambda(R)$-ary relation of M, called the **interpretation of** R **in** $\mathscr{M}$. Observe that for different $R_1, R_2 \in \mathscr{R}$ we may have $R_1^{\mathscr{M}} = R_2^{\mathscr{M}}$.

Formally, $(R^{\mathscr{M}} \mid R \in \mathscr{R})$ is a map $\mathscr{R} \longrightarrow \bigcup_{n\in\mathbb{N}} \mathcal{P}(M^n)$ such that the image $R^{\mathscr{M}}$ of $R \in \mathscr{R}$ under this map is a subset of $M^{\lambda(R)}$.

(S3) A family $(F^{\mathscr{M}} \mid F \in \mathscr{F})$ of functions, where for $F \in \mathscr{F}$, $F^{\mathscr{M}} : M^{\mu(F)} \longrightarrow M$. Hence $F^{\mathscr{M}}$ is a $\mu(F)$-ary function of M, called the **interpretation of** F **in** $\mathscr{M}$. Observe that for different $F_1, F_2 \in \mathscr{F}$ we may have $F_1^{\mathscr{M}} = F_2^{\mathscr{M}}$.

Formally, $(F^{\mathscr{M}} \mid F \in \mathscr{F})$ is a map $\mathscr{F} \longrightarrow \bigcup_{n\in\mathbb{N}} \mathrm{Maps}(M^n, M)$ such that the image $F^{\mathscr{M}}$ of $F \in \mathscr{F}$ under this map is a function $M^{\mu(F)} \longrightarrow M$.

(S4) A family $(c^{\mathscr{M}} \mid c \in \mathscr{C})$ of elements of M. Hence $c^{\mathscr{M}}$ is an element of M, called the **interpretation of** c **in** $\mathscr{M}$. Observe that for different $c_1, c_2 \in \mathscr{C}$ we may have $c_1^{\mathscr{M}} = c_2^{\mathscr{M}}$.

Formally, $(c^{\mathscr{M}} \mid c \in \mathscr{C})$ is simply a map $\mathscr{C} \longrightarrow M$.

$\mathscr{M}$ is called **finite**/**countable**/**uncountable**/**infinite** if its universe $|\mathscr{M}|$ is finite/countable/uncountable/infinite. If $\mathscr{M}$ is finite, we say $\mathscr{M}$ is **of size** $k \in N$ if $|\mathscr{M}|$ is of size k.

1.22. **Definition.** An **assignment** or a **valuation** of an $\mathscr{L}$-structure $\mathscr{M}$ is a map

$$h : \mathrm{Vbl} \longrightarrow |\mathscr{M}|.$$

In the literature, also the pair (M, h) is called a valuation. Given an assignment h of $\mathscr{M}$, a variable x and an element $a \in |\mathscr{M}|$ we denote by $h\binom{x}{a}$ the assignment of $\mathscr{M}$ which differs from h only at the variable x, with value a at x:

$$h\binom{x}{a}(y) = \begin{cases} h(y) & \text{if } y \neq x \\ a & \text{if } y = x. \end{cases}$$

1.23. **Definition.** Let $\mathscr{M}$ be an $\mathscr{L}$-structure with domain M.

(A) We define by induction on the complexity of an $\mathscr{L}$-term t an element $t^{\mathscr{M}}[h] \in M$ for each assignment h of $\mathscr{M}$ as follows:

(i) If $c(t) = 0$, then

$$t^{\mathscr{M}}[h] = \begin{cases} t^{\mathscr{M}} & \text{if } t \in \mathscr{C} \\ h(t) & \text{if } t \in \mathrm{Vbl}\,. \end{cases}$$

(ii) If $t_1, ..., t_n$ are $\mathscr{L}$-terms and $F \in \mathscr{F}$ with $\mu(F) = n$, then we define

$$F(t_1, ..., t_n)^{\mathscr{M}}[h] := F^{\mathscr{M}}(t_1^{\mathscr{M}}[h], ..., t_n^{\mathscr{M}}[h]).$$

(B) We define by induction on the complexity of an $\mathscr{L}$-formula φ and each assignment h of $\mathscr{M}$, the expression φ **holds in** $\mathscr{M}$ **at** h, or φ **is valid in** $\mathscr{M}$ **at** h, or $\mathscr{M}$ **satisfies** φ **at** h, denoted by

$$\mathscr{M} \models \varphi[h],$$

as follows:

(i) If φ is of the form $t_1 \doteq t_2$ with $\mathscr{L}$-terms t_1, t_2 then

$$\mathscr{M} \models t_1 \doteq t_2 \ [h] \iff t_1^{\mathscr{M}}[h] = t_2^{\mathscr{M}}[h].$$

If φ is of the form $R(t_1, ..., t_n)$ with $R \in \mathscr{R}$ of arity n and $t_1, ..., t_n \in \operatorname{tm}\mathscr{L}$ then

$$\mathscr{M} \models R(t_1, ..., t_n) \ [h] \iff (t_1^{\mathscr{M}}[h], ..., t_n^{\mathscr{M}}[h]) \in R^{\mathscr{M}}.$$

(ii) For the induction step we take $\varphi, \psi \in \operatorname{Fml}\mathscr{L}$, $x \in \operatorname{Vbl}$ and define

- $\mathscr{M} \models (\varphi \to \psi)[h] \iff$ if $\mathscr{M} \models \varphi[h]$ then $\mathscr{M} \models \psi[h]$,

- $\mathscr{M} \models (\neg\varphi)[h] \iff \mathscr{M} \not\models \varphi[h]$ i.e. $\mathscr{M} \models \varphi[h]$ does not hold

and

- $\mathscr{M} \models (\forall x \varphi)[h] \iff$ for all $a \in |\mathscr{M}|$ we have $\mathscr{M} \models \varphi[h\binom{x}{a}]$.

(C) Let $\Sigma \subseteq \operatorname{Fml}\mathscr{L}$ and let h be an assignment of $\mathscr{M}$. $\mathscr{M}$ is called a **model of** Σ **at** h if

$$\mathscr{M} \models \sigma[h] \text{ for all } \sigma \in \Sigma.$$

We denote this by

$$\mathscr{M} \models \Sigma[h].$$

Some authors also use $\mathscr{M} \models_h \Sigma$ instead of $\mathscr{M} \models \Sigma[h]$. We say that Σ has a model if it has a model at some assignment. In this case, Σ is called **satisfiable** or **consistent**.

1.24. **Lemma.** *If $\Sigma \subseteq \operatorname{Fml}\mathscr{L}$ and h, h' are assignments of an $\mathscr{L}$-structure $\mathscr{M}$ such that*

$$h(x) = h'(x) \text{ for all variables } x \text{ that occur freely in some } \sigma \in \Sigma,$$

then

$$\mathscr{M} \models \Sigma[h] \iff \mathscr{M} \models \Sigma[h'].$$

In particular, if Σ is a set of $\mathscr{L}$-sentences we may define

$$\mathscr{M} \models \Sigma \iff \mathscr{M} \models \Sigma[h] \text{ for some assignment } h \text{ of } \mathscr{M}.$$

1.25. **Lemma.** *Let $t, t' \in \operatorname{tm}\mathscr{L}$, $\varphi \in \operatorname{Fml}\mathscr{L}$ and $x \in \operatorname{Vbl}$. Let h be an assignment of an $\mathscr{L}$-structure $\mathscr{M}$ and let $a = t^{\mathscr{M}}[h]$. Then*

(i) $(t'(x/t))^{\mathscr{M}}[h] = t'^{\mathscr{M}}[h\binom{x}{a}]$.

(ii) If x is free in φ for t then $\mathscr{M} \models \varphi(x/t) \ [h] \iff \mathscr{M} \models \varphi[h\binom{x}{a}]$.

1.3. **The compactness theorem.**

The compactness theorem says that every **finitely satisfiable** set of $\mathscr{L}$-formulas is satisfiable. This statement stands at the beginning of model theory and is used everywhere. In order to make this crystal clear, let us write out again precisely what it means:

Let $\mathscr{L} = (\lambda : \mathscr{R} \longrightarrow \mathbb{N}, \mu : \mathscr{F} \longrightarrow \mathbb{N}, \mathscr{C})$ be a formal language and let Σ be a set of $\mathscr{L}$-formulas.

IF Σ is finitely satisfiable (i.e. for every finite subset Σ_0 of Σ, there is an $\mathscr{L}$-structure $\mathscr{M}$ and an assignment h of $\mathscr{M}$ such that $\mathscr{M} \models \Sigma_0[h]$),

THEN Σ itself is satisfiable, i,e, there is an $\mathscr{L}$-structure $\mathscr{M}$ and an assignment h of $\mathscr{M}$ such that $\mathscr{M} \models \Sigma[h]$.

Some remarks about the proof. Traditionally, this is an easy consequence of Gödel's completeness theorem (finite satisfiability implies that Σ is consistent in the proof theoretic sense by the soundness theorem).

There are other proofs which avoid the syntactic machinery of predicate logic, but still non-trivial work has to be invested in these alternative proofs. In section 1.4 we give a proof using ultraproducts and the so-called theorem of Łoś (read "Wosh"), see 1.34.

1.4. **Ultraproducts and proof of the compactness theorem.**

This section is not examinable and is provided for those who are interested.

1.26**. Remark on the proof using the Completeness Theorem of Predicate Logic** The proof of the Compactness Theorem given in this section goes via so called *ultraproducts* (see 1.33 and 1.35 below) and does not depend on a proof system. However, it should be mentioned that this path is available, similar to what happens in Propositional Logic. More concretely, for each language $\mathscr{L}$ there is a notion of a formal proof in predicate logic, which consist of a set of explicit $\mathscr{L}$-sentences, called $\mathscr{L}$-*axioms* (considered as tautologies of $\mathscr{L}$), together with rules about which finite sequences of $\mathscr{L}$-sentences are formal proofs from a given set of $\mathscr{L}$-sentences. The *completeness theorem* of first order logic then says that for every unsatisfiable set T of $\mathscr{L}$-sentences, there is a formal proof of a contradiction (i.e., the negation of an axiom) from T. Since proofs are finite, this formal proof is then also a proof from a finite subset of T and so T has a finite unsatisfiable subset. This implies the Compactness Theorem.

For our proof we first need a combinatorial statement about

Filters and Ultrafilters

1.27**. Filters.** Let S be a set. A **filter** of S is a set of subsets $\mathcal{F}$ of S (hence $\mathcal{F} \subseteq \mathcal{P}(S)$) that satisfies

(F1) $S \in \mathcal{F}$.
(F2) If $X, Y \in \mathcal{F}$, then $X \cap Y \in \mathcal{F}$.
(F3) If $X \in \mathcal{F}$ and $X \subseteq Y \subseteq S$, then $Y \in \mathcal{F}$.

1.28. *Examples.*

(i) For example $\{S\}$ is the smallest filter of S and $\mathcal{P}(S)$ is the largest filter of S. Any filter that is different from $\mathcal{P}(S)$ is called a **proper filter**. By (F3) a filter is proper if and only if it does not contains $\emptyset$.

(ii) If $T \subseteq S$, then
$$\mathcal{F}_T := \{X \subseteq S \mid T \subseteq X\}$$
is a filter. A filters of this form is called **principal filter**. Convince yourself that $\mathcal{F}_T$ is indeed a filter. Obviously $\mathcal{F}_T$ is proper if and only if $T \neq \emptyset$.

(iii) An important role plays the **cofinite filter** of an infinite S defined as
$$\mathcal{F} = \{X \subseteq S \mid S \setminus X \text{ is finite}\}.$$
Convince yourself that the cofinite filter of an infinite set is indeed a proper filter.

1.29**. Ultrafilters.** A filter $\mathcal{U}$ on a set S is called an **ultrafilter** of S if for all $X \subseteq S$ we have
$$X \notin \mathcal{U} \iff S \setminus X \in \mathcal{U}.$$
Observe that every ultrafilter is proper by (F1). For example a principal filter $\mathcal{F}_T$ is an ultrafilter if and only if T is of the form $\{t\}$ for some $t \in S$. On the other hand, the cofinite filter on an infinite set S is not an ultrafilter, because S has subsets X which are infinite and whose complement $S \setminus X$ is also infinite.

Here is a very useful characterisation of ultrafilters. For a filter $\mathcal{U}$ on a set S, the following are equivalent:

(i) $\mathcal{U}$ is an ultrafilter.

(ii) $\mathcal{U}$ is a maximal proper filter; this means $\mathcal{U}$ is a proper filter and if $\mathcal{F}$ is any other proper filter with $\mathcal{U} \subseteq \mathcal{F}$, then $\mathcal{F} = \mathcal{U}$.

In poset terminology this is saying that $\mathcal{U}$ is a maximal element in the poset of all proper filters of X (partially ordered by the inclusion relation).

(iii) $\mathcal{U}$ is a proper filter and for all $X, Y \subseteq S$ we have
$$X \cup Y \in \mathcal{U} \quad \Longrightarrow \quad X \in \mathcal{U} \text{ or } Y \in \mathcal{U}.$$

Proof. (i)⇒(ii). Suppose $\mathcal{U} \subsetneq \mathcal{F}$ and $\mathcal{F}$ is a filter. Then there is some $X \in \mathcal{F} \setminus \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter we get $S \setminus X \in \mathcal{U}$. But then $S \setminus X \in \mathcal{F}$ and so $\emptyset = X \cap (S \setminus X) \in \mathcal{F}$ (by (F2)). Thus $\mathcal{F}$ is not a proper filter. This shows (ii).

(ii)⇒(iii). Assume $\mathcal{U}$ is a maximal proper filter and suppose for a contradiction that there are $X, Y \subseteq S$ with $X \cup Y \in \mathcal{U}$ and $X \notin \mathcal{U}, Y \notin \mathcal{U}$. Consider the set
$$\mathcal{F} = \{Z \subseteq S \mid \exists U \in \mathcal{U} : U \cap X \subseteq Z\}.$$
Then one checks easily conditions (F1)-(F3) of a filter for $\mathcal{F}$ and so $\mathcal{F}$ is a filter with $X \in \mathcal{F}$ and $\mathcal{U} \subseteq \mathcal{F}$. Since $X \notin \mathcal{U}$, the maximality of $\mathcal{U}$ implies that $\mathcal{F}$ is not proper, i.e., $\emptyset \in \mathcal{F}$. Hence there is some $U \in \mathcal{U}$ with $U \cap X = \emptyset$. Similarly (and using $Y \notin \mathcal{U}$), there is some $V \in \mathcal{U}$ with $V \cap Y = \emptyset$. On the other hand $X \cup Y \in \mathcal{U}$, hence by (F2), $(X \cup Y) \cap U \cap V \in \mathcal{U}$. However,
$$(X \cup Y) \cap U \cap V = (X \cap U \cap V) \cup (Y \cap U \cap V) = \emptyset$$
and this contradicts the assumption that $\mathcal{U}$ is proper.

(iii)⇒(i). We need to show
$$X \notin \mathcal{U} \iff S \setminus X \in \mathcal{U} \text{ for all } X \subseteq S.$$
⇐. Suppose $S \setminus X \in \mathcal{U}$ but also $X \in \mathcal{U}$. By (F2) then $\emptyset \in \mathcal{U}$ and by (F3), $\mathcal{U}$ is not proper, a contradiction.

⇒. Suppose $X \notin \mathcal{U}$. Since $X \cup (S \setminus X) = S \in \mathcal{U}$, condition (iii) implies $S \setminus X \in \mathcal{U}$ as required. □

The crucial statement about filters needed for the compactness theorem is the following:

1.30**. Ultrafilter Theorem.**
Let S be a set. Every proper filter of S is contained in an ultrafilter of S.

Proof. Let $\mathcal{F}$ be our proper filter. We will apply Zorn's lemma and show that the set $\mathfrak{K}$ of proper filters of S containing $\mathcal{F}$ has maximal elements. By 1.29, this is enough to confirm that $\mathcal{U}$ is an ultrafilter. $\mathfrak{K}$ is partially ordered by inclusion (because every set of sets has this property) and $\mathfrak{K}$ is nonempty because $\mathcal{F} \in \mathfrak{K}$. Hence by Zorn's lemma it suffices to show that every chain $\mathfrak{C}$ in $\mathfrak{K}$ has an upper bound in $\mathfrak{K}$. We claim that

$$\mathcal{H} = \bigcup_{\mathcal{G}\in\mathfrak{C}} \mathcal{G}$$

is such an upper bound. Since $\mathcal{G} \subseteq \mathcal{H}$ for all $\mathcal{G} \in \mathfrak{C}$ we only need to confirm that $\mathcal{H}$ is a proper filter. Conditions (F1) and (F3) are readily verified as all $\mathcal{G} \in \mathfrak{C}$ are filters. For condition (F2) we need that $\mathfrak{C}$ is a chain: Take $X, Y \in \mathcal{H}$. Hence there are $\mathcal{G}_1, \mathcal{G}_2 \in \mathfrak{C}$ with $X \in \mathcal{G}_1$ and $Y \in \mathcal{G}_2$. Now as $\mathfrak{C}$ is a chain it follows $\mathcal{G}_1 \subseteq \mathcal{G}_2$ or $\mathcal{G}_2 \subseteq \mathcal{G}_1$. By symmetry we may assume that $\mathcal{G}_1 \subseteq \mathcal{G}_2$. But then $X, Y \in \mathcal{G}_2$ and as $\mathcal{G}_2$ is a filter we get $X \cap Y \in \mathcal{G}_2$. Consequently $X \cap Y \in \mathcal{H}$ as well. It remains to show that the filter $\mathcal{H}$ is proper and we only need to check that $\emptyset \notin \mathcal{H}$. But $\emptyset$ is not in any of the filters from $\mathfrak{C}$, hence it is not in $\mathcal{H}$ either. □

We now start with the model theoretic part of the proof of the Compactness Theorem.

Reduced products

Throughout we work with a language $\mathscr{L}$.

1.31**. Lemma and Definition** Let I be a nonempty set and for each $i \in I$ let $\mathscr{M}_i$ be an $\mathscr{L}$-structure. Let $\mathcal{F}$ be a filter of I. We define the **reduced product** $\prod_{i\in I} \mathscr{M}_i/\mathcal{F}$ as follows:

First define a binary relation $\sim$ on $\prod_{i\in I} |\mathscr{M}_i|$[1] as follows:

$$(a_i)_{i\in I} \sim (b_i)_{i\in I} \iff \{i \in I \mid a_i = b_i\} \in \mathcal{F}.$$

Then $\sim$ is an equivalence relation on $\prod_{i\in I} |\mathscr{M}_i|$.

Proof. (a) $\sim$ is reflexive, since $I \in \mathcal{F}$ by the filter condition (F1), in other words $(a_i)_{i\in I} \sim (a_i)_{i\in I}$ for all $(a_i)_{i\in I}$.

(b) $\sim$ is obviously symmetric.

(c) $\sim$ is transitive: To see this, assume $(a_i)_{i\in I} \sim (b_i)_{i\in I}$ and $(b_i)_{i\in I} \sim (c_i)_{i\in I}$. By definition of $\sim$, this means that

$$F_1 = \{i \in I \mid a_i = b_i\} \in \mathcal{F} \text{ and}$$
$$F_2 = \{i \in I \mid b_i = c_i\} \in \mathcal{F}.$$

By the filter condition (F2) we know $F_1 \cap F_2 \in \mathcal{F}$. Since

$$F_1 \cap F_2 \subseteq \{i \in I \mid a_i = c_i\},$$

---

[1] If I is any set and X_i is a set for each $i \in I$, then the product $\prod_{i\in I} X_i$ is defined as the set of all maps $a : I \longrightarrow \bigcup_{i\in I} X_i$ with the property that $a(i) \in X_i$ for all i. Normally these functions are written as $(a_i)_{i\in I}$ (so $a_i = a(i)$) and thought of as sequences of elements $a_i \in X_i$. For example, a sequence of real numbers is an element of $\prod_{n\in\mathbb{N}} \mathbb{R}$, so here $X_n = \mathbb{R}$ for all n.

the filter condition (F3) implies that $\{i \in I \mid a_i = c_i\} \in \mathcal{F}$. Thus $(a_i)_{i\in I} \sim (c_i)_{i\in I}$ as required.

□

Now we are ready to define $\mathscr{M} = \prod_{i\in I} \mathscr{M}_i/\mathcal{F}$:

(i) The universe of $\mathscr{M}$ is the set of equivalence classes of $\sim$. We denote the equivalence class of $(a_i)_{i\in I}$ by $[(a_i)_{i\in I}]$.

(ii) If R is an n-ary relation symbol of $\mathscr{L}$ then we define

$$R^{\mathscr{M}}\Big([(a_{1i})_{i\in I}], \dots, [(a_{ni})_{i\in I}]\Big) \iff \{i \in I \mid R^{\mathscr{M}_i}(a_{1i}, \dots, a_{ni})\} \in \mathcal{F}.$$

This is well defined: Assume $(a_{1i})_{i\in I} \sim (b_{1i})_{i\in I}, \dots, (a_{ni})_{i\in I} \sim (b_{ni})_{i\in I}$. Then

$$J = \{i \in I \mid a_{1i} = b_{1i} \ \& \ \dots \ \& \ a_{ni} = b_{ni}\} = \bigcap_{k=1}^{n} \{i \in I \mid a_{ki} = b_{ki}\} \in \mathcal{F}$$

and consequently

$$\begin{aligned} \{i \in I \mid R^{\mathscr{M}_i}(a_{1i}, \dots, a_{ni})\} \in \mathcal{F} &\iff \{i \in I \mid R^{\mathscr{M}_i}(a_{1i}, \dots, a_{ni})\} \cap J \in \mathcal{F} \\ &\iff \{i \in I \mid R^{\mathscr{M}_i}(b_{1i}, \dots, b_{ni})\} \cap J \in \mathcal{F} \\ &\iff \{i \in I \mid R^{\mathscr{M}_i}(b_{1i}, \dots, b_{ni})\} \in \mathcal{F}. \end{aligned}$$

(iii) If F is an n-ary relation symbol of $\mathscr{L}$ then we define

$$F^{\mathscr{M}}\Big([(a_{1i})_{i\in I}], \dots, [(a_{ni})_{i\in I}]\Big) = \Big(F^{\mathscr{M}_i}(a_{1i}, \dots, a_{ni})\Big)_{i\in I}.$$

This is well defined using the same reasoning as in (ii).

(iv) If c is a constant symbol then we define

$$c^{\mathscr{M}} = (c^{\mathscr{M}_i})_{i\in I}.$$

1.32. *Example.* A prominent example of reduced products is the case when $\mathcal{F}$ is the trivial filter $\{I\}$. Then the equivalence relation $\sim$ in 1.31 is just identity and the universe of $\prod_{i\in I} \mathscr{M}_i/\mathcal{F}$ is just $\prod_{i\in I} |\mathscr{M}_i|$. The structure $\prod_{i\in I} \mathscr{M}_i/\mathcal{F}$ then is the **product** of the $\mathscr{M}_i$. For example, if each $\mathscr{M}_i$ is a group (in the language of groups), this defines the ordinary product of groups.

1.33**.** In model theory, reduced products play an important role when the filter is an ultrafilter. In this case the reduced product is called **ultraproduct**. The main theorem on ultraproducts is

1.34**. Łoś's theorem.**
Let I be an index set and let $\mathcal{U}$ be an ultrafilter of I. For each $i \in I$ let $\mathscr{M}_i$ be an $\mathscr{L}$-structure. Let $\mathscr{M} = \prod_{i\in I} \mathscr{M}_i/\mathcal{U}$. If $\varphi(x_1, \dots, x_n)$ is an $\mathscr{L}$-formula then

(∗) for all $a_1, \dots, a_n \in \prod_{i\in I} \mathscr{M}_i$ we have
$\mathscr{M} \models \varphi([a_1], \dots, [a_n]) \iff \{i \in I \mid \mathscr{M}_i \models \varphi(a_{1i}, \dots, a_{ni})\} \in \mathcal{U}$.

(Here we write $a_1 = (a_{1i})_{i\in I}, \dots, a_n = (a_{ni})_{i\in I}$.)

Proof. The proof is by induction on the complexity of φ. For atomic formulas condition (∗) follows by routine checking from the interpretation of the non-logical symbols in 1.31; this is left to the reader. For the induction step we have to consider various cases.

Case 1. Suppose $\varphi(x_1,\dots,x_n,y)$ is an $\mathscr{L}$-formula and we know already $(*)$ for φ. We show $(*)$ for $\exists y\ \varphi(x_1,\dots,x_n,y)$.

Take $a_1,\dots,a_n\in\prod_{i\in I}\mathscr{M}_i$. Then

$$\begin{aligned}\mathscr{M}\models\exists y\ \varphi([a_1],\dots,[a_n],y) &\iff \text{there is } b\in\prod_{i\in I}\mathscr{M}_i : \mathscr{M}\models\varphi([a_1],\dots,[a_n],[b])\\ &\iff \text{there is } b\in\prod_{i\in I}\mathscr{M}_i : \{i\in I\mid \mathscr{M}_i\models\varphi(a_{1i},\dots,a_{ni},b_i)\}\in\mathcal{U}, \text{ by induction}\\ &\iff \{i\in I\mid \text{there is } c\in\mathscr{M}_i : \mathscr{M}_i\models\varphi(a_{1i},\dots,a_{ni},c)\}\in\mathcal{U}, \text{ using (F3) for } \mathcal{U}\\ &\iff \{i\in I\mid \mathscr{M}_i\models\exists y\ \varphi(a_{1i},\dots,a_{ni},y)\}\in\mathcal{U},\end{aligned}$$

as required for case 1.

Case 2. Suppose $\varphi(x_1,\dots,x_n),\psi(x_1,\dots,x_n)$ are $\mathscr{L}$-formulas and we know already $(*)$ for φ and ψ. We show $(*)$ for $\varphi\wedge\psi$.

Take $a_1,\dots,a_n\in\prod_{i\in I}\mathscr{M}_i$. Then

$$\begin{aligned}&\mathscr{M}\models\varphi([a_1],\dots,[a_n])\wedge\psi([a_1],\dots,[a_n])\\ &\iff \mathscr{M}\models\varphi([a_1],\dots,[a_n]) \text{ and } \mathscr{M}\models\psi([a_1],\dots,[a_n])\\ &\iff \{i\in I\mid \mathscr{M}_i\models\varphi(a_{1i},\dots,a_{ni})\}\in\mathcal{U} \text{ and}\\ &\qquad \{i\in I\mid \mathscr{M}_i\models\psi(a_{1i},\dots,a_{ni})\}\in\mathcal{U}, \text{ by induction}\\ &\iff \{i\in I\mid \mathscr{M}_i\models\varphi(a_{1i},\dots,a_{ni})\wedge\psi(a_{1i},\dots,a_{ni})\}\in\mathcal{U}, \text{ using (F2) for } \mathcal{U},\end{aligned}$$

as required for case 2.

Case 3. Suppose $\varphi(x_1,\dots,x_n)$ is an $\mathscr{L}$-formula and we know already $(*)$ for φ. We show $(*)$ for $\neg\varphi$. (It is here and only here where we need that $\mathcal{U}$ is a maximal proper filter.)

Take $a_1,\dots,a_n\in\prod_{i\in I}\mathscr{M}_i$. Then

$$\begin{aligned}\mathscr{M}\models\neg\varphi([a_1],\dots,[a_n]) &\iff \mathscr{M}\not\models\varphi([a_1],\dots,[a_n])\\ &\iff \{i\in I\mid \mathscr{M}_i\models\varphi(a_{1i},\dots,a_{ni})\}\notin\mathcal{U}, \text{ by induction}\\ &\iff \{i\in I\mid \mathscr{M}_i\models\neg\varphi(a_{1i},\dots,a_{ni})\}\in\mathcal{U}, \text{ since } \mathcal{U} \text{ is an ultrafilter,}\end{aligned}$$

as required for case 3. □

1.35. **Proof of the Compactness Theorem.** Let $\mathscr{L}$ be any language and let T be a set of $\mathscr{L}$-sentences such that any finite subset of T has a model. We need to show that T itself has a model.

Let I be the set of finite subsets of T. For each $\varphi\in T$ let

$$I_\varphi=\{i\in I\mid \varphi\in i\}.$$

Then the set $\mathcal{C}=\{I_\varphi\mid\varphi\in T\}\subseteq\mathcal{P}(I)$ has the finite intersection property, i.e. for all $n\in\mathbb{N}$ and all $\varphi_1,\dots,\varphi_n\in T$ we have $I_{\varphi_1}\cap\ldots\cap I_{\varphi_n}\neq\emptyset$; e.g. $i=\{\varphi_1,\dots,\varphi_n\}$ is in this intersection. It follows that the filter $\mathcal{F}=\{F\subseteq I\mid\exists C\in\mathcal{C}:C\subseteq F\}$ is proper. By the Ultrafilter Theorem 1.30, applied to the proper filter $\mathcal{F}$ there is an ultrafilter $\mathcal{U}$ of I containing $\mathcal{F}$, in particular $\mathcal{U}$ contains each I_φ.

For $i\in I$ choose a model $\mathscr{M}_i$ of i. We claim that the ultraproduct

$$\mathscr{M}:=\prod_{i\in I}\mathscr{M}_i/\mathcal{U}$$

is a model of T. Otherwise there is some $\varphi \in T$ with $\mathscr{M} \models \neg\varphi$. By Łoś's theorem 1.34 then, the set

$$J := \{j \in I \mid \mathscr{M}_j \models \neg\varphi\}$$

is in $\mathcal{U}$. Since also $I_\varphi \in \mathcal{U}$ we have $J \cap I_\varphi \in \mathcal{U}$ and there is some $j \in J \cap I_\varphi$. By choice of $\mathscr{M}_j$ we have $\mathscr{M}_j \models j$. But $j \in I_\varphi$ means $\varphi \in j$, thus $\mathscr{M}_j \models \varphi$ in contradiction to $j \in J$. □

1.5. **Theories and logical equivalence.**

Throughout, $\mathscr{L} = (\lambda : \mathscr{R} \longrightarrow \mathbb{N}, \mu : \mathscr{F} \longrightarrow \mathbb{N}, \mathscr{C})$ denotes a formal language.

1.36. **Definition.** Let Σ and Ψ be sets of $\mathscr{L}$-formulas. We say that Σ **implies** Ψ and write

$$\Sigma \models \Psi$$

If for every assignment h of every $\mathscr{L}$-structure $\mathscr{M}$ we have:

$$\mathscr{M} \models \Sigma[h] \Rightarrow \mathscr{M} \models \Psi[h].$$

Σ and Ψ are **logical equivalent** if $\Sigma \models \Psi$ and $\Psi \models \Sigma$; we also say that Ψ is an **axiom system** for Σ, or Σ is **axiomatised** by Ψ . We write $\Sigma \equiv \Psi$ in this case.

If $\psi \in \operatorname{Fml}\mathscr{L}$ then we write $\Sigma \models \psi$ for $\Sigma \models \{\psi\}$. Moreover we write $\models \Psi$ instead of $\emptyset \models \Psi$.

1.37. **Theorem.** *(Prenex Normal Form Theorem)*
Every $\mathscr{L}$-formula is logically equivalent to a formula in **prenex normal form**. *This means, for every formula φ of $\mathscr{L}$, there are $n \in \mathbb{N}_0$, variables $x_1, ..., x_n$, $(x_i \neq x_j)$ $(i \neq j)$, a quantifier-free formula χ with $\operatorname{Fr}\chi = \{x_1, ..., x_n\} \cup \operatorname{Fr}\varphi$ and letters $Q_1, ..., Q_n \in \{\forall, \exists\}$ such that*

$$\models \varphi \leftrightarrow Q_1 x_1 ... Q_n x_n \ \chi.$$

1.38. **Definition.** Let $\mathscr{L}$ be a language and let $\Sigma \subseteq \operatorname{Sen}\mathscr{L}$.

(i) The **deductive closure** of Σ is the set

$$\operatorname{Ded}_{\mathscr{L}} \Sigma := \{\varphi \in \operatorname{Sen}\mathscr{L} \mid \Sigma \models \varphi\}.$$

If $\mathscr{L}$ is clear from the context we will write $\operatorname{Ded}\Sigma$ instead of $\operatorname{Ded}_{\mathscr{L}}\Sigma$.

(ii) If $\Sigma = \operatorname{Ded}\Sigma$, then Σ is called **deductively closed**.

1.39. **Definition.** A subset T of $\operatorname{Sen}\mathscr{L}$ is called an **$\mathscr{L}$-theory** or simply a **theory**, if T is consistent and deductively closed.

An $\mathscr{L}$-theory T which is maximally consistent (i.e. no proper superset of T in $\operatorname{Sen}(\mathscr{L})$ is a theory) is called a **complete theory** .

1.40. **Observation.** *Let $\mathscr{L}$ be a language and let $\Sigma \subseteq \operatorname{Sen}\mathscr{L}$.*

(i) Σ is consistent if and only if $\operatorname{Ded}\Sigma \neq \operatorname{Sen}\mathscr{L}$.
(ii) $\operatorname{Ded}\Sigma$ is deductively closed.
(iii) An arbitrary (nonempty) intersection of $\mathscr{L}$-theories is again an $\mathscr{L}$-theory.

Also observe that for $\mathscr{L}$-theories T and T' we have $T \equiv T' \iff T = T'$.

1.41. **Proposition.** *The following are equivalent for every $\mathscr{L}$-theory T:*

(i) T is a complete theory.
(ii) For all $\varphi \in \operatorname{Sen}(\mathscr{L})$ with $\varphi \notin T$ we have $\neg\varphi \in T$.
(iii) For all $\varphi, \psi \in \operatorname{Sen}(\mathscr{L})$ with $T \models \varphi \vee \psi$ we have $\varphi \in T$ or $\psi \in T$.
(iv) There is an $\mathscr{L}$-structure $\mathscr{M}$ such that $T = \operatorname{Th}(\mathscr{M})$, where $\operatorname{Th}(\mathscr{M})$ denotes the **theory of** *$\mathscr{M}$:*

$$\operatorname{Th}(\mathscr{M}) = \{\varphi \in \operatorname{Sen}(\mathscr{L}) \mid M \models \varphi\}.$$

Moreover, every satisfiable set of $\mathscr{L}$-sentences is contained in a complete theory.

1.42. **Definition.** Two $\mathscr{L}$-structures are called **elementary equivalent** if $\mathrm{Th}(\mathscr{M}) = \mathrm{Th}(\mathscr{N})$, in symbols:

$$\mathscr{M} \equiv \mathscr{N}.$$

Then, another way of saying that a theory is complete is to say that all models of T are elementarily equivalent.

1.6. **Extensions of languages.**

1.43. **Definition.** Let $\mathscr{L} = (\lambda : \mathscr{R} \longrightarrow \mathbb{N}, \mu : \mathscr{F} \longrightarrow \mathbb{N}, \mathscr{C})$ and $\mathscr{L}^+ = (\lambda^+ : \mathscr{R}^+ \longrightarrow \mathbb{N}, \mu^+ : \mathscr{F}^+ \longrightarrow \mathbb{N}, \mathscr{C}^+)$ be languages. $\mathscr{L}^+$ is called an **extension of** $\mathscr{L}$ and $\mathscr{L}$ is called a **sublanguage** of $\mathscr{L}^+$ if the following conditions hold:

- $\mathscr{R} \subseteq \mathscr{R}^+$ and $\lambda^+ \restriction \mathscr{R} = \lambda$.
- $\mathscr{F} \subseteq \mathscr{F}^+$ and $\mu^+ \restriction \mathscr{F} = \mu$.
- $\mathscr{C} \subseteq \mathscr{C}^+$.

If $\mathscr{L}^+$ is an extension of $\mathscr{L}$ with $\mathscr{R}^+ = \mathscr{R}$ and $\mathscr{F}^+ = \mathscr{F}$ (hence also $\lambda^+ = \lambda$ and $\mu^+ = \mu$), then $\mathscr{L}^+$ is called an **extension by constants** of $\mathscr{L}$. In this case we also write $\mathscr{L}^+ = \mathscr{L}(\mathscr{D})$, where $\mathscr{D} = \mathscr{C}^+ \setminus \mathscr{C}$.

1.44. *Remark and Definition.* Let $\mathscr{L}^+ = (\lambda^+ : \mathscr{R}^+ \longrightarrow \mathbb{N}, \mu^+ : \mathscr{F}^+ \longrightarrow \mathbb{N}, \mathscr{C}^+)$ be an extension of the language $\mathscr{L} = (\lambda : \mathscr{R} \longrightarrow \mathbb{N}, \mu : \mathscr{F} \longrightarrow \mathbb{N}, \mathscr{C})$.

(1) The set of $\mathscr{L}$-terms is the set of all $\mathscr{L}^+$-terms which are $\mathscr{L}$-strings. The set of $\mathscr{L}$-formulas is the set of all $\mathscr{L}^+$-formulas which are $\mathscr{L}$-strings.

(2) If $\mathscr{M}^+$ is an $\mathscr{L}^+$-structure then there is a unique $\mathscr{L}$ structure $\mathscr{M}$ with $|\mathscr{M}| = |\mathscr{M}^+|$, $R^{\mathscr{M}} = R^{\mathscr{M}^+}$ $(R \in \mathscr{R})$, $F^{\mathscr{M}} = F^{\mathscr{M}^+}$ $(F \in \mathscr{F})$ and $c^{\mathscr{M}} = c^{\mathscr{M}^+}$ $(c \in \mathscr{C})$. $\mathscr{M}$ is called the **restriction of** $\mathscr{M}^+$ **to** $\mathscr{L}$ and $\mathscr{M}^+$ is called an **expansion** of $\mathscr{M}$ to $\mathscr{L}^+$. $\mathscr{M}$ is also called a **reduct of** $\mathscr{M}^+$. We write $\mathscr{M} = \mathscr{M}^+ \restriction \mathscr{L}$.

(3) If $\mathscr{M}$ is an $\mathscr{L}$ structure then $\mathscr{M}$ can be expanded to $\mathscr{L}^+$ (in several ways if $\mathscr{L}^+ \neq \mathscr{L}$). Simply choose an arbitrary interpretation of the symbols from $\mathscr{L}^+$ which are not symbols of $\mathscr{L}$.

1.45. **Lemma.** *Let $\mathscr{L}^+$ be an extension of the language $\mathscr{L}$. If $\mathscr{M}$ is the restriction of the $\mathscr{L}^+$-structure $\mathscr{M}^+$ to $\mathscr{L}$, $t \in \mathrm{tm}\,\mathscr{L}$, $\varphi \in \mathrm{Fml}\,\mathscr{L}$ and $h : \mathrm{Vbl} \longrightarrow |\mathscr{M}|$ (hence h is an assignment of $\mathscr{M}$ and of $\mathscr{M}^+$), then*

$$t^{\mathscr{M}}[h] = t^{\mathscr{M}^+}[h] \text{ and } \mathscr{M} \models \varphi[h] \iff \mathscr{M}^+ \models \varphi[h].$$

Proof. This is a straightforward induction on the complexity of terms and formulas. □

1.46. **Corollary.** *Let $\mathscr{L}^+$ be an extension of $\mathscr{L}$ and let $\mathscr{M}$ be the restriction of the $\mathscr{L}^+$-structure $\mathscr{M}^+$ to $\mathscr{L}$. Let $t(x_1, ..., x_n) \in \mathrm{tm}\,\mathscr{L}$, $\varphi(x_1, ..., x_n) \in \mathrm{Fml}\,\mathscr{L}$ and let $d_1, ..., d_n$ be pairwise distinct constants of $\mathscr{L}^+$. If $h : \mathrm{Vbl} \longrightarrow |\mathscr{M}|$ with $h(x_i) = d_i^{\mathscr{M}^+}$ then*

$$t^{\mathscr{M}}[h] = t(x_1/d_1, ..., x_n/d_n)^{\mathscr{M}^+} \text{ and } \mathscr{M} \models \varphi[h] \iff \mathscr{M}^+ \models \varphi(x_1/d_1, ..., x_n/d_n).$$

Proof. From 1.45 and 1.25. □

1.47. **Theorem.** *Let $\mathscr{L}$ be language and let $\mathscr{D} = \{d_0, d_1, ...\}$ be a countable set of new constants. For $\varphi(v_0, ..., v_n) \in \mathrm{Fml}\,\mathscr{L}$ let $\varphi_{\mathscr{D}}$ be the $\mathscr{L}(\mathscr{D})$-sentence*

$$\varphi_{\mathscr{D}} \;=\; \varphi(v_0/d_0, ..., v_n/d_n).$$

Let $\Sigma \subseteq \operatorname{Fml}\mathscr{L}$ *and* $\varphi \in \operatorname{Fml}\mathscr{L}$. *Then*

$$\Sigma \models_{\mathscr{L}} \varphi \iff \Sigma_{\mathscr{D}} \models_{\mathscr{L}(\mathscr{D})} \varphi_{\mathscr{D}},$$

where $\Sigma_{\mathscr{D}} = \{\sigma_{\mathscr{D}} \mid \sigma \in \Sigma\}$.

2. Comparing structures

Firstly we simplify notation. Let $\mathscr{M}$ be an $\mathscr{L}$-structure, $t(x_1, ..., x_n) \in \text{tm}\,\mathscr{L}$, $\varphi(x_1, ..., x_n) \in \text{Fml}\,\mathscr{L}$ and let $a_1, ..., a_n \in |\mathscr{M}|$ (not necessarily $a_i \neq a_j$ for $i \neq j$). We write $t^{\mathscr{M}}[a_1, ..., a_n]$ instead of $t^{\mathscr{M}}[h]$ and $\mathscr{M} \models \varphi[a_1, ..., a_n]$ instead of $\mathscr{M} \models \varphi[h]$, where $h : \text{Vbl} \longrightarrow |\mathscr{M}|$ with $h(x_1) = a_1, ..., h(x_n) = a_n$. Recall from 1.24 that this is well defined.

2.1. Formulas preserved by maps.

2.1.1. **Definition.** Let $\mathscr{M}, \mathscr{N}$ be $\mathscr{L}$-structures. A **map between $\mathscr{M}$ and $\mathscr{N}$** is a map $f : |\mathscr{M}| \longrightarrow |\mathscr{N}|$. We write $f : \mathscr{M} \longrightarrow \mathscr{N}$ instead of $f : |\mathscr{M}| \longrightarrow |\mathscr{N}|$.

A formula $\varphi(x_1, ..., x_n) \in \text{Fml}\,\mathscr{L}$ is **preserved by a map** $f : \mathscr{M} \longrightarrow \mathscr{N}$, or f **respects** φ, or f is a φ**-morphism**, if for all $a_1, ..., a_n$ we have

$$\mathscr{M} \models \varphi[a_1, ..., a_n] \Rightarrow \mathscr{N} \models \varphi[f(a_1), ..., f(a_n)].$$

2.1.2. *Examples.*

(i) If x, y are different variables, then the formula $\neg x \doteq y$ is preserved by $f : \mathscr{M} \longrightarrow \mathscr{N}$ if and only if f is injective.

(ii) If $f : \mathscr{M} \longrightarrow \mathscr{N}$ is bijective (i.e. $f : |\mathscr{M}| \longrightarrow |\mathscr{N}|$ is bijective), then we also have a map $f^{-1} : \mathscr{N} \longrightarrow \mathscr{M}$ and for every formula φ,

$$\varphi \text{ is preserved by } f \iff \neg\varphi \text{ is preserved by } f^{-1}.$$

(iii) If $\mathscr{L} = \{\emptyset, \{\circ\}, \{e\}\}$ is the language of monoids, then a map $f : \mathscr{M} \longrightarrow \mathscr{N}$ between groups preserves the formulas $y = x_1 \cdot x_2$ if and only if $f(a_1 \cdot a_2) = f(a_1) \cdot f(a_2)$ for all $a_1, a_2 \in |\mathscr{M}|$. I.e. we are describing group homomorphisms here.

(iv) Similarly, an order preserving map between partially ordered sets $(M, \leq_M) \longrightarrow (N, \leq_N)$ is the same as a map $\mathscr{M} \longrightarrow \mathscr{N}$, which preserves the formula $x \leq y$, where $\mathscr{M} = (M, \leq_M)$ and $\mathscr{N} = (N, \leq_N)$ are considered as structures in the language $\mathscr{L} = \{\leq\}$ of a binary relation symbol.

2.1.3. **Definition.** A map $f : \mathscr{M} \longrightarrow \mathscr{N}$ between $\mathscr{L}$-structures is called an **$\mathscr{L}$-homomorphism** if f respects all atomic formulas.

2.1.4. **Lemma.** *Let $f : \mathscr{M} \longrightarrow \mathscr{N}$ be a map between $\mathscr{L}$-structures. The following are equivalent:*

(i) f is an $\mathscr{L}$-homomorphism.

(ii) f satisfies each of the following conditions:

(a) For all $R \in \mathscr{R}$ of arity n and all $a_1, ..., a_n \in |\mathscr{M}|$ we have

$$(a_1, ..., a_n) \in \mathscr{R}^{\mathscr{M}} \Rightarrow (f(a_1), ..., f(a_n)) \in \mathscr{R}^{\mathscr{N}}.$$

(b) For all $F \in \mathscr{F}$ of arity n and all $a_1, ..., a_n \in |\mathscr{M}|$ we have

$$f(F^{\mathscr{M}}(a_1, ..., a_n)) = F^{\mathscr{N}}(f(a_1), ..., f(a_n)).$$

(c) For all $c \in \mathscr{C}$ we have

$$f(c^{\mathscr{M}}) = c^{\mathscr{N}}.$$

(iii) f respects each of the following formulas:

(a) all formulas of the form $R(v_1,...,v_n)$*, where* $R\in\mathscr{R}$ *is a relation symbol of* $\mathscr{L}$ *of arity* n*.*

(b) all formulas of the form $v_0 \doteq F(v_1,...,v_n)$*, where* $F\in\mathscr{F}$ *is a function symbol of* $\mathscr{L}$ *of arity* n*.*

(c) all formulas of the form $v_0 \doteq c$*, where* $c\in\mathscr{C}$ *is a constant symbol of* $\mathscr{L}$

If this is the case, then for every $\mathscr{L}$*-term* $t(x_1,...,x_n)\in \mathrm{tm}\,\mathscr{L}$ *and all* $a_1,...,a_n\in|\mathscr{M}|$ *we have*

$$f(t^{\mathscr{M}}[a_1,...,a_n]) = t^{\mathscr{N}}(f(a_1),...,f(a_n)).$$

Proof. (i)⇒(iii) and (iii)⇒(ii) are trivial. We assume now (ii) and show (i) as well as the additional statement. A straightforward induction on the complexity of $t(x_1,...,x_n)\in \mathrm{tm}\,\mathscr{L}$ using (ii)(b) and (ii)(c) shows that $f(t^{\mathscr{M}}[a_1,...,a_n]) = t^{\mathscr{N}}(f(a_1),...,f(a_n))$ for all $a_1,...,a_n\in|\mathscr{M}|$.

Clearly this, together with (ii)(a) proves (i). □

2.1.5. **Definition.** A map $f:\mathscr{M}\longrightarrow\mathscr{N}$ between $\mathscr{L}$-structures is called an **embedding** if f preserves every quantifier-free formula. If $|\mathscr{M}|\subseteq|\mathscr{N}|$ and the inclusion $|\mathscr{M}|\hookrightarrow|\mathscr{N}|$ is an embedding then $\mathscr{M}$ is called a **substructure** of $\mathscr{N}$. If in addition $|\mathscr{M}|\neq|\mathscr{N}|$, then $\mathscr{M}$ is called a **proper substructure** of $\mathscr{N}$.

2.1.6. **Lemma.** *Let* $f:\mathscr{M}\longrightarrow\mathscr{N}$ *be a map between* $\mathscr{L}$*-structures. The following are equivalent:*

(i) f *is an embedding.*

(ii) f *is an* <u>*injective*</u> $\mathscr{L}$ *homomorphism such that for all* $R\in\mathscr{R}$ *of arity* n *and all* $a_1,...,a_n\in|\mathscr{M}|$ *we have*

$$(a_1,...,a_n)\in\mathscr{R}^{\mathscr{M}} \Leftrightarrow (f(a_1),...,f(a_n))\in\mathscr{R}^{\mathscr{N}}.$$

(iii) For all $\varphi(x_1,...,x_n)\in \text{at-Fml}(\mathscr{L})$ *and all* $a_1,...,a_n\in|\mathscr{M}|$ *we have*

$$\mathscr{M}\models\varphi[a_1,...,a_n] \iff \mathscr{N}\models\varphi[f(a_1),...,f(a_n)].$$

Proof. (i)⇒(ii). By 2.1.4 and 2.1.2(i), f is an injective $\mathscr{L}$-homomorphism. If $R\in\mathscr{R}$ of arity n and $a_1,...,a_n\in|\mathscr{M}|$ then we know

$$(a_1,...,a_n)\in\mathscr{R}^{\mathscr{M}} \Leftarrow (f(a_1),...,f(a_n))\in\mathscr{R}^{\mathscr{N}},$$

since f respects the quantifier-free formula $\neg R(v_1,...,v_n)$. Since f also respects $R(v_1,...,v_n)$, this proves (ii).

(ii)⇒(iii). Let $\varphi(x_1,...,x_n)\in \text{at-Fml}(\mathscr{L})$ and $a_1,...,a_n\in|\mathscr{M}|$.

Case 1. φ is of the form $t\doteq s$ with $t(x_1,...,x_n), s(x_1,...,x_n)\in \mathrm{tm}\,\mathscr{L}$.

Since f is an $\mathscr{L}$-homomorphism, f respects φ. Furthermore, by 2.1.4 , we have

$$f(t^{\mathscr{M}}[a_1,...,a_n]) = t^{\mathscr{N}}(f(a_1),...,f(a_n)) \text{ and}$$
$$f(s^{\mathscr{M}}[a_1,...,a_n]) = s^{\mathscr{N}}(f(a_1),...,f(a_n)).$$

Hence if $\mathscr{N}\models\varphi[f(a_1),...,f(a_n)]$, then

$$f(s^{\mathscr{M}}[a_1,...,a_n]) = s^{\mathscr{N}}(f(a_1),...,f(a_n)) = t^{\mathscr{N}}(f(a_1),...,f(a_n)) = f(t^{\mathscr{M}}[a_1,...,a_n])$$

and since f is assumed to be injective we obtain $\mathscr{M}\models\varphi[a_1,...,a_n]$.

Case 2. φ is of the form $R(t_1,...,t_k)$ where $R\in\mathscr{R}$ is of arity n and $t_1,...,t_k\in \mathrm{tm}\,\mathscr{L}$ with free variables in $\{x_1,...,x_n\}$.

Again by 2.1.4, we have

$$f(t_i^{\mathscr{M}}[a_1,...,a_n]) = t_i^{\mathscr{N}}(f(a_1),...,f(a_n)) \ (1\le i\le k).$$

Together with the equivalence assumed in (ii) this shows

$$\mathscr{M} \models R(t_1, ..., t_k)[a_1, ..., a_n] \iff \mathscr{N} \models R(t_1, ..., t_k)[f(a_1), ..., f(a_n)].$$

(iii)⇒(i) holds since the equivalence in (iii) is preserved by negation and conjunction (i.e. if the equivalence holds for φ and ψ then it also holds for $\neg\varphi$ and $\varphi \wedge \psi$). Since every quantifier-free formula is provably equivalent to a formula which built up from atomic formulas using negations and conjunctions, this shows (i). □

2.1.7. **Corollary.** *Let $\mathscr{M}$ be an $\mathscr{L}$-structure and let $A \subseteq |\mathscr{M}|$. If $c^{\mathscr{M}} \in A$ $(c \in \mathscr{C})$ and for each n-ary function symbol F of $\mathscr{L}$, the function $F^{\mathscr{M}}$ maps A^n to A, then A is the universe of a (unique!) substructure $\mathscr{A}$ of $\mathscr{M}$, which is called the* **substructure of $\mathscr{M}$ induced on** *A: $\mathscr{A}$ has by definition universe A and interprets the non-logical symbols as follows:*

- *$R^{\mathscr{A}} = R^{\mathscr{M}} \cap A^n$ for all $R \in \mathscr{R}$ of arity n.*
- *$F^{\mathscr{A}}(a_1, ..., a_n) = F^{\mathscr{M}}(a_1, ..., a_n)$ for all $F \in \mathscr{F}$ of arity n.*
- *$c^{\mathscr{A}} = c^{\mathscr{M}}$ for all $c \in \mathscr{C}$.*

Observe that (ii) and (iii) make sense by the assumption on A.

Proof. Immediate from 2.1.6. □

2.1.8. **Definition.** A map $f : \mathscr{M} \longrightarrow \mathscr{N}$ between $\mathscr{L}$-structures is called an **elementary embedding** if f preserves all formulas. If $|\mathscr{M}| \subseteq |\mathscr{N}|$ and the inclusion $|\mathscr{M}| \hookrightarrow |\mathscr{N}|$ is an elementary embedding then $\mathscr{M}$ is called a an **elementary substructure** of $\mathscr{N}$, denoted by $\mathscr{M} \prec \mathscr{N}$ and $\mathscr{N}$ is called an **elementary extension** of $\mathscr{M}$,

At the moment we have only one (rather trivial) class of examples of elementary embeddings:

2.1.9. **Definition.** A map $f : \mathscr{M} \longrightarrow \mathscr{N}$ is called an ($\mathscr{L}$-)**isomorphism** if f is a bijective embedding. Two $\mathscr{L}$-structures are called **isomorphic** if there is an isomorphism $\mathscr{M} \longrightarrow \mathscr{N}$. An isomorphism $\mathscr{M} \longrightarrow \mathscr{M}$ is called an **automorphism of $\mathscr{M}$**.

2.1.10. *Warning.* In general, a bijective $\mathscr{L}$-homomorphism is not an embedding. E.g. if $\mathscr{L}$ has a unique non-logical symbol, namely a unary relation symbol R. Let $\mathscr{N}$ be an $\mathscr{L}$-structure with $R^{\mathscr{N}} = |\mathscr{N}|$ and let $\mathscr{M}$ be the $\mathscr{L}$-structure with universe $|\mathscr{N}|$ and $R^{\mathscr{M}} = \emptyset$. Then the identity map $|\mathscr{M}| \longrightarrow |\mathscr{N}|$ is a bijective $\mathscr{L}$-homomorphism. But this map is not an embedding!

On the other hand, a bijective $\mathscr{L}$-homomorphism $f : \mathscr{M} \longrightarrow \mathscr{N}$ is an isomorphism if and only if the inverse map $f^{-1} : \mathscr{N} \longrightarrow \mathscr{M}$ is a homomorphism (as follows from 2.1.6(i)⇔(iii)).

2.1.11. **Lemma.** *Every $\mathscr{L}$-isomorphism is an elementary embedding.*

Proof. Let $f : \mathscr{M} \longrightarrow \mathscr{N}$ be an $\mathscr{L}$-isomorphism. It is straightforward to show by induction on the complexity of $\varphi(x_1, ..., x_n) \in \mathrm{Fml}\,\mathscr{L}$ that

$$\mathscr{M} \models \varphi[a_1, ..., a_n] \iff \mathscr{N} \models \varphi[f(a_1), ..., f(a_n)].$$

The case of atomic formulas holds by 2.1.6. For quantification use the surjectivity of f. □

2.2. **Naming elements of structures: Maps and diagrams.**

Let $\mathscr{L} = (\lambda : \mathscr{R} \longrightarrow \mathbb{N}, \mu : \mathscr{F} \longrightarrow \mathbb{N}, \mathscr{C})$ be a language and let $\mathscr{M}$ be an $\mathscr{L}$-structure. Let A be a subset of $|\mathscr{M}|$. We want to have a constant symbol for the elements in A available, in order to name these elements in the language. We do the following.

Let $\underline{A}$ be a set of constants, new w.r.t. $\mathscr{L}$, which is in bijection with A: So $\underline{A} = \{\underline{a} \mid a \in A\}$ and the map $A \longrightarrow \underline{A}$ that sends a to $\underline{a}$ is bijective. We denote by $\mathscr{L}(A)$ the extension by constants $\mathscr{L}(\underline{A})$ of $\mathscr{L}$. We expand $\mathscr{M}$ to an $\mathscr{L}(A)$-structure, denoted by $(\mathscr{M}, A)$ via

$$(\underline{a})^{(\mathscr{M},A)} := a.$$

(Observe that the expression '$(\mathscr{M}, A)$ is an expansion of $\mathscr{M}$' leaves no choice for the interpretation of the $\mathscr{L}$-symbols in $(\mathscr{M}, A)$: they have to be interpreted as in $\mathscr{M}$; similarly, this expression also clarifies the universe of $(\mathscr{M}, A)$: it is the universe of $\mathscr{M}$).

This looks like a weird undertaking. What has happened? Let us look at an example: Let $\mathscr{L} = (\leq, +, \cdot, -, \underline{0}, \underline{1})$ be the language of ordered rings and let $\mathscr{M}$ be the real field (with its natural definition of the $\mathscr{L}$-symbols). Let $A = \mathbb{R}$, so here A is all of the universe of $\mathscr{M}$. In $(\mathscr{M}, A)$ we have the following terms and formulas:

- Polynomials with coefficients in $\mathbb{R}$ are named by terms: Given $a_0, ..., a_d \in \mathbb{R}$ the expression $\underline{a_d} \cdot x^d + ... + \underline{a_1} \cdot x + \underline{a_0}$ is a term in the variable x and the function $t^{(\mathscr{M},\mathbb{R})} : \mathbb{R} \longrightarrow \mathbb{R}$ is the polynomial function that is given by the ordinary polynomial $a_d T^d + ... + a_0$.
- If $t(x)$ is the term from the previous item, then the $\mathscr{L}(\mathbb{R})$-formula $\varphi(x)$ defined as

$$\forall \varepsilon\ \varepsilon > 0 \rightarrow (\forall y\ |x - y| < \varepsilon \rightarrow |t(x) - t(y)| < \varepsilon)$$

  (where $|z| < u$ is shorthand for $z < u \wedge -z < u$) expresses a strong continuity property of $t^{(\mathscr{M},\mathbb{R})}$ at x.

Such terms and formulas obviously are subject of interest in the analysis of the structure $\mathscr{M}$. So in order to incorporate elements $a \in |\mathscr{M}|$ in the study of $\mathscr{M}$ we introduce names for them as new constant symbols $\underline{a}$.

2.2.1. *Remark.* A simple but extremely powerful observation is the following: Let $\mathscr{M}$ be an $\mathscr{L}$-structure with universe M. We take $A = M$ in the definition above. What are the $\mathscr{L}(A)$-structures?

By definition, an $\mathscr{L}(A)$-structure $\mathscr{N}^+$ is just an $\mathscr{L}$-structure $\mathscr{N}$ together with an interpretation of the new constants $\underline{a}$ for each $a \in A$. Such an interpretation is simply given by a map $\underline{A} \longrightarrow |\mathscr{N}|$ and by composing this map with the bijection $A \longrightarrow \underline{A}$ we get a map $f : A \longrightarrow |\mathscr{N}|$,

$$f(a) = (\underline{a})^{\mathscr{N}^+}.$$

Conversely, if $g : \mathscr{M} \longrightarrow \mathscr{N}$ is a map between $\mathscr{L}$-structures, then the expansion $\mathscr{N}^+$ of $\mathscr{N}$ by

$$(\underline{a})^{\mathscr{N}^+} := g(a)$$

defines an $\mathscr{L}(A)$ structure that gives g back when we run through the construction of the previous paragraph.

In other words, an $\mathscr{L}(M)$-structure is the same as a map from $\mathscr{M}$ to some $\mathscr{L}$-structure! For those who likes things more formally: The function

$$\mathscr{L}(M)\text{-structures} \longrightarrow \text{Pairs } (\mathscr{N}, f),\ \mathscr{N} \text{ an } \mathscr{L}\text{-structure and } f : |\mathscr{M}| \longrightarrow |\mathscr{N}|;$$

which maps $\mathscr{N}^+$ to $(\mathscr{N}^+ \restriction \mathscr{L}, a \mapsto \underline{a}^{\mathscr{N}^+})$, is a bijection (between classes).

We will therefore denote $\mathscr{L}(\mathscr{M})$-structures by $(\mathscr{N}, f)$. Carrying on with this set up we can now read maps that preserve a formula as models of certain theories: Let $\mathscr{M}$ be an $\mathscr{L}$-structure and let $\varphi(x_1, ..., x_n) \in \mathrm{Fml}\,\mathscr{L}$. Let A be the universe of $\mathscr{M}$ and consider in the language $\mathscr{L}(A)$ the set

$$\Phi := \{\varphi(x_1/\underline{a_1}, ..., x_n/\underline{a_n}) \mid a_1, ..., a_n \in A \text{ and } \mathscr{M} \models \varphi[a_1, ..., a_n]\}.$$

So Φ is a (generally infinite) set of $\mathscr{L}(A)$-sentences: the free variables $x_1, ..., x_n$ of φ have been replaced by the (new) constant symbols $\underline{a_1}, ..., \underline{a_n}$.

Let $\mathscr{N}^+$ be an $\mathscr{L}(A)$.structure, given by the map $f : \mathscr{M} \longrightarrow \mathscr{N} = \mathscr{N}^+ \restriction \mathscr{L}$. When is $\mathscr{N}^+$ a model of Φ? By definition this is the case if and only if for all $a_1, ..., a_n \in A$ with $\mathscr{M} \models \varphi[a_1, ..., a_n]$ we have $\mathscr{N}^+ \models \varphi(x_1/\underline{a_1}, ..., x_n/\underline{a_n})$. Now by 1.46, the latter condition just means $\mathscr{N}^+ \models \varphi[\underline{a_1}^{\mathscr{N}^+}, ..., \underline{a_n}^{\mathscr{N}^+}]$ and by 1.45 this in turn is equivalent to $\mathscr{N} \models \varphi[f(a_1), ..., f(a_n)]$ (recall that $\underline{a_i}^{\mathscr{N}^+} = f(a_i)$ by the choice of $\mathscr{N}^+$ and f). In summary:

An $\mathscr{L}(A)$-structure $\mathscr{N}^+$, given by the map $f : \mathscr{M} \longrightarrow \mathscr{N}$, is a model of Φ if and only if for all $a_1, ..., a_n \in |\mathscr{M}|$ $(= A)$ we have

$$\mathscr{M} \models \varphi[a_1, ..., a_n] \Rightarrow \mathscr{N} \models \varphi[f(a_1), ..., f(a_n)].$$

But this simply says: f preserves φ!

What have we achieved? We see that under the bijection of $\mathscr{L}(A)$.structures (where again $A = |\mathscr{M}|$) and maps from $\mathscr{M}$ to $\mathscr{L}$-structures, the φ-preserving maps correspond precisely to the models of Φ. So 'morphisms' have been identified with models of theories. Looking back at our model theoretic tools so far, this opens the possibility to construct morphisms of all kind with the aid of the compactness theorem. As we are particulary interested in homomorphisms, embeddings and elementary embeddings, the theories that encode this type of maps get their own names:

2.2.2. **Definition.** Let $\mathscr{M}$ be an $\mathscr{L}$-structure with universe M. We define the following sets of $\mathscr{L}(M)$-sentences:

$$\mathrm{diag}_+(\mathscr{M}) = \{\varphi(x_1/\underline{a_1}, ..., x_n/\underline{a_n}) \mid n \in \mathbb{N}, \varphi(x_1, ..., x_n) \in \text{at-Fml}(\mathscr{L}) \text{ and } \mathscr{M} \models \varphi[a_1, ..., a_n]\}$$

$\mathrm{diag}_+(\mathscr{M})$ is called the **atomic diagram of** $\mathscr{M}$.

$$\mathrm{diag}(\mathscr{M}) = \{\varphi(x_1/\underline{a_1}, ..., x_n/\underline{a_n}) \mid n \in \mathbb{N}, \varphi(x_1, ..., x_n) \in \text{qf-Fml}(\mathscr{L}) \text{ and } \mathscr{M} \models \varphi[a_1, ..., a_n]\}$$

$\mathrm{diag}(\mathscr{M})$ is called the **(quantifier-free) diagram of** $\mathscr{M}$.

$$\mathrm{diag}_\infty(\mathscr{M}) = \{\varphi(x_1/\underline{a_1}, ..., x_n/\underline{a_n}) \mid n \in \mathbb{N}, \varphi(x_1, ..., x_n) \in \mathrm{Fml}(\mathscr{L}) \text{ and } \mathscr{M} \models \varphi[a_1, ..., a_n]\}$$

$\mathrm{diag}_\infty(\mathscr{M})$ is called the **elementary** (or **complete**) **diagram of** $\mathscr{M}$.

Observe that $\mathrm{diag}_+(\mathscr{M}) \subseteq \mathrm{diag}(\mathscr{M}) \subseteq \mathrm{diag}_\infty(\mathscr{M}) = \mathrm{Th}((\mathscr{M}, M))$ and $(\mathscr{M}, M)$ is a model of all of these sets.

2.2.3. **Proposition.** *Let $\mathscr{M}$ be an $\mathscr{L}$-structure with universe M and let $f : \mathscr{M} \longrightarrow \mathscr{N}$ be a map to another $\mathscr{L}$-structure $\mathscr{N}$. Let $\mathscr{N}^+$ be the unique expansion of $\mathscr{N}$ to an $\mathscr{L}(M)$-structure via*

$$\underline{a}^{\mathscr{N}^+} = f(a) \ (a \in M).$$

Then

(i) f is a homomorphism if and only if $\mathscr{N}^+ \models \mathrm{diag}_+(\mathscr{M})$.
(ii) f is an embedding if and only if $\mathscr{N}^+ \models \mathrm{diag}(\mathscr{M})$.
(iii) f is an elementary embedding if and only if $\mathscr{N}^+ \models \mathrm{diag}_\infty(\mathscr{M})$.

Proof. The proof is a repetition of the argument given above that φ-morphisms correspond to models of Φ (as defined above). This is left as an exercise. □

We can now easily produce arbitrary large elementary extensions of infinite structures:

2.2.4. **Corollary.** *If $\mathscr{M}$ is an infinite $\mathscr{L}$-structure, then $\mathscr{M}$ has elementary extensions of arbitrary large cardinalities.*

Proof. Let $M = |\mathscr{M}|$ and let $\mathscr{D}$ be a set of constants, new w.r.t. the language $\mathscr{L}(M)$. We will show that $\mathscr{M}$ has an elementary extension that possesses an injection from $\mathscr{D}$ into its universe. This will prove the corollary as there is no restriction on the cardinality of $\mathscr{D}$ and we may have chosen $\mathscr{D}$ of cardinality as big as we like.

We work in the language $\mathscr{L}(M)(\mathscr{D})$. Consider the following set Σ of $\mathscr{L}(M)(\mathscr{D})$-sentences:

$$\Sigma = \mathrm{diag}_\infty(\mathscr{M}) \cup \{\neg\, d \doteq e \mid d, e \in \mathscr{D} \text{ with } d \neq e\}.$$

We claim that Σ has a model. By the compactness theorem it suffices to show that every finite subset of Σ has a model. However each such finite set is contained in a set of the form

$$\Gamma = \mathrm{diag}_\infty(\mathscr{M}) \cup \{\neg\, d \doteq e \mid d, e \in \mathscr{E} \text{ with } d \neq e\},$$

for some finite subset $\mathscr{E}$ of $\mathscr{D}$, say $\mathscr{E} = \{d_1, ..., d_n\}$ is of size n. In order to find a model of Γ we expand $(\mathscr{M}, M)$ to an $\mathscr{L}(M)(\mathscr{D})$-structure $\mathscr{M}^*$, by choosing n elements $a_1, ..., a_n \in M$ and define for $d \in \mathscr{D}$:

$$d^{\mathscr{M}^*} = \begin{cases} a_i & \text{if } d = d_i \text{ for some } i \in \{1, ..., n\} \\ \text{arbitrary} & \text{otherwise.} \end{cases}$$

Since $(\mathscr{M}, M)$ is a model of $\mathrm{diag}_\infty(\mathscr{M})$ it is then clear that $\mathscr{M}^*$ is a model of Γ. This shows that Σ is finitely satisfiable and so Σ has a model $\mathscr{N}^*$. Now $\mathscr{N}^*$ is an expansion by constants of an $\mathscr{L}(\mathscr{M})$-structure $\mathscr{N}^+$ and by 2.2.3, $\mathscr{N}^+$ is given by the elementary embedding $f : \mathscr{M} \longrightarrow \mathscr{N}$, $f(a) = \underline{a}^{\mathscr{N}^+}$, where $\mathscr{N} = \mathscr{N}^+ \restriction \mathscr{L}$.

After identifying $\mathscr{M}$ with its image under f we may then also assume that $\mathscr{N}$ is an elementary extension of $\mathscr{M}$ 8and f is the inclusion map $M \hookrightarrow |\mathscr{N}|$.

It remains to show that there is an injection $\mathscr{D} \hookrightarrow |\mathscr{N}|$. However, remember that $\mathscr{N}$ is the restriction of $\mathscr{N}^*$ to $\mathscr{L}$ and $\mathscr{N}^*$ is a model of Σ. Therefore the map which sends $d \in \mathscr{D}$ to $d^{\mathscr{N}^*}$ is an injection. □

What happens with the statement 2.2.4 in the case of a finite $\mathscr{L}$-structure $\mathscr{M}$?

2.3. **The Tarski-Vaught test and the Skolem-Löwenheim theorems.**

2.3.1. **Tarski-Vaught Test**

Let $\mathscr{M}$ be an $\mathscr{L}$-structure and let $A \subseteq |\mathscr{M}|$. The following are equivalent:

(i) A is the universe of an elementary substructure of $\mathscr{M}$. Since on each subset of $\mathscr{M}$ there can only live at most one substructure of $\mathscr{M}$, this property is sometimes referred to as "A **is** a substructure of $\mathscr{M}$".

(ii) For every $\mathscr{L}$-formula $\varphi(x,\bar{y})$ and all $\bar{a} \in A^{\bar{y}}$, if $\mathscr{M} \models \exists x\ \varphi(x,\bar{a})$ (to be correct we should actually write $\mathscr{M} \models (\exists x\ \varphi)[\bar{a}]$) , then there is some $b \in A$ with $\mathscr{M} \models \varphi[b,\bar{a}]$.

Proof. (i)$\Rightarrow$(ii) is clear. For the converse we first observe that under assumption (ii), A is closed under all $F^{\mathscr{M}}$ ($F \in \mathscr{F}$) and A contains all $c^{\mathscr{M}}$ with $c \in \mathscr{C}$: To see this, take an n-ary function symbol F. Applying condition (ii), where $\varphi(x,\bar{y})$ is the formula $F(\bar{y}) \doteq x$ we see that $F^{\mathscr{M}}(\bar{a}) \in A$ for all $\bar{a} \in A^{\bar{y}}$. Similarly, using the formula $x \doteq c$ (so, no other variables $\bar{y}$ are present in φ), we see that $c^{\mathscr{M}} \in A$ for all $c \in \mathscr{C}$.

Hence by 2.1.7 we know that A together with the structure induced from $\mathscr{M}$ is a substructure of $\mathscr{M}$ which we denote by $\mathscr{A}$. We now show (and this will prove (i)) by induction on the complexity of an $\mathscr{L}$-formula $\varphi(x_1,...,x_k)$:

$$\text{For all } \bar{a} \in A^k \text{ we have } \mathscr{A} \models \varphi[\bar{a}] \overset{(*)}{\iff} \mathscr{M} \models \varphi[\bar{a}].$$

If φ is quantifier-free this certainly holds true, since $\mathscr{A}$ is a substructure of $\mathscr{M}$. Moreover $(*)$ holds for $\neg\varphi$ if it holds for φ and $(*)$ holds for $\varphi \wedge \psi$ if it holds for φ and for ψ. Thus, for the induction step, it is enough to assume that $\varphi(x_1,...,x_k)$ is $\exists y\ \psi(y,x_1,...,x_k)$ where $\psi(y,x_1,...,x_k) \in \operatorname{Fml}\mathscr{L}$ and $(*)$ holds for ψ and all $(k+1)$-tuples from A. Take $\bar{a} \in A^k$.

"$\Rightarrow$". If $\mathscr{A} \models \varphi[\bar{a}]$, then there is some $b \in A$ with $\mathscr{A} \models \psi[b,\bar{a}]$ and by the induction hypothesis we have $\mathscr{M} \models \psi[b,\bar{a}]$, thus $\mathscr{M} \models \varphi[\bar{a}]$.

"$\Leftarrow$". If $\mathscr{M} \models \varphi[\bar{a}]$ then $\mathscr{M} \models \exists y\ \psi(y,\bar{a})$. By (ii), there is some $b \in A$ with $\mathscr{M} \models \psi[b,\bar{a}]$. By the induction hypothesis we get $\mathscr{A} \models \psi[b,\bar{a}]$, thus $\mathscr{A} \models \varphi[\bar{a}]$. ☐

2.3.2. **Skolem-Löwenheim downwards**

Let $\mathscr{M}$ be an $\mathscr{L}$-structure and let $A \subseteq |\mathscr{M}|$. Then there is an elementary substructure $\mathscr{N}$ of $\mathscr{M}$ with $A \subseteq |\mathscr{N}|$ such that $\operatorname{card}|\mathscr{N}| \leq \operatorname{card}(A) + \operatorname{card}\mathscr{L} + \aleph_0$.

Proof. The strategy is the following: We want to apply the Tarski-Vaught test 2.3.1. By this test, the only thing we need to do is to add elements to A so that condition (ii) of the test is satisfied for the resulting set. We then have to count the number of elements we have added to get the cardinality estimates. Here the details.

We use the following ad-hoc notation for each $\mathscr{L}$-formula $\varphi(x,\bar{y})$ and all $\bar{a} \in S^{\bar{y}}$: If $\mathscr{M} \models \exists x\ \varphi(x,\bar{a})$, choose an element $b_{\varphi,\bar{a}} \in |\mathscr{M}|$ with

$$\mathscr{M} \models \varphi(b_{\varphi,\bar{a}},\bar{a}).$$

If $\mathscr{M} \not\models \exists x\ \varphi(x,\bar{a})$, then choose $b_{\varphi,\bar{a}} \in |\mathscr{M}|$ arbitrarily.

For an arbitrary subset S of $|\mathscr{M}|$ we denote by S^* the following subset of $|\mathscr{M}|$:

$$S^* = \{b_{\varphi,\bar{a}} \mid \varphi(x,\bar{y}) \in \operatorname{Fml}\mathscr{L} \text{ and } \bar{a} \in S^{\bar{y}}\}.$$

Observe that S^* contains S (if we choose φ as $x \doteq y$ and $a \in S$, then $b_{\varphi,a} = a$). Moreover, S^* is the image under the map

$$D \longrightarrow |\mathscr{M}|;\ (\varphi, \bar{a}) \mapsto b_{\varphi,\bar{a}},$$

where $D = \{(\varphi(x,\bar{y}), \bar{a}) \mid \bar{a} \in S^{\bar{y}}\}$. As $D \subseteq \mathrm{Fml}(\mathscr{L}) \times \bigcup_{n\in\mathbb{N}} S^n$ we see that the cardinality of S^* is at most

$$(\dagger) \qquad \mathrm{card}(\mathrm{Fml}(\mathscr{L}) \times \bigcup_{n\in\mathbb{N}} S^n) \leq \aleph_0 + \mathrm{card}\, S + \mathrm{card}\, \mathscr{L}.$$

By choice of all the $b_{\varphi,\bar{a}}$ we know that all tests performed on S in condition (ii) of the Tarski-Vaught test are solvable in S^*. However, we have introduced new elements and so the test does not give an elementary substructure on S^*.

The trick now is to iterate the construction. So we define (and return to our set A) $A_0 := A$ and by induction on $n \in \mathbb{N}$: $A_n = (A_{n-1})^*$. Finally we take $B := \bigcup_n A_n$ and claim that B satisfies the Tarksi-Vaught test: Take $\varphi(x,\bar{y}) \in \mathrm{Fml}\,\mathscr{L}$ and $\bar{a} \in B^{\bar{y}}$. Since $\bar{a}$ is a finite tuple it is contained in some A_n. But then $b_{\varphi,\bar{a}} \in A_{n+1} \subseteq B$ and so B satisfies condition (ii) of the Tarski-Vaught test.

Finally we take $\mathscr{N}$ as the structure on B given by the Tarski-Vaught test and we need to estimate the cardinality of B: By induction on n using ($\dagger$), we see that $\mathrm{card}\, A_n \leq \aleph_0 + \mathrm{card}\, A + \mathrm{card}\, \mathscr{L}$. Then the countable union B of these sets also has cardinality $\leq \aleph_0 + \mathrm{card}\, A + \mathrm{card}\, \mathscr{L}$. □

2.3.3. Skolem-Löwenheim upwards
Let $\mathscr{M}$ be an infinite $\mathscr{L}$-structure and let κ be a cardinal with $\kappa \geq \mathrm{card}\,|\mathscr{M}| + \mathrm{card}\,\mathscr{L}$. Then there is an elementary extension $\mathscr{N}$ of $\mathscr{M}$ with $\mathrm{card}\,|\mathscr{N}| = \kappa$.

Proof. By 2.2.4, there is an elementary extension $\mathscr{M}'$ of $\mathscr{M}$ of cardinality $\geq \kappa$. Now choose a subset A of $\mathscr{M}'$ of cardinality κ containing $|\mathscr{M}|$ and apply Skolen-Löwenheim downwards for A and $\mathscr{M}'$: We obtain an elementary substructure $\mathscr{N} \prec \mathscr{M}'$ of cardinality at most $\aleph_0 + \mathrm{card}\, A + \mathrm{card}\, \mathscr{L}$. Since $\kappa \geq \mathrm{card}\,|\mathscr{M}| + \mathrm{card}\,\mathscr{L} \geq \aleph_0$, this means that $\mathscr{N}$ has cardinality $\leq \kappa$ and as $A \subseteq |\mathscr{N}|$ we get $\mathrm{card}\,\mathscr{N} = \kappa$ as required. Finally, observe that $\mathscr{N}$ is an elementary extension of $\mathscr{M}$ by exercise 5(ii). □

We conclude this section with another very useful application of the diagram method.

2.3.4. **Proposition.** *Let $\mathscr{M}.\mathscr{N}$ be $\mathscr{L}$-structures. The following are equivalent.*

(i) There is an embedding $\mathscr{M} \longrightarrow \mathscr{N}'$ for some elementary extension $\mathscr{N}'$ of $\mathscr{N}$.
(ii) If $\chi(\bar{x})$ is a quantifier-free $\mathscr{L}$-formula and $\mathscr{M} \models \exists\bar{x}\chi$, then $\mathscr{N} \models \exists\bar{x}\chi$.
(iii) If $\chi(\bar{x})$ is a quantifier-free $\mathscr{L}$-formula and $\mathscr{N} \models \forall\bar{x}\chi$, then $\mathscr{M} \models \forall\bar{x}\chi$.

Proof. (i)⇒(ii) is left as an exercise.

The equivalence of (ii) and (iii) follows by considering negations and contrapositives.

(iii)⇒(i) By 2.2.3 it suffices to find a model of $T^* := \mathrm{diag}(\mathscr{M}) \cup \mathrm{diag}_\infty(\mathscr{N})$, in the language $\mathscr{L}^*$ which has constant symbols c_m, d_n for $m \in |\mathscr{M}|, n \in |\mathscr{N}|$ with the convention that $c_m \neq d_n$.

Suppose there is no such model. Then T^* is inconsistent and therefore there are $\mathscr{L}$-formulas $\chi(\bar{x})$, $\varphi(\bar{y})$, where $\bar{x}, \bar{y}$ are disjoint tuples of variables such that χ is quantifier free and $\bar{a} \in |\mathscr{M}|^{\bar{x}}$, $\bar{b} \in |\mathscr{N}|^{\bar{y}}$ such that $\mathscr{M} \models \chi(\bar{a})$, $\mathscr{N} \models \varphi(\bar{b})$ and $\{\chi(\bar{a}), \varphi(\bar{b})\}$ is inconsistent, when considered as sentences in the language $\mathscr{L}^*$, in other words $\models \varphi(\bar{b}) \to \neg\chi(\bar{a})$ (in $\mathscr{L}^*$). By 1.47, this implies $\models \varphi(\bar{x}) \to \neg\chi(\bar{y})$ (in $\mathscr{L}$). But this means $\models \forall\bar{x},\bar{y}(\varphi(\bar{x}) \to \neg\chi(\bar{y}))$. Since $\bar{x},\bar{y}$ are disjoint tuples of variables we obtain $\models (\exists\bar{x}\varphi(\bar{x})) \to \forall\bar{y}\neg\chi(\bar{y})$. In particular $\mathscr{N} \models (\exists\bar{x}\varphi(\bar{x}) \to \forall\bar{y}\neg\chi(\bar{y}))$. Since $\mathscr{N} \models \varphi(\bar{b})$ we get $\mathscr{N} \models \forall\bar{y}\neg\chi(\bar{y}))$. Now (iii) applied to $\neg\chi$ entails $\mathscr{M} \models \forall\bar{y}\neg\chi(\bar{y}))$, which contradicts $\mathscr{M} \models \chi(\bar{a})$. □

2.3.5. **Corollary.** *Let T be an $\mathscr{L}$-theory. We write $T_\forall$ for the set of all universal sentences that are true in all models of T. If $\mathscr{M}$ is an $\mathscr{L}$-structure, then $\mathscr{M} \models T_\forall$ if and only if $\mathscr{M}$ is a substructure of a model of T.*

[In 3.6.5 we will see a similar statement for sentences with higher quantifier complexity.]

Proof. Since truth of universal sentences in any structure is inherited by all of its substructures, we know that every substructure of a model of T is a model of $T_\forall$.

Conversely suppose $\mathscr{M} \models T_\forall$. Then $T \cup \{\neg\sigma \mid \sigma$ a universal sentence, $\mathscr{M} \models \neg\sigma\}$ is consistent. Take any model $\mathscr{N}$ of this set. Then condition (iii) of 2.3.4 is satisfied and by 2.3.4(iii)(i) there is an elementary extension $\mathscr{N}'$ of $\mathscr{N}$ and an embedding $\mathscr{M} \hookrightarrow \mathscr{N}$. Since $\mathscr{N}' \succ \mathscr{N}$ we get $\mathscr{N}' \models T$. By assumption, also $\mathscr{M} \models T$, as required. □

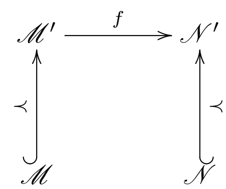
2.3.6. **Corollary.** *An $\mathscr{L}$-theory T is* **universal** *(i.e. T and $T_\forall$ have the same models) if and only if every substructure of a model of T is again a model of T.*

Proof. Apply 2.3.5. □

2.4. **Categoricity by examples.**

2.4.1. *Example.* Let $\mathscr{L} = \emptyset$ be the language of sets. If $\mathscr{M}$ and $\mathscr{N}$ are infinite sets, then they are elementary equivalent as $\mathscr{L}$-structures. Another way of saying this is: The theory of infinite sets is complete.

Proof. Choose a cardinal κ of at least the cardinality of $\mathscr{M}$ and $\mathscr{N}$. By 2.3.3, there are elementary extensions $\mathscr{M}' \succ \mathscr{M}$ and $\mathscr{N}' \succ \mathscr{N}$ both of cardinality κ. In particular $\mathscr{M}'$ and $\mathscr{N}'$ are of the same cardinality and there is a bijective map $f : \mathscr{M}' \longrightarrow \mathscr{N}'$. So we are in the following situation:



where the vertical maps are the inclusion (they are elementary). Since the vertical maps are elementary we get $\mathscr{M} \equiv \mathscr{M}'$ and $\mathscr{N} \equiv \mathscr{N}'$. Since f is a bijection and $\mathscr{L}$ has no non-logical symbols, f is an isomorphism $\mathscr{M}' \longrightarrow \mathscr{N}'$ and we know that $\mathscr{M}' \equiv \mathscr{N}'$ in this case. Altogether we get $\mathscr{M} \equiv \mathscr{M}' \equiv \mathscr{N}' \equiv \mathscr{N}$ as desired. □

The strategy in the proof of 2.4.1 is indeed also applicable in more complicated situations. The next example we are dealing with is vector spaces.

2.4.2. *Example.* Given a field F we first set up the language for F-vector spaces: The language of F-vector spaces, denoted by $\mathscr{L}_{F-\text{vec.sp.}}$ has no relation symbols, one constant symbol $\underline{0}$, one binary function symbol $\underline{+}$ and for each $a \in F$ a unary function symbol λ_a.

If V is an F-vector space then V is turned (naturally) into an $\mathscr{L}_{F-\text{vec.sp.}}$-structure $\mathscr{V}$, by interpreting

- $a \underline{+}^{\mathscr{V}} b := a + b$ (the sum in V).
- $\underline{0}^{\mathscr{V}} := 0$ (the neutral element in V w.r.t. $+$)
- $\lambda_a^{\mathscr{V}}(v) := a \cdot v$, where $v \in V$ and $a \cdot v$ is the scalar multiplication in V of $a \in F$ with v.

The theory $T_{F-\text{vec.sp.}}$ of F-vector spaces is defined to be the deductive closure of the axioms of vector spaces in the language $\mathscr{L}_{F-\text{vec.sp.}}$: These are the axioms of abelian groups (in the language $(+,-,0)$) together with the following sentences:

- for all $a \in F$ the sentence $\forall xy\ \lambda_a(x+y) = \lambda_a(x) + \lambda_b(x)$.
- for all $a, b \in F$ the sentence $\forall x\ \lambda_{a+b}(x) = \lambda_a(x) + \lambda_b(x)$.
- for all $a, b \in F$ the sentence $\forall x\ \lambda_a(\lambda_b(x)) = \lambda_{a \cdot b}(x)$
- the sentence $\forall x\ \lambda_1(x) = x$.

Now we claim that all infinite F-vector spaces are elementary equivalent. In other words, the theory of infinite F-vector spaces is complete.

Proof. Let V, W be our F-vector spaces and let κ be an uncountable cardinal that is at least the cardinality of V and W. Moreover we choose κ strictly bigger than the cardinality of the field F. By Skolem-Löwenheim upwards (using that V and W are infinite), there are elementary extensions $V' \succ V$ and $W' \succ W$ of cardinality κ.

As in 2.4.1 it now suffices to show that V' and W' are isomorphic. Consider the following counting argument from linear algebra:

Let U be an F-vector space and let B be a basis of U, i.e. B is a maximally linearly independent subset of U. Then every element of U is a finite linear combination of elements from B and we get a surjection

$$S := \{\mathrm{Maps}(E, F) \mid E \subseteq B \text{ finite}\} \twoheadrightarrow U,$$

which sends $\sigma : E \longrightarrow F$ to $\sum_{e \in E} \sigma(e) \cdot e$. In particular, the cardinality of U is at most the cardinality of the set S on the left hand side. We estimate the cardinality of S: For $E \subseteq B$ finite, the cardinality of $\mathrm{Maps}(E, F)$ is at most $\max\{\aleph_0, \mathrm{card}(F)\}$. Hence

$$\mathrm{card}\, S \leq (\max\{\aleph_0, \mathrm{card}(F)\}) \cdot \mathrm{card}\{E \subseteq B \mid E \text{ finite}\}.$$

Now $\mathrm{card}\{E \subseteq B \mid E \text{ finite}\} \leq \max\{\aleph_0, \mathrm{card}(B)\}$ and so

$$\mathrm{card}\, S \leq \max\{\aleph_0, \mathrm{card}(F), \mathrm{card}(B)\}.$$

Altogether it follows

$$(*) \qquad \mathrm{card}(B) \leq \dim U \leq \mathrm{card}\, S \leq \max\{\aleph_0, \mathrm{card}(F), \mathrm{card}(B)\}.$$

Hence if $\mathrm{card}\, U$ is bigger than $\aleph_0$ and bigger than $\mathrm{card}\, F$, then

$$\max\{\aleph_0, \mathrm{card}(F), \mathrm{card}(B)\} = \mathrm{card}(B)$$

and the estimation $(*)$ becomes $\mathrm{card}\, B = \mathrm{card}\, U$, in other words the dimension of U is the cardinality of U.

Returning to our vector spaces V' and W', this explains our choice of κ: κ is the dimension of V' and of W'. However, vector spaces of the same dimension are isomorphic (if you have some doubts here, because we are working with infinite dimensional vector spaces, then consider this as an exercise). □

2.4.3. *Examples.* Let $\mathscr{L} = (\leq)$ be the language of po-sets (partially ordered sets) and let T be the $\mathscr{L}$-theory of **densely, totally ordered sets without endpoints**. This theory is axiomatised by the following $\mathscr{L}$-sentences:

- Axioms for po-sets: $\forall x \leq x$, $\forall xy\ (x \leq y \wedge y \leq x \rightarrow x \doteq y)$, $\forall xyz\ (x \leq y \wedge y \leq z \rightarrow x \leq z)$
- The axiom of totality (or linearity) $\forall xy\ (x \leq y \vee y \leq x)$.
- The density axiom $\forall xy\ (x < y \rightarrow \exists z\ x < z < y)$, where $x < y$ stands for $x \leq y$ and $x \neq y$.
- Axioms which say that models have no endpoint: $\forall x\ \exists yz\ y < x < z$.

We will show that all models of T are elementary equivalent, in other words, T is complete.

Proof. This time we will use Skolem-Löwenheim downwards instead of Skolem-Löwenheim upwards:

Firstly, convince yourself that all models of T are infinite. If $\mathscr{M}$ and $\mathscr{N}$ are models of T, then by the Skolem-Löwenheim downwards theorem 2.3.2 says that there are countable elementary substructures $\mathscr{M}' \prec \mathscr{M}$ and $\mathscr{N}' \prec \mathscr{N}$. We will show below that all countable models of T are isomorphic. Choosing an isomorphism

$f : \mathscr{M}' \longrightarrow \mathscr{N}'$ we proceed as follows: We have

$$\begin{array}{ccc} \mathscr{M} & & \mathscr{N} \\ \uparrow \prec & & \uparrow \prec \\ \mathscr{M}' & \xrightarrow[f]{} & \mathscr{N}' \end{array}$$

where the vertical maps are the inclusion (they are elementary). It follows

$$\mathscr{M} \overset{\text{as } \mathscr{M}' \prec \mathscr{M}}{\equiv} \mathscr{M}' \overset{\text{as } f \text{ is an isomorphism}}{\equiv} \mathscr{N}' \overset{\text{as } \mathscr{N}' \prec \mathscr{N}}{\equiv} \mathscr{N}$$

It remains to show that two countable, densely ordered sets without endpoints are isomorphic. This will be explained in detail now, because the method in the proof has a vast generalisation and more consequences than in our situation. It will be studied later for structures in arbitrary languages. The method is the **back-and-forth technique**: Let us denote our countable, densely ordered sets without endpoints by $X = (X, \leq)$ and $Y = (Y, \leq)$ (using the ordinary abuse of mathematical language to increase readability).

We enumerate the set X by $x_1, x_2, x_3, ...$ and the set Y by $y_1, y_2, y_3, ...$ Our goal is to construct by induction on n the following objects:

(1) a finite subset X_n of X containing X_{n-1} and containing $x_1, ..., x_k$ provided $n = 2k$ is even (so if n is odd, the only requirement is $X_{n-1} \subseteq X_n$)
(2) a finite subset Y_n of Y containing Y_{n-1} and containing $y_1, ..., y_k$, provided $n = 2k - 1$ is odd (so if n is even, the only requirement is $Y_{n-1} \subseteq Y_n$)
(3) an isomorphism $f_n : X_n \longrightarrow Y_n$ which extends $f_{n-1} : X_{n-1} \longrightarrow Y_{n-1}$. Here by isomorphism we mean an isomorphism between the induced (total) orders.

To start with, we take $X_1 = \{x_1\}$, $Y = \{y_1\}$ and f_1 the unique map $X_1 \longrightarrow Y_1$. Suppose we have already constructed X_{n-1}, Y_{n-1} and $f : X_{n-1} \longrightarrow Y_{n-1}$.

Case 1. $n = 2k$ is even (The "forth"-construction):

We already know by induction that $x_1, ..., x_{k-1} \in X_{n-2} \subseteq X_{n-1}$ (provided $n > 2$).

Case 1.1. $x_k \in X_{n-1}$. Then we take $X_n = X_{n-1}$, $Y_n = Y_{n-1}$ and $f_n = f_{n-1}$.

Case 1.2. $x_k < X_{n-1}$. Since Y_{n-1} is finite and Y has no smallest element, there is some $y \in Y$ with $y < Y_{n-1}$. We take $X_n = X_{n-1} \cup \{x_k\}$, $Y_n = Y_{n-1} \cup \{y\}$ and extend f_{n-1} to $f_n : X_n \longrightarrow Y_n$ by $f_n(x_k) = y$.

Case 1.3. $X_{n-1} < x_k$. Since Y_{n-1} is finite and Y has no largest element, there is some $y \in Y$ with $Y_{n-1} < y$. We take $X_n = X_{n-1} \cup \{x_k\}$, $Y_n = Y_{n-1} \cup \{y\}$ and extend f_{n-1} to $f_n : X_n \longrightarrow Y_n$ by $f_n(x_k) = y$.

Case 1.4. There are $x, x' \in X_{n-1}$ with $x < x_k < x'$ such that x' is the immediate successor of x in X_{n-1} (recall that by induction X_{n-1} is finite, so this is indeed the remaining case).

Since $f_{n-1} : X_{n-1} \longrightarrow Y_{n-1}$ is an isomorphism of po-sets, $f_{n-1}(x')$ is also the immediate successor of $f_{n-1}(x)$ in Y_{n-1}. Since Y is densely ordered, there is some $y \in Y$ with $f_{n-1}(x) < y < f_{n-1}(x')$. We take $X_n = X_{n-1} \cup \{x_k\}$, $Y_n = Y_{n-1} \cup \{y\}$ and extend f_{n-1} to $f_n : X_n \longrightarrow Y_n$ by $f_n(x_k) = y$.

In all 4 cases it is straightforward to check that f_n is an isomorphism $X_n \longrightarrow Y_n$ that extends f_{n-1}. This finishes the "forth"-construction.

Case 2. $n = 2k - 1$ is odd (The "back"-construction):

This is the same as case 1 where the roles of X, Y, X_{n-1}, Y_{n-1} and f_{n-1}, f_{n-1}^{-1} are swapped:

We already know by induction that $y_1, ..., y_{k-1} \in Y_{n-2} \subseteq Y_{n-1}$ (provided $n > 2$).
Case 2.1. $y_k \in Y_{n-1}$. Then we take $Y_n = Y_{n-1}$, $X_n = X_{n-1}$ and $f_n = f_{n-1}$.
Case 2.2. $y_k < Y_{n-1}$. Since X_{n-1} is finite and X has no smallest element, there is some $x \in X$ with $x < X_{n-1}$. We take $Y_n = Y_{n-1} \cup \{y_k\}$, $X_n = X_{n-1} \cup \{x\}$ and extend f_{n-1} to $f_n : Y_n \longrightarrow X_n$ by $f_n(x) = y_k$.
Case 2.3. $Y_{n-1} < y_k$. Since X_{n-1} is finite and X has no largest element, there is some $x \in X$ with $X_{n-1} < x$. We take $Y_n = Y_{n-1} \cup \{y_k\}$, $X_n = X_{n-1} \cup \{x\}$ and extend f_{n-1} to $f_n : Y_n \longrightarrow X_n$ by $f_n(x) = y_k$.
Case 2.4. There are $y, y' \in Y_{n-1}$ with $y < y_k < y'$ such that y' is the immediate successor of y in Y_{n-1} (recall that by induction Y_{n-1} is finite, so this is indeed the remaining case).

Since $f_{n-1} : Y_{n-1} \longrightarrow X_{n-1}$ is an isomorphism of po-sets, $f_{n-1}^{-1}(y')$ is also the immediate successor of $f_{n-1}^{-1}(y)$ in X_{n-1}. Since X is densely ordered, there is some $x \in X$ with $f_{n-1}^{-1}(y) < x < f_{n-1}^{-1}(y')$. We take $Y_n = Y_{n-1} \cup \{y_k\}$, $X_n = X_{n-1} \cup \{y\}$ and extend f_{n-1} to $f_n : Y_n \longrightarrow X_n$ by $f_n(x) = y_k$.

In all 4 cases it is straightforward to check that f_n is an isomorphism $Y_n \longrightarrow X_n$ that extends f_{n-1}. This finishes the "back"-construction.

So we have constructed the family $f_n : X_n \longrightarrow Y_n$ as indicated. Since f_n extends f_{n-1} for all n, we can take a common extension $f : \bigcup_n X_n \longrightarrow \bigcup_n Y_n$. Clearly f is an isomorphism. Finally, $\bigcup_n X_n = X$ by (1) and $\bigcup_n Y_n = Y$ by (2). □

Obviously in the previous three examples a general property of the class of models of a theory is responsible for the completeness of a theory. This is called **categoricity**: An $\mathscr{L}$-theory T without finite models is **categorical in an infinite cardinal** κ if all models of T of cardinality κ are isomorphic. We have

2.4.4. **Theorem.** *If T has no finite models and T is categorical in some infinite cardinal $\geq \operatorname{card} \mathscr{L}$, then T is complete.*

Proof. Let T be categorical in the cardinal $\kappa \geq \operatorname{card}(\mathscr{L})$. We have to show that all models of T are elementary equivalent. Take $\mathscr{M}, \mathscr{N} \models T$. By Skolem-Löwenheim upwards OR Skolem-Löwenheim downwards, there are $\mathscr{L}$-structures $\mathscr{M}'$ and $\mathscr{N}'$ with $\mathscr{M} \equiv \mathscr{M}'$,$\mathscr{N} \equiv \mathscr{N}'$ such that $\mathscr{M}'$ and $\mathscr{N}'$ are of cardinality κ. By assumption, $\mathscr{M}'$ and $\mathscr{N}'$ are isomorphic, in particular they are elementary equivalent. It follows $\mathscr{M} \equiv \mathscr{M}' \equiv \mathscr{N}' \equiv \mathscr{N}$ as required. □

Another example where this theorem applies is the theory of algebraically closed fields. Recall that a field K is algebraically closed if every non-constant polynomial $p(T) \in K[T]$ in one indeterminate has a zero in K. For example the field of complex numbers is algebraically closed.

2.4.5. *Example.* Let $\mathscr{L} = (+, -, \cdot, 0, 1)$ be the language of rings. Let ACF be the $\mathscr{L}$-theory of algebraically closed fields. This theory is axiomatised by the axioms of fields together with all the axioms

$$\forall x_0 \dots x_{d-1} \; \exists y \; y^d + x_{d-1} y^{d-1} + \dots + x_0 = 0,$$

where $d \in \mathbb{N}$.

If p is a prime number or 0, then ACF_p denotes the $\mathscr{L}$-theory of algebraically closed fields of characteristic p, which is axiomatized by ACF and

- $p \cdot 1 \doteq 0$, if $p \neq 0$, OR
- all the statements $n \cdot 1 \neq 0$ ($n \in \mathbb{N}$), if $p = 0$.

We claim that ACF_p is complete. In fact ACF_p is categorical in all uncountable cardinals, hence completeness follows with 2.4.4.

Proof. Let κ be an uncountable cardinal. The categoricity of ACF_p is a statement of algebra that you might or might not have seen. It says:

If K, L are algebraically closed fields of the same characteristic, then they are isomorphic if and only if they have the same **transcendence degree**. The transcendence degree of an arbitrary field K is the cardinality of a maximal algebraically independent subset B of K. The situation is similar to the one in vector spaces: If we replace "linear independence" by "algebraic independence" this gives a notion of dimension for fields (called 'transcendence degree') and a proof similar to the one given in example 2.4.2 shows that the cardinality of an uncountable field is equal to the cardinality of some/any maximally algebraically independent subset. Hence if K and L are uncountable fields of the same cardinality, then they have the same transcendence degree. If in addition K and L have the same characteristic and both are algebraically closed, then they are isomorphic. This is not difficult to prove, in particular if you have seen some introduction to algebra. We refer to [Lang2002, chap. V, §1–2]. $\square$

2.5. **Chains.**

Let I be a totally ordered index set and let $\mathscr{M}_i$ be an $\mathscr{L}$-structure for each $i \in I$. Suppose for all $i \leq j$ from I, $\mathscr{M}_i$ is a substructure of $\mathscr{M}_j$. Then we call the family $(\mathscr{M}_i \mid i \in I)$ a **chain of $\mathscr{L}$-structures**. Given such a chain, we define a new structure $\mathscr{M}$, called the **union of the $\mathscr{M}_i$** as follows:

- The universe M of $\mathscr{M}$ is $\bigcup_{i \in I} |\mathscr{M}_i|$.
- For an n-ary relation symbol R and elements $a_1, ..., a_n \in M$ we define
$$(a_1, ..., a_n) \in R^{\mathscr{M}} \iff \text{there is some } i \in I \text{ with } (a_1, ..., a_n) \in R^{\mathscr{M}_i}.$$
Observe that for $i \leq j$ from I and $(a_1, ..., a_n) \in |\mathscr{M}_i|$ we have $(a_1, ..., a_n) \in R^{\mathscr{M}_i} \iff (a_1, ..., a_n) \in R^{\mathscr{M}_j}$.
- For an n-ary function symbol F and elements $a_1, ..., a_n \in M$ we define
$$F^{\mathscr{M}}(a_1, ..., a_n) = F^{\mathscr{M}_i}(a_1, ..., a_n) \text{ whenever } i \in I \text{ with } (a_1, ..., a_n) \in |\mathscr{M}_i|.$$
Observe that for $i \leq j$ from I and $(a_1, ..., a_n) \in |\mathscr{M}_i|$ we have
$$F^{\mathscr{M}_i}(a_1, ..., a_n) = F^{\mathscr{M}_j}(a_1, ..., a_n),$$
so this indeed makes sense.
- For a constant symbol we define $c^{\mathscr{M}}$ as $c^{\mathscr{M}_i}$ for some/every i.

We write $\bigcup_{i \in I} \mathscr{M}_i$ for this structure.

2.5.1**. Elementary chain lemma**

We again write $\mathscr{M} = \bigcup_{i \in I} \mathscr{M}_i$. For every $i \in I$, $\mathscr{M}_i$ is an $\mathscr{L}$-substructure of $\mathscr{M}$. If in addition all extensions $\mathscr{M}_i \hookrightarrow \mathscr{M}_j$ are elementary $(i \leq j \in I)$, then also $\mathscr{M}_i \hookrightarrow \mathscr{M}$ is elementary for all $i \in I$.

Proof. $\mathscr{M}_i$ is a substructure of $\mathscr{M}$, since by definition, the inclusion $\mathscr{M}_i \hookrightarrow \mathscr{M}$ is a homomorphism and by the remark in the definition of $R^{\mathscr{M}}$ this inclusion is an embedding (use 2.1.6). So we only need to prove that $\mathscr{M}_i \prec \mathscr{M}$ for all i. We take $\varphi \in \mathrm{Fml}\,\mathscr{L}$ in prenex normal form and show by induction on the number of quantifiers that

$$(*) \qquad \text{for all } i \in I, \text{ the inclusion } \mathscr{M}_i \hookrightarrow \mathscr{M} \text{ preserves } \varphi.$$

If φ is quantifier-free then we know this already, since all $\mathscr{M}_i$ are substructures of $\mathscr{M}$. If φ is of the form $\exists u \psi$ and we know $(*)$ for ψ, then it is easy to see that $(*)$ holds for φ (exercise!). So the substantial case here is when φ is of the form $\forall u \psi$, where we may assume that $(*)$ holds for $\psi(u, \bar{x}) \in \mathrm{Fml}\,\mathscr{L}$. Take an $\bar{x}$-tuple $\bar{a}$ with entries in $\mathscr{M}_i$ and assume $\mathscr{M}_i \models \forall u \psi[\bar{a}]$. We must show $\mathscr{M} \models \forall u \psi[\bar{a}]$. So take some $b \in |\mathscr{M}|$ and choose $j \in I$ with $b \in \mathscr{M}_j$. If $j \leq i$, then we may replace j by i as $\mathscr{M}_j \subseteq \mathscr{M}_i$ in this case. So we may assume that $i \leq j$, hence $\mathscr{M}_i \subseteq \mathscr{M}_j$. As $\mathscr{M}_i \prec \mathscr{M}_j$ by assumption and $\mathscr{M}_i \models \forall u \psi[\bar{a}]$ we know $\mathscr{M}_j \models \forall u \psi[\bar{a}]$, in particular $\mathscr{M}_j \models \psi[b, \bar{a}]$. Now by induction $\mathscr{M} \models \psi[b, \bar{a}]$, as desired. □

There will be more results about chains presented in 3.6.6.

2.6. **Intersections and generated substructures.**

Let $\mathscr{L} = (\lambda : \mathscr{R} \longrightarrow \mathbb{N}, \mu : \mathscr{F} \longrightarrow \mathbb{N}, \mathscr{C})$ be a language.

In the case $\mathscr{C} = \emptyset$, it is convenient to extend the definition of a structure also to the empty set. We have no choice and must interpret all relation symbols and all function symbols as the empty set. We call this structure the **empty structure** and denote it by $\emptyset$ again. Observe that $\emptyset$ is a substructure of every $\mathscr{L}$-structure.

2.6.1. **Definition.** Let $\mathscr{N}$ be an $\mathscr{L}$-structure and let $(\mathscr{M}_i \mid i \in I)$ be an arbitrary family of substructures of $\mathscr{N}$. Using 2.1.7 it is clear that the intersection of the universes of all the $\mathscr{M}_i$ is again the universe of a substructure $\mathscr{P}$ of $\mathscr{N}$. $\mathscr{P}$ is called the **intersection of the** $\mathscr{M}_i$ and denoted by

$$\bigcap_{i \in I} \mathscr{M}_i.$$

2.6.2. **Definition.** Let $\mathscr{M}$ be an $\mathscr{L}$-structure and let $A \subseteq |\mathscr{M}|$. The **substructure generated by** A is defined as the intersection of all substructures $\mathscr{N}$ of $\mathscr{M}$ with the property $A \subseteq |\mathscr{N}|$ and it is denoted by

$$\langle A \rangle_{\mathscr{M}}.$$

2.6.3. **Proposition.**

$$\begin{aligned}\langle A \rangle_{\mathscr{M}} &= \bigcup \{ t^{\mathscr{M}}(A^n) \mid t(x_1, ..., x_n) \textit{ is an } \mathscr{L}\textit{-term} \} = \\ &= \{ t^{\mathscr{M}}(\bar{a}) \mid t(\bar{x}) \textit{ is an } \mathscr{L}\textit{-term and } \bar{a} \in A^{\bar{x}} \}.\end{aligned}$$

Proof. This is again immediate from 2.1.7. □

2.7. **Back and Forth equivalence.**

2.7.1. **Definition.** Let $\mathscr{M}$ and $\mathscr{N}$ be $\mathscr{L}$-structures. A **partial isomorphism** $\mathscr{M} \longrightarrow \mathscr{N}$ is a triple (p, A, B) such that

(1) $A \subseteq |\mathscr{M}|$,
(2) $B \subseteq |\mathscr{N}|$,
(3) $p : A \longrightarrow B$ is a bijection and
(4) for every quantifier-free $\mathscr{L}$-formula $\varphi(x_1, ..., x_n)$ and all $a_1, ..., a_n \in A$ we have

$$\mathscr{M} \models \varphi[a_1, ..., a_n] \iff \mathscr{N} \models \varphi[p(a_1), ..., p(a_n)].$$

2.7.2. *Remarks.*

(i) If (p, A, B) is a partial isomorphism $\mathscr{M} \longrightarrow \mathscr{N}$, then obviously, (p^{-1}, B, A) is a partial isomorphism $\mathscr{N} \longrightarrow \mathscr{M}$.
(ii) In the definition of a partial isomorphism, A and B may be empty. In this case p is empty, too. This partial isomorphism (provided it exists) is called the **empty partial isomorphism** $\mathscr{M} \longrightarrow \mathscr{N}$.
(iii) The notation (p, A, B), strictly speaking is slightly ambiguous because the definition a priori depends also on $\mathscr{M}$ and $\mathscr{N}$. However, given $A \subseteq |\mathscr{M}|$, $B \subseteq |\mathscr{N}|$ and a map $p : A \longrightarrow B$ the following are equivalent.
    (i) (p, A, B) is a partial isomorphism $\mathscr{M} \longrightarrow \mathscr{N}$.
    (ii) There is an isomorphism $f : \langle A \rangle_{\mathscr{M}} \longrightarrow \langle B \rangle_{\mathscr{N}}$ with the property $f(a) = p(a)$ for every $a \in A$.

Proof. Exercise 12. □

2.7.3. **Definition.** A **back and forth system** between infinite $\mathscr{L}$-structures $\mathscr{M}$ and $\mathscr{N}$ is a non-empty family $((p_i, A_i, B_i) \mid i \in I)$ of partial isomorphisms $\mathscr{M} \longrightarrow \mathscr{N}$ satisfying the following conditions:

Forth: For all $i \in I$ and every $a \in |\mathscr{M}|$ there is some $j \in I$ with $a \in A_j$ such that p_j extends p_i; hence also $A_i \subseteq A_j$ and $B_i \subseteq B_j$.

Back: For all $i \in I$ and every $b \in |\mathscr{N}|$ there is some $j \in I$ with $b \in B_j$ such that p_j extends p_i.

If there is a Back and Forth system between $\mathscr{M}$ and $\mathscr{N}$, then $\mathscr{M}$ and $\mathscr{N}$ are called **back and forth equivalent**.

Observe that many times the empty partial isomorphism $\mathscr{M} \longrightarrow \mathscr{N}$ is in the system, which explicitly is allowed. Also, if we have a Back and Forth system we may always add the empty partial isomorphism $\mathscr{M} \longrightarrow \mathscr{N}$ to get again a Back and Forth system $\mathscr{M} \longrightarrow \mathscr{N}$.

A trivial but useful observation is the following.

2.7.4. **Observation.** *If $((p_i, A_i, B_i) \mid i \in I)$ is a Back and Forth system between $\mathscr{M}$ and $\mathscr{N}$, then $((p_i^{-1}, B_i, A_i) \mid i \in I)$ is a Back and Forth system between $\mathscr{N}$ and $\mathscr{M}$.*

2.7.5. *Example.* In example 2.4.3 we have constructed a Back and Forth system between any two countable, densely, totally ordered sets without endpoints. There, we actually deduced that the involved structures are isomorphic!

Moreover, in this example any two partial isomorphisms (f_i, X_i, Y_i) have the property that f_i extends f_j or vice versa. This is in general not the case and in general, also structures of different cardinalities can be Back and Forth equivalent: Consider again densely, totally ordered sets $(X, \leq)$ and $(Y, \leq)$ without endpoints, but this time without any cardinality restriction. We show that $(X, \leq)$ and $(Y, \leq)$ are Back and Forth equivalent. We define the following family of partial isomorphisms. Let the index set I be defined as

$$I = \{(A, B) \mid A \subseteq X, B \subseteq Y,\ A, B \text{ finite of the same size}\}.$$

For $i = (A_i, B_i) \in I$, let $p_i : A_i \longrightarrow B_i$ be the unique bijection which preserves the orders induced by X and Y on A_i, B_i respectively. Then the same case by case analysis as in 2.4.3 shows that $((p_i, A_i, B_i) \mid i \in I)$ is a Back and Forth system between X and Y.

2.7.6. **Theorem.** *If $\mathscr{M}$ and $\mathscr{N}$ are back and forth equivalent $\mathscr{L}$-structures, then they are elementary equivalent.*

Proof. We show by induction on the complexity of $\mathscr{L}$-formulas $\varphi(x_1, ..., x_n)$ the following:

$$(*) \qquad \begin{gathered} \text{If } i \in I \text{ and } a_1, ..., a_n \in A_i \text{ then} \\ \mathscr{M} \models \varphi[a_1, ..., a_n] \iff \mathscr{N} \models \varphi[p_i(a_1), ..., p_i(a_n)]. \end{gathered}$$

If φ is quantifier-free then this holds true by the definition of partial isomorphisms. For the induction step, observe that the case of boolean connectives is obvious: If $(*)$ holds for ψ and γ, then $(*)$ also holds for $\psi \wedge \gamma$ and for $\neg\psi$.

Therefore, it remains to show that $(*)$ holds for $\exists y \psi(\bar{x}, y)$ if it holds for $\psi(\bar{x}, y)$. So let us assume that $(*)$ holds for $\psi(\bar{x}, y)$. We prove $(*)$ for $\exists y \psi(\bar{x}, y)$:

$\underline{\Rightarrow}$: We have $\mathscr{M} \models \exists y \psi[a_1, ..., a_n]$, i.e. there is some $a \in |\mathscr{M}|$ with $\mathscr{M} \models \psi[a_1, ..., a_n, a]$. Now by the "Forth"-condition for our Back and Forth system, there is some $j \in I$ such that $a \in A_j$ and p_j extends p_i. We may now apply the implication "$\Rightarrow$" of the induction hypothesis for p_j and $a_1, ..., a_n, a$ and $\psi(x_1, ..., x_n, y)$ to obtain $\mathscr{N} \models \psi[p_j(a_1), ..., p_j(a_n), p_j(a)]$. As p_j extends p_i we get

$$\mathscr{N} \models \psi[p_i(a_1), ..., p_i(a_n), p_j(a)],$$

in particular $\mathscr{N} \models \exists y \psi[p_i(a_1), ..., p_i(a_n)]$, as desired.

$\underline{\Leftarrow}$: Similar to "$\Rightarrow$", using the "Back" condition for our Back and Forth system. Another proof is the following: The implication "$\Leftarrow$" for our Back and Forth system is the implication "$\Rightarrow$" for the inverse Back and Forth system $((p_i^{-1}, B_i, A_i) \mid i \in I)$.

This finishes the induction and so we know that $(*)$ holds true for all formulas. Now by assumption our Back and Forth system contains at least one element, i.e. we can actually apply $(*)$!! Doing this for sentences (and any i) shows

$$\mathscr{M} \models \varphi \iff \mathscr{N} \models \varphi$$

for all $\mathscr{L}$-sentences φ. Thus $\mathscr{M} \equiv \mathscr{N}$. □

2.8. **The Elementary Joint Embedding Theorem.**

2.8.1. **Elementary Joint Embedding Theorem**

Let I be an arbitrary index set and for $i \in I$ let $\mathscr{M}_i$ be an $\mathscr{L}$-structure. If all $\mathscr{M}_i$ are elementary equivalent, then there is an $\mathscr{L}$-structure $\mathscr{P}$ in which all $\mathscr{M}_i$ embed, i.e. there are elementary embeddings $\mathscr{M}_i \longrightarrow \mathscr{P}$ for every i.

Proof. We will first show that two elementary equivalent $\mathscr{L}$-structures $\mathscr{M}$ and $\mathscr{N}$ can be elementarily embedded into another $\mathscr{L}$-structure. We choose $\overline{\text{disjoint sets}}$ of new (w.r.t. $\mathscr{L}$) constants $\underline{M} = \{\underline{m} \mid m \in |\mathscr{M}|\}$, $\underline{N} = \{\underline{n} \mid n \in |\mathscr{N}|\}$ and work in the language $\mathscr{L}(\underline{M} \cup \underline{N})$. So this language is the "union" of the languages $\mathscr{L}(\mathscr{M})$ and $\mathscr{L}(\mathscr{N})$, where it is important to notice that the newly introduced constants are disjoint.

By 2.2.3(iii) it is enough to show that

$$\Sigma := \text{diag}_\infty(\mathscr{M}) \cup \text{diag}_\infty(\mathscr{N})$$

is satisfiable. Suppose this is not the case. Then by compactness there are finite subsets $\Gamma \subseteq \text{diag}_\infty(\mathscr{M})$ and $\Delta \subseteq \text{diag}_\infty(\mathscr{N})$ such that

$$(*) \qquad \Gamma \cup \Delta \text{ is not satisfiable.}$$

Since all elementary diagrams are closed under finite conjunctions we may assume that $\Gamma = \{\gamma\}$ and $\Delta = \{\delta\}$ with $\gamma \in \text{diag}_\infty(\mathscr{M})$ and $\delta \in \text{diag}_\infty(\mathscr{N})$.

From $(*)$ it follows that

$$(+) \qquad \gamma \models \neg\delta.$$

By definition, δ is of the form $\psi(y_1/\underline{n_1}, ..., y_l/\underline{n_l})$, for some $\mathscr{L}$-formula $\psi(y_1, ..., y_l)$ and some $n_1, ..., n_l \in |\mathscr{N}|$. By choice of the new constant symbols, the constants symbols $\underline{n_1}, ..., \underline{n_l}$ are new w.r.t. the language $\mathscr{L}(\mathscr{M})$, whereas δ is an $\mathscr{L}(\mathscr{M})$-sentences. Using 1.46 we therefore deduce from $(+)$ that

$$(++) \qquad \gamma \models \forall y_1...y_l\ \neg\psi.$$

(Convince yourself that this is indeed a consequence of 1.46).

Now $(++)$ implies

$$(+++) \qquad \models \gamma \to \forall y_1...y_l\ \neg\psi.$$

By definition, γ is of the form $\varphi(x_1/\underline{m_1}, ..., x_l/\underline{m_k})$, for some $\mathscr{L}$-formula $\varphi(x_1, ..., x_k)$ and some $m_1, ..., m_k \in |\mathscr{M}|$. Clearly $(+++)$ implies

$$(\dagger) \qquad \models (\exists x_1, ...x_k\varphi) \to \forall y_1...x_l\ \neg\psi.$$

Since $\gamma \in \text{diag}_\infty(\mathscr{M})$ we know from 1.46 that $\mathscr{M} \models \exists x_1, ...x_k\varphi$. As $\mathscr{N} \equiv \mathscr{M}$, also $\mathscr{N} \models \exists x_1, ...x_k\varphi$. Together with $(\dagger)$ we infer $\mathscr{N} \models \forall y_1...y_l\ \neg\psi$.

However, $\delta = \psi(y_1/\underline{n_1}, ..., y_l/\underline{n_l})$ is in $\text{diag}_\infty(\mathscr{N})$ which is equivalent to $\mathscr{N} \models \psi[n_1, ..., n_l]$, a contradiction.

This finishes the proof in the case of two models. For the general case, we work in the language $\mathscr{L}^+$ which expands $\mathscr{L}$ and has new constant symbols $\underline{a}$ for every $a \in \mathscr{M}_i$, where we assume that $\underline{a} \neq \underline{b}$, if $a \in |\mathscr{M}_i|$ and $b \in |\mathscr{M}_j|$ with $i \neq j$.

By 2.2.3(iii) the assertion of the theorem is equivalent to the satisfiability of

$$\Sigma := \bigcup_{i \in I} \text{diag}_\infty(\mathscr{M}_i).$$

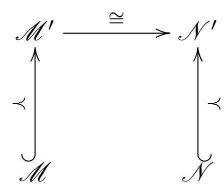
By compactness, we may then assume that I in fact is finite. Now the finite case follows by a trivial induction on the size k of I once we know it for $k = 2$. But this is what we have shown already. □

2.8.2. **Corollary.** *The following are equivalent for all $\mathscr{L}$-structures $\mathscr{M}$ and $\mathscr{N}$.*

(i) $\mathscr{M}$ and $\mathscr{N}$ are elementary equivalent

(ii) There is an elementary embedding $\mathscr{M} \longrightarrow \mathscr{P}$ into some elementary extension of $\mathscr{N}$.

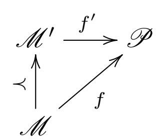
(iii) there are elementary extensions $\mathscr{M}' \succ \mathscr{M}$ and $\mathscr{N}' \succ \mathscr{N}$ which are isomorphic:



Proof. (iii)⇒(ii)⇒(i) are obvious.

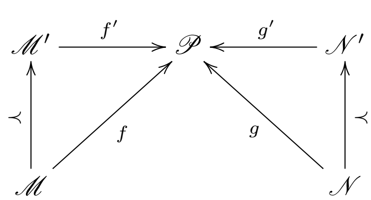
(i)⇒(iii). By the Elementary Joint Embedding Theorem there is an $\mathscr{L}$-structure $\mathscr{P}$ and elementary embeddings $f : \mathscr{M} \longrightarrow \mathscr{P}$ and $g : \mathscr{N} \longrightarrow \mathscr{P}$.

We construct $\mathscr{M}'$ according to the "identification process" of $\mathscr{M}$ in $\mathscr{P}$ via f. To be precise we construct an $\mathscr{L}$-structure $\mathscr{M}' \succ \mathscr{M}$ and an isomorphism $f' : \mathscr{M}' \longrightarrow \mathscr{P}$ such that the following diagram commutes:



Take a set A that is disjoint from $|\mathscr{M}|$ and in bijection $f' : A \longrightarrow |\mathscr{P}| \setminus f(|\mathscr{M}|)$. Define $M' = |\mathscr{M}| \cup A$ and extend f' to a bijection $M' \longrightarrow |\mathscr{P}|$ by mapping $m \in |\mathscr{M}|$ to $f(m)$. Now transfer the $\mathscr{L}$-structure of $\mathscr{P}$ via the bijection f' to M' and call the resulting structure $\mathscr{M}'$. Then by definition, f' is an isomorphism $\mathscr{M}' \longrightarrow \mathscr{P}$. Since f is an elementary embedding, also $(f')^{-1} \circ f$ is an elementary embedding. However, $(f')^{-1} \circ f$ is just the inclusion $|\mathscr{M}| \longrightarrow M'$ by definition of M' and f'.

We do the same with the elementary embedding $g : \mathscr{N} \longrightarrow \mathscr{P}$ and obtain $\mathscr{N}' \succ \mathscr{N}$ and an isomorphism $g' : \mathscr{N}' \longrightarrow \mathscr{P}$. Hence we are in the following situation:



and we can define $h = (g')^{-1} \circ f'$ to obtain an isomorphism $\mathscr{M}' \longrightarrow \mathscr{N}'$ as required. □

3. Types and definable sets

We will now simplify the notation further and denote $\mathscr{L}$-structures by $M, N, P, \ldots$. That is, we will not distinguish the structures from their universe when this is not necessary. This is the standard in mathematics, e.g. in the expression

"let G be a group and let $a \in G$"

it is well understood that the first occurrence of "G" refers to a structure and the second refers to the universe of this structure.

3.1. **Definable sets.**

3.1.1. **Definition.** Let M be an $\mathscr{L}$-structure. A subset S of M^n is called **definable** if there is some $\mathscr{L}$-formula $\varphi(x_1, ..., x_n, y_1, ..., y_k)$ and a k-tuple $\bar{a} \in M^k$ such that

$$S = \varphi[M^n, \bar{a}] := \{(m_1, ..., m_n) \in M^n \mid M \models \varphi[m_1, ..., m_n, a_1, ..., a_k]\}.$$

One should think of such a definable set S as the

"solution set of the formula $\varphi(x_1, ..., x_n, a_1, ..., a_k)$"

This is ambiguous because strictly speaking $\varphi(x_1, ..., x_n, a_1, ..., a_k)$ is not an $\mathscr{L}$-formula. So to be correct we should say $\varphi(x_1, ..., x_n, a_1, ..., a_k)$ is an $\mathscr{L}(M)$-formula and then it is shorthand for $\varphi(x_1, ..., x_n, y_1/\underline{a_1}, ..., y_k/\underline{a_k})$. This can not be misunderstood and we will use this notation from now on.

For example in the structure $M = (\mathbb{R}, \leq, +, -, \cdot)$, φ might be the formula

$$x^8 + y_7 \cdot x^7 + ... + y_1 \cdot x + y_0 \geq y_8$$

and $\bar{a}$ might be $(\pi, 1, 2, 3, 4, 5, 6, \sqrt{2}, 0)$. So $\varphi(x_1, ..., x_n, a_1, ..., a_k)$ is

$$x^8 + \sqrt{2} \cdot x^7 + ... + 1 \cdot x + \pi \geq 0$$

and $S \subseteq \mathbb{R}$ is the solution set of this inequality. It is clear that we are interested in the analysis of such sets when we are analysing the reals.

The elements $a_1, ..., a_k$ play the role of "coefficients" or "parameters" and it is suitable to think of them in this way, also in the general situation.

One might answer the question "What is Model Theory?" by saying "Model Theory is the analysis of structures in terms of first order logic". More precisely, we want to understand the shape of its definable sets! In this sense we have arrived at a central notion of Model Theory.

For example in the case of the real field we might ask topological questions about definable sets (are they connected, open, closed, or finite unions of such sets?). Or we might ask whether there is a formula in the language of rings which describes the graph of a Peano curve (i.e. a continuous <u>and surjective</u> function $\mathbb{R} \longrightarrow \mathbb{R}^2$).

In an arbitrary structure M, the following sets are always definable:

1. Finite and cofinite subsets of M^n (recall that **cofinite** means: the complement in M^n is finite)
2. For every $\bar{a} \in M^n$, the set $\{\bar{a}\} \times M^k \subseteq M^n \times M^k$.
3. The diagonal $\{(a, a) \mid a \in M\} \subseteq M^2$.

It is important to keep track of the parameters that are necessary to define a set. We extend definition 3.1.1 as follows:

3.1.2. **Definition.** Let $\mathscr{M}$ be an $\mathscr{L}$-structure and let $A \subseteq M$ be a set. A subset S of $|\mathscr{M}|^n$ is called A**-definable** or **definable with parameters from** A if there is some $\mathscr{L}$-formula $\varphi(x_1, ..., x_n, y_1, ..., y_k)$ and a k-tuple $\bar{a} \in A^k$ such that

$$S = \{(m_1, ..., m_n) \in M^n \mid M \models \varphi[m_1, ..., m_n, a_1, ..., a_k]\}.$$

Hence "definable" means definable with parameters from M.

For example, in the structure $(\mathbb{R}, +, -, \cdot, 0, 1)$, the singleton $\{\sqrt{2}\}$ ($\sqrt{2}$ denotes the positive square root of 2) is $\emptyset$-definable by the formula

$$x^2 \doteq 2 \wedge \exists u\ u^2 \doteq x.$$

Whereas in the structure $(\mathbb{C}, +, -, \cdot, 0, 1)$, the singleton $\{\sqrt{2}\}$ is <u>not</u> $\emptyset$-definable (cf. exercise 19)

3.1.3. **Proposition.** *Let M be an $\mathscr{L}$-structure and let $A \subseteq M$.*

(i) If S, T are A-definable subsets of M^n, then also $S \cap T$, $S \cup T$ and $S \setminus T$ are A-definable. If p is the projection $M^n \longrightarrow M^k$ and S is an A-definable subset of M^n, then $p(S)$ is an A-definable subset of M^k.

(ii) If $f : M \longrightarrow N$ is an isomorphism between $\mathscr{L}$-structures and $S \subseteq M^n$ is defined by $\varphi(\bar{x}, \bar{a})$, then $f(S)$ is defined by $\varphi(\bar{x}, f(\bar{a}))$ (here we also consider f as a map $M^n \longrightarrow N^n$ obtained from f by applying f coordinate wise; thus $f(S) \subseteq N^n$ and $f(\bar{a}) \in N^n$).

(iii) If $S \subseteq M^n$ is A-definable and $f : M \longrightarrow M$ is an automorphism of M that fixes A pointwise (i.e. $f(a) = a$ for all $a \in A$), then M fixes S setwise (i.e. $f(S) = S$).

Proof. Exercise. □

3.1.4. *Example.* Let M be a set considered as a structure in the empty language. Then every definable subset of M is finite or cofinite.

Proof. Suppose $S \subseteq M$ is infinite and $M \setminus S$ is infinite, too. It is clear that for every finite set $A \subseteq M$, there is a bijection $f : M \longrightarrow M$ which fixes A pointwise, but does not fix S setwise, i.e. $f(S) \not\subseteq S$. Hence by 3.1.3(iii), S is not A-definable. Since A was arbitrary, S is not definable either. □

3.1.5. *Remark.* Firstly a warning: Statement (ii) of 3.1.3 is not true anymore if we replace 'isomorphism' by 'elementary embedding': If $M \prec N$ is an elementary extension and $S \subseteq M$ is definable, then in general, S (which is $f(S)$ for the inclusion $f : M \longrightarrow N$) in general is not definable in N. For example the set M itself is definable in M, but in general not definable in N: take example 3.1.4 (and your favorite Skolem-Löwenheim theorem to produce $M \prec N$ as required).

Despite of the warning above, given an elementary extension $M \prec N$, there is a natural way to attach a definable set of N to each definable set of M:

If $S \subseteq M^n$ is definable, then for every elementary extension $N \succ M$ we define a set $S_N \subseteq M^n$ as follows:

Pick a formula $\varphi(\bar{x}, \bar{a})$ that defines S and let S_N be the set defined by $\varphi(\bar{x}, \bar{a})$ in N^n. Of course we have to confirm that this is well defined: Exercise 16.

A particularly interesting class of definable sets is given through properties of functions. First a definition.

3.1.6. **Definition.** Let M be an $\mathscr{L}$-structure, $A \subseteq M$ and let $S \subseteq M^n$. A function $f : S \longrightarrow M^k$ is called A-definable (in M) if its graph is an A-definable subset of $M^n \times M^k$.

As a (trivial example): If F is an n-ary function symbol of $\mathscr{L}$, then clearly $F^M : M^n \longrightarrow M$ is $\emptyset$-definable. However, not every $\emptyset$-definable function is of this form. For example in the structure $M = (\mathbb{Z}, +, \cdot)$, the successor function is $\emptyset$-definable.

3.1.7. *Remark.* Let M be an $\mathscr{L}$-structure, $A \subseteq M$ and let $S \subseteq M^n$. Let $f : S \longrightarrow M^k$ be a function.

(i) f is A-definable if and only if each component of f is an A-definable map $S^n \longrightarrow M$.
(ii) f is A-definable, then S and the image of f are A-definable.
(iii) The composition of A-definable maps is A-definable.

Proof. Exercise. □

For example, look at $M = (\mathbb{R}, +, \cdot)$ and a definable (i.e. M-definable) map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^k$. The definition from real analysis of differentiability of such a function can be used to show that the set D of all points where f is differentiable, is again definable in M.

3.1.8. **Definition.** An element s of an $\mathscr{L}$-structure M is called A-definable (where $A \subseteq M$), if $\{s\}$ is A-definable. The set of all elements of M that are A-definable is called the **definable closure of** A (in M) and denoted by

$$\mathrm{dcl}_M(A).$$

The set A is called **definably closed** (in M) if $A = \mathrm{dcl}_M(A)$.

Warning. The notion "definable closure" is slightly misleading in the sense that it indicates that $\mathrm{dcl}_M(A)$ might be "the smallest definable subset of M containing A". However this is in general not the case and in exercise 18 you should give examples which show

- In general, there is no smallest definable subset of M containing A.
- $\mathrm{dcl}_M(A)$ is in general not the intersection of all (A-)definable subsets of M.

3.1.9. **Proposition.** *Let A be a subset of a structure M. Then $A \subseteq \mathrm{dcl}_M(A)$ and*

(i) A subset S of M^n is A-definable if and only if S is $\mathrm{dcl}_M(A)$-definable. In particular $\mathrm{dcl}_M(A)$ is definably closed.
(ii) $\mathrm{dcl}_M(A)$ is closed under all A-definable functions, hence if $f : S^n \longrightarrow M$ is A-definable, then for every n-tuple $\bar{b} \in S$ with entries in $\mathrm{dcl}_M(A)$, also $f(\bar{b}) \in \mathrm{dcl}_M(A)$.
(iii) $\mathrm{dcl}_M(A)$ is a substructure of M.
(iv) If $M \prec N$, then $\mathrm{dcl}_N(A) = \mathrm{dcl}_M(A)$.

Proof. Obviously $A \subseteq \mathrm{dcl}_M(A)$.

(i) The implication $\Rightarrow$ is clear. Conversely assume $S = \varphi[M^n, \bar{b}]$ with $b_1, ..., b_k \in \mathrm{dcl}_M(A)$. Take $\varphi_1(y, \bar{z}), ..., \varphi_k(y, \bar{z}) \in \mathrm{Fml}\,\mathscr{L}$ and $\bar{a}_1, ..., \bar{a}_k \in A^{\bar{z}}$ such that

$$\{b_i\} = \varphi_i[M, \bar{a}_i] \ (1 \leq i \leq k).$$

Then S is defined by

$$\exists y_1, ..., y_k \ \left(\varphi(\bar{x}, y_1, ..., y_k) \wedge \bigwedge_{i=1}^{k} \varphi_i(y_i, \bar{a}_i)\right)$$

and so S is A-definable.

This proves the equivalence. It follows that $\mathrm{dcl}_M(A)$ is definably closed: every element that is $\mathrm{dcl}_M(A)$-definable is also A-definable.

(ii) Let the graph of f be defined by $\gamma(\bar{x}, y, \bar{a})$ with $\bar{a}$ a tuple with entries from A. Then $f(\bar{b})$ is defined by $\gamma(\bar{b}, y, \bar{a})$. Consequently $f(\bar{b})$ is $\mathrm{dcl}_M(A)$-definable and by (i) therefore A-definable.

(iii) To see that $\mathrm{dcl}_M(A)$ is a substructure of M (note that this strictly speaking means: $\mathrm{dcl}_M(A)$ is the universe of a substructure of M) we only need to show that $c^M \in \mathrm{dcl}_M(A)$ and that F^M maps $\mathrm{dcl}_M(A)^n$ into $\mathrm{dcl}_M(A)$ for every n-ary function symbol F of $\mathscr{L}$. However, c^M and F^M are $\emptyset$ definable functions, so this holds by (ii).

(iii). Let $\varphi(y, \bar{x})$ be an $\mathscr{L}$-formula and let $\bar{a} \in A^{\bar{x}}$.

If $b \in \mathrm{dcl}_M(A)$ and $\{b\}$ is defined by $\varphi(y, \bar{a})$ in M, then b the unique element of M with the property $M \models \varphi[b, \bar{a}]$. Since $M \prec N$ we have

- $N \models \varphi[b, \bar{a}]$ and
- $N \models \exists! y \ \varphi[y, \bar{a}]$. Here $\exists! y$ is an abbreviation for "there exists a unique y".

But this means $\{b\}$ is defined by $\varphi(y, \bar{x})$ in N.

Conversely if $c \in N$ is defined by $\varphi(y, \bar{a})$ in N, then b the unique element of N with the property $N \models \varphi[b, \bar{a}]$. Since $M \prec N$ we have

$$(*) \qquad M \models \exists! y \ \varphi[y, \bar{a}].$$

Take some $b \in M$ with $M \models \varphi[b, \bar{a}]$. As $M \prec N$ we also have $N \models \varphi[b, \bar{a}]$. However, in N there is only one element with this property and this element is c. Thus $c = b$. From $M \models \varphi[b, \bar{a}]$ and $(*)$ we see that $c = b \in \mathrm{dcl}_M(A)$. □

3.1.10. *Example.* Let M be a set considered as a structure in the empty language. Take $A \subseteq M$. We compute $\mathrm{dcl}_M(A)$ and claim that $\mathrm{dcl}_M(A) = A$ if and only if $M \setminus A$ has at least two elements or M is infinite. In the remaining case, when M is finite and $M \setminus A$ is a singleton we have $\mathrm{dcl}_M(A) = M$. To show this we do a case by case analysis.

Case 1. If $A = M$, then trivially $\mathrm{dcl}_M(A) = M$.

Case 2. If $M \setminus A$ contains at least 2 elements.

Then take a bijection $f : M \setminus A \longrightarrow M \setminus A$ without fixed point and extend f to M by $f(a) = a$ for $a \in A$. Then f is an automorphism of M that fixes A, but does not fix any element outside A. Hence by 3.1.3(iii), no element outside A is definable over A. Hence $A = \mathrm{dcl}_M(A)$.

Case 3. If M is finite and $M \setminus A$ is a singleton, say $A = \{a_1, ..., a_n\}$, then the unique element in $M \setminus A$ is A-definable by the formula

$$x \neq a_1 \wedge x \neq a_2 \wedge ... \wedge x \neq a_n.$$

Case 4. The remaining case is when M is infinite and $M \setminus A$ is a singleton.

Here we can not apply the automorphism argument from case 2, since every bijection $M \longrightarrow M$ that fixes A pointwise also fixes the missing point. However we can apply 3.1.9(iv): Since M is infinite, M has proper elementary extensions N. In particular $N \setminus A$ contains at least 2 elements. So by case 2 we know $\mathrm{dcl}_N(A) = A$. By 3.1.9(iv) we get $\mathrm{dcl}_M(A) = \mathrm{dcl}_N(A) = A$.

3.2. n-types of structures (warm up).

Let M be an $\mathscr{L}$-structure. In subsection 3.1 we have looked at definable subsets of M^n and we have seen that knowledge obtained about these definable sets is a key tool to understand the structure M. We will now introduce topology into our study and get a new and very fruitful way of looking at definable sets and first order structures. To do this we will add points to M^n and install a topology on the extended set. Then we will see that definable subsets of M^n can be characterised by those subsets of the space which are closed and open. The resulting space will be compact due to the compactness theorem! This is where the name comes from. In what follows it is a good idea to have a concrete model M in mind, For example one might think of M as a one-dimensional vector space over a field or as a densely linearly ordered set without endpoints.

We fix $n \in \mathbb{N}$ and distinct variables $x_1, ..., x_n$. We write $\bar{x} = (x_1, ..., x_n)$. The new points "code" properties of n-tuples from elementary extensions of M. If N is an elementary extension of M and $\bar{\alpha} \in N^n$, then we define the **type of $\bar{\alpha}$ in N over M** by

$$\text{tp}^N(\bar{\alpha}/M) = \{\varphi(\bar{x}) \in \text{Fml}\,\mathscr{L}(M) \mid (N, M) \models \varphi(\bar{\alpha})\}.$$

(Note that in this definition, φ has parameters from M, which are interpreted in (N, M) by the elements they name, see section 2.2)

Every such subset (N and $\bar{\alpha}$ may vary) of formulas of $\mathscr{L}(M)$ with free variables among the $x_1, ..., x_n$ is called an **n-type over M**. The n-tuple $\bar{\alpha}$ is called a **realization of p in N** if $N \succ M$ and $p = \text{tp}^N(\bar{\alpha}/M)$. These types are our new points and we define the **space of n-types over M**

$$S_n(M) = \{p \mid p \text{ is an } n\text{-type over } M\}.$$

Well, at the moment this is just a set and it is not clear why we 'added' points to M^n. Moreover the name 'space' is not yet justified as we still need to define a topology. We address the first issue now:

3.2.1. **Observation.** *The map*

$$\iota : M^n \longrightarrow S_n(M); \quad \iota(\bar{a}) = \text{tp}^M(\bar{a}/M)$$

is injective and its image is exactly the set of all types that are realised in M.

Proof. If $\bar{a} \neq \bar{b} \in M^n$, then for some $i \in \{1, ..., n\}$ we have $a_i \neq b_i$. Thus the $\mathscr{L}(M)$-formula $x_i \doteq a_i$ is in $\text{tp}^M(\bar{a}/M)$ but not in $\text{tp}^M(\bar{b}/M)$.

It is clear that the image of ι is exactly the set of all types that are realised in M. □

In the proof above we should have written $x_i \doteq \underline{a_i}$, because this formula is an $\mathscr{L}(M)$-formula. However, we will drop the underlining now, since it will never be unambiguous which language we are talking about; therefore $x_i \doteq a_i$ will also be unambiguous.

The injection in 3.2.1 explains in which way we see the types as new points: If we consider n-tuples of M^n as the types they realise in $S_n(M)$, all the other types are added to M^n.

Now we are heading towards the topology on $S_n(M)$. Firstly, we extend our intuition of formulas φ of $\mathscr{L}(M)$ with free variables among the $x_1, ..., x_n$, and the way they are related to each other and to points in M^n, to types: The formula φ

defines the subset $S = \varphi[M^n]$ of M^n and an n-tuple $\bar{a}$ might or might not be in S. If we consider the set S as the 'geometric incarnation' of the formula φ, the relation between an n-tuple $\bar{a}$ and φ is given by the membership

$$\bar{a} \in S, \text{ which is equivalent to } (M, M) \models \varphi(\bar{a}).$$

However $(M, M) \models \varphi(\bar{a})$ is equivalent to $\varphi \in \mathrm{tp}^M(\bar{a}/M)$. So wee see that

$$(+) \qquad \bar{a} \in \varphi[M^n] \iff \varphi \in \mathrm{tp}^M(\bar{a}/M).$$

We define

$$\langle \varphi \rangle = \{p \in S_n(M) \mid \varphi \in p\} \subseteq S_n(M)$$

and obtain

$$(++) \qquad p \in \langle \varphi \rangle \iff \varphi \in p.$$

You should verify that indeed condition (++) boils down to condition (+) if p is realised in M.

Thus we have extended the membership question $\bar{a} \in S = \varphi[M^n]$ to the larger set $S_n(M)$: the set S is replaced by the set $\langle \varphi \rangle$ and the relation of $\langle \varphi \rangle$ to the new points is exactly the same as the relation of $\varphi[M^n]$ to the old points.

3.2.2. **Definition.** We define a topology on $S_n(M)$ which has all the sets $\langle \varphi \rangle$ above (so φ now ranges over all formulas φ of $\mathscr{L}(M)$ with free variables among the $x_1, ..., x_n$) as a sub-basis of open sets. $S_n(M)$ with this topology now deserves the attribute 'space'.

The interesting thing about this space is the following:

3.2.3. **Theorem.** *$S_n(M)$ is a compact Hausdorff space and the sets $\langle \varphi \rangle$ are exactly those subsets of $S_n(M)$ that are at the same time open and closed.*

The interest consist here in the introduction of topology to the subject, which gives a new point of view of first order logic; e.g. we may now say that formulas with parameters in M "are" subsets of $S_n(M)$ which are at the same time open and closed. We will prove 3.2.3 in 3.4.3 below; for applications we need and we will prove it in greater generality. We finish this section with a look back to the original set M^n and how it sits inside $S_n(M)$ (we don't need 3.2.3 for this):

3.2.4. **Proposition.** *The image of $\iota : M^n \longrightarrow S_n(M)$ is exactly the set of all types p which are isolated (i.e. the set $\{p\}$ is open in $S_n(M)$). Moreover the image of ι is dense in $S_n(M)$ (i.e. its closure is all of $S_n(M)$).*

Proof. We first show that the image of ι is dense in $S_n(M)$: We take a nonempty open subset O of $S_n(M)$ and we have to find some $\bar{a} \in M^n$ with $\iota(\bar{a}) \in O$.

Take an arbitrary $p \in O$. Since the sets $\langle \varphi \rangle$ are a sub-basis of open sets of $S_n(M)$ and O is open, there are $\varphi_1, ..., \varphi_k \in \mathrm{Fml}\,\mathscr{L}(M)$, all with free variables among the $x_1, ..., x_n$ such that

$$p \in \langle \varphi_1 \rangle \cap ... \cap \langle \varphi_k \rangle \subseteq O.$$

Take a realization $\bar{\alpha}$ of p in $N \succ M$ (thus $\bar{\alpha} \in N^n$ for some $N \succ M$ and $p = \mathrm{tp}^N(\bar{\alpha}/M)$). Then $p \in \langle \varphi_i \rangle$ means

$$(N, M) \models \varphi_i[\bar{\alpha}] \ (1 \leq i \leq k)$$

and therefore

$$(N, M) \models \exists x_1...x_n \ (\varphi_1 \wedge ... \wedge \varphi_k).$$

Since $M \prec N$ we have

$$(M, M) \models \exists x_1 ... x_n \ (\varphi_1 \wedge ... \wedge \varphi_k)$$

which means that for some $\bar{a} \in M^n$ we have

$$(M, M) \models \varphi_1(\bar{a}) \wedge ... \wedge \varphi_k(\bar{a}).$$

Another way of saying this is $\mathrm{tp}^M(\bar{a}/M) \in \langle \varphi_i \rangle$ for each $i \in \{1, ..., k\}$. This shows

$$\iota(\bar{a}) = \mathrm{tp}^M(\bar{a}/M) \in \langle \varphi_1 \rangle \cap ... \cap \langle \varphi_k \rangle \subseteq O,$$

as required.

So we know that the image of ι is dense in $S_n(M)$. In particular, if p is an isolated point of $S_n(M)$, then $\{p\}$ is open and so p has to be in this image already.

It remains to show that conversely, all types, realised in M are isolated: Let $\bar{a} \in S_n(M)$. We show that $\mathrm{tp}^M(\bar{a}/M)$ is isolated. Take φ to be the formula

$$x_1 \doteq a_1 \wedge ... \wedge x_n \doteq a_n.$$

We claim that $\langle \varphi \rangle = \{\mathrm{tp}^M(\bar{a}/M)\}$. The inclusion $\supseteq$ holds true, since $(M, M) \models \varphi[\bar{a}]$. To see the other inclusion take $p \in \langle \varphi \rangle$ and let $\bar{\alpha} \in N^n$ for some $N \succ M$ such that $p = \mathrm{tp}^N(\bar{\alpha}/M)$. Since $p \in \langle \varphi \rangle$ we have $(N, M) \models \varphi[\bar{\alpha}]$. But this means

$$\alpha_1 = a_1 \text{ and } ... \text{ and } \alpha_n = a_n.$$

Hence $p = \mathrm{tp}^N(\bar{\alpha}/M) = \mathrm{tp}^M(\bar{\alpha}/M)$ (as $M \prec N$). □

The interest in 3.2.4 is that we can topologically recover the set M^n from the $S_n(M)$ and that the set M^n is topologically "big" in $S_n(M)$. Another consequence is

3.2.5. **Corollary.** *The relative topology induced by $S_n(M)$ on M^n (after identifying M^n with its image under ι) is discrete. (i.e. all subsets of M^n are open).*

Proof. Since all points of M^n in the relative topology are open, this is clear. □

3.3. **Types.**

We will restart and introduce type spaces in full generality. This looks slightly artificial at the beginning, but the general set up simplifies proofs and we will see shortly that the space $S_n(M)$ constructed in section 3.2 is an instance of what we define now:

Let Σ be a set of $\mathscr{L}$-sentences. The set of **types of** Σ (in $\mathscr{L}$) is

$$S(\Sigma) := \{p \subseteq \mathrm{Sen}(\mathscr{L}) \mid p \text{ is a complete } \mathscr{L}\text{-theory with } \Sigma \subseteq p\}.$$

For an $\mathscr{L}$-sentence φ we define

$$\langle \varphi \rangle = \{p \in S(\Sigma) \mid \varphi \in p\},$$

so that $p \in \langle \varphi \rangle \iff \varphi \in p$. If we have to specify Σ we write $\langle \varphi \rangle_\Sigma$ instead of $\langle \varphi \rangle$.

We will see (and actually exploit) that $S(\Sigma)$ is very language dependent, i.e. if we enlarge $\mathscr{L}$, $S(\Sigma)$ will change dramatically. If it is unclear which language we are working in we add a superscript $\mathscr{L}$ and write $S^{\mathscr{L}}(\Sigma)$, $\langle \varphi \rangle^{\mathscr{L}}$, etc. In this section however, $\mathscr{L}$ is fixed throughout.

3.3.1. **Lemma.** *For $\varphi, \psi \in \mathrm{Sen}(\mathscr{L})$ we have*

(i) $\langle \varphi \rangle = \emptyset \iff \Sigma \cup \{\varphi\}$ *is not satisfiable.*
(ii) $\langle \varphi \rangle = S(\Sigma) \iff \Sigma \models \varphi$.
(iii) $\langle \varphi \rangle \cap \langle \psi \rangle = \langle \varphi \wedge \psi \rangle$.
(iv) $\langle \varphi \rangle \cup \langle \psi \rangle = \langle \varphi \vee \psi \rangle$.
(v) $S(\Sigma) \setminus \langle \varphi \rangle = \langle \neg\varphi \rangle$.
(vi) $\langle \varphi \rangle \subseteq \langle \psi \rangle \iff \Sigma \models \varphi \to \psi$.
(vii) $\langle \varphi \rangle = \langle \psi \rangle \iff \Sigma \models \varphi \leftrightarrow \psi$.

Proof. Recall from 1.41 that

(∗) a complete $\mathscr{L}$-theory is the same thing as the theory of an $\mathscr{L}$-structure.

(i) $\langle \varphi \rangle = \emptyset$ says that $\Sigma \cup \{\varphi\}$ is not contained in any complete theory. Using (∗) this means $\Sigma \cup \{\varphi\}$ is not satisfiable.

(ii) $\langle \varphi \rangle = S(\Sigma)$ says that $\Sigma \cup \{\varphi\}$ is contained in any complete theory containing Σ. Using (∗) this means $\Sigma \models \varphi$.

(iii) holds since $\varphi \wedge \psi \in p \iff \varphi \in p$ and $\psi \in p$ for every (complete) $\mathscr{L}$-theory p.

(iv) holds since $\varphi \vee \psi \in p \iff \varphi \in p$ or $\psi \in p$ for every complete $\mathscr{L}$-theory p (use (∗)).

(v) holds since for every complete $\mathscr{L}$-theory p, exactly of φ or $\neg\varphi$ is in p (use (∗)).

(vi) Using (∗), $\langle \varphi \rangle \subseteq \langle \psi \rangle$ is the definition of $\Sigma \models \varphi \to \psi$.

(vii) is obvious from (vi). □

Item (vii) of 3.3.1 says that if we are interested in $\mathscr{L}$-sentences up to logical equivalence, assuming Σ as set of axioms, then these sentences can be studied entirely via the subsets $\langle \varphi \rangle$.

By 3.3.1, the set

$$B(\Sigma) = \{\langle \varphi \rangle \mid \varphi \in \mathrm{Sen}(\mathscr{L})\}$$

is a **boolean algebra** w.r.t. intersection, union and complement, called the **Tarski-Lindenbaum algebra** of Σ. What this means is that $B(\Sigma)$ is closed under finite intersections, unions and complement.

In particular $B(\Sigma)$ is a basis of open sets of a topology on $S(\Sigma)$. This means that the collection

$$\mathfrak{O} = \{ \text{ arbitrary unions of sets from } B(\Sigma) \ \}$$

is (the collection of open sets of) a topology on $S(\Sigma)$. What has to be checked here is that $\mathfrak{O}$ is closed under finite intersections and this is what follows from 3.3.1(iii).

3.3.2. **Definition.** The topological space $S(\Sigma)$ is called the **type space of** Σ

From now on we will consider $S(\Sigma)$ equipped with this topology. Hence if we say 'let $O \subseteq S(\Sigma)$ be open', then we mean open with respect to the topology defined above.

The statement that gives the compactness theorem its name is:

3.3.3. **Theorem.** *$S(\Sigma)$ is compact.*

Proof. By definition of 'compact' we need to show that for every set $\{O_i \mid i \in I\}$ of open subsets of $S(\Sigma)$ with the property that

$$(+) \qquad S(\Sigma) = \bigcup_{i \in I} O_i,$$

(we say that $\{O_i \mid i \in I\}$ **covers** $S(\Sigma)$) there is a finite **subcover**, that is, there is a finite subset J of I with

$$(++) \qquad S(\Sigma) = \bigcup_{j \in J} O_j.$$

By the definition of open sets before 3.3.2, every O_i is the union of all $\langle\varphi\rangle$ contained in it (see the definition of open sets before 3.3.2). In other words, if we define

$$\Gamma = \{\gamma \in \mathrm{Sen}(\mathscr{L}) \mid \langle\gamma\rangle \text{ is contained in } O_i \text{ for some } i \in I\},$$

then every $p \in S(\Sigma)$ is contained in some O_i (by (+)) and so p contains some $\gamma \in \Gamma$. But then the set $\neg\Gamma$ (defined as $\{\neg\gamma \mid \gamma \in \Gamma\}$) can not be contained in any type of Σ. In other words $\Sigma \cup \neg\Gamma$ is not satisfiable.

Now by the compactness theorem, for some finite subset Δ of $\neg\Gamma$, also $\Sigma \cup \Delta$ is not satisfiable. We unwind this: there is a finite subset Γ_0 of Γ such that $\Delta = \{\neg\gamma \mid \gamma \in \Gamma_0\}$. This in turn means that for every $p \in S(\Sigma)$, p must contain some $\gamma \in \Gamma_0$, in other words

$$(\dagger) \qquad S(\Sigma) = \bigcup_{\gamma \in \Gamma_0} \langle\gamma\rangle.$$

For each $\gamma \in \Gamma_0$, take an index $i(\gamma) \in I$ with $\langle\gamma\rangle \subseteq O_{i(\gamma)}$ and let J be the finite set $\{i(\gamma) \mid \gamma \in \Gamma_0\}$. Now clearly (†) implies (++) for J. □

We can now recover $\mathscr{L}$-sentences (modulo Σ, see 3.3.1(vii) and the remark after this lemma) topologically:

3.3.4. **Corollary.** *A subset $S(\Sigma)$ is of the form $\langle\varphi\rangle$ for some $\varphi \in \mathrm{Sen}(\mathscr{L})$ if and only if $\langle\varphi\rangle$ is* **clopen** *(i.e. closed and open).*

Proof. Since $S(\Sigma) = \langle\varphi\rangle \dot\cup \langle\neg\varphi\rangle$ and both sets are open by definition, $\langle\varphi\rangle$ is clopen.

Conversely if $X \subseteq S(\Sigma)$ is clopen, then it is compact, since closed subsets of compact spaces are compact and $S(\Sigma)$ is compact by 3.3.3. On the other hand X is also open, so X is a union of sets of the form $\langle\varphi\rangle$ by definition. Since X is compact it is a finite union of sets of the form $\langle\varphi\rangle$. Thus there are $\varphi_1, ..., \varphi_n \in \mathrm{Sen}(\mathscr{L})$ such that

$$X = \langle\varphi_1\rangle \cup ... \cup \langle\varphi_n\rangle.$$

By 3.3.1(iv) we see that

$$X = \langle\varphi_1 \vee ... \vee \varphi_n\rangle.$$

□

3.3.5. **Corollary.** *$S(\Sigma)$ Hausdorff and $S(\Sigma)$ has a basis of clopen sets. It follows that $S(\Sigma)$ is also* **totally disconnected***, i.e. the only connected subsets of $S(\Sigma)$ are singletons.*

Proof. $S(\Sigma)$ is Hausdorff because for $p, q \in S(\Sigma)$ with $p \neq q$ there is some $\varphi \in p \setminus q$ (note that the complete theories p and q are not comparable w.r.t. inclusion if they are different). Hence $p \in \langle\varphi\rangle$, $q \in \langle\neg\varphi\rangle$ and $\langle\varphi\rangle$, $\langle\neg\varphi\rangle$ are nonempty and disjoint open sets as required for the Hausdorff property.

Since $B(\Sigma)$ is a basis of the topology and all sets in $B(\Sigma)$ are clopen, $S(\Sigma)$ has a basis of clopen sets.

Concerning the last statement, recall that a subset X of a topological space is **connected** if for all open and disjoint open subsets U, V of that space with $X \subseteq U \cup V$ we have $X \subseteq U$ or $X \subseteq V$. In our situation, if $X \subseteq S(\Sigma)$ is a set containing at least two points p, q, then take a clopen subset U of $S(\Sigma)$ with $p \in U \not\ni q$. Then with $V = S(\Sigma) \setminus U$ we have $X \subseteq U \cup V$, $X \not\subseteq U$ and $X \not\subseteq V$. This shows that X is not connected. □

We can also code $\mathscr{L}$-theories containing Σ topologically. Firstly, if $\Gamma \subseteq \mathrm{Sen}(\mathscr{L})$, then the set

$$\mathfrak{C}(\Gamma) := \bigcap_{\gamma\in\Gamma} \langle\gamma\rangle$$

is a closed subset of $S(\Sigma)$ (since all sets of the form $\langle\varphi\rangle \in B(\mathscr{L})$ are closed and since the intersection of closed sets is closed).

Secondly, if X is a subset of $S(\Sigma)$, then

$$\mathfrak{T}(X) := \{\varphi \in \mathrm{Sen}(\mathscr{L}) \mid X \subseteq \langle\varphi\rangle\}$$

is a deductively closed subset of $\mathrm{Fml}\,\mathscr{L}$ containing Σ as follows readily from 3.3.1

3.3.6. **Proposition.**

(i) For every $\Gamma \subseteq \mathrm{Sen}(\mathscr{L})$ the set

$$\mathfrak{T}(\mathfrak{C}(\Gamma)) \text{ is the deductive closure of } \Sigma \cup \Gamma.$$

(ii) For every subset X of $S(\Sigma)$, the set

$$\mathfrak{C}(\mathfrak{T}(X)) \text{ is the closure of } X \text{ in } S(\Sigma).$$

(iii) $\mathfrak{C}$ and $\mathfrak{T}$ define bijections

$$\mathfrak{C} : \{\mathscr{L}\text{-theories containing } \Sigma\} \longrightarrow \{\text{nonempty closed subsets of } S(\Sigma)\}$$

and

$$\mathfrak{T} : \{\text{nonempty closed subsets of } S(\Sigma)\} \longrightarrow \{\mathscr{L}\text{-theories containing } \Sigma\},$$

which are inverse to each other.

Proof. (i) That $\mathfrak{T}(\mathfrak{C}(\Gamma))$ contains the deductive closure of $\Sigma\cup\Gamma$ is left as an exercise. For the other inclusion, take $\varphi \in \mathfrak{T}(\mathfrak{C}(\Gamma))$, i.e.

$$\bigcap_{\gamma\in\Gamma}\langle\gamma\rangle = \mathfrak{C}(\Gamma) \subseteq \langle\varphi\rangle.$$

Hence

$$S(\Sigma) = \langle\varphi\rangle \cup (S(\Sigma) \setminus \bigcap_{\gamma\in\Gamma}\langle\gamma\rangle) = \langle\varphi\rangle \cup \bigcup_{\gamma\in\Gamma}\langle\neg\gamma\rangle$$

Since $S(\Sigma)$ is compact there is a finite subset $\Gamma_0 \subseteq \Gamma$ with

$$S(\Sigma) = \langle\varphi\rangle \cup \bigcup_{\gamma\in\Gamma_0}\langle\neg\gamma\rangle.$$

This in turn means

$$\bigcap_{\gamma\in\Gamma_0}\langle\gamma\rangle \subseteq \langle\varphi\rangle$$

and using 3.3.1 we see that $\Sigma \models (\bigwedge\Gamma_0) \to \varphi$. Hence φ is in the deductive closure of $\Sigma \cup \Gamma$.

(ii) That $\mathfrak{C}(\mathfrak{T}(X))$ contains the closure of X is left as an exercise. For the other inclusion, take $p \in \mathfrak{C}(\mathfrak{T}(X))$, i.e.

$$(*) \qquad p \in \bigcap_{\gamma\in\mathfrak{T}(X)}\langle\gamma\rangle.$$

We must show that every open subset of $S(\Sigma)$ containing p also contains points from X. Since $B(\Sigma)$ is a basis of the topology we may assume that our open set is of the form $\langle\varphi\rangle$. If $X\cap\langle\varphi\rangle$ were empty, then $X \subseteq \langle\neg\varphi\rangle$, in other words $\neg\varphi \in \mathfrak{T}(X)$. However, by $(*)$, we then had $p \in \langle\neg\varphi\rangle$, in contradiction to $p \in \langle\varphi\rangle$.

(iii) That both maps are well defined is left as an exercise. By (i) we have $\mathfrak{T}(\mathfrak{C}(T)) = T$ for every $\mathscr{L}$-theory T containing Σ. By (ii) we have $\mathfrak{C}(\mathfrak{T}(C)) = C$ for every closed subset C of $S(\Sigma)$. This shows (iii). □

Here a summary of what we have shown in this section:

3.3.7. **Summary.** *Let* $\Sigma \subseteq \mathrm{Sen}(\mathscr{L})$.

(i) $S(\Sigma)$ *is a* **boolean space**, *i.e.* $S(\Sigma)$ *is*
  - *(a) compact,*
  - *(b) Hausdorff and*
  - *(c) has a basis of clopen sets.*

  In particular $S(\Sigma)$ *is totally disconnected.*

(ii) The Tarski-Lindenbaum algebra $B(\Sigma)$ *of all sets of the form* $\langle\varphi\rangle$ *is the set of all clopen subsets of* $S(\Sigma)$.

(iii) The closed subsets of $S(\Sigma)$ *are in bijection with the* $\mathscr{L}$*-theories containing* Σ. □

It should be said that every boolean space is of the form $S(\Sigma)$ for some Σ and some language $\mathscr{L}$. We will not use this and omit the proof.

Warning. The notion "type" is used in various different ways (in model theory). If you read a model theory text make sure you first check what the author(s) mean by "type".

3.4. n-types of theories and structures.

We will now define three type spaces as instances of the space $S(\Sigma)$ from section 3.3. Let $n \in \mathbb{N}$.

(A) For an $\mathscr{L}$-theory T, we define the space of **n-types of** T, denoted by $S_n(T)$ as the space $S(T)$ in the language $\mathscr{L}(\bar{c})$, where $\bar{c} = (c_1, ..., c_n)$ is an n-tuple of new (w.r.t. $\mathscr{L}$) constant symbols. In compact form:

$$S_n(T) = S^{\mathscr{L}(\bar{c})}(T).$$

(B) (this is an instance of (A)). For an $\mathscr{L}$-structure M and a subset A of M we define the space of n-types of M with parameters in A as the space of n-types of the $\mathscr{L}(A)$-theory $\mathrm{Th}(M, A)$. In symbols

$$S_n(M, A) := S_n(Th(M, A)).$$

(C) (this is an instance of (B)). For an $\mathscr{L}$-structure M we define

$$S_n(M) = S_n(M, M).$$

Remark on the notation. In the literature the notion "type" most of the time is used for n-types in the sense of (A), (B) or (C). Sometimes, "type" refers only to "incomplete types" which are just a collection of formulas (in some language); so in these texts, the elements of the various type spaces that we have defined now are called "complete types".

Of course we need to check that the notation in (C) is compatible with what we have done in section 3.2. In fact it is convenient to rephrase the spaces above in terms of formulas.

Throughout we fix n distinct variables $x_1, ..., x_n$ and $c_1, ..., c_n$, new constant symbols. We write $\mathrm{Fml}\,\mathscr{L}_n$ for the set of all $\mathscr{L}$-formulas with free variables among $\{x_1, ..., x_n\}$.

First recall from 1.47 that the injection

$$\begin{array}{rcl} C : \mathrm{Fml}\,\mathscr{L}_n & \longrightarrow & \mathrm{Sen}\,\mathscr{L}(\bar{c}) \\ \varphi & \longmapsto & \varphi(x_1/c_1, ..., x_n/c_n) \end{array}$$

also respects logical implication.

C is 'essentially surjective' in the sense that for every $\psi \in \mathrm{Sen}(\mathscr{L}(\bar{c}))$, there is some $\varphi \in \mathrm{Fml}\,\mathscr{L}_n$ such that $\models C(\varphi) \leftrightarrow \psi$: Take variables $y_1, ..., y_n$ not occurring in ψ and different from $x_1, ..., x_n$ and let ψ' be the result of replacing x_i by y_i in the string ψ. It is clear that $\models \psi \leftrightarrow \psi'$. Moreover, the $\mathscr{L}$-formula φ obtained from ψ' by replacing each c_i with x_i now satisfies $C(\varphi) = \psi'$, thus $\models C(\varphi) \leftrightarrow \psi$.

Now fix an $\mathscr{L}$-theory T. It follows that for each $q \in S_n(T)$, the set $C^{-1}(q) \subseteq \mathrm{Fml}\,\mathscr{L}_n$ uniquely determines q, i.e.

$$C(C^{-1}(q)) \models q.$$

Note that

$$C^{-1}(q) = \{\varphi(\bar{x}) \mid \varphi \in \mathrm{Fml}\,\mathscr{L}_n \text{ and } \varphi(\bar{x}/\bar{c}) \in q\}$$

and

$$C(C^{-1}(q)) = \{\varphi(\bar{x}/\bar{c}) \mid \varphi \in \mathrm{Fml}\,\mathscr{L}_n \text{ and } \varphi(\bar{x}/\bar{c}) \in q\}.$$

3.4.1. **Proposition.** *A subset p of $\mathrm{Fml}\,\mathscr{L}_n(M)$ is of the form $C^{-1}(q)$ for some $q \in S_n(T)$ if and only if there is a model M of T and a so-called* **realization** $\bar{\alpha}$ **of** p **in** M*, i.e. $\bar{\alpha} \in M^n$ and p is equal to*

$$\mathrm{tp}^M(\bar{\alpha}) := \{\varphi(\bar{x}) \in \mathrm{Fml}\,\mathscr{L}_n \mid M \models \varphi[\bar{\alpha}]\},$$

called the **type of** $\bar{\alpha}$ **in** M*.*

Proof. If $q \in S_n(T)$, then pick a model M^+ of q. Let $M := M \restriction \mathscr{L}$ and let $\bar{\alpha} = (c_1^{M^+}, ..., c_n^{M^+})$. Then by 1.46, $M^+ \models q$ implies $p = C^{-1}(q) = \mathrm{tp}^M(\bar{\alpha})$.

Conversely, if $p = \mathrm{tp}^M(\bar{\alpha})$, then again by 1.46, the $\mathscr{L}(\bar{c})$-structure $M^+ = (M, \bar{\alpha})$ satisfies $\varphi(\bar{x}/\bar{c})$ if and only if $\varphi \in p$ for every $\varphi \in \mathrm{Fml}\,\mathscr{L}_n$. But this means $p = C^{-1}(q)$ where q is the $\mathscr{L}(\bar{c})$-theory of M^+. □

By 3.4.1 it is enough to talk about n-types of theories in terms of the formulas $\varphi \in \mathrm{Fml}\,\mathscr{L}_n$ that occur in these types after we have replaced the variables by constants. In this sense we will refer to n-types as such sets of formulas.

As a matter of fact, a good way of thinking about the entire matter is to think of $\mathrm{Sen}\,\mathscr{L}(\bar{c})$ as the image of C (the other sentences all being presented up to logical equivalence), and then to use $\mathscr{L}$-formulas with free variables among $x_1, ..., x_n$ interchangeably with $\mathscr{L}(\bar{c})$-sentences.

3.4.2. **Corollary.** *Let M be an $\mathscr{L}$-structure, $A \subseteq M$ and let $n \in \mathbb{N}$. Let $p \subseteq \mathrm{Fml}\,\mathscr{L}_n(A)$. The following are equivalent:*

(i) $p \in S_n(M, A)$ (formulated in the rigorous setup above we should say $p = C^{-1}(q)$ for some $q \in S_n(M, A)$, where now C is the replacement map $\mathrm{Fml}\,\mathscr{L}_n(A) \longrightarrow \mathrm{Fml}\,\mathscr{L}(A)(\bar{c})$)

(ii) There is an elementary extension $N \succ M$ and a so-called **realization** $\bar{\alpha}$ **of** p **in** N*, i.e. $\bar{\alpha} \in N^n$ and p is equal to*

$$\mathrm{tp}^N(\bar{\alpha}/A) := \{\varphi(\bar{x}) \in \mathrm{Fml}\,\mathscr{L}(A) \mid (N, A) \models \varphi[\bar{\alpha}]\},$$

called the **type of** $\bar{\alpha}$ **over** A **in** N*.*

Proof. This is 3.4.1 applied to the language $\mathscr{L}(A)$ and the $\mathscr{L}(A)$-theory $T = \mathrm{Th}(M, A)$. By 3.4.1 we know that $p \in S_n(M, A)$ if and only if

$$(*) \qquad p = \mathrm{tp}^{P^+}(\bar{\beta}) \text{ for some } \mathscr{L}(A)\text{-structure } P^+ \text{ and some } \bar{\beta} \in P^+.$$

Hence it suffices to show that $(*)$ is equivalent to (ii).

(ii)$\Rightarrow$ $(*)$. If $N \succ M$ and $\bar{\alpha} \in N^n$ with $p = \mathrm{tp}^N(\bar{\alpha}/A)$, then $(N, A) \models \mathrm{Th}(M, A)$ and we may take $P^+ = (N, A)$ (now use 1.46 again).

$(*)$ $\Rightarrow$(ii) By the elementary joint embedding theorem (or rather 2.8.2), there is an elementary extension $N^+ \succ (M, A)$ (in the language $\mathscr{L}(A)$) and an elementary embedding $f : P^+ \longrightarrow N^+$. Since p is the type of $\bar{\beta}$ in P^+, p is also the type of $f(\bar{\beta})$ in N^+. We take $N = N \restriction \mathscr{L}(A)$ and $\bar{\alpha} = f(\bar{\beta})$. And see that

$$p = \mathrm{tp}^{P^+}(\bar{\beta}) = \mathrm{tp}^{N^+}(\bar{\alpha}) = \mathrm{tp}^N(\bar{\alpha}/A).$$

□

3.4.3. **Corollary.** *The definition of $S_n(M)$ in item (C) above is justified and theorem 3.2.3 is an instance of 3.3.7.* □

3.4.4. *Remark.* Let $M \prec N$ be $\mathscr{L}$-structures and let $A \subseteq M$. It then follows directly from the definition of $S_n(M, A)$, that

$$S_n(N, A) = S_n(M, A).$$

Proof. Exercise □

3.4.5. *Remark.* Let M be an $\mathscr{L}$-structure and let $A \subseteq B \subseteq M$. If $q \in S_n(M, B)$, then one can see directly from the definition or use the characterisation in 3.4.2 that $q \restriction A$, defined as

$$q \restriction A = q \cap \operatorname{Fml} \mathscr{L}_n(A)$$

is in $S_n(M, A)$. Moreover every realization of q is a realization of $q \restriction A$.

Conversely, if we start with $p \in S_n(M, A)$ it follows directly from the definition or using the characterisation in 3.4.2 that there is some $q \in S_n(M, B)$ with $p = q \restriction A$.

Proof. Exercise. □

3.5. Realizing types: Saturated structures.

3.5.1. **Lemma.** *Every infinite $\mathscr{L}$-structure M possesses an elementary extension N such that for all $n \in \mathbb{N}$ and all $A \subseteq M$, every $p \in S_n(M, A)$ is realized in N.*

Proof. By 3.4.5 it suffices to find $N \succ M$ in which all types of $S_n(M)$ for all n are realised. For $n \in \mathbb{N}$ and $p \in S_n(M)$ choose $N_p \succ M$ and a realization $\bar{\alpha}_p \in N_p^n$ of p. In particular for each p, the theory of the $\mathscr{L}(M)$-structure (N_p, M) is $\mathrm{Th}(M, M)$. Hence all the (N_p, M) are elementary equivalent and by the elementary joint embedding theorem 2.8.1, there are an $\mathscr{L}(M)$-structure N^+ and elementary embeddings $f_p : (N_p, M) \longrightarrow N^+$ in the language $\mathscr{L}(M)$ for all $p \in S_n(M), n \in \mathbb{N}$.

Using 2.8.2 we may then also assume that N^+ is an elementary extension of (M, M), in other words $M \prec N := N^+ \restriction \mathscr{L}$.

Since $f_p : (N_p, M) \longrightarrow (N, M)$ is elementary in $\mathscr{L}(M)$, this implies f_p fixes M pointwise. Therefore, the realization $\bar{\alpha}_p$ of p is mapped onto the realization $f_p(\bar{\alpha})$ of p in N. Hence N has the required properties. □

3.5.2. **Definition.** Let κ be an infinite cardinal. An infinite $\mathscr{L}$-structure M is called **κ-saturated** if for any $A \subseteq M$ of cardinality strictly less than κ and every $n \in \mathbb{N}$, every n-type from $S_n(M, A)$ is realized in M.

If you don't know cardinals: we only need one instances here:

An infinite $\mathscr{L}$-structure M is called **$\aleph_0$-saturated** if for any finite $A \subseteq M$ and every $n \in \mathbb{N}$, every n-type from $S_n(M, A)$ is realized in M.

If you know cardinals, it might be interesting to notice that a **saturated structure** is defined to be an infinite structure which is saturated in its own cardinality.

3.5.3. **Theorem.** *For every cardinal κ, every infinite $\mathscr{L}$-structure M has a κ^+-saturated elementary extension of size 2^κ.*

Hence if you believe in the generalized continuum hypothesis, then this elementary extension is saturated. If you don't believe in the generalized continuum hypothesis: the existence of saturated elementary extensions of a given structure can in general not be shown from ZFC alone.

Proof. For us it is only important to have this theorem for $\kappa = \aleph_0$ - and we do not need a bound on the size of N.

Hence we only need the following statement:

(∗) Every $\mathscr{L}$-structure M has an $\aleph_0$-saturated elementary extension.

We simply iterate 3.5.1: By 3.5.1 there is an elementary chain $(M_i \mid i \in \mathbb{N})$ starting at $M = M_1$ such that for each $i \in \mathbb{N}$, every n-type over all $A \subseteq M_i$ is realised in M_{i+1}.

We take $N = \bigcup_i M_i$ and use the elementary chain lemma 2.5.1: If $A \subseteq N$ is finite, then there is some $i \in \mathbb{N}$ such that $A \subseteq M_i$. Take $p \in S_n(N, A)$. By 3.4.4, $S_n(M_i, A) = S_(N, A_i)$. But p is realised in $M_{i+1} \prec N$ by some $\bar{\alpha} \in M_{i+1}^n$. So then p is also realized by $\bar{\alpha}$ in N. This finishes the proof of (∗).

Remark: For those who are acquainted with cardinal arithmetic, the proof of (∗) can be easily amended to a proof of the full statement by choosing a "longer" chain, namely a chain of length κ^+. The cardinal bound comes for free if we do careful bookkeeping - already in 3.5.1. □

In saturated structures we can do restricted compactness arguments. This means the following. Let M be κ-saturated ($\aleph_0$-saturated) and let $A \subseteq M$ such that $\text{card}(A) < \kappa$ (A is finite). If I is an arbitrary index set and $(X_i \mid i \in I)$ is a collection of subsets X_i of M^n, <u>all defined over A</u>, then the Heine-Borel covering property is satisfied for this collection, i.e.:

$$\text{IF } M^n = \bigcup_{i \in I} X_i,$$
$$\text{THEN there is a finite subset } J \text{ of } I \text{ with } M^n = \bigcup_{j \in J} X_j.$$

By taking complements and taking into account that a subset of M^n is definable over A if and only if its complement in M^n is definable over A, this property s equivalent to the following.

If I is an arbitrary index set and $(X_i \mid i \in I)$ is a collection of subsets X_i of M^n, <u>all defined over A</u>, such that $(X_i \mid i \in I)$ has the **finite intersection property** (abbreviated as **FIP**), then

$$\bigcap_{i \in I} X_i \neq \emptyset.$$

Proof. Since each X_i is definable over A, there is a formula $\varphi_i(x_1, ..., x_n)$ in the language $\mathscr{L}(A)$ that defines X_i, i.e.

$$X_i = \{\bar{b} \in M^n \mid (M, A) \models \varphi_i[\bar{b}]\}.$$

Now $(X_i \mid i \in I)$ has the FIP if and only if the set

$$\Sigma := \text{Th}(M, A) \cup \{\varphi_i(\bar{x}) \mid i \in I\} \subseteq \text{Fml}\,\mathscr{L}(A)$$

is finitely satisfiable. This is the case if and only if there is a type $p \in S_n(M, A)$ containing Σ.

Now we use the assumption that M is κ-saturated and $\text{card}(A) < \kappa$ (or in the countable case, that M is $\aleph_0$-saturated and A is finite): p is realised in M^n, i.e. there is some $\bar{b} \in M^n$ such that $(M, A) \models \varphi_i[\bar{b}]$ for all $i \in I$. Now this means $\bar{b} \in \bigcap_{i \in I} X_i$ as desired. $\square$

3.6. **Existentially closed models and model-completeness.**

3.6.1. **Definition.** Let $\mathscr{L}$ be a language.

(i) An **existential $\mathscr{L}$-formula** is a formula of the form $\exists\bar{y}\chi(\bar{x},\bar{y})$, where $\chi(\bar{x},\bar{y})$ is a quantifier-free $\mathscr{L}$-formula. The set of all existential $\mathscr{L}$-formulas is denoted by $\exists\text{-}\operatorname{Fml}(\mathscr{L})$.

(ii) A **universal $\mathscr{L}$-formula** is a formula of the form $\forall\bar{y}\chi(\bar{x},\bar{y})$, where $\chi(\bar{x},\bar{y})$ is a quantifier-free $\mathscr{L}$-formula. The set of all universal $\mathscr{L}$-formulas is denoted by $\forall\text{-}\operatorname{Fml}(\mathscr{L})$.

(iii) If $\mathscr{A}$ is an $\mathscr{L}$-structure, $n \in \mathbb{N}_0$ and $\bar{a} \in |\mathscr{A}|^n$ then we define the **$\exists$-type** of $\bar{a}$ in $\mathscr{A}$ as

$$\operatorname{tp}_\exists(\mathscr{A},\bar{a}) = \{\delta(v_1,\dots,v_n) \mid \delta \in \exists\text{-}\operatorname{Fml}(\mathscr{L}),\ \mathscr{A} \models \delta(\bar{a})\}.$$

(iv) An $\mathscr{L}$-structure $\mathscr{A}$ is said to be **existentially closed in an $\mathscr{L}$-structure $\mathscr{B}$** if it is a substructure of $\mathscr{B}$ and if for every quantifier-free $\mathscr{L}$-formula $\chi(\bar{x},\bar{y})$ and every $\bar{a} \in |\mathscr{A}|^{\bar{x}}$ we have

$$\mathscr{B} \models \exists\bar{x}\chi(\bar{x},\bar{a}) \Longrightarrow \mathscr{A} \models \exists\bar{x}\chi(\bar{x},\bar{a}).$$

(v) If $\mathfrak{C}$ is a class of $\mathscr{L}$-structures, then a structure $\mathscr{A}$ is called **existentially closed in $\mathfrak{C}$** if $\mathscr{A} \in \mathfrak{C}$ and $\mathscr{A}$ is existentially closed in $\mathscr{B}$ for all $\mathscr{B} \in \mathfrak{C}$ with $\mathscr{A} \subseteq \mathscr{B}$.

(vi) If T is an $\mathscr{L}$-theory, then an **existentially closed model of** T is an existentially closed structure in the class of all models of T.

We abbreviate the expression *existentially closed* by **e.c.**

3.6.2. **Observation.** *Let $\mathscr{A} \subseteq \mathscr{B}$ be an extension of $\mathscr{L}$-structures. The following are equivalent.*

(i) $\mathscr{A}$ is existentially closed in $\mathscr{B}$

(ii) The inclusion map $\mathscr{A} \hookrightarrow \mathscr{B}$ preserves all universal formulas in the sense of 2.1.1.

(iii) For all $n \in \mathbb{N}$ and all $\bar{a} \in \mathscr{A}^n$ we have $\operatorname{tp}_\exists(\mathscr{A},\bar{a}) = \operatorname{tp}_\exists(\mathscr{B},\bar{a})$[2]

Proof. (i) is equivalent to (ii) by taking contrapositives and negation. (iii) is just a reformulation of (i). □

3.6.3. **Proposition.** *Let $\mathscr{A} \subseteq \mathscr{B}$ be an extension of $\mathscr{L}$-structures. The following are equivalent.*

(i) $\mathscr{A}$ is existentially closed in $\mathscr{B}$.

(ii) $\mathscr{B}$ is a substructure of an elementary extension of $\mathscr{A}$.

Further characterizations are given in 3.6.8 and in 3.6.17.

Proof. The implication (ii)⇒(i) is clear. For the converse we proceed very similar to the proof of 2.3.4.

Claim. $\Sigma := \operatorname{diag}_\infty(\mathscr{A}) \cup \operatorname{diag}(\mathscr{B})$ in the language $\mathscr{L}(\mathscr{B})$ (which has a new constant symbol c_b for every $b \in |\mathscr{B}|$) is consistent.

Proof. Otherwise Σ^* is inconsistent and therefore there are $\mathscr{L}$-formulas $\chi(\bar{x},\bar{y})$, $\varphi(\bar{x})$, where $\bar{x},\bar{y}$ are disjoint tuples of variables such that χ is quantifier free and

[2]Notice that $\operatorname{tp}_\exists(\mathscr{A},\bar{a}) \subseteq \operatorname{tp}_\exists(\mathscr{B},\bar{a})$ is true for any extension $\mathscr{B}$ of $\mathscr{A}$.

$\bar{a} \in |\mathscr{A}|^{\bar{x}}$, $\bar{b} \in |\mathscr{B}|^{\bar{y}}$ such that $\mathscr{B} \models \chi(\bar{a},\bar{b})$, $\mathscr{A} \models \varphi(\bar{a})$ and $\{\chi(\bar{a},\bar{b}),\varphi(\bar{a})\}$ is inconsistent, when considered as sentences in the language $\mathscr{L}(\mathscr{B})$, in other words $\models \varphi(\bar{a}) \to \neg\chi(\bar{a},\bar{b})$ (in $\mathscr{L}(\mathscr{B})$). By 1.47 this implies $\models \varphi(\bar{x}) \to \neg\chi(\bar{x},\bar{y})$ (in $\mathscr{L}$). We thus obtain $\models \forall\bar{x},\bar{y}(\varphi(\bar{x}) \to \neg\chi(\bar{x},\bar{y}))$. Since $\bar{x},\bar{y}$ are disjoint tuples of variables this says $\models \forall\bar{x}(\varphi(\bar{x}) \to \forall\bar{y}\neg\chi(\bar{x},\bar{y}))$. In particular $\mathscr{A} \models \forall\bar{x}(\varphi(\bar{x}) \to \forall\bar{y}\neg\chi(\bar{x},\bar{y}))$. Since $\mathscr{A} \models \varphi(\bar{a})$ we get $\mathscr{A} \models \forall\bar{y}\neg\chi(\bar{a},\bar{y}))$.

Since $\mathscr{A}$ is e.c. closed in $\mathscr{B}$ we get $\mathscr{B} \models \forall\bar{y}\neg\chi(\bar{a},\bar{y}))$, in contradiction to $\mathscr{B} \models \chi(\bar{a},\bar{b})$. ◇

Now take a model $\mathscr{M}^*$ of Σ and let $\mathscr{M}$ be its reduct to $\mathscr{L}$. Then the map $f : \mathscr{A} \longrightarrow \mathscr{M}$, $f(a) = c_a^{\mathscr{M}^*}$ is an elementary embedding, witnessed by $\mathscr{M}^* \models \mathrm{diag}_\infty(\mathscr{A})$, and the map $g : \mathscr{B} \longrightarrow \mathscr{M}$, $g(b) = c_b^{\mathscr{M}^*}$ is an embedding, witnessed by $\mathscr{M}^* \models \mathrm{diag}(\mathscr{B})$ (also see 2.2.3). Hence g extends f and after appropriate identifications we see that $\mathscr{B}$ is a substructure of the elementary extension $\mathscr{M}$ of $\mathscr{A}$. □

3.6.4. **Definition.** An $\mathscr{L}$-formula is called an $\forall\exists$**-formula** if it is logically equivalent to a formula of the from $\forall\bar{x}\exists\bar{y}\chi$ for some quantifier-free $\mathscr{L}$-formula χ. If T is an $\mathscr{L}$-theory, then $T_{\forall\exists}$ denotes the set of all $\forall\exists$-sentences φ with $T \models \varphi$.

Similar to 2.3.5 we have

3.6.5. **Proposition.** *Let T be an $\mathscr{L}$-theory. If $\mathscr{M}$ is an $\mathscr{L}$-structure, then $\mathscr{M} \models T_{\forall\exists}$ if and only if $\mathscr{M}$ is e.c. in a model of T.*

Proof. Since truth of $\forall\exists$- sentences in any structure $\mathscr{M}$ is inherited by all of its substructures that are e.c. in $\mathscr{M}$, we know that every substructure of a model $\mathscr{M}$ of T that is e.c. in $\mathscr{M}$ is a model of $T_{\forall\exists}$. Conversely suppose $\mathscr{M} \models T_{\forall\exists}$.

Claim. $T \cup \mathrm{diag}_\forall(\mathscr{M})$ is consistent in the language $\mathscr{L}(\mathscr{M})$, where $\mathrm{diag}_\forall(\mathscr{M}) = \{\sigma(\bar{a}) \mid \sigma(\bar{x}) \in \forall\text{-}\mathrm{Fml}(\mathscr{L}),\ \bar{a} \in |\mathscr{M}|^{\bar{x}},\ \mathscr{M} \models \sigma(\bar{a})\}$.

Proof. Otherwise there are $\sigma(\bar{x}) \in \forall\text{-}\mathrm{Fml}(\mathscr{L})$ and $\bar{a} \in |\mathscr{M}|^{\bar{x}}$ with $\mathscr{M} \models \sigma(\bar{a})$ such that $T \models \neg\sigma(\bar{a})$ (in $\mathscr{L}(\mathscr{M})$). This implies $T \models \forall\bar{x}\neg\sigma(\bar{x})$ and so $\forall\bar{x}\neg\sigma(\bar{x}) \in T_{\forall\exists}$. But then $\mathscr{M} \models \forall\bar{x}\neg\sigma(\bar{x})$ in contradiction to $\mathscr{M} \models \sigma(\bar{a})$. ◇

The claim implies that the reduct $\mathscr{N}$ to $\mathscr{L}$ of a model $\mathscr{N}^*$ of $T \cup \mathrm{diag}_\forall(\mathscr{M})$ is a model of T such that the map $\mathscr{M} \longrightarrow \mathscr{N}$, $a \mapsto c_a^{\mathscr{N}^*}$ is an isomorphism onto a substructure of $\mathscr{N}$ that is e.c. in $\mathscr{N}$, as required. □

3.6.6. **Corollary.** *An $\mathscr{L}$-theory T is* **inductive***, i.e. it is axiomatized by $\forall\exists$-sentences if and only if for every chain $(\mathscr{M}_i \mid i \in I)$ of models of T, the union $\bigcup_{i\in I}\mathscr{M}_i$ is again a model of T.*

Proof. The easier implication here has already been shown in question 10. For the converse we need to show that T is inductive provided that the union of every chain of models of T is again a model of T. It suffices to show that every model $\mathscr{M}$ of $T_{\forall\exists}$ is a model of T.

By 3.6.5, $\mathscr{M}$ is e.c. in a model $\mathscr{N}_0$ of T. By 3.6.3, $\mathscr{N}_0$ is a substructure of an elementary extension $\mathscr{M}_1$ of $\mathscr{M}$. Then $\mathscr{M}_1 \models T_{\forall\exists}$ again and we can repeat the two constructions. We get a chain of $\mathscr{L}$-structures

$$\mathscr{M} = \mathscr{M}_0 \subseteq \mathscr{N}_0 \subseteq \mathscr{M}_1 \subseteq \mathscr{N}_1 \subseteq \mathscr{M}_2 \subseteq \mathscr{N}_2 \subseteq \dots$$

such that $\mathscr{M}_i \prec \mathscr{M}_{i+1}$ and $\mathscr{N}_i \models T$. Let $\mathscr{N}$ be the union of this chain. By assumption applied to the chain $(\mathscr{N}_i)_{i\in\mathbb{N}_0}$ we know that $\mathscr{N} \models T$ and by the elementary chain lemma 2.5.1 applied to the chain $(\mathscr{M}_i)_{i\in\mathbb{N}_0}$ we know $\mathscr{M} \prec \mathscr{N}$, which implies that $\mathscr{M} \equiv \mathscr{N}$. Hence $\mathscr{M} \models T$ as required. □

We need the following application of 2.3.4.

3.6.7. **Proposition.** *Let $\mathscr{M}, \mathscr{N}$ be $\mathscr{L}$-structures and let $\bar{a} \in |\mathscr{M}|^n$, $\bar{b} \in |\mathscr{N}|^n$. The following are equivalent.*

(i) $\mathrm{tp}_\exists(\mathscr{M}, \bar{a}) \subseteq \mathrm{tp}_\exists(\mathscr{N}, \bar{b})$.
(ii) There is an elementary extension $\mathscr{N}'$ of $\mathscr{N}$ and an embedding $f : \mathscr{M} \longrightarrow \mathscr{N}'$ with $f(\bar{a}) = \bar{b}$.
(iii) There is an extension $\mathscr{N}'$ of $\mathscr{M}$ and an elementary embedding $g : \mathscr{N} \longrightarrow \mathscr{N}'$ with $g(\bar{b}) = \bar{a}$.

Proof. The equivalence of (ii) and (iii) follows easily from appropriate identifications. The implication (ii)⇒(i) is clear. We show (i)⇒(ii).

We work in the language $\mathscr{L}^* = \mathscr{L}(c_1, \ldots, c_n)$ with new constant symbols $c_1, \ldots, c_n$ and consider $(\mathscr{M}, \bar{a})$, $(\mathscr{N}, \bar{b})$ as structures in this language. Then (i) says that condition (ii) of 2.3.4 is satisfied for $(\mathscr{M}, \bar{a})$, $(\mathscr{N}, \bar{b})$. Hence by 2.3.4(i) there is an elementary extension $\mathscr{N}^*$ of $(\mathscr{N}, \bar{b})$ together with an $\mathscr{L}^*$-embedding $f : (\mathscr{M}, \bar{a}) \longrightarrow \mathscr{N}^*$. Now $\mathscr{N}^* = (\mathscr{N}', \bar{b})$, where $\mathscr{N}'$ is the reduct of $\mathscr{N}^*$ to $\mathscr{L}$. Then $\mathscr{N}'$ is an elementary extension of $\mathscr{N}$ and f is an embedding $\mathscr{M} \longrightarrow \mathscr{N}'$ with $f(\bar{a}) = \bar{b}$. □

3.6.8. **Theorem.** *Let T be any $\mathscr{L}$-theory. The following are equivalent for every model $\mathscr{M}$ of T.*

(i) $\mathscr{M}$ is an e.c. model of T.
(ii) For every $n \in \mathbb{N}$ and all $\bar{a} \in |\mathscr{M}|^n$, the $\exists$-type $\mathrm{tp}_\exists(\mathscr{M}, \bar{a})$ is maximal for inclusion among all $\exists$-types of the form $\mathrm{tp}_\exists(\mathscr{N}, \bar{b})$, $\mathscr{N} \models T$ and $\bar{b} \in |\mathscr{N}|^n$.

Proof. (ii)⇒(i): Take $\mathscr{N} \models T$ with $\mathscr{M} \subseteq \mathscr{N}$ and some $\bar{a} \in |\mathscr{M}|^n$. Then $\mathrm{tp}_\exists(\mathscr{M}, \bar{a}) \subseteq \mathrm{tp}_\exists(\mathscr{N}, \bar{a})$. By (ii) we have $\mathrm{tp}_\exists(\mathscr{M}, \bar{a}) = \mathrm{tp}_\exists(\mathscr{N}, \bar{a})$, as required (see 3.6.2(iii))

(i)⇒(ii). By 3.6.7, there is an elementary extension $\mathscr{N}'$ of $\mathscr{N}$ and an embedding $f : \mathscr{M} \longrightarrow \mathscr{N}'$ with $f(\bar{a}) = \bar{b}$. Hence $\mathscr{N}' \models T$ and (upon identifying $\mathscr{M}$ with the image of $\mathscr{M}$) the assumption in (i) implies that $\mathrm{tp}_\exists(\mathscr{M}, \bar{a}) = \mathrm{tp}_\exists(\mathscr{N}', \bar{b})$. As $\mathscr{N} \prec \mathscr{N}'$ we get $\mathrm{tp}_\exists(\mathscr{N}', \bar{b}) = \mathrm{tp}_\exists(\mathscr{N}, \bar{b})$. Altogether we obtain $\mathrm{tp}_\exists(\mathscr{M}, \bar{a}) = \mathrm{tp}_\exists(\mathscr{N}, \bar{b})$, as required. □

3.6.9**. Robinson Test for Model-Completeness** The following are equivalent for every $\mathscr{L}$-theory T.

(i) T is **model-complete**, i.e. for every $\mathscr{L}$ formulas $\varphi(\bar{x})$ there is an existential $\mathscr{L}$-formula $\delta(\bar{x})$ such that $T \models \forall \bar{x}(\varphi \leftrightarrow \delta)$.
(ii) For every existential $\mathscr{L}$ formulas $\delta(\bar{x})$ there is a universal $\mathscr{L}$-formula $\sigma(\bar{x})$ such that $T \models \forall \bar{x}(\delta \leftrightarrow \sigma)$.
(iii) Every extension $\mathscr{A} \subseteq \mathscr{B}$ of models of T is elementary.
(iv) Every model of T is an e.c. model of T.

Proof. (i)⇒(iii) By (i), every formula is equivalent to an existential $\mathscr{L}$-formula modulo T. Since every embedding preserves every existential formula, every embedding between models of T preserves all formulas. This is what (iii) says.

(iii)⇒(iv) is a weakening.

(iv)⇒(ii). There are proofs using diagrams for example see [Hodges1993, Theorem 8.3.1]; we use 3.6.8 for a topological argument in the setup of [DiScTr2019, chapter 14]. By 3.6.8, every $\exists$-type of T in n variables is maximal. Hence every point of the spectral space of all $\exists$-types of T in n variables[3] is closed. By [DiScTr2019, Proposition 1.3.20] these spaces are then Boolean. This assertion is equivalent to (ii).

(ii)⇒(i) is straightforward from the prenex normal form theorem 1.37. □

3.6.10. **Corollary.** *Every model-complete theory T is inductive, hence is axiomatised by $\forall\exists$-sentences.*

Proof. By 3.6.9, every chain of models of T is an elementary chain. By the elementary chain lemma 2.5.1, for every chain $(\mathscr{M}_i \mid i \in I)$ of models of T, the union $\bigcup_{i\in I} \mathscr{M}_i$ is again a model of T. By 3.6.6 we see that T is inductive. □

3.6.11. **Proposition.** *Suppose T is an inductive $\mathscr{L}$-theory. Then every model $\mathscr{M}$ of T is a substructure of an e.c. model $\mathscr{N}$ of T of cardinality $\leq \max\{\operatorname{card}(\mathscr{L}), \operatorname{card}(\mathscr{M})\}$.*

Proof. Take a model $\mathscr{M}$ of T and enumerate the pairs $(\delta_i(\bar{x}), \bar{a}_i)_{i<\lambda}$ where $\delta_i(\bar{x}) \in \exists\text{-}\operatorname{Fml}(\mathscr{L})$ and $\bar{a}_i \in |\mathscr{M}|^{\bar{x}}$, as follows: Take $\mathscr{M}_0 = \mathscr{M}$ and at limit ordinals take unions (by 3.6.6, using that T is inductive, these unions will again be models of T). For the induction step: If there is a model $\mathscr{N}$ of T such that $\mathscr{N} \models \delta_i(\bar{a}_i)$, then take $\mathscr{M} \subseteq \mathscr{M}_{i+1} \prec \mathscr{N}$ with $\operatorname{card}(\mathscr{M}_i + 1) \leq \max\{\operatorname{card}(\mathscr{L}), \operatorname{card}(\mathscr{M})$; otherwise take $\mathscr{M}_{i+1} = \mathscr{M}_i$. The union of the chain is denoted by $\mathscr{M}^*$ and is still a model of T (by 3.6.6 again).

$\mathscr{M}^*$ then has the following property by construction: If $\mathscr{N} \models T$ is an extension of $\mathscr{M}^*$, and $\delta_i(\bar{x}) \in \exists\text{-}\operatorname{Fml}(\mathscr{L})$ and $\bar{a}_i \in |\mathscr{M}|^{\bar{x}}$ such that $\mathscr{N} \models \delta_i(\bar{a}_i)$, then also $\mathscr{M}^* \models \delta_i(\bar{a}_i)$. Thus if we iterate this construction we get a chain $\mathscr{M} \subseteq \mathscr{M}^* \subseteq \mathscr{M}^{**} \subseteq \mathscr{M}^{***} \ldots$ and its union $\mathscr{N}$ is an e.c. model of T. The cardinality estimate is left as an exercise. □

3.6.12. *Remark.* Without the assumption that T is inductive, the conclusion of 3.6.11 fails badly, see 3.6.14.

We obtain first examples of model-complete theories:

3.6.13**. Lindström's test** If T is an inductive $\mathscr{L}$-theory without finite models and T is λ-categorical for some $\lambda \geq \operatorname{card}(\mathscr{L})$, then T is model-complete.

Proof. Otherwise, by 3.6.9 there is an extension $\mathscr{M} \subseteq \mathscr{N}$ of models of T such that $\mathscr{M}$ is not e.c. in $\mathscr{N}$. Using that $\mathscr{M}$ is infinite and $\lambda \geq \operatorname{card}(\mathscr{L})$, an application of the compactness theorem to the theory of the pair $(\mathscr{N}, \mathscr{M})$ shows that we may assume that $\operatorname{card}(\mathscr{M}) = \operatorname{card}(\mathscr{N}) = \lambda$ (we only need that $\operatorname{card}(\mathscr{M}) = \lambda$). Since $\mathscr{M}$ is not e.c. in $\mathscr{N}$, the T-model $\mathscr{M}$ is not an e.c. model of T. However, since T is inductive, 3.6.11 says that there is an e.c. model $\mathscr{N}'$ of T of cardinality λ. By assumption $\mathscr{N}' \cong \mathscr{M}$, showing that $\mathscr{M}$ is an e.c. model of T, a contradiction. □

3.6.14. *Remark.* The assumption that T is inductive in 3.6.13 is necessary. There is an $\aleph_1$-categorical theory T in a countable language that is not model-complete: See [Hodges1993, exercise 10 for section 8.3, p. 381].

[3]This space consist of the $\exists$-types $\operatorname{tp}_\exists(\mathscr{M}, \bar{a})$, where $\mathscr{M} \models T$ and $\bar{a} \in |\mathscr{M}|^n$ and a basis is given by all the sets $\{p \mid p \notin \delta\}$ with $\delta(x_1, \ldots, x_n) \in \exists\text{-}\operatorname{Fml}(\mathscr{L})$.

This also shows that T does not have any uncountable e.c. models: Suppose $\mathscr{N}'$ would be such a model and let λ be its cardinality. By Morley's theorem, T is also λ-categorical. Take $\mathscr{M}, \mathscr{N} \models T$ such that $\mathscr{M} \subseteq \mathscr{N}$ but $\mathscr{M}$ is not e.c. in $\mathscr{N}$. As in the proof of 3.6.13 we may then produce such a situation where in addition $\operatorname{card}(\mathscr{M}) = \operatorname{card}(\mathscr{N}')$. Since T is λ-categorical we get $\mathscr{M} \cong \mathscr{N}'$, a contradiction.

Resultants

3.6.15. **Definition.** Let T be an $\mathscr{L}$-theory and let $\delta(\bar{x}) \in \exists\text{-}\operatorname{Fml}(\mathscr{L})$. The **resultant** of δ (for T) is the set $\operatorname{Res}_\delta^T$ of all $\sigma(\bar{x}) \in \forall\text{-}\operatorname{Fml}(\mathscr{L})$ with the property $T \models \forall\bar{x}(\delta \to \sigma)$. [Notice that the sentence $\forall\bar{x}(\delta \to \sigma)$ is universal and thus $T \models \forall\bar{x}(\delta \to \sigma)$ is the same as $\forall\bar{x}(\delta \to \sigma) \in T_\forall$. In other words $\operatorname{Res}_\delta^T$ only depends on $T_\forall$]

The crucial importance of the resultant of $\delta(\bar{x}) \in \exists\text{-}\operatorname{Fml}(\mathscr{L})$ is the following:

3.6.16. **Theorem.** *[Hodges1985, Theorem 3.1.1], [Hodges1993, Theorem 8.2.4.] If $\mathscr{A}$ is an $\mathscr{L}$-structure (not necessarily a model of T) and $\bar{a} \in \mathscr{A}^{\bar{x}}$, then the following are equivalent:*

(i) There is some $\mathscr{N} \models T$ with $\mathscr{A} \subseteq \mathscr{N}$ such that $\mathscr{N} \models \delta(\bar{a})$.
(ii) $\mathscr{A} \models \sigma(\bar{a})$ for all σ from the resultant of δ for T.

Proof. We show the contrapositive, i.e. we show the equivalence of the following conditions.

(a) For all $\mathscr{N} \models T$ with $\mathscr{A} \subseteq \mathscr{N}$ we have $\mathscr{N} \models \neg\delta(\bar{a})$. In other words: $T \cup \operatorname{diag}(\mathscr{A}) \vdash \neg\delta(\bar{a})$.
(b) $\mathscr{A} \models \neg\sigma(\bar{a})$ for some σ from the resultant of δ for T. In other words: There is some $\sigma(\bar{x}) \in \forall\text{-}\operatorname{Fml}(\mathscr{L})$ with $T \vdash \delta \to \sigma$ such that $\mathscr{A} \models \neg\sigma(\bar{a})$

Proof of (b)⇒(a): If $\mathscr{A} \models \neg\sigma(\bar{a})$ for some $\sigma \in \operatorname{Res}_\delta^T$ and $\mathscr{N} \models T$ with $\mathscr{A} \subseteq \mathscr{N}$, then $\mathscr{N} \models \neg\delta(\bar{a})$, otherwise $\mathscr{N} \models \delta(\bar{a})$ and as $T \vdash \delta \to \sigma$ we have $\mathscr{N} \models \sigma(\bar{a})$. But σ is universal and so $\mathscr{A} \models \sigma(\bar{a})$, a contradiction.

Proof of (a)⇒(b): Condition (a) says that for some quantifier-free $\mathscr{L}$-formula $\chi(\bar{x}, \bar{y})$ and some $\bar{b} \in \mathscr{A}$ disjoint from $\bar{a}$ with $\mathscr{A} \models \chi(\bar{a}, \bar{b})$ we have $T \vdash \chi(\bar{a}, \bar{b}) \to \neg\delta(\bar{a})$, i.e. $T \vdash \delta(\bar{a}) \to \neg\chi(\bar{a}, \bar{b})$. Since $\bar{a}$ and $\bar{b}$ are disjoint, this is equivalent – by standard diagram arguments – to $T \vdash \delta(\bar{a}) \to \forall\bar{y}\neg\chi(\bar{a}, \bar{y})$ and so $\sigma(\bar{x})$ defined as $\forall\bar{y}\neg\chi(\bar{x}, \bar{y})$ is in the resultant of δ.

But now $\mathscr{A} \models \neg\sigma(\bar{a})$. Otherwise $\mathscr{A} \models \sigma(\bar{a})$, in particular $\mathscr{A} \models \neg\chi(\bar{a}, \bar{b})$, in contradiction to the choice of $\chi(\bar{a}, \bar{b})$. □

3.6.17. **Corollary.** *If follows that a model $\mathscr{M}$ of T is e.c. if and only if for every $\delta(\bar{x}) \in \exists\text{-}\operatorname{Fml}(\mathscr{L})$ and each $\bar{a} \in \mathscr{M}^{\bar{x}}$, if $\mathscr{M}$ realizes $\operatorname{Res}_\delta^T$ at $\bar{a}$, then $\mathscr{M} \models \delta(\bar{a})$.* □

3.6.18. **Corollary.** *If T is inductive, then an $\mathscr{L}$-structure $\mathscr{A}$ is an e.c. model of T if and only if $\mathscr{A}$ is e.c. in an e.c. model of T.*

Proof. This is clear in one direction. So assume that $\mathscr{A}$ is e.c. in $\mathscr{M}$ and $\mathscr{M}$ is an e.c. model of T. Since T is inductive we know $\mathscr{A} \models T$ and we use 3.6.17 to verify that $\mathscr{A}$ is e.c. Take $\delta(\bar{x}) \in \exists\text{-}\operatorname{Fml}(\mathscr{L})$ and $\bar{a} \in \mathscr{A}^{\bar{x}}$, such that $\mathscr{A}$ realizes $\operatorname{Res}_\delta^T$ at $\bar{a}$. We need to show that $\mathscr{A} \models \delta(\bar{a})$.

Since $\operatorname{Res}_\delta^T \subseteq \forall\text{-}\operatorname{Fml}(\mathscr{L})$ and $\mathscr{A}$ is e.c. in $\mathscr{M}$ we also know that $\mathscr{M}$ realizes $\operatorname{Res}_\delta^T$ at $\bar{a}$. As $\mathscr{M}$ is an e.c. model of T we get $\mathscr{M} \models \delta(\bar{a})$ from 3.6.17. As $\mathscr{A}$ is e.c. in $\mathscr{M}$ we get $\mathscr{A} \models \delta(\bar{a})$ as required. □

3.7. **Omitting types.**

The omitting types theorem can be seen as a feature of the term model of a consistent theory as constructed in the Henkin proof of the completeness theorem. We emphasize this point here and give a detailed account to this point of view. Many proofs in the literature go down the same route, but in a more compressed form. There are other proofs, references are given in 3.7.9.

3.7.1. **Definition.** Given a language $\mathscr{L}$ we define $\mathrm{Fml}(\mathscr{L})(1)$ as the set of all $\mathscr{L}$-formulas with at most one free variable.

A **system of witnesses for** $\mathscr{L}$ is just a map

$$\zeta : \mathrm{Fml}(\mathscr{L})(1) \longrightarrow \mathscr{C},$$

where $\mathscr{C}$ is the set of constant symbols of $\mathscr{L}$. For such a map we define

$$H(\zeta) := \{\ (\exists x\varphi)\ \rightarrow\ \varphi(x/\zeta(\varphi))\mid \varphi(x)\in \mathrm{Fml}(\mathscr{L})(1)\ \}$$

3.7.2. **Lemma.** *If $\mathscr{M}\models H(\zeta)$, then the set $A := \{c^{\mathscr{M}}\mid c\in\mathscr{C}\}$ is (the universe of) an elementary substructure of $\mathscr{M}$.*

If $\mathscr{N}$ is another model of $H(\zeta)$ with prime substructure B (i.e., B is the smallest substructure of $\mathscr{N}$), then

$$A\cong B \iff A \text{ and } B \text{ satisfy the same atomic sentences.}$$

It follows that the isomorphism classes of prime substructures of models of $H(\zeta)$ is in bijection with the complete $\mathscr{L}$-theories containing $H(\zeta)$.

Proof. This is a direct consequence of the Tarski-Vaught test: Let $\varphi(\bar x, y)$ be an $\mathscr{L}$-formula and let $\bar a\in A^{\bar x}$, say $a_1 = c_1^{\mathscr{M}},...,a_n = c_n^{\mathscr{M}}$, with $\mathscr{M}\models\exists y\varphi[\bar a]$. By the Tarski-Vaught test we only need to find some $b\in A$ with $\mathscr{M}\models\exists y\varphi[\bar a, b]$. Let $\psi := \varphi(x_1/c_1,...,x_n/c_n)$. Then $\psi(y)\in\mathrm{Fml}(\mathscr{L})(1)$ and $\mathscr{M}\models\exists y\varphi[\bar a]$ reads as $\mathscr{M}\models\exists y\psi$. As $\mathscr{M}\models H(\zeta)$ we get $\mathscr{M}\models\psi(\zeta(\psi))$, so we may take $b := \zeta(\psi)^{\mathscr{M}}$.

The equivalence is clear □

It is of course not true that $H(\zeta)$ is consistent in general.

3.7.3. **Definition.** Let $\mathscr{L}$ be any language, let $T\subseteq\mathrm{Sen}(\mathscr{L})$ and let $\Sigma\subseteq\mathrm{Fml}(\mathscr{L})$. We say that Σ **is supported by** T if there is a formula φ such that $T\cup\{\varphi\}$ is consistent and

$$T\models\varphi\rightarrow\sigma \text{ for all } \sigma\in\Sigma.$$

Otherwise for every formula φ such that $T\cup\{\varphi\}$ is consistent there is some $\sigma\in\Sigma$ such that $T\cup\{\varphi,\neg\sigma\}$ is consistent, and we say that Σ **is unsupported by** T.

3.7.4. *Remarks.* Let $T\subseteq\mathrm{Sen}(\mathscr{L})$, $\Sigma\subseteq\mathrm{Fml}(\mathscr{L})$.

(i) Notice that T is consistent if it supports some Σ, i.e. if T is inconsistent, then every Σ is unsupported by T.

(ii) If T is consistent and does not support Σ, then $\Sigma\neq\emptyset$.

(iii) If X is a set of variables and the free variables of any $\sigma\in\Sigma$ are in X, then Σ is already unsupported by T if for every formula φ with free variables in X such that $T\cup\{\varphi\}$ is consistent there is some $\sigma\in\Sigma$ such that $T\cup\{\varphi,\neg\sigma\}$ is consistent.

The reason is that for a formula $\varphi(\bar x,\bar y)$ with $\bar x\subseteq X$ and $\bar y$ disjoint from X, such that $T\cup\{\varphi\}$ is consistent, also $T\cup\{\exists\bar y\varphi\}$ is consistent, hence there is some $\sigma\in\Sigma$ such that $T\cup\{\exists\bar y\varphi,\neg\sigma\}$ is consistent and because $\bar y$ does not occur freely in σ we get that $T\cup\{\varphi,\neg\sigma\}$ is consistent.

3.7.5. **Proposition.** *Let $\mathscr{L}$ be any language and let $\zeta : \mathrm{Fml}(\mathscr{L})(1) \longrightarrow \mathscr{D}$ be a bijection onto a set $\mathscr{D}$ which is disjoint from the alphabet of $\mathscr{L}$. Let $\mathscr{L}'$ be the language $\mathscr{L}(D)$ and let*

$$H(\zeta) := \{\ (\exists x\varphi)\ \rightarrow\ \varphi(x/\zeta(\varphi))\ \mid \varphi(x) \in \mathrm{Fml}(\mathscr{L})(1)\ \}$$

Let $T \subseteq \mathrm{Sen}(\mathscr{L})$ be consistent. Then

(i) Every model of T can be expanded to a model of $T \cup H(\zeta)$, in particular $T \cup H(\zeta)$ is consistent.

(ii) If $\Sigma \subseteq \mathrm{Fml}(\mathscr{L})$ is unsupported by T, then Σ is also unsupported by $T \cup H(\zeta)$ (in the language $\mathscr{L}'$).

`Remark:` Item (i) here is straightforward as one can see from its proof below. The crucial statement is (ii), which is the core technical argument of the omitting types theorem 3.7.9.

Proof. (i) Let $\mathscr{M} \models T$ and expand $\mathscr{M}$ to an $\mathscr{L}'$ structure $\mathscr{M}'$ by

$$\zeta(\varphi)^{\mathscr{M}'} = \begin{cases} a & \text{for some } a \in |\mathscr{M}| \text{ with } \mathscr{M} \models \varphi(a) \text{ if } \mathscr{M} \models \exists x\varphi, \\ \text{arbitrarily} & \text{if } \mathscr{M} \models \neg\exists x\varphi. \end{cases}$$

for every $\varphi(x) \in \mathrm{Fml}(\mathscr{L})(1)$. Obviously $\mathscr{M}' \models T \cup H(\zeta)$.

(ii) Let ρ be an $\mathscr{L}'$-formula and assume $T \cup H(\zeta) \cup \{\rho\}$ is consistent. Then ρ is of the form $\varphi(\bar{x}, \zeta(\varphi_1), \dots, \zeta(\varphi_k))$ for some $\mathscr{L}$-formula $\varphi(\bar{x}, \bar{y})$, $\bar{y} = y_1, \dots, y_k$ and some $\varphi_1, \dots, \varphi_k \in \mathrm{Fml}(\mathscr{L})(1)$ with $\varphi_i \neq \varphi_j$ for $i \neq j$. By assumption there is an $\mathscr{L}$-structure $\mathscr{M}$ and some $A \subseteq |\mathscr{M}|$ with $(\mathscr{M}, A) \models T \cup H(\zeta)$ together with some $\bar{b} \in |\mathscr{M}|^{\bar{x}}$ such that $(\mathscr{M}, A) \models \varphi(\bar{b}, \zeta(\varphi_1), \dots, \zeta(\varphi_k))$. Let $v_1, \dots, v_k$ be distinct variables not occurring in φ and not occurring in any φ_i and let $I \subseteq \{1, \dots, k\}$ be the set of all indices with $\mathscr{M} \models \exists v_i \varphi_i(v_i)$. Then $\mathscr{M} \models \exists \bar{x} \hat{\varphi}(\bar{x})$, where

$$\hat{\varphi}(\bar{x}) \text{ is } \forall w \bigwedge_{j \in \{1,\dots,k\} \setminus I} \neg\varphi_j(w)\ \wedge\ \exists v_1, \dots, v_k \bigwedge_{i \in I} \varphi_i(v_i) \wedge \varphi(\bar{x}, v_1, \dots, v_k).$$

Since $\hat{\varphi}(\bar{x})$ is an $\mathscr{L}$-formula, the assumption in (ii) says that there is some $\sigma(\bar{x}, \bar{z}) \in \Sigma$ such that $T \cup \{\hat{\varphi}(\bar{x}) \wedge \neg\sigma(\bar{x}, \bar{z})\}$ is consistent. Take a model $\mathscr{N}$ of T and some $(\bar{\beta}, \bar{\gamma}) \in |\mathscr{N}|^{\bar{x}} \times |\mathscr{N}|^{\bar{z}}$ with $\mathscr{N} \models \hat{\varphi}(\bar{\beta}) \wedge \neg\sigma(\bar{\beta}, \bar{\gamma})$. In particular there are $\varepsilon_1, \dots, \varepsilon_k \in |\mathscr{N}|$ with

$$\mathscr{N} \models \bigwedge_{i \in I} \varphi_i(\varepsilon_i) \wedge \varphi(\bar{\beta}, \varepsilon_1, \dots, \varepsilon_k).$$

We expand $\mathscr{N}$ to an $\mathscr{L}'$-structure $\mathscr{N}'$ as follows. If $\psi \in \mathrm{Fml}(\mathscr{L})(1)$, then define

$$\zeta(\psi)^{\mathscr{N}'} = \begin{cases} \varepsilon_i & \text{if } \psi = \varphi_i \text{ for some } i \in \{1, \dots, k\}, \\ \tau & \text{if } \psi \neq \varphi_i \text{ for all } i \in \{1, \dots, k\},\ \mathscr{N} \models \exists w \psi(w) \text{ and} \\ & \quad \tau \in |\mathscr{N}| \text{ is some realization of } \psi, \\ \text{arbitrarily} & \text{otherwise.} \end{cases}$$

As $\mathscr{N} \models \hat{\varphi}(\bar{\beta})$ it follows that $\mathscr{N}' \models T \cup H(\zeta)$ and from $\zeta(\psi)^{\mathscr{N}'} = \varepsilon_i$ for $i \in \{1, \dots, k\}$ we see that $\mathscr{N}' \models \varphi(\bar{\beta}, \zeta(\varphi_1), \dots, \zeta(\varphi_k)) \wedge \neg\sigma(\bar{\beta}, \bar{\gamma})$. Hence $\mathscr{N}'$ witnesses the consistency of $T \cup H(\zeta) \cup \{\varphi(\bar{x}, \zeta(\varphi_1), \dots, \zeta(\varphi_k)) \wedge \neg\sigma(\bar{x}, \bar{z})\} = T \cup H(\zeta) \cup \{\rho, \neg\sigma\}$, as required. □

3.7.6. **Theorem.** *Let* $\mathscr{L} = (\lambda : \mathscr{R} \longrightarrow \mathbb{N}, \mu : \mathscr{F} \longrightarrow \mathbb{N}, \mathscr{C})$ *be a language. Then there is an extension by constants* $\mathscr{L}^*$ *of* $\mathscr{L}$ *by* $\operatorname{card}(\operatorname{Fml}(\mathscr{L}))$ *many constants and a subset* H^* *of* $\mathscr{L}^*$*-sentences such that*

(i) H^* *possess a system of witnesses (in* $\mathscr{L}^*$*).*

(ii) For every consistent $T \subseteq \operatorname{Sen}(\mathscr{L})$ *the set* $T \cup H^*$ *is consistent (note that any system of witnesses for* H^* *is also a system of witnesses for* $T \cup H^*$*).*

(iii) If $\Sigma \subseteq \operatorname{Fml}(\mathscr{L})$ *is unsupported by* T*, then* Σ *is also unsupported by* $T \cup H^*$ *(in the language* $\mathscr{L}^*$*).*

Proof. We iterate 3.7.5 on the syntactic side only and get $\mathscr{L}$ and H^* as the union of $\mathscr{L}, \mathscr{L}', \mathscr{L}'', \ldots$ and $H(\zeta) \cup H(\zeta)(\zeta) \cup \ldots$ respectively. In the first step we know that $T \cup H(\zeta)$ is consistent and for all $\psi \in \operatorname{Fml}(\mathscr{L}')$ with $T \cup H(\zeta) \cup \{\psi\}$ consistent, there is some $\sigma \in \Sigma$ such that $T \cup H(\zeta) \cup \{\psi, \neg\sigma\}$ is consistent. Now replace $\mathscr{L}$ by $\mathscr{L}'$ and T by $T \cup H(\zeta)$ and apply 3.7.5 again to the new data. Continue in this way to obtain $\mathscr{L}^*$ and H^* with the required properties. □

3.7.7. **Corollary.** *Let* $\mathscr{L}$ *be a language. Then for each infinite cardinal* $\kappa \geq \operatorname{card}(\mathscr{L})$ *there is an extension by constants* $\mathscr{L}^*$ *of* $\mathscr{L}$ *and an* $\mathscr{L}^*$*-theory* H^* *that has the following properties (where we write* C *for the set of constants of* $\mathscr{L}^*$*).*

(i) For every consistent set T *of* $\mathscr{L}$*-sentences, the set* $T \cup H^*$ *is consistent in* $\mathscr{L}^*$*.*

(ii) Every model $\mathscr{M}^*$ *of* H^* *induces an* $\mathscr{L}^*$*-elementary substructure on* $C^{\mathscr{M}^*}$ *(the set of interpretations in* $\mathscr{M}^*$ *of the constants from* C*) of size* κ*.*

(iii) If $\Sigma \subseteq \operatorname{Fml}(\mathscr{L})$ *is unsupported by* T*, then* Σ *is also unsupported by* $T \cup H^*$ *(in the language* $\mathscr{L}^*$*).*

Proof. Let D be a set of new constants for $\mathscr{L}$ of size κ and let $\mathscr{L}^*$, H^* be the language and theory assigned to $\mathscr{L}(D)$ by 3.7.6. We then know that H^* has a system of witnesses and that (i) holds.

(ii) holds by 3.7.2.

(iii) holds by 3.7.6. □

3.7.8. **Lemma.** *Let* $T \subseteq \operatorname{Sen}(\mathscr{L})$*,* $\Sigma \subseteq \operatorname{Fml}(\mathscr{L})$*. If* Σ *is unsupported by* T *and* $\rho \in \operatorname{Sen}(\mathscr{L})$*, then* Σ *is unsupported by* $T \cup \{\rho\}$*.*

Proof. Suppose Σ is supported by $T \cup \{\rho\}$, i.e. there is a formula φ such that $T \cup \{\rho\} \cup \{\varphi\}$ is consistent and

$$T \cup \{\rho\} \models \varphi \to \sigma \text{ for all } \sigma \in \Sigma.$$

But then $T \cup \{\rho \wedge \varphi\}$ is consistent and

$$T \models (\rho \wedge \varphi) \to \sigma \text{ for all } \sigma \in \Sigma,$$

proving that Σ is supported by T. □

3.7.9. **Omitting Types Theorem** *[Hodges1993, Theorem 7.2.1],[TenZie2012, Theorem 4.1.2]. There is also a proof using model theoretic forcing in [Hodges1985, section 5.1]. A generalization may be found in [Marker2016, Theorem 4.2.1]. A variant for $\forall$-types may be found in [Hodges1993, Theorem 8.2.6].*

Let $\mathscr{L}$ be a countable language, let T be a consistent $\mathscr{L}$-theory and for each $n \in \mathbb{N}$ let $\Sigma_n(v_1, \dots, v_n)$ be a set of $\mathscr{L}$-formulas in at most n free variables. Suppose every Σ_n is unsupported by T, i.e. for each $n \in \mathbb{N}$ and every $\mathscr{L}$-formula φ that is consistent with T there is some $\sigma \in \Sigma_n$ such that $T \cup \{\varphi, \neg\sigma\}$ is consistent. Then there is a model $\mathscr{M}$ of T that **omits** *all Σ_n, i.e., for every $n \in \mathbb{N}$, the set Σ is not realized in $\mathscr{M}$.*

Proof. Using 3.7.7 we can assume that we work with a theory T in a countable language $\mathscr{L}$ that has a system of witnesses.

Let $(\bar{c}_i)_{i<\omega}$ be an enumeration all of tuples of arbitrary but finite length of constant symbols of $\mathscr{L}$. We write $|\bar{c}_i|$ for the length of the tuple $\bar{c}_i$.

By induction on i we define formulas $\sigma_i \in \Sigma_{|\bar{c}_i|}$ such that $T_i := T \cup \{\neg\sigma_j(\bar{c}_j) \mid j < i\}$ is consistent and does not support any Σ_n:

We take $T_0 = T$, which is consistent and does not support any Σ_n by assumption. For the induction step assume T_i has already been defined, T_i is consistent and does not support any Σ_n. Let $k = |\bar{c}_i|$. Then $T_i \cup \{\bar{v} = \bar{c}_i\}$ is consistent, where $\bar{v} = (v_1, \dots, v_k)$. As T_i does not support Σ_k there is some $\sigma_i \in \Sigma_k$ such that $T_i \cup \{\bar{v} = \bar{c}_i, \neg\sigma_i\}$ is consistent. But this means $T_{i+1} = T_i \cup \{\neg\sigma_i(\bar{c}_i)\}$ is consistent and by 3.7.8, each Σ_n is again unsupported by T_{i+1}.

This finishes the definition of the σ_i. Now we see that the set $\bigcup_{i<\omega} T_i = T \cup \{\neg\sigma_j(\bar{c}_j) \mid j < \omega\}$ is consistent and thus has a model $\mathscr{M}$. As T has a system of witnesses we know from 3.7.2 that we may choose $\mathscr{M}$ such that all of its elements are interpretations of constant symbols of $\mathscr{L}$. We show that this model omits every Σ_n: Take $n \in \mathbb{N}$ and let $\bar{a} \in |\mathscr{M}|^n$. Then $\bar{a}$ is the interpretation of some n-tuple $\bar{c}_i$ of constants. By construction $\sigma_i \in \Sigma_n$ and $\mathscr{M} \models \neg\sigma_i(\bar{c}_i)$. Hence $\mathscr{M}$ omits Σ_n. □

3.7.10. *Warning.* The sets Σ_n in the omitting types theorem 3.7.9 are all countable, because a countable language has only countably many formulas. One might wonder whether every consistent theory T in an arbitrary language $\mathscr{L}$ that does not support a given countable set $\Sigma(v)$ in a single variable, has a model that omits Σ. However this is not the case.

A prominent application of the Omitting Types Theorem is the following.

3.7.11. **Ryll-Nardzewski Theorem** *A theory T without finite models in a countable language is $\aleph_0$-categorical if and only if T is complete and $S_n(T)$ is finite for all $n \in \mathbb{N}$.*

Proof. First suppose T is $\aleph_0$-categorical. The Skolem-Löwenheim downwards theorem then implies that T is complete. In order to show that $S_n(T)$ is finite it suffices to show that each $p \in S_n(T)$ is isolated (because $S_n(T)$ is compact). Suppose p is not isolated. This is equivalent to saying that p is unsupported by T. By the Omitting Types Theorem, using crucially that the language is countable, there is a model $\mathscr{M}$ of T omitting p. By Skolem-Löwenheim downwards we may then assume that $\mathscr{M}$ is countable. On the other hand, p is realized in some model $\mathscr{N}$ of T and by Skolem-Löwenheim downwards again we may also assume that $\mathscr{N}$ is countable. However T is $\aleph_0$-categorical, hence $\mathscr{M} \cong \mathscr{N}$, a contradiction.

Conversely suppose T is complete and $S_n(T)$ is finite for all $n \in \mathbb{N}$. Then for every finite subset $A = \{a_1, \ldots, a_n\}$ of a model $\mathscr{M}$ of T one verifies that $S_1(\mathscr{M}, A)$ is finite by mapping $S_1(\mathscr{M}, A)$ injectively into $S_{n+1}(T)$, where $p = \mathrm{tp}^{\mathscr{M}}(\beta/A)$ is mapped to $\mathrm{tp}^{\mathscr{M}}(\beta, a_1, \ldots, a_n)$. Since $S_1(\mathscr{M}, A)$ is finite and a Hausdorff space, we may use 3.3.7 to obtain

(†) for each $p \in S_1(\mathscr{M}, A)$ there is some $\varphi \in p$ such that $\langle\varphi\rangle = \{p\}$.

Now take countable models $\mathscr{M}$ and $\mathscr{N}$ of T. Consider the set of bijective maps $p : \{a_1, \ldots, a_n\} \longrightarrow \{b_1, \ldots, b_n\}$, $p(a_i) = b_i$, where $n \in \mathbb{N}_0$, $a_i \in |\mathscr{M}|$ and $b_i \in |\mathscr{N}|$ such that $\mathrm{tp}^{\mathscr{M}}(a_1, \ldots, a_n) = \mathrm{tp}^{\mathscr{N}}(b_1, \ldots, b_n)$. Using (†) one then checks that these maps form a back and forth system between $\mathscr{M}$ and $\mathscr{N}$. Note that the empty map occurs here since T is a complete theory.

Since $\mathscr{M}$ and $\mathscr{N}$ are countable we may then use this back and forth system to construct an isomorphism $\mathscr{M} \longrightarrow \mathscr{N}$ just in the same way we did in 2.4.3. □

4. Quantifier elimination

4.0.1. **Definition.** An $\mathscr{L}$-theory T is said to have **quantifier elimination** (**QE** for short) if for each $\mathscr{L}$-formula φ there is a quantifier-free $\mathscr{L}$-formula χ such that

$$T \models \varphi \leftrightarrow \chi$$

If we find a theory with quantifier elimination in a "natural" language, then we know a lot about the models of T and can extract a lot of information on T. Care has to be taken when applying this philosophy, because one can artificially introduce symbols in a language so that the given theory has quantifier elimination. So it is immanently important to specify the language when talking about this concept.

In order to demonstrate the strength of this notion for analysability of structures and theories let us prove the following

4.0.2. **Proposition.** *If T has quantifier elimination, then every embedding between $\mathscr{L}$-structures is an elementary embedding.*

Proof. Exercise. □

It must be mentioned that the criterion in 4.0.2 is necessary, but not sufficient for quantifier elimination. For example the theory of the real field in the language $\mathscr{L} = \{+, -, \cdot, 0, 1\}$ of rings also has the property that all embedding between $\mathscr{L}$-structures is an elementary embedding. But this theory does not have quantifier elimination in this language. We will see both statements later on. The property of an $\mathscr{L}$-theory T discussed here is called model-completeness, cf. 3.6.9 and plays a central role in model theory. Also this will come up later.

In the next section we will introduce the main model theoretic techniques to test quantifier elimination. However, in some cases, the quantifiers have actually been removed by "hand". After that we shall demonstrate the usefulness of these tests.

4.1. **The main tests for quantifier elimination.**

The first test is of theoretical interest. It will be used to prove the applicable test 4.1.2.

4.1.1. **Proposition.** *Let T be an $\mathscr{L}$-theory. The following are equivalent.*

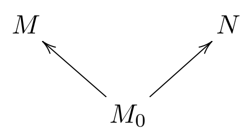
(i) *T has quantifier elimination.*

(ii) *If M_0 is a finitely generated substructure of a model of T, then $T \cup \operatorname{diag}(M_0)$ axiomatises a complete $\mathscr{L}(M_0)$-theory (i.e. the deductive closure of $T \cup \operatorname{diag}(M_0)$ in $\operatorname{Sen}\mathscr{L}(M_0)$ is complete).*

(iii) *If M_0 is a substructure of a model of T, then $T \cup \operatorname{diag}(M_0)$ axiomatises a complete $\mathscr{L}(M_0)$-theory.*

Proof. (i)⇒(iii). We show that all models of $T \cup \operatorname{diag}(M_0)$ are elementary equivalent to (M, M_0). Let N^+ be an $\mathscr{L}(M_0)$-structure and a model of $T \cup \operatorname{diag}(M_0)$. By 2.2.3, N^+ is of the form (N, f) for some $\mathscr{L}$-structure N and some map $M_0 \longrightarrow N$. As $(N, f) \models \operatorname{diag}(M_0)$, f is indeed an embedding of M_0 onto a substructure of N. By replacing the image of f in N by M_0 and modifying N accordingly, we may therefore also assume that M_0 is a substructure of N and so $N^+ = (N, M_0)$.

Thus we have two $\mathscr{L}$-structures M and N which are models of T and which have a common substructure M_0:



We need to show that $(M, M_0) \equiv (N, M_0)$. Take a sentence φ in the language $\mathscr{L}(M_0)$. We must show $(M, M_0) \models \varphi \iff (N, M_0) \models \varphi$. Recall that φ is of the form $\psi(x_1/a_1, ..., x_n/a_n)$ (not showing the underlines) for some $\mathscr{L}$-formula ψ and some $a_1, ..., a_n \in M_0$, $n \in \mathbb{N}_0$. Without loss of generality we may assume that the free variable x_1 actually occurs in ψ (otherwise replace ψ by $\psi \wedge x_1 \doteq x_1$).

By assumption, T has quantifier elimination. Hence there is a quantifier-free $\mathscr{L}$-formula $\chi(\bar{x})$ such that $T \models \psi \leftrightarrow \chi$. Now

$$
\begin{aligned}
(M, M_0) \models \varphi &\overset{\text{by } 1.47}{\iff} M \models \psi[\bar{a}] \overset{\text{as } M \models T}{\iff} M \models \chi[\bar{a}] \\
&\overset{\text{as } M_0 \text{ is a substructure of } M \text{ and } \chi \text{ is quantifier-free}}{\iff} M_0 \models \chi[\bar{a}] \\
&\overset{\text{as } M_0 \text{ is a substructure of } N \text{ and } \chi \text{ is quantifier-free}}{\iff} N \models \chi[\bar{a}] \\
&\overset{\text{as } N \models T}{\iff} N \models \psi[\bar{a}] \overset{\text{by } 1.47}{\iff} (N, M_0) \models \varphi,
\end{aligned}
$$

as required

(iii)⇒(ii) is a weakening.

(ii)⇒(i). This is the crucial statement of the proposition.

Let $\varphi(\bar{x})$ be an $\mathscr{L}$-formula, where $\bar{x} = (x_1, \dots, x_n)$. We may assume that at least one free variable occurs in φ, otherwise we replace φ by $\varphi \wedge x \doteq x$. Hence $n \geq 1$. We need to find a quantifier-free $\mathscr{L}$-formula $\chi(\bar{x})$ with $T \models \varphi \leftrightarrow \chi$. We follow a commonly used strategy here: We place ourselves into the context of the type space $S_n(T)$ and rephrase the problem there: We are given the clopen subset $\langle \varphi \rangle$ of $S_n(T)$ and we are looking for a quantifier-free formula $\chi \in \operatorname{Fml}\mathscr{L}_n$ which defines the same set, i.e. $\langle \varphi \rangle = \langle \chi \rangle$. Suppose we can show the following

Claim. For each $p \in S_n(T)$ there is some quantifier-free formula $\chi_p \in \mathrm{Fml}\,\mathscr{L}_n$ with the property

$$(*) \qquad p \in \langle \chi_p \rangle \subseteq \langle \varphi \rangle.$$

Then we get χ as follows: By $(*)$ we have

$$\langle \varphi \rangle = \bigcup_{p \in \langle \varphi \rangle} \langle \chi_p \rangle.$$

Since $\langle \varphi \rangle$ is compact and all $\langle \chi_p \rangle$ are open, there are $p_1, ..., p_k \in \langle \varphi \rangle$ with

$$\langle \varphi \rangle = \langle \chi_{p_1} \rangle \cup ... \cup \langle \chi_{p_k} \rangle.$$

Now $\langle \chi_{p_1} \rangle \cup ... \cup \langle \chi_{p_k} \rangle = \langle \chi_{p_1} \vee ... \vee \chi_{p_k} \rangle$ and we can choose $\chi = \chi_{p_1} \vee ... \vee \chi_{p_k}$.
Hence we only need to show the claim and this is where the assumption (ii) is used: Pick $p \in \langle \varphi \rangle$ and take a realization $\bar{a} \in M^n$ of p in some model M of T. Let M_0 be the substructure generated by $a_1, ..., a_n$ in M_0. Then M_0 is finitely generated and by assumption $T \cup \mathrm{diag}(M_0)$ axiomatises a complete $\mathscr{L}(M_0)$-theory. Since M_0 is an $\mathscr{L}$-substructure of M, the $\mathscr{L}(M_0)$-structure (M, M_0) is a model of $T \cup \mathrm{diag}(M_0)$. Therefore

$$(\dagger) \qquad T \cup \mathrm{diag}(M_0) \models \mathrm{Th}(M, M_0).$$

Since $p \in \langle \varphi \rangle$ and $p = \mathrm{tp}^M(\bar{a})$ we have $\varphi(\bar{a}) \in \mathrm{Th}(M, M_0)$ (note that $\varphi(\bar{a})$ is an $\mathscr{L}(M_0)$-sentence). Hence from $(\dagger)$ (and the compactness theorem) we obtain a finite subset Γ of $\mathrm{diag}(M_0)$ with $T \cup \Gamma \models \varphi(\bar{a})$.

We now have to make Γ explicit. By definition, each $\gamma \in \Gamma$ is of the form $\psi(y_1/b_1, ..., y_l/b_l)$ (we drop the underlining of the b_i) for some quantifier-free $\mathscr{L}$-formula $\psi(y_1, ..., y_l)$ and some $b_1, ..., b_l \in M_0$ with $M_0 \models \psi[\bar{b}]$. By replacing the variables in these ψ's and re-ordering all the b_i we may therefore assume that Γ is of the form

$$\{\psi_1(y_1/b_1, ..., y_l/b_l), ..., \psi_m(y_1/b_1, ..., y_l/b_l)\}.$$

But then we may actually replace Γ by $\psi(y_1/b_1, ..., y_l/b_l)$, where $\psi = \psi_1 \wedge ... \wedge \psi_m$ and $M_0 \models \psi[\bar{b}]$. We simplify the situation further by looking at the $b_i \in M_0$. By definition, M_0 is the $\mathscr{L}$-substructure of M generated by $a_1, ..., a_n$. From 2.6.3 we then know that each b_j $(1 \leq j \leq l)$ is of the form

$$b_j = t_j(\bar{a})$$

for some $\mathscr{L}$-term $t_j(\bar{x})$. Now we define χ_p as

$$\chi_p(\bar{x}) = \psi(t_1(\bar{x}), ..., t_l(\bar{x})).$$

Then $M_0 \models \psi[\bar{b}]$ says $M_0 \models \chi_p[\bar{a}]$.
Moreover from $T \cup \psi(\bar{b}) \models \varphi(\bar{a})$ we obtain

$$T \cup \chi_p(\bar{a}) \models \varphi(\bar{a}).$$

(Exercise!). However, this means $T \models \chi_p(\bar{a}) \to \varphi(\bar{a})$ and by 1.47 we get

$$T \models \chi_p(\bar{x}) \to \varphi(\bar{x}).$$

Thus we have $\langle \chi_p \rangle \subseteq \langle \varphi \rangle$ (in $S_n(T)$). Since $M_0 \models \chi_p[\bar{a}]$ and χ_p is quantifier-free we have $M \models \chi_p[\bar{a}]$, in other words $p \in \langle \chi_p \rangle$, as required for the claim. □

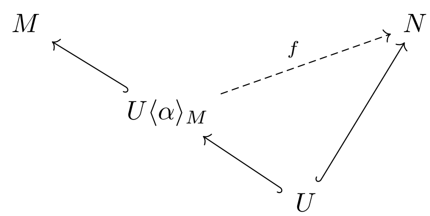
4.1.2**. Shoenfield-Blum test for quantifier elimination**
The following are equivalent for every $\mathscr{L}$-theory T without finite models.

(I) T has quantifier elimination.

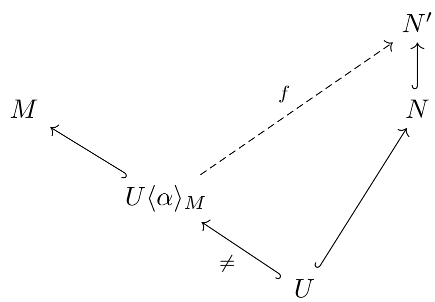
(II) Given models M, N of T, a finitely generated common substructure U of M, N, a quantifier-free $\mathscr{L}$-formula $\chi(y_1, ..., y_n, x)$ and some $\bar{a} \in U^n$, the following implication holds:

$$M \models \exists x\, \chi(\bar{a}, x) \;\Rightarrow\; N \models \exists x\, \chi(\bar{a}, x).$$

(III) If M, N are models of T such that N is $\aleph_0$-saturated and U is a finitely generated common substructure of M, N, then for all $\alpha \in M$, there is an embedding $f : U\langle\alpha\rangle_M \longrightarrow N$ over U (i.e. $f \restriction U = \mathrm{id}_U$). Here $U\langle\alpha\rangle_M$ denotes the substructure generated by $U \cup \{\alpha\}$ in M. In a diagram:



(IV) If M, N are models of T and U is a common substructure of M, N, with $U \neq M$, then there are some $\alpha \in M \setminus U$, an elementary extension N' of N and an embedding $f : U\langle\alpha\rangle_M \longrightarrow N'$ over U (i.e. $f|_U = \mathrm{id}_U$). Here $U\langle\alpha\rangle_M$ denotes the substructure generated by $U \cup \{\alpha\}$ in M. In a diagram:



(If N is $(\mathrm{card}(M) + |\mathscr{L}|)^+$-saturated then we may choose $N' = N$.)

(V) Let M, N be models of T. Then the following two conditions hold:
- ($\models$) If U is a common substructure of M, N, then there are a substructure P of M containing U with $P \models T$, an elementary extension N' of N and an embedding $P \longrightarrow N'$ over U.
- (†) If $P \models T$ is a common substructure of M, N with $P \neq M$, then there are some $\alpha \in M \setminus P$, an elementary extension N' of N and an embedding $f : P\langle\alpha\rangle_M \longrightarrow N'$ over P.

(If N is $(\mathrm{card}(M) + |\mathscr{L}|)^+$-saturated then we may choose $N' = N$ in both conditions.)

Proof. (I)⇒(II) follows from 4.1.1(i)⇒(ii) (Exercise).

(II)⇒(III) Note that condition (III) is not true without the assumption that N is $\aleph_0$-saturated.

Take M, N, U and α as in (III). Let

$$\Sigma = \{\varphi \in \mathrm{tp}^M(\alpha/U) \mid \varphi(x) \text{ is equal to } \chi(a_1, ..., a_n, x) \text{ for some } a_1, ..., a_n \in U, \text{ some quantifier-free } \mathscr{L}\text{-formula } \chi(y_1, ..., y_n, x) \text{ and some } n \in \mathbb{N}\}$$

We claim that $\mathrm{Th}(N, U) \cup \Sigma$ is finitely satisfiable. To see this, take $\varphi_1, ..., \varphi_k \in \Sigma$ and quantifier-free $\mathscr{L}$-formulas $\chi_i(y_1, ..., y_n, x)$ with

$$\varphi_i(x) = \chi(a_1, ..., a_n, x) \ (1 \leq i \leq k).$$

Observe that we can choose n, the variables occurring in the χ_i and the elements $a_i \in U$ so that all free variables occurring in any χ_i is among the y_i with $i \in \{1, ..., n\}$. Since $\varphi_i \in \mathrm{tp}^M(\alpha/U)$ we have

$$(M, U) \models \exists x \ (\chi_1(\bar{a}, x) \wedge ... \wedge \chi_k(\bar{a}, x)).$$

By (II) we then also have

$$(N, U) \models \exists x \ (\chi_1(\bar{a}, x) \wedge ... \wedge \chi_k(\bar{a}, x)).$$

As $(N, U) \models \mathrm{Th}(N, U)$ this shows that that $\mathrm{Th}(N, U) \cup \Sigma$ is finitely satisfiable. Hence Σ is contained in a 1-type q of N. Since N is $\aleph_0$-saturated and U is finitely generated, there is some element $\beta \in N$ that realises q. We will map α to β and obtain f:

Every element in $U\langle\alpha\rangle_M$ is of the form $t^M(\alpha, \bar{a})$ for some $\mathscr{L}$-term $t(x, \bar{y})$ and some $\bar{a} \in U^n$. We define

$$f(t^M(\alpha, \bar{a})) = t^N(\beta, \bar{a}).$$

This is well defined, since $t_1^M(\alpha, \bar{a}_1) = t_2^M(\alpha, \bar{a}_2)$ is witnessed in $\mathrm{tp}^M(\alpha/U)$: we have

$$t_1(x, \bar{a}_1) \doteq t_2(x, \bar{a}_2) \in \mathrm{tp}^M(\alpha/U), \text{ thus}$$
$$t_1(x, \bar{a}_1) \doteq t_2(x, \bar{a}_2) \in \Sigma \subseteq q,$$

and so $t_1^N(\beta, \bar{a}_1) = t_2^N(\beta, \bar{a}_2)$. Similarly, f is seen to be an embedding and $f(a) = a$ for all $a \in U$.

(III)⇒(I) We use 4.1.1 and show that for every substructure U of a model of T, the theory $T \cup \mathrm{diag}(U)$ is complete. Let M^+, N^+ be models of $T \cup \mathrm{diag}(U)$. Thus $M^+ = (M, g)$ and $N^+ = (N, h)$ for some models M, N of T and maps $g : U \longrightarrow M$, $h : U \longrightarrow N$. The condition $M^+, N^+ \models \mathrm{diag}(U)$ says that g, h are embeddings defined on U and after replacing the images of g, h in M, N with U, respectively and altering M, N accordingly, we may assume that U is a common substructure of M and N (and so $M^+ = (M, U)$, $N^+ = (N, U)$).

In order to show that $(M, U) \equiv (N, U)$ we may replace M, N by elementary extensions which are $\aleph_0$-saturated (use 3.5.3; here we need that M and N are infinite, because T does not have finite models by assumption; also observe that $(M, U) \equiv (M', U)$ for every elementary extension M' of M and similarly for N). Thus, we may assume right away that M and N are $\aleph_0$-saturated.

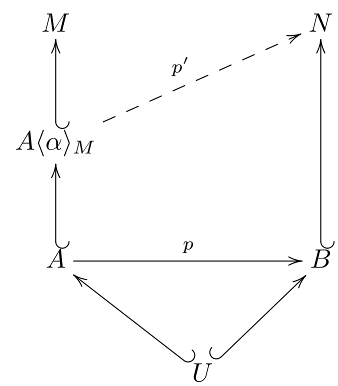
We prove $(M, U) \equiv (N, U)$ by setting up a Back and Forth system and then use 2.7.6: We define

$$\mathfrak{S} = \{(p, A, B) \mid A \text{ is a finitely generated substructure of } M \text{ containing } U, B \text{ is a finitely generated substructure of } N \text{ containing } U, \text{ and } p : A \longrightarrow B \text{ is an } \mathscr{L}\text{-isomorphism over } U\}$$

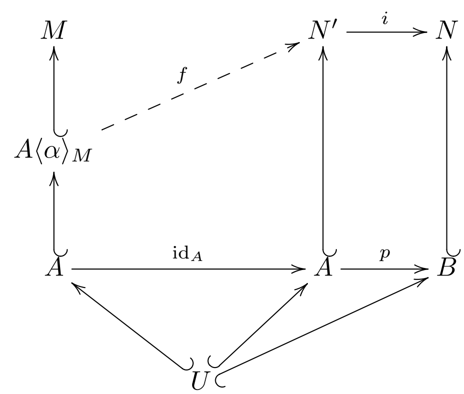
Claim. $\mathfrak{S}$ is a Back and Forth system $(M,U) \longrightarrow (N,U)$.

To see this, note first that for each $(p,A,B) \in \mathfrak{S}$, the map p is an $\mathscr{L}(U)$-isomorphism, because p is the identity on U and an $\mathscr{L}$-isomorphism. Furthermore, $\mathfrak{S}$ is not empty, since $(\mathrm{id}_U, U, U) \in \mathfrak{S}$. So we need to check the Forth and the Back condition for this system. As the situation is symmetric in M and N, we just need to show the Forth condition:

Take $(p,A,B) \in \mathfrak{S}$ and an element $\alpha \in M$. We have to find some $(p',A',B') \in \mathfrak{S}$ such that $\alpha \in A'$ and p' extends p. We will now use condition (III) of our theorem, applied for A instead of U: We take $A' = A\langle\alpha\rangle_M$ and we are looking for an embedding $p' : A' \longrightarrow N$ extending p. We are in the following situation:



If we manage to find such a p', then we can take $B' = p'(A')$ which again is a finitely generated substructure of N (generated by B and $p'(\alpha)$) and so $(p',A',B') \in \mathfrak{S}$ as desired. However, such a p' exists by (III): Just identify A with B and apply condition (III) directly. For those who want to have this written in details: Let N' be an $\mathscr{L}$-structure containing A as a substructure together with an isomorphism $i : N' \longrightarrow N$ extending $p : A \longrightarrow B$:



Then by (III) (observe that N' is $\aleph_0$-saturated) applied to M, N', A and α, there is an embedding $f : A' = A\langle\alpha\rangle_M \longrightarrow N'$ over A. Now take $p' = i \circ f$.

Hence we know that (I), (II) and (III) are equivalent.

(I)$\Rightarrow$(V) follows from 4.1.1(i)$\Rightarrow$(iii) and 2.8.1.

(V)⇒(IV): Take M, N, U as in (IV). If U is a model of T, then we may deploy assumption (†) in (V). If U is not a model of T, then choose P as in condition ($\models$) of (V). Then $P \neq U$ and the assertion follows.

(IV)⇒(II) Let $M, N \models T$, take a finitely generated common substructure U of M, N. Let $\kappa = \mathrm{card}(\mathscr{L}) + \mathrm{card}(M)$ and let N^* be a κ^+-saturated elementary extension of N. We claim that there is an embedding $M \longrightarrow N^*$ over U. Since $N \prec N^*$, this readily implies (II). To find the embedding consider the set

$$\mathfrak{S} = \{(f, A) \mid A \text{ is a substructure of } M \text{ containing } U \text{ and } f : A \longrightarrow N^* \text{ is an embedding over } U\},$$

partially ordered by $(f, A) \leq (g, B) \iff A \subseteq B$ and $g|_A = f$. This defines a partial order, which obviously is inductive. Hence by Zorn's lemma there is a maximal element (f, A) in $\mathfrak{S}$ and it suffices to show that $A = M$. Suppose this is not the case. We may assume that f is the inclusion map, hence A is a common substructure of M and N^*. By Skolem-Löwenheim downwards, there is an elementary restriction N_0 of N^* of size $\leq \kappa$ containing A. Since $A \neq M$ we may deploy (IV), hence there are $b \in M \setminus A$ and an embedding of $A\langle b\rangle_M$ over A into some elementary extension of N_0. As N^* is κ^+-saturated there is also such an embedding into N^* (use Skolem-Löwenheim downwards again and then 2.8.1). By maximality of (f, A) in $\mathfrak{S}$ this is impossible. □

In applications the following consequence is frequently used.

4.1.3. **Corollary.** *Suppose T is an $\mathscr{L}$-theory without finite models that has prime models over substructures, i.e., for all T-models M and every substructure A of M there is a substructure $P \subseteq M$ containing A with $P \models T$ such that for every $N \models T$ containing A as a substructure, there is an embedding of P into N over A. (This for example is trivially the case when T is a universal theory.)*

Then T has quantifier elimination if and only if T is model-complete if and only if condition (†) of 4.1.2(V) holds.

Proof. The existence of prime models obviously implies condition ($\models$) of 4.1.2(V). Consequently, under our assumption about the existence of prime models, the equivalence (V) $\iff$ (I) of 4.1.2 says that condition (†) in 4.1.2(V) is equivalent to quantifier elimination. Now the corollary follows because the implications

$$T \text{ has quantifier elimination} \Rightarrow T \text{ is model-complete} \Rightarrow T \text{ satisfies (†) of 4.1.2(V)}$$

hold true already without any assumption on T. □

4.1.4. *Remark.* A theory with quantifier elimination does not necessarily have prime models over substructures. In order to compare 4.1.3 with the equivalence (V) $\iff$ (I) of 4.1.2, consider the following statements about a theory T without finite models:

(a) T has prime models over substructures.
(b) If U is a common substructure of $M, N \models T$, then there is a substructure $P \subseteq M$ containing U with $P \models T$ and an embedding $P \longrightarrow N$ over U.

Then

(i) Obviously (a) $\Longrightarrow$ (b) $\Longrightarrow$ condition ($\models$) in 4.1.2(V).
(ii) If the language is countable, then (a) is actually equivalent to (b). The proof involves the omitting types theorem 3.7.9.

4.2. **Some theories with quantifier elimination.**
We return to the theories from section 2.4 and we show that they all have quantifier elimination.

4.2.1. *Example.* The theory T of infinite sets in the empty language has quantifier elimination.[4]

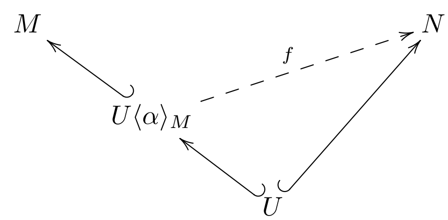
Proof. This we can prove directly from 4.1.1: Take a nonempty subset M_0 of an infinite set M (then M_0 is already a substructure). We must show that $T\cup\text{diag}(M_0)$ is complete. This can be shown as in 2.4.1: If N_1 and N_2 are infinite sets, containing M_0, then a minor amendment of the proof given for 2.4.1 shows that (N_1, M_0) and (N_2, M_0) are elementary equivalent.

Alternatively, one can run the Shoenfield-Blum test (III) . □

4.2.2. *Example.* Given a field F, the theory T of infinite F-vector spaces in the language $\mathscr{L}_{F-\text{vec.sp.}}$ has quantifier elimination. See 2.4.2 for the definition of this language.

Proof. We apply the Shoenfield-Blum test (III).

Let M, N be infinite F-vector spaces such that N is $\aleph_0$-saturated and let U be a finitely generated common substructure of M, N. Take $\alpha \in M$. We must find an embedding $f : U\langle\alpha\rangle_M \longrightarrow N$ over U:



Of course we may assume that $\alpha \notin U$. Firstly, the substructure generated by $U \cup \{\alpha\}$ in M is the F-vector space $U + \alpha{\cdot}F$ generated by U and α in M, as can be seen readily. (Observe that U might not be a model of T!) Since N is $\aleph_0$-saturated and U is finitely generated, N is different from U (convince yourself that this is true; note that U itself might be infinite). Take any element $\beta \in N \setminus U$. Then both $U\langle\alpha\rangle_M$ and $U\langle\beta\rangle_N$ are 1-dimensional F-vector spaces over U (i.e. their quotient by U is of dimension 1). But then we know from linear algebra, that $U\langle\alpha\rangle_M$ and $U\langle\beta\rangle_N$ are isomorphic over U. Explicitly, define

$$\begin{aligned} f : U\langle\alpha\rangle_M &\longrightarrow U\langle\beta\rangle_N \\ u + \alpha{\cdot}x &\longmapsto u + \beta{\cdot}x \end{aligned}$$

and check directly, that f is an embedding as requested. □

4.2.3. *Example.* Let $\mathscr{L} = (\leq)$ be the language of po-sets and let T be the $\mathscr{L}$-theory of densely, totally ordered sets without endpoints. Then T has quantifier elimination.[5] Recall the precise definition of this theory from 2.4.3.

Proof. We apply the Shoenfield-Blum test (III) (obviously, T does not have finite models).

[4] Notice that there are no quantifier-free sentences in this language.
[5] As in 4.2.1, there are no quantifier-free sentences in this language.

Let M, N be densely, totally ordered sets without endpoints such that N is $\aleph_0$-saturated and let U be a finitely generated common substructure of M, N. Take $\alpha \in M$. We must find an embedding $f : U\langle\alpha\rangle_M \longrightarrow N$ over U. Of course we may assume that $\alpha \notin U$. Firstly, since $\mathscr{L}$ is a language without function symbols and without constant symbols, every nonempty subset of any $\mathscr{L}$-structure is (the universe of) a substructure of that structure. Hence in our situation, $U\langle\alpha\rangle_M = U \cup \{\alpha\}$. Also, the set U is finite, since it is finitely generated.

Moreover, an $\mathscr{L}$-embedding is a map f satisfying $a \leq b \iff f(a) \leq f(b)$ for all a, b in the domain of f.

We list the elements of U, say U consist of n elements

$$u_1 <^M \dots <^M u_n \text{ with } n \in \mathbb{N}.$$

Since U is also a substructure of N, this means

$$u_1 <^N \dots <^N u_n.$$

Now for the element $\alpha \in M$, there is exactly one $k \in \{0, 1, \dots, n\}$ such that for all $i \in \{1, \dots, n\}$ we have $\alpha < u_i \iff i < k$.

Using that fact that N is densely, totally ordered sets without endpoints we can now find some $\beta \in N$, which is positioned w.r.t. U exactly as α is positioned w.r.t. U. Hence there is some $\beta \in N$ such that for all $i \in \{1, \dots, n\}$ we have $\beta < u_i \iff i < k$.

Hence if we extend the identity $U \longrightarrow U$ to $f : U \cup \{\alpha\} \longrightarrow N$ by $f(\alpha) = \beta$ we get an embedding as required. □

4.2.4. *Example.* Let $\mathscr{L} = (+, -, \cdot, 0, 1)$ be the language of rings. Let ACF be the $\mathscr{L}$-theory of algebraically closed fields as defined in 2.4.5. Then ACF has quantifier elimination.

Proof. Again we apply the Shoenfield-Blum test (III). Recall from algebra that ACF does not have finite models, i.e. every algebraically closed field is infinite.

Let M, N be algebraically closed fields such that N is $\aleph_0$-saturated and let U be a finitely generated common substructure of M, N. Take $\alpha \in M$. We must find an embedding $f : U\langle\alpha\rangle_M \longrightarrow N$ over U. Again, we may assume that $\alpha \notin U$. Now we first notice:

- a substructure of a (algebraically closed) field in our language is a subring of that field containing the multiplicative unit of that field. In particular, U is a common subring of M, N and $U\langle\alpha\rangle_M$ is the subring of M generated by U and α. We denote it by $U[\alpha]$ as usual.
- an embedding is just an injective ring homomorphism.

So we are looking for an embedding $U[\alpha] \longrightarrow N$ over U. In order to do this, we first reduce to the case, where U is a common subfield of M, N. Let K, L be the subfields generated by U in M, N respectively. The both K and L are isomorphic over U to the so-called field of fractions of U. If you are not acquainted with this construction, we can do the following: We have $K = \{\frac{a}{b} \in M \mid a, b \in U, b \neq 0\}$ and $L = \{\frac{a}{b} \in N \mid a, b \in U, b \neq 0\}$. Note that K and L "look" equal, but the operation of inverting elements is a priori performed in different fields. However, the map

$$\begin{aligned} g : K &\longrightarrow L \\ \frac{a}{b} &\longmapsto \frac{a}{b} \end{aligned}$$

is well-defined and therefore an isomorphism over U. We identify K with L (similarly to the identification process at the end of the proof of 4.1.2) and then continue to work with U.

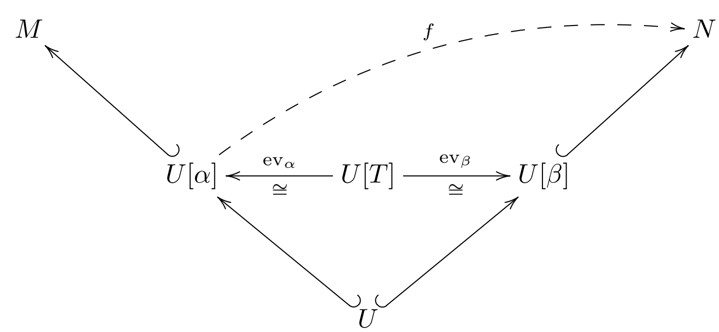
Hence we may assume that U is a common subfield of M, N. We use a ring homomorphism, called evaluation map, defined as follows: Let $U[T]$ be the polynomial ring over U in one indeterminate T. Then there is a unique ringhomomorphism

$$\begin{aligned} \mathrm{ev}_\alpha : U[T] &\longrightarrow U[\alpha] \\ P(T) &\longmapsto P(\alpha), \end{aligned}$$

called "evaluation at α", which fixes U pointwise and maps T to α (all this can be verified readily). We do two cases.

Case 1. α is transcendental over U (i.e. α is not the zero of any polynomial $P(T) \in U[T]$, unless P is the zero polynomial).

So in this case evaluation at α is injective, and therefore ev_α an isomorphism. If we manage to find an element $\beta \in N$ which is transcendental over U, then the same argument shows that also evaluation at β is an isomorphism. Hence we get a commutative diagram



and $f := \mathrm{ev}_\beta \circ \mathrm{ev}_\alpha^{-1}$ has the required properties.

We can find such β using our assumption that N is $\aleph_0$-saturated (and without it, the argument would break down here, because N might simply be equal to U). The idea is to list all polynomials $P(T) \in U[T]$, except the null polynomial and to check that for all finite sub-lists there is an element $\gamma \in N$ which is not a zero of any of the polynomials from that finite list (use that N is infinite as is implied by $\aleph_0$-saturation). Then we use saturation (and the assumption that U is the fraction field of a finitely generated ring) to obtain $\beta \in N$, transcendental over U. Explicitly, we find $\beta \in N$ realising the following set of $\mathscr{L}(N, U)$-formulas in at most one free variable x:

$$\mathrm{Th}(N, U) \cup \{P(x) \doteq 0 \mid P(T) \in U[T] \setminus \{0\}\}.$$

The details are left as an exercise. Note that we have not yet used that N is algebraically closed!

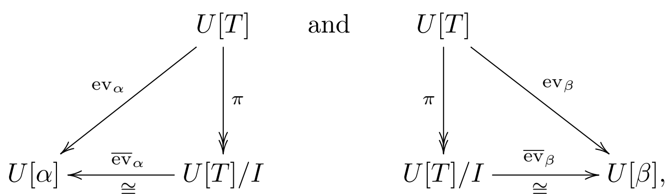
Case 2. α is algebraic over U (i.e. α is the zero of some polynomial $P(T) \in U[T] \setminus \{0\}$).

In this case, the evaluation map $\mathrm{ev}_\alpha : U[T] \longrightarrow U[\alpha]$ is not injective and so has non-trivial kernel: Let $I : \{P(T) \in K[T] \mid P(\alpha) = 0\}$ be the preimage of 0 under ev_α.

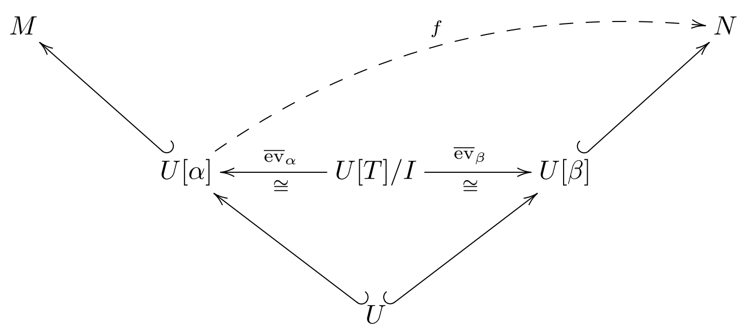
We now use some basic information from algebra (and here we also need that U is a field), namely: the ring $U[T]$ is a principal ideal domain, i.e. every ideal of

$U[T]$ is generated by a single element. The set I is an ideal of $U[T]$ and the minimal polynomial $\mu_\alpha(T) \in U[T]$ is defined as the unique monic (i.e. leading coefficient is 1) polynomial that generates I. Alternatively one can say: $\mu_\alpha(T)$ is the monic polynomial of least degree that annihilates α.

Here (and only here) we use the assumption that N is algebraically closed. There is a zero β of $\mu_\alpha(T)$ in N. Let $\mu_\beta(T) \in U[T]$ be the minimal polynomial of β over U. It is then clear that the degree of μ_β is at most the degree of μ_α. Since $\mu_\alpha(\beta) = 0$ we also know that μ_β divides μ_α in $U[T]$, i.e. $\mu_\alpha(T) = \mu_\beta(T)\cdot Q(T)$ for some $Q(T) \in U[T]$. Since μ_β has degree at least 1, the degree of $Q(T)$ is strictly less than the degree of μ_α. Since $\mu_\alpha(\alpha) = 0$, the minimality assumption on μ_α implies that also $\mu_\beta(\alpha) = 0$. Thus the degree of μ_α is at most the degree of μ_β. This proves that $\mu_\alpha = \mu_\beta$. From the isomorphism theorem for rings we obtain commutative diagrams



where π is the residue class map $U[T] \twoheadrightarrow U[T]/I$. Inserting the bottom isomorphisms of these two diagrams into our embedding problem gives a commutative diagram



and we can choose $f := \overline{\mathrm{ev}}_\beta \circ \overline{\mathrm{ev}}_\alpha^{-1}$ to solve our embedding problem. □

4.3. **Tarski's Theorem.**

In the previous section we have seen that every subset of $\mathbb{C}^n$ that is definable in the field $\mathbb{C}$ can be described by quantifier-free formulas in the language of rings. Moreover we know already for a long time an explicit (and a recursive list for those who know the term) list of axioms which imply all true statements about the field $\mathbb{C}$.

In this section we want to carry out the same fundamental model theoretic analysis for the real field. Throughout $\mathscr{L}_{ri}$ denotes the language of rings $\{+,-,\cdot,0,1\}$. One might wonder whether the quantifier elimination result for the complex field above is still true for the real field. However: The formula $\exists u \ x = u^2$ is not quantifier-free definable in the field $\mathbb{R}$. Exercise!

So in order to eliminate quantifiers we have to add at least one new relation symbol to the language. We will add a binary relation symbol $\leq$ (intended to denote the order in structures). We will mainly work with $\mathscr{L}_{ri}$ and the extension $\mathscr{L}_{ri}(\leq)$.

Ordered and real closed fields

An **ordered field** F is an $\mathscr{L}_{ri}(\leq)$-structure expanding a field and satisfying the following sentences:

OF0: $\leq$ is a total order on the universe of F

OF1: $\forall x,y,z \ (x \leq y \ \to x+z \leq y+z)$

OF2: $\forall x,y,z \ (x \leq y \wedge 0 \leq z \ \to x\cdot z \leq y\cdot z)$

The $\mathscr{L}_{ri}(\leq)$-theory axiomatised by the field axioms and OF0,OF1,OF2 is called the **theory of ordered fields**.

Obviously, both $\mathbb{Q}$ and $\mathbb{R}$ with the standard interpretation of the $\mathscr{L}_{ri}(\leq)$-symbols are ordered fields. However, the following "axiom" is not true in $\mathbb{Q}$:

IVT: Every polynomial in one variable that changes sign in some interval has a zero in that interval.

In fact **IVT** is an axiom scheme. For each $d \in \mathbb{N}$ let $f_d(x) := y_d\cdot x^d+\ldots+y_1\cdot x+y_0$ be the **general polynomial of degree** d. Then **IVT** is the set of axioms

$$\forall y_0,\ldots,y_d,u,w \left(u \leq w \wedge f_d(u)\cdot f_d(w) \leq 0 \ \to \ \exists v \ (u \leq v \leq w \wedge f_d(v) \doteq 0)\right).$$

4.3.1. **Definition.** A **real closed field** is an ordered field which satisfies **IVT**. The $\mathscr{L}_{ri}(\leq)$-theory axiomatised by the theory of ordered fields and **IVT** is denoted by RCF.

Our aim is to show

4.3.2. **Theorem.** *(Tarski)*
RCF has quantifier elimination.

4.3.3. **Corollary.** *RCF is complete and decidable. In particular, the field* $\mathbb{R}$ *is decidable*

If you don't know what decidable means: just ignore it for the moment. Up to definitions this corollary is an easy consequence of Tarski's theorem.

Proof. To see that RCF is complete we must show that all real closed fields are elementary equivalent. We use the following observation:

If F is an arbitrary ordered field, then the smallest substructure of F is isomorphic to the ring of integers expanded by the natural order. Another way of saying this is that the map

$$\begin{array}{rcl} f:\mathbb{Z} & \longrightarrow & F \\ k & \longmapsto & k\cdot 1^F \end{array}$$

is an isomorphism of $(\mathbb{Z}, \leq, +, -, \cdot, 0, 1)$ onto the smallest substructure of F. This is straightforward from the axioms of ordered fields and left as an exercise.

Now if M, N are real closed fields, then using this observation we see that the smallest substructures in M and in N are isomorphic (to $(\mathbb{Z}, \leq, +, -, \cdot, 0, 1)$). After identifying these two substructures we may therefore assume that M and N have a common substructure U. Since RCF has quantifier elimination by 4.3.2 we may use 4.1.1 and get $(M, U) \equiv (N, U)$. But then in particular $M \equiv N$. □

Before we can prove Tarski's theorem we have to study some algebraic tools used in the proof.

Real algebraic tools

4.3.4. **Proposition.** *Let R be a real closed field.*

(i) If $a \in R$, then $a \geq 0 \iff$ there is some $b \in R$ with $a = b^2$. Hence

$$\mathrm{RCF} \models x \leq y \leftrightarrow \exists z\ y - x = z^2.$$

(ii) Every polynomial $P(T) \in R[T]$ in one indeterminate of odd degree with coefficients in R has a root in R, i.e. there is some $a \in R$ with $P(a) = 0$.

Remark: As a matter of fact, the converse is also true, i.e. every ordered field that has these two properties is real closed. We will not use this here, but it is too important to be omitted.

Proof. (i). We have $b \geq 0$ or $-b \geq 0$ for all $b \in R$. In any case $b^2 = (-b)^2 \geq 0$. Conversely if $a \geq 0$, then the polynomial $P(T) = T^2 - a$ changes sign: We have $1 + a + a^2 \geq 0$ from $a \geq 0$, hence

$$P(0)\cdot P(1 + a) = (-a)\cdot(1 + a + a^2) \leq 0.$$

Hence by IVT, P has a zero b as required.

(ii) Let $P(T) = a_{2d+1}T^{2d+1} + ... + a_0$ with $a_i \in R$ and $a_{2d+1} \neq 0$. We must find a root of $P(T)$ in R and we may divide by a_{2d+1}. Hence we may assume that

$$P(T) = T^{2d+1} + a_{2d}T^{2d} + ... + a_0.$$

Such polynomials change sign (actually for arbitrary ordered fields). the proof goes as in the case $\mathbb{R}$: If a tends to $+\infty$, then $P(a)$ tends to $+\infty$, where as if a tends to $-\infty$, then $P(a)$ tends to $-\infty$. The details for ordered fields are left as an exercise. Hence $P(T)$ changes sign in R and by IVT, $P(T)$ has a zero as required. □

4.3.5. **Corollary.** *Every ringhomomorphism between real closed fields is order preserving*

Proof. Ringhomomorphisms map squares onto squares. Now use 4.3.4 (i). □

To apply the Shoenfield-Blum test we shall also need some purely Galois theoretic lemma. Since this has a surprising link to type spaces we expand this a little bit:

4.3.6. **Proposition.** *Let $K \subseteq L$ be fields and let G be the group of K-automorphisms of L (L does not need to be algebraic over K; however if L is a Galois extension of K, then G is the Galois group of L over K). We work in the language of rings $\mathscr{L}_{\rm ri}$ and expand this language by two constants for every element of L. So we choose disjoint sets C and D which are in bijection with L and we write $C = \{c_b \mid b \in L\}$, $D = \{d_b \mid b \in L\}$. In the language $\mathscr{L}_{\rm ri}(C \dot\cup D)$ we have two copies of the diagram of L, the first one is in* $\mathrm{Sen}(\mathscr{L}(C))$, *we denote it by Δ_C, and the second one is in* $\mathrm{Sen}(\mathscr{L}(D))$, *we denote it by Δ_D. Let*

$$\Sigma := \mathrm{ACF} \cup \Delta_C \cup \Delta_D \cup \{c_a \doteq d_a \mid a \in K\}.$$

Then Σ is a set of $\mathscr{L}_{\rm ri}(C \dot\cup D)$-sentences and our goal here is to show that $S(\Sigma)$ is in natural bijection with G provided L is a Galois extension of K.

(i) An $\mathscr{L}_{\rm ri}(C \dot\cup D)$-structure is an $\mathscr{L}_{\rm ri}$-structure M, together with two map $f, g : L \longrightarrow M$ which describe the interpretation of the new constant symbols in the structure: $f(b)$ is the interpretation of c_b and $g(b)$ is the interpretation of d_b.

(ii) A model of Σ is an $\mathscr{L}_{\rm ri}(C \dot\cup D)$-structure (M, f, g), such that M is an algebraically closed field and f and g are embeddings $L \longrightarrow M$ and $f \restriction K = g \restriction K$ (the latter because Σ contains all the sentences $c_a \doteq d_a$ for $a \in K$).

(iii) Let Ω be the algebraic closure of L. For every K-embedding $\sigma : L \longrightarrow \Omega$, the $\mathscr{L}_{\rm ri}(C \dot\cup D)$-structure

$$\Omega(\sigma) := (\Omega, \mathrm{id}_L, \sigma)$$

is a model of Σ, and its theory is therefore an element of $S(\Sigma)$, which we denote by $p(\sigma)$:

$$p(\sigma) = \mathrm{Th}(\Omega(\sigma)) = \mathrm{Th}((\Omega, \mathrm{id}_L, \sigma)) \in S(\Sigma).$$

(iv) The map

$$\begin{array}{rcl} \Theta : G & \longrightarrow & S(\Sigma) \\ \sigma & \longmapsto & p(\sigma) \end{array}$$

is injective.

(v) If L/K is a normal extension, then Θ is bijective. A normal extension is an algebraic extension defined by the following property (at least we choose this definition): Every K-embedding $L \longrightarrow \Omega$ is an automorphism of L; this in particular is the case if L/K is Galois, for example if $L = \Omega$.

(vi) If L/K is a normal extension, then the bijection Θ induces a topology on G and a basis of neighborhoods of the identity is given by the sets

$$\mathrm{Stab}(F) := \{\sigma \in G \mid \sigma \restriction F = \mathrm{id}_F\}, \text{ where } F \subseteq L \text{ is finite}$$

and all these sets are clopen in G. A basis of neighborhoods of $\sigma \in G$ is given by all the $\sigma \cdot \mathrm{Stab}(F)$, where $F \subseteq L$ is finite.

G together with this topology is a so-called profinite group (which by definition is a projective limit $\varprojlim G_i$ of finite groups G_i, equipped with the profinite topology induced from the discrete topologies on the G_i; in Galois theory, this topology is called the Krull topology).

Proof. (i)-(iii) are clear.
(iv). Let $\sigma \neq \tau \in G$. Then for some $b \in L$ we have $\sigma(b) \neq \tau(b)$. Then

$$\Omega(\sigma) \models c_{\sigma(b)} \doteq d_b$$

and

$$\Omega(\tau) \not\models c_{\sigma(b)} \doteq d_b$$

by definition of the interpretation of the new constant symbols. Hence $c_{\sigma(b)} \doteq d_b \in p(\sigma) \notin p(\tau)$, i.e. $p(\sigma) \neq p(\tau)$.
(v). Now assume L/K is normal. Take $p \in S(\Sigma)$ and a model (M, f, g) of p. Then f, g are embeddings of L into the algebraically closed field M and $K_0 := f(K) = g(K)$. Since L is algebraic over K, $f(L)$ and $g(L)$ are algebraic over K_0 and so both are contained in the algebraic closure M_0 of K_0 in M. Now $g \circ f^{-1}$ is a K_0-embedding $f(L) \longrightarrow M_0$. Since L/K is normal, also $f(L)/K_0$ is normal, which implies that $g \circ f^{-1}$ is a K_0-automorphism of $f(L)$. Hence $g(L) = f(L)$ and we may transfer this automorphism back to L via the isomorphism f: we get an automorphism $\sigma \in G$. In symbols:

$$\sigma = f^{-1} \circ (g \circ f^{-1}) \circ f = f^{-1} \circ g.$$

We claim that $\Omega(\sigma)$ is a model of p (and therefore $\Theta(\sigma) = p$ as desired). Here we make use of quantifier elimination for ACF: Since M_0 is a substructure of M and both structures are models of ACF, quantifier elimination of ACF implies that M_0 is an elementary substructure of M. It is then straightforward to see that $(M_0, f, g) \prec (M, f, g)$. Let $F : \Omega \longrightarrow M_0$ be an isomorphism extending $f : L \longrightarrow f(L)$ (such an isomorphism exists by the basic theory of algebraically closed fields). We claim that

$$F : \Omega(\sigma) = (\Omega, \mathrm{id}_L, \sigma) \longrightarrow (M_0, f, g)$$

is an isomorphism. For this it is enough to show that F respects the interpretation of the new constant symbols. This mean for $b \in L$ we have:

- $F(b) = f(b)$ which holds true, as F extends f and
- $F(\sigma(b)) = g(b)$ which holds true, as F extends f and $\sigma(b) = f^{-1} \circ g(b)$.

Thus

$$\Omega(\sigma) \cong (M_0, f, g) \prec (M, f, g) \models p$$

and therefore $\Omega(\sigma) \models p$ as claimed.
(vi). It is a straightforward matter to show that

- the topology defined on G in item (iv) is Hausdorff and
- the map Θ is continuous.

This is left as an exercise. Since very continuous bijection from a compact space onto a Hausdorff space is a homeomorphism, we get (iv). □

4.3.7. **Lemma.** *Let L, L' be fields which have a common subfield K and suppose L/K is algebraic. Suppose*

> *for ever finite field extension E of K inside L there is an embedding $E \longrightarrow L'$ over K.*

Then there is an embedding $L \longrightarrow L'$ over K, too.

Note: 'finite' means here that E is a finite dimensional K-vector space. In our situation - where L/K is algebraic - this is equivalent to say that E is finitely generated over K as a field, equivalently: as a ring.

Proof. Since L/K is algebraic, every K-embedding from a subfield of L containing K into L' is algebraic over K. Therefore we may replace L' by the algebraic closure of K in L', hence we may assume that also L'/K is algebraic. Let Ω be the algebraic closure of L. Since L'/K is algebraic, there is a K-embedding of L' into Ω over K and so we may assume that also $L' \subseteq \Omega$.

Let G be the absolute Galois group of K, i.e. G is the set of K-automorphisms of Ω with the composition of maps as group operation.

For a subset F of L, let

$$X(F) := \{\sigma \in G \mid \sigma(F) \subseteq L'\}.$$

We now use 4.3.6 and the fact that the topology described in (iv) of 4.3.6 is compact. Each of the sets $X(F)$ is closed, because $X(F) = \bigcap_{a\in F} X(a)$ and $X(a)$ is closed (observe that the set $\{\sigma(a) \mid \sigma \in G\}$ is finite).

We have

(a) $X(L) = \bigcap_{F\subseteq L \text{ finite}} X(F)$,

(b) $X(F_1) \cap X(F_1) = X(F_1 \cup F_2)$,

(c) $X(F) \neq \emptyset$ by the assumption in our lemma.

Thus, $\{X(F) \mid F \subseteq L \text{ finite}\}$ is a set of closed subsets of G with the finite intersection property. Since G is compact by 4.3.6 (it is homeomorphic to a type space), $X(L) \neq \emptyset$, as required. □

4.3.8. **Proposition.** *Let S be a real closed field and let $R \subseteq S$ be a subfield that is algebraically closed <u>in S</u> (i.e. every element of S that is algebraic over R is already contained in R). Then*

(i) R (together with the induced order from S) is again real closed.

(ii) If $\chi(x, \bar{y})$ is a quantifier-free $\mathscr{L}_{\mathrm{ri}}(\leq)$-formula, $\bar{y} = (y_1, ..., y_k)$, $\bar{a} \in R^k$ and if there is some $s \in S$ with $S \models \chi[s, \bar{a}]$, then there is some $r \in R$ with $R \models \chi[r, \bar{a}]$.

Note: The converse for (i) is also true, i.e. if R (together with the induced order from S) is real closed, then R is algebraically closed in S. We will come to this later.

Proof. (i). We need to check the intermediate value theorem for polynomials over R. So let $P(T) \in R[T]$ and let $a, b \in R$ with $P(a)\cdot P(b) \leq 0$. By the intermediate value theorem for S, we then know that $P(s) = 0$ for some $s \in S$ with $a \leq s \leq b$. In particular s is algebraic over R. Now by assumption, $s \in R$ and we are done.

(ii). Up to logical equivalence of the theory of ordered fields, χ is a finite disjunction of formulas of the form

$$f(x, \bar{y}) = 0 \wedge g_1(x, \bar{y}) > 0 \wedge ... \wedge g_n(x, \bar{y}) > 0,$$

where f and all g_i are polynomials over $\mathbb{Z}$. We may therefore assume that χ itself is of this form. Of course we may also assume that $s \notin R$ and by assumption this means s is transcendental over R. Since $f(t, \bar{a}) = 0$, it follows that $f(T, \bar{a})$ is the null polynomial of $R[T]$. But then it suffices to find some $r \in R$ with $g_1(r, \bar{a}) > 0 \wedge ... \wedge g_n(r, \bar{a}) > 0$. Let $b_1 < ... < b_l$ be the set of all roots of all $g_i(x, \bar{a})$ in R and fix some $i \in \{0, ..., l\}$ with $b_i < s < b_{i+1}$, where we set $b_0 := -\infty$ and $b_{l+1} := +\infty$. It is now enough to show that $g_1(r, \bar{a}) > 0 \wedge ... \wedge g_n(r, \bar{a}) > 0$ for all $r \in R$ with $b_i < r < b_{i+1}$.

Otherwise $g_j(r,\bar{a}) \leq 0$ for some $r \in (b_i, b_{i+1}) \subseteq R$ and some $j \in \{1,...,n\}$. Since $g_j(s,\bar{a}) > 0$, the intermediate value theorem for S implies that $g_j(x,\bar{a})$ has a zero in S between r and s. As R is algebraically closed in S, this zero must be in R. However $b_i < r, s < b_{i+1}$, hence by the choice of the b_i, $g_j(x,\bar{a})$ does not have a zero between r and s. □

Counting real zeroes of polynomials.

We now enter the heart of the proof of Tarski's theorem on quantifier elimination of real closed fields.

4.3.9. **Proposition.** *Let K be an ordered field and let $P(T)$ be a polynomial in one variable over K. If $P(T)$ has a zero in some real closed field containing K (as an ordered subfield), then $P(T)$ has a zero in all real closed fields containing K.*

There are various paths to prove this. We will obtain it from Sturm's theorem, which we postpone for now. Instead we use 4.3.9 to prove 4.3.2

What we gain from 4.3.9 (and 4.3.7) is the following

4.3.10. **Corollary.** *Let R, S be real closed fields and let K be a common ordered subfield of R and S. If R is algebraic over K, then there is an order preserving embedding $R \longrightarrow S$ over K*

Proof. By 4.3.5 it suffices to find a ring homomorphism $R \longrightarrow S$ over K. By 4.3.7, it suffices to show that for every subfield E of R containing K and finite over K, there is an embedding $E \longrightarrow S$ over K. Since E is finite, E is simply generated, because all our fields have characteristic 0: This is the so-called theorem of the primitive element from Galois theory (true for all finite separable field extensions).

Hence we have $E = K(\alpha)$. Let $P(T)$ be the minimal polynomial of α over K. Since P has a root in R, P also has a root β in S by 4.3.9.

But then we know that there is a (unique) K-isomorphism $K(\alpha) \longrightarrow K(\beta)$. Thus we get the K-embedding

$$E = K(\alpha) \longrightarrow K(\beta) \hookrightarrow S$$

as desired. □

Proof of quantifier elimination for real closed fields.

We want to show that the $\mathscr{L}_{\mathrm{ri}}(\leq)$-theory RCF has quantifier elimination and we use the Shoenfield-Blum test (II) to confirm this:

So let M, N be real closed fields and let U be a finitely generated common substructure. Let $\chi(y_1,...,y_n,x)$ be a quantifier-free $\mathscr{L}_{\mathrm{ri}}(\leq)$-formula and let $\bar{a} \in U^n$. We assume

$$M \models \exists x\, \chi(\bar{a},x)$$

and we must show

$$N \models \exists x\, \chi(\bar{a},x).$$

Let R be the algebraic closure of K in M. By 4.3.8(i), R itself is real closed (and has K as an ordered subfield). By 4.3.8(ii), our assumption $M \models \exists x\, \chi(\bar{a},x)$ implies $R \models \exists x\, \chi(\bar{a},x)$. Take $b \in R$ with $R \models \chi[\bar{a},b]$. By 4.3.10, there is an ordered K-embedding $f : R \longrightarrow N$. Applying this embedding we see that $N \models \chi[\bar{a},f(b)]$, as desired.

It remains to fill the gap and to prove 4.3.9. We show Sturm's theorem which gives a much better result than 4.3.9:

4.3.11. **Definition.** Let K be an ordered field and let $c := (c_0, ..., c_n) \in K^{n+1}$. We define the **variance** of the tuple c to be

$$\mathrm{var}(c) = \mathrm{card}\{i \in \{0, ..., n-1\} \mid \exists j > i : c_i \cdot c_j < 0 \text{ and } c_k = 0\ (i < k < j)\}.$$

Hence $\mathrm{var}(c)$ is the number of sign changes in $(c_0, ..., c_n)$ after crossing out all c_i which are zero.

Observe that

(+) $\mathrm{var}(c) = \mathrm{var}(c_0, ..., c_k) + \mathrm{var}(c_k, ..., c_n)$ whenever $k \in \{1, ..., n-1\}$ and $c_k \neq 0$.

4.3.12. **Definition.** Let K be a field and let $f(X) \in K[X]$ be a polynomial. The Sturm sequence $\mathfrak{f}$ of f is the following tuple $\mathfrak{f} := (f_0, f_1, ..., f_d)$ of polynomials $f_i \in K[X]$:
$f_0 := f$, $f_1 := f'$ and for each $i > 1$ let f_{i+1} be the negative of the remainder if we divide f_{i-1} by f_i. Hence

$$\begin{array}{rcl} f_0 &=& f \\ f_1 &=& f' \\ f_0 &=& q_1 \cdot f_1 - f_2 \text{ with } q_1 \in K[X],\ \deg f_2 < \deg f_1 \\ &\vdots& \\ f_{i-1} &=& q_i \cdot f_i - f_{i+1} \text{ with } q_i \in K[X],\ \deg f_{i+1} < \deg f_i \\ &\vdots& \\ f_{d-1} &=& q_d \cdot f_d \text{ with } q_i \in K[X] \end{array}$$

By induction we see that the natural number d as well as the polynomials $f_0, ..., f_d$ are well defined. The construction of $\mathfrak{f}$ differs from the euclidean algorithm applied to f and f' only in the choice of the sign of the remainders. The proof that the euclidean algorithm applied for f and f' computes the greatest common divisor of f and f' can be literally copied in order to see

$$f_d = \gcd(f, f').$$

□

4.3.13. **Theorem.** *(Sturm, 1829)*
Let R be a real closed field. Let $f(X) \in R[X]$ with $f \neq 0$ and let $(f_0, ..., f_d)$ be the Sturm sequence of f. If $a < b$ are elements from R such that $f(a), f(b) \neq 0$ then the number of different roots (so we don't count multiplicities) of f in (a, b) is

$$\mathrm{var}(f_0(a), ..., f_d(a)) - \mathrm{var}(f_0(b), ..., f_d(b))$$

Proof. For $i \in \{0, ..., d\}$ let $h_i := \frac{f_i}{f_d} \in R[X]$. Observe that by the definition of the Sturm sequence $f_0, ..., f_d$ we have

$$f_{i-1} = q_i \cdot f_i - f_{i+1} \text{ with } q_i \in R[X],\ \deg f_{i+1} < \deg f_i$$

and therefore

$$(*) \qquad h_{i-1} = q_i \cdot h_i - h_{i+1},\ \deg h_{i+1} < \deg h_i\ (1 \leq i < d).$$

Moreover for each $i \in \{1, ..., d\}$,

(†) h_{i-1} and h_i do not have common zeroes in R,

otherwise (∗) implies that h_d has a zero; but $h_d = 1$. For $x \in R$ let

$$W(x) := \mathrm{var}(h_0(x), ..., h_d(x)).$$

Claim 1. If $c \in R$ with $f(c) \neq 0$, then $W(c) = \mathrm{var}(f_0(c), ..., f_d(c))$.

This is so, since $f(c) \neq 0$ implies $f_d(c) \neq 0$ and therefore

$$\mathrm{var}(f_0(c), ..., f_d(c)) = \mathrm{var}(f_d(c)h_0(c), ..., f_d(c)h_d(c)) = W(x).$$

Claim 2. h_0 and f have the same zero set in R.

To see claim 2 it is enough to prove $h_0(c) = 0$ for each zero c of f. Let $k > 0$ and $g(X) \in R[X]$, $g(X) \neq 0$ with $f(X) = (X - c)^k \cdot g(X)$. Since $k > 0$ we have

$$f'(X) = (X - c)^{k-1} \cdot (kg(X) + (X - c)g'(X)).$$

As $g(c) \neq 0$ the multiplicity of $X - c$ is $k - 1$ in f'. Since $f_d = \gcd(f, f')$ this shows that $X - c$ divides $h_0 = f/f_d$, in other words $h_0(c) = 0$.

Since $f(a), f(b) \neq 0$, claim 1 and claim 2 reduce the problem to show that the number of different zeroes of h_0 in (a, b) is equal to $W(a) - W(b)$. Let

$$h := h_0 \cdot ... \cdot h_d.$$

Claim 3. If $c < d$ are elements from R and h does not vanish in the interval $[c, d]$, then $W(X)$ is constant on $[c, d]$.

Claim 3 holds by the intermediate value property for polynomials.

Claim 4. If $i \in \{1, ..., d - 1\}$ and $c \in R$ is a zero of h_i, then there is some $\varepsilon > 0$ such that $\mathrm{var}(h_{i-1}(x), h_i(x), h_{i+1}(x)) = 1$ for $x \in (c - \varepsilon, c + \varepsilon)$.

As $h_i(c) = 0$, we get $h_{i-1}(c) = -h_{i+1}(c)$ from (∗). Since not both, h_i and h_{i+1} are zero in c, it follows $h_{i-1}(c) = -h_{i+1}(c) \neq 0$ and we may choose ε so that $\mathrm{sign}\, h_{i-1}(x) = -\mathrm{sign}\, h_i(x) \neq 0$ for all $x \in (c - \varepsilon, c + \varepsilon)$. Then, no matter what $h_i(x)$ is in $(c - \varepsilon, c + \varepsilon)$, we always have $\mathrm{var}(h_{i-1}(x), h_i(x), h_{i+1}(x)) = 1$ for $x \in (c - \varepsilon, c + \varepsilon)$.

For $i \in \{0, ..., d - 1\}$ let

$$W_i(x) := \mathrm{var}(h_i(x), ..., h_d(x)).$$

Claim 5. If $c \in R$ and $i \in \{0, ..., d - 1\}$ with $h_i(c) \neq 0$, then there is some $\varepsilon > 0$ such that $W_i(X)$ is constant on $(c - \varepsilon, c + \varepsilon)$.

Let $j_1 < ... < j_l$ be an enumeration of those indices $j \in \{i, ..., d\}$ such that $h_j(c) \neq 0$. Take ε so that

(a) $\mathrm{sign}\, h_{j_\alpha}(x) = \mathrm{sign}\, h_{j_\alpha}(c) \neq 0$ for all $x \in (c - \varepsilon, c + \varepsilon)$ and all $\alpha \in \{1, ..., l\}$.

By claim 4 we may shrink ε such that

(b) for each $j \in \{i + 1, ..., d - 1\}$ with $h_j(c) = 0$, $\mathrm{var}(h_{j-1}(x), h_j(x), h_{j+1}(x)) = 1$ for $x \in (c - \varepsilon, c + \varepsilon)$.

As $h_i(c) \neq 0$ by assumption and $h_d(c) \neq 0$ we have $j_1 = i$ and $j_l = d$. Thus by (+) (before 4.3.12) and (a),

$$W_i(X) = w_1(x) + ... + w_{l-1}(x), \text{ with}$$
$$w_\alpha(x) := \mathrm{var}(h_{j_\alpha}(x), ..., h_{j_{\alpha+1}}(x)) \ (x \in (c - \varepsilon, c + \varepsilon)).$$

By (†), $j_{\alpha+1} \leq j_\alpha + 2$. Hence, either $w_\alpha(x) = \text{var}(h_{j_\alpha}(x), h_{j_\alpha+1}(x))$ ($x \in (c-\varepsilon, c+\varepsilon), \alpha \in \{1,...,l-1\}$), which is constant on $(c-\varepsilon, c+\varepsilon)$ by (a), or,

$$w_\alpha(x) = \text{var}(h_{j_\alpha}(x), h_{j_\alpha+1}(x), h_{j_\alpha+2}(x)) \ (x \in (c-\varepsilon, c+\varepsilon), \alpha \in \{1,...,l-1\}),$$

which is constant on $(c-\varepsilon, c+\varepsilon)$ by (b).

Since $W_i(X) = w_1(x) + ... + w_{l-1}(x)$, with $w_\alpha(x)$ for $x \in (c-\varepsilon, c+\varepsilon)$, this shows claim 5.

Claim 6. If $c \in R$ is a zero of h_0, then there is some $\varepsilon > 0$ such that

$$W(x) = W(y) + 1 \text{ for all } x, y \text{ with } c - \varepsilon < x < c < y < c + \varepsilon.$$

Since $h_0(c) = 0$ we have $h_1(c) \neq 0$ by (†). Choose $\varepsilon > 0$ such that

(i) $W_1(X)$ is constant on $(c-\varepsilon, c+\varepsilon)$ (this is possible by claim 5),
(ii) $\text{sign}\, h_1(x) = \text{sign}\, h_1(c) \neq 0$ ($x \in (c-\varepsilon, c+\varepsilon)$) (this is possible, since $h_1(c) \neq 0$).
(iii) c is the unique zero of f in $(c-\varepsilon, c+\varepsilon)$.

Let $k > 0$ and $g(X) \in R[X]$, $g(X) \neq 0$ with $f(X) = (X-c)^k \cdot g(X)$. Since $k > 0$ we have

$$f'(X) = (X-c)^{k-1} \cdot (kg(X) + (X-c)g'(X)).$$

For $x \in (c, c+\varepsilon)$ we have $\text{sign}\, f(x) = \text{sign}\, g(x)$ and $\text{sign}\, f'(x) = \text{sign}(kg(x) + (X-c)g'(x))$. By shrinking ε if necessary and since $g(c) \neq 0$ we see that $\text{sign}\, f'(x) = \text{sign}\, g(x)$ ($x \in (c, c+\varepsilon)$). It follows that $\text{sign}\, h_0(x) = \text{sign}\, h_1(x) \neq 0$ for all $x \in (c, c+\varepsilon)$, in other words

$$(**) \qquad \text{var}(h_0(x), h_1(x)) = 0 \text{ for all } x \in (c, c+\varepsilon).$$

As $g(c) \neq 0$ the multiplicity of $X - c$ is $k-1$ in f'. Since $f_d = \gcd(f, f')$ and $h_0 = \frac{f}{f_d}$ the multiplicity of $X - c$ in h_0 is 1. Hence h_0 changes sign in c. By $(**)$ and (ii) we get

$$(***) \qquad \text{var}(h_0(x), h_1(x)) = 1 \text{ for all } x \in (c-\varepsilon, c).$$

Now for $y \in (c, c+\varepsilon)$ we have $W(y) \overset{(+),(ii)}{=} \text{var}(h_0(y), h_1(y)) + W_1(y) \overset{(**)}{=} W_1(y)$. Whereas for $x \in (c-\varepsilon, c)$ we have $W(x) \overset{(+),(ii)}{=} \text{var}(h_0(x), h_1(x)) + W_1(x) \overset{(**)}{=} W_1(x) + 1$. Since $W_1(x)$ is constant on $(c-\varepsilon, c+\varepsilon)$ by (i), this shows claim 6.

Now we prove the Theorem. Let $c_1 < ... < c_m$ be the enumeration of the zeroes of h in $[a, b]$. Choose $\varepsilon > 0$ such that for each $j \in \{1, ..., m\}$ the following conditions are satisfied:

(1) If $h_0(c_j) = 0$, then $W(x) = W(y) + 1$ for all x, y with $c - \varepsilon < x < c < y < c + \varepsilon$. This is possible by claim 6.
(2) If $h_0(c_j) \neq 0$, then $W(X)$ is constant on $(c_j - \varepsilon, c_j + \varepsilon)$. This is possible by claim 5 applied to $i = 0$.
(3) $c_j + \varepsilon < c_{j+1} - \varepsilon$ ($j \in \{1, ..., m-1\}$).

Choose $d_j \in (c_j - \varepsilon, c_j)$ and $e_j \in (c_j, c_j + \varepsilon)$ ($1 \leq j \leq m$), in particular

$$d_1 < c_1 < e_1 < d_2 < c_2 < e_2 < ... < d_m < c_m < e_m.$$

By enlarging d_1 and shrinking e_m if necessary, we may assume that all zeros of h in $[d_1, e_m]$ are among the $c_1, ..., c_m$ (note that a or b might be zeros of h).

By the choice of ε in (1), $W(x)$ decreases in the interval $[d_j, e_j]$ by 1 if and only c_j is a zero of h_0, whereas in in all other intervals $[d_j, e_j]$, $W(x)$ is constant by the choice of ε in (2). Finally $W(x)$ is constant in every interval $[e_i, d_{i+1}]$ $(1 \leq i < m)$ by claim 3.

Thus $W(d_1) - W(e_m)$ is the number of zeroes of h_0 in (d_1, e_m).

Since $f(a) \neq 0$, also $h_0(a) \neq 0$ and by our choice of d_1, h_0 does not have zeroes in the closed interval between a and d_1. Thus $W(d_1) = W(a)$. Similarly $W(e_m) = W(b)$. Hence $W(a) - W(b)$ is the number of zeroes of h_0 in (d_1, e_m), which is the number of zeroes of h_0 in (a, b). □

The proof of 4.3.9 from 4.3.13 is left as an exercise.

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