

## MODEL COMPLETENESS OF O-MINIMAL STRUCTURES EXPANDED BY DEDEKIND CUTS

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**§1. Introduction.** Let  $M$  be a totally ordered set. A (Dedekind) cut  $p$  of  $M$  is a couple  $(p^L, p^R)$  of subsets  $p^L, p^R$  of  $M$  such that  $p^L \cup p^R = M$  and  $p^L < p^R$ , i.e.,  $a < b$  for all  $a \in p^L, b \in p^R$ . In this article we are looking for model completeness results of o-minimal structures  $M$  expanded by a set  $p^L$  for a cut  $p$  of  $M$ . This means the following. Let  $M$  be an o-minimal structure in the language  $\mathcal{L}$  and suppose  $M$  is model complete. Let  $\mathcal{D}$  be a new unary predicate and let  $p$  be a cut of (the underlying ordered set of)  $M$ . Then we are looking for a natural, definable expansion of the  $\mathcal{L}(\mathcal{D})$ -structure  $(M, p^L)$  which is model complete.

The first result in this direction is a theorem of Cherlin and Dickmann (cf. [Ch-Dic]) which says that a real closed field expanded by a convex valuation ring has a model complete theory. This statement translates into the cuts language as follows. If  $Z$  is a subset of an ordered set  $M$  we write  $Z^+$  for the cut  $p$  with  $p^R = \{a \in M \mid a > Z\}$  and  $Z^-$  for the cut  $q$  with  $q^L = \{a \in M \mid a < Z\}$ . We call  $Z^+$  the **upper edge** of  $Z$  and  $Z^-$  the **lower edge** of  $Z$ . Then the Cherlin-Dickmann theorem says that  $(M, p^L)$  is model complete if  $p$  is the upper edge of a convex valuation ring of a real closed field  $M$ . This theorem has been generalized by van den Dries and Lewenberg in [vdD-Lew], for o-minimal expansions  $M$  of real closed fields and so called  $T$ -convex subrings of  $M$  (where  $T$  is the theory of  $M$ ; a  $T$ -convex valuation ring is the convex hull of an elementary restriction of  $M$ , cf. 4.3).

If  $p$  is not an edge of a convex valuation ring of a real closed field  $M$  then one can show that the  $\mathcal{L}(\mathcal{D})$ -theory (where  $\mathcal{L}$  is the language of ordered rings) of  $(M, p^L)$  is not model complete (cf. [T1], §16). So for model completeness we actually have to extend the language  $\mathcal{L}(\mathcal{D})$ .

We do the following. Let  $M$  be again an o-minimal expansion of a real closed field and let  $p$  be a cut of  $M$ . Let  $G(p) := \{a \in M \mid a + p = p\}$ , where  $a + p := (a + p^L, a + p^R)$ .  $G(p)$  is a convex subgroup of  $(M, +, \leq)$ , called the invariance group of  $p$ . Moreover  $V(p) := \{b \in M \mid b \cdot G(p) \subseteq G(p)\}$  is a convex valuation ring of  $M$ , called the invariance ring of  $p$ . Suppose there is a  $T$ -convex valuation ring  $V$  of  $M$ , such that  $V(p)$  is definable in  $(M, V)$  (this is no restriction if  $M$  is a pure real closed field). Then our results 3.8 and 7.4 imply that we get model completeness of  $Th(M, p^L)$  if we expand the language  $\mathcal{L}(\mathcal{D})$  by a unary predicate

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for the group  $G(p)$ , a unary predicate for the ring  $V(p)$  and unary predicates for the sets  $\{a \in M \mid a \pm G(p)^+ = p\}$  and  $\{a \in M \mid a \cdot V(p) = G(p) \text{ or } a \cdot \mathfrak{m}(p) = G(p)\}$ , where  $\mathfrak{m}(p)$  denotes the maximal ideal of  $V(p)$ .

This shows that the basic invariants of the cut  $p$  are  $G(p)$  and  $V(p)$ . Indeed if  $M$  is a real closed field (or more general a polynomially bounded o-minimal structure with archimedean prime model) and  $A$  is a subset of  $M$ , then the theory of  $(M, p^L)$  over the parameter set  $A$  is entirely determined by saying where  $p$  is located when shifting the cuts  $G(p)^+$  and  $V(p)^+$  with certain maps  $M \rightarrow M$  of the form  $a \cdot x^q + b$ , where  $a, b \in A$  and  $q$  is an exponent of  $M$  (hence  $q \in \mathbb{Q}$  if  $M$  is a real closed field). This will be done in a sequel to this paper [T2].

We describe now, how the paper is organized and mention some important intermediate results. In order not to overload notations at the moment, we assume that  $T$  is an o-minimal expansion of groups, which has quantifier elimination, in the language  $\mathcal{L}$ .

The results described above are based on a general description of so called ‘‘heirs’’ (cf. 2.6) of cuts in o-minimal expansions of groups, proved in 2.9 below. It implies the following fixpoint criterion:

Let  $M \prec N$  be models of  $T$ , let  $q$  be a cut of  $N$  and let  $p$  be the restriction of  $q$  on  $M$  such that  $p^L$  is not definable in  $M$ . Then the  $\mathcal{L}(\mathcal{D})$ -structure  $(M, p^L)$  is existentially closed in  $(N, q^L)$  (this is the definition of ‘‘ $q$  is an heir of  $p$ ’’) if and only if for every definable function  $f : M \rightarrow M$  with  $f(p) = p$  we have  $f(q) = q$ . How  $f$  is applied to  $p$  and  $q$  is explained in 2.4.

For some cuts, this fixpoint criterion already implies that the theory of  $(M, p^L)$  has quantifier elimination in the language  $\mathcal{L}(\mathcal{D})$ , if  $T$  has a universal axiomatization (cf. 3.8). More precisely, this holds for cuts  $p$  such that  $p$  is not the fixpoint of any definable map  $f : M \rightarrow M$  which is different from the identity, close to  $p$ . These cuts are called ‘‘dense’’, cf. 3.7. For example, if  $M$  is a subset of  $\mathbb{R}$  then the cut induced by  $\alpha \in \mathbb{R} \setminus M$  on  $M$  has these property.

In order to treat non-dense cuts let us take a cut  $p$  of a model  $M$  of  $T$  and consider the following two cases.

First suppose  $p$  is a translate of an edge of a convex subgroup, so  $p = a \pm G^+$  for some  $a \in M$  and some convex subgroup  $G$  of  $M$ . Then clearly every model completeness result concerning  $(M, G)$  implies a similar model completeness result for  $(M, p^L)$  and we are reduced to solving this last problem.

In the second case — if  $p$  is not such a translate — then we prove that a certain condition on  $p$  allows an application of model completeness results concerning  $(M, G)$  to  $(M, p^L)$ . This is the main goal in Section 3, cf. 3.19. The criterion in question — called the **signature alternative** (cf. 3.16) — is a weakening of the valuation property for polynomially bounded structures (cf. [vdD-S],(9.2)). The signature alternative for a cut  $p$  implies the following. If  $p$  is not a translate of an edge of a convex subgroup of  $M$ , then there is no definable map  $f : M \rightarrow M$  with  $f(p) = G(p)^+$ . Hence the only way of shifting the cut  $p$  into the upper edge of its invariance group by a definable function is by applying a map  $a \pm x$ . In view of the fixpoint criterion above this property reduces inheritance of cuts to inheritance of edges of subgroups. An important result is Theorem 7.2, which gives a large variety of cuts having the signature alternative.

If the signature alternative holds for a cut  $p$  (e.g., if  $T$  is polynomially bounded), then both cases above say that we have to understand convex subgroups of  $M$  if we want to understand  $p$ . For this, we go to expansions of fields now, so  $T$  will denote an o-minimal expansion of fields for the rest of the introduction.

Let  $G$  be a convex subgroup of  $M \models T$ . In order to understand the structure  $(M, G)$  we apply the same strategy as explained above for  $p$ , to the cut  $G^+$  but now viewed as a cut of the multiplicative group  $(M^{>0}, \cdot, \leq)$ .  $M$  induces in a natural way an o-minimal expansion  $M^{>0}$  on this group, so the techniques above are indeed applicable. An important step is to prove the signature alternative for these cuts  $G^+$  w.r.t. to the structure  $M^{>0}$ . This is the aim of Section 5, cf. 5.4.

What happens if we go from  $G^+$  to its invariance group w.r.t.  $M^{>0}$ ? We get the positive units of the invariance ring  $V(G)$  of  $G$  as defined above. So we are back to the Cherlin-Dickmann and the van den Dries-Lewenberg work. Provided,  $V(G)$  is definable from  $M$  enlarged by a  $T$ -convex valuation ring! Hence the results on  $T$ -convexity from [vdD-Lew] will be the starting point in Section 5, for the proof of our results in Section 7.

Finally I want to point out that Theorem 7.3 below implies the substantial part of almost everything which is described above. It is of technical nature and will be our starting point for the elementary theory of Dedekind cuts in polynomially bounded structures, treated in [T2].

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**§2. Heirs.** First we recall some general model theoretic notations and tools.

**DEFINITION 2.1.** Let  $\mathcal{L}$  be a first order language, let  $M$  be an  $\mathcal{L}$ -structure and let  $A \subseteq M$ . A subset  $X$  of  $M^n$  is called  **$A$ -definable** if  $X$  is the set of all realizations in  $M$  of an  $\mathcal{L}$ -formula with parameters from  $A$ . If  $N$  is an elementary extension of  $M$ , then this formula also defines a subset of  $N^n$  and we denote this set by  $X_N$ . If there is no ambiguity we also write  $X$  for  $X_N$ . One may think of  $X$  as the formula defining  $X$ . Note that  $X$  is defined by several formulas, but  $X_N$  does not depend on that representation.

A map  $f : X \rightarrow M^k$  is called  **$A$ -definable**, if the graph of  $f$  is an  $A$ -definable subset of  $M^n \times M^k$ . If  $A = \emptyset$ , then we say 0-definable instead of  $A$ -definable. Again, if  $N \succ M$  is an elementary extension, then  $f_N$  denotes the map  $X_N \rightarrow N^k$  defined by a formula which defines the graph of  $f : X \rightarrow M^k$ . We'll also write  $f$  for  $f_N$  if there is no ambiguity.

**DEFINITION 2.2.** The **definable closure**  $\text{dcl}(A)$  of  $A$  (in  $M$ ), is the set of all points  $x \in M$ , which are  $A$ -definable, i.e., the set  $\{x\}$  is  $A$ -definable.

Observe that

$$\text{dcl}(A) = \{f(a_1, \dots, a_n) \mid n \in \mathbb{N}, f : M^n \rightarrow M \text{ is } 0\text{-definable and } a_1, \dots, a_n \in A\}$$

and for all  $b_1, \dots, b_n \in A$  we have

$$\text{dcl}(A \cup \{b_1, \dots, b_n\}) = \{f(b_1, \dots, b_n) \mid f : M^n \rightarrow M \text{ is } A\text{-definable}\}.$$

**DEFINITION 2.3.** Let  $\mathcal{L}$  be a first order language. If  $M \prec N$  is an elementary extension of  $\mathcal{L}$ -structures and  $\bar{a} \in N^n$ , then we write  $\text{tp}(\bar{a}/M)$  for the  $n$ -type  $p$  of  $\bar{a}$  over  $M$ . Hence  $p$  is the set of all  $\mathcal{L}$ -formulas in  $n$  fixed variables with parameters

from  $M$  which hold true at  $\bar{\alpha}$  in  $N$ . The tuple  $\bar{\alpha}$  is called a **realization** of  $p$  and we write  $\bar{\alpha} \models p$ .

Recall that the set  $S_n(M) := \{\text{tp}(\bar{\alpha}/M) \mid \bar{\alpha} \in N^n, N \succ M\}$  of all  $n$ -types of  $M$  is a Boolean space and the set  $M^n$  is in 1-1 correspondence with the isolated points of  $S_n(M)$  via  $\bar{a} \mapsto \text{tp}(\bar{a}/M)$  ( $\bar{a} \in M^n$ ). We will always consider  $M^n$  as a subset of  $S_n(M)$  in this way.

Moreover the  $M$ -definable subsets  $X$  of  $M^n$  are in 1-1-correspondence with the clopen subsets of  $S_n(M)$  via  $X \mapsto \{p \in S_n(M) \mid X \in p\}$ . Here,  $X \in p$  stands for “a formula defining  $X$  is in  $p$ ”. Therefore, geometrically, the notation  $X \in p$  reads better as  $p \in X$  (i.e.,  $p$  is in the clopen defined by  $X$ ).

**DEFINITION 2.4.** Let  $p \in S_n(M)$ , let  $X \subseteq M^n$  be  $M$ -definable and let  $f : X \rightarrow M^k$  be an  $M$ -definable map. Then

$\{\varphi(y_1, \dots, y_k) \mid [\exists y_1, \dots, y_k (y_1, \dots, y_k) = f(x_1, \dots, x_n) \wedge \varphi(y_1, \dots, y_k)] \in p\}$  is certainly a  $k$ -type of  $M$ , which we denote by  $f(p)$ . We get a map  $\{p \in S_n(M) \mid X \in p\} \rightarrow S_k(M)$  and this map extends the original map  $f : X \rightarrow M^k$ .

Note that if  $\bar{\alpha}$  is a realization of  $p$ ,  $X \in p$ ,  $\bar{\alpha}$  from some elementary extension  $N$  of  $M$ , then  $f(\bar{\alpha}) (= f_N(\bar{\alpha}))$  is a realization of  $f(p)$  and  $f(X) \in f(p)$ .

We now turn to cuts. Let  $X = (X, \leq)$  be a totally ordered set. A cut  $p$  of  $X$  is called **principal** if  $p$  is of the form  $a_X^+ := \{a\}^+$ ,  $a_X^- := \{a\}^-$  ( $a \in X$ ),  $+\infty_X := X^+ = \emptyset^-$  or  $-\infty_X := X^- = \emptyset^+$ . Thus a cut  $p$  is principal if and only if the subset  $p^L$  of  $X$  is quantifier free definable (with parameters) in the ordered set  $X$  considered as a first order structure in the language  $\{\leq\}$ .

If  $p$  is a cut of  $X$  and  $x \in X$  then we also write  $x \leq p$ ,  $p \leq x$  for  $x \in p^L$ ,  $x \in p^R$ , respectively. This definition extends the given order on  $X$  to a partial order on the **Dedekind completion**  $D := X \cup \{\text{the cuts of } X\}$ . Finally we write  $p \leq q$  if  $p^L \subseteq q^L$  for cuts  $p, q$  of  $X$ . Observe that the elements of  $X$  are not considered as cuts in our setup, hence  $x \leq p$  is the same as  $x < p$  if  $x \in X$  and  $p$  is a cut of  $X$ . Now  $D$  is a totally ordered set extending  $X$  and  $D$  is **Dedekind complete**, i.e., every cut of  $D$  is principal.

An **interval** of  $X$  is a convex subset  $I$  of  $X$  such that  $I^+$  and  $I^-$  are principal. A **neighborhood**  $I$  of the cut  $p$  is an interval of  $X$  with  $I^- \leq p \leq I^+$ ; we write  $p \in I$  if  $I$  is a neighborhood of  $p$ . Observe that every cut has an open neighborhood, i.e., a neighborhood which is open in the order topology of  $X$ .

Let  $p$  be a cut of  $X$  and let  $Y \supseteq X$  be another totally ordered set. If  $q$  is a cut of  $Y$  we say  $q$  **extends**  $p$  or  $q$  **lies over**  $p$  or  $q$  is an **extension** of  $p$  and write  $p \subseteq q$  or  $p = q \upharpoonright X$  if  $q^L \cap X = p^L$ . Observe that  $p$  always has a least and a largest extension on  $Y$ ; the least extension is the upper edge of  $p^L$  in  $Y$  and the largest extension is the lower edge of  $p^R$  in  $Y$ .

A **realization**  $\alpha$  of  $p$  in  $Y$  is an element  $\alpha \in Y$  such that  $p^L < \alpha < p^R$ . In particular  $\alpha \notin X$ .

**From now on, let  $T$  be an o-minimal extension of the theory of dense linear ordered sets without endpoints in the language  $\mathcal{L}$ .** Recall from [vdD-Lew], (3.2) that  $T$  is equipped with a dimension; if  $p \in S_n(M)$ ,  $M \models T$ , then  $\dim p$  denotes the dimension of  $M \cup \{\bar{\alpha}\}$  over  $M$  with respect to this dimension, where  $\bar{\alpha}$  is a realization of  $p$  from some elementary extension of  $M$ .

Let  $M$  be a model of  $T$ . Then for each cut  $p$  of  $M$  there is a 1-type  $P$  of  $M$  such that  $a < x \in P \iff a \in p^R$  ( $a \in M$ ). Since  $M$  is o-minimal, the type  $P$  is uniquely determined by  $p$  in this way. Therefore **we'll frequently consider a cut  $p$  as the corresponding 1-type  $P$**  which allows us to use the type set-up for cuts.

By o-minimality, a cut  $p$  of  $M$  is principal if and only if  $p^L$  is an  $M$ -definable subset of  $M$  if and only if the 1-type corresponding to  $p$  is definable in the sense of type theory (cf. [Poi], chapter 11).

Moreover if  $N \succ M$  is an elementary extension of  $M$ , then an element  $\alpha \in N$  is a realization of the cut  $p$  of  $M$  if and only if  $\alpha$  is a realization of the 1-type determined by  $p$ . Therefore we write  $\alpha \models p$  if  $\alpha$  is a realization of the cut  $p$ .

Following [vdD-Lew], (2.5), if  $M \prec N$  are models of  $T$  and  $Z \subseteq N$ , then  $M\langle Z \rangle$  denotes the definable closure of  $M \cup Z$  in  $N$ . Note that if  $T$  expands the theory DOAG of nontrivial, divisible ordered abelian groups and  $A$  is any subset of a model  $N$  of  $T$ , which contains an element different from 0, then the definable closure of  $A$  in  $N$  is an elementary substructure of  $N$  (cf. [vdD-Lew], (2.4) and (2.5)).

Let  $M$  be o-minimal, let  $p \in S_1(M)$  and let  $f : X \rightarrow M$ ,  $X \subseteq M$  be  $M$ -definable with  $X \in p$ . If  $p$  is a cut, (i.e.,  $p \notin M$ ), then the application of  $f$  to  $p$  does the obvious thing: by the monotonicity theorem [vdD3], 3, (1.2), there is an open interval  $I$  of  $M$  with  $p \in I \subseteq X$  such that  $f$  is constant on  $I$  or a strictly monotonous homeomorphism onto  $f(I)$ . By o-minimality  $f(I)$  is again an open interval.

Suppose  $f$  is strictly increasing on  $I$ . Then  $f(p)$  is the cut  $f(p^L \cap I)^+$  if  $p^L \cap I \neq \emptyset$  and  $f(p^R \cap I)^-$  otherwise.

Furthermore—by our definition of “cut”— $f(p)$  is a cut if and only if  $f$  is nonconstant in each neighborhood of  $p$ .

**LEMMA 2.5.** *If  $p$  is a cut of an o-minimal structure  $M$  and  $f : M \rightarrow M$  is  $M$ -definable with  $f(p) = p$ , then there is a neighborhood  $I$  of  $p$  such that  $f|_I$  is strictly increasing.*

**PROOF.** Since  $f(p)$  is a cut, there is an open interval  $I$  of  $M$  with  $p \in I$  such that  $f$  is a strictly monotonous homeomorphism onto the open interval  $f(I)$ .  $I$  is of the form  $(a, b)$  with  $a < b$  from  $M \cup \{\pm\infty\}$ . We assume that  $a, b \in M$ , the other cases are similar. If the formula  $f(x) = x$  is in the type  $p$ , then we may shrink  $I$ , so that  $f$  is the identity on  $I$  and we are done. Hence we may assume that the formula  $f(x) = x$  is not in the type  $p$ , so  $f(x) \neq x$  is in the type  $p$ . By o-minimality we can shrink  $I$  so that  $f(x) \neq x$  for all  $x \in I$ .

Suppose  $f$  is strictly decreasing in  $(a, b)$ . Then  $f$  maps the cuts  $r$  of  $M$  with  $a < r < b$  bijectively and order reversing onto the cuts  $r$  of  $M$  with  $r \in f(I)$ .

Consequently  $f(a^+)$  must be different from  $f(p)$ , otherwise  $a^+ = p$ , thus  $f(a^+) = f(p) = p = a^+$ ; but  $f(a^+) = f(I)^+$  which must be different from  $a^+$ , as  $f(I)$  is open. It follows  $f(a^+) > f(p)$ .

The same argument shows  $f(p) > f(b^-)$ , thus  $f(a^+) > f(p) = p > f(b^-)$ . With these inequalities it follows that the graph of  $f$  meets the diagonal of  $M$  in a point  $c \in (a, b)$ , contradicting the assumption  $f(x) \neq x$  for all  $x \in (a, b)$ .  $\dashv$

**DEFINITION 2.6.** Let  $M, N$  be models of  $T$  and let  $A \prec M, N$  be a common elementary substructure. Let  $p_1, \dots, p_n$  be pairwise distinct cuts of  $M$  and let  $q_1, \dots, q_n$  be pairwise distinct cuts of  $N$ . Let  $\mathcal{D}_1, \dots, \mathcal{D}_n$  be new unary predicates. We

say the tuple  $(q_1, \dots, q_n)$  is an **heir** of  $(p_1, \dots, p_n)$  over  $A$  if the following condition holds: if  $\varphi(x_1, \dots, x_k)$  is an  $\mathcal{L}$ -formula with parameters from  $A$ ,  $\psi(x_1, \dots, x_k)$  is a quantifier free  $\mathcal{L}(\mathcal{D}_1, \dots, \mathcal{D}_n)$ -formula with parameters from  $A$  and if

$$(N, q_1^L, \dots, q_n^L) \models \exists \bar{x} \varphi(\bar{x}) \wedge \psi(\bar{x}), \text{ then}$$

$$(M, p_1^L, \dots, p_n^L) \models \exists \bar{x} \varphi(\bar{x}) \wedge \psi(\bar{x}).$$

Observe that  $p_i \upharpoonright A = q_i \upharpoonright A$  ( $1 \leq i \leq n$ ) in this case.

**REMARK 2.7.** If  $A = M$  in 2.6, then we say ‘heir’ instead of ‘heir over  $M$ ’. It is easy to see that a single cut  $q$  of  $N$  is an heir of  $p := q \upharpoonright M$  in the sense of 2.6 if and only if the 1-type corresponding to the cut  $q$  is an heir of the 1-type corresponding to the cut  $p$  in the sense of [Poi], 11.a. More generally, the tuple  $(q_1, \dots, q_n)$  is an heir of  $(p_1, \dots, p_n)$  if and only if for each  $n$ -type  $p$  over  $M$  with projections  $p_1, \dots, p_n$  there is an  $n$ -type  $q$  over  $N$  with projections  $q_1, \dots, q_n$  which is an heir of  $p$  in the sense of [Poi]. Moreover a compactness argument as in [Poi], 11.01, implies that for given  $A \prec M, N$  and cuts  $p_1, \dots, p_n$  of  $M$  there is always an heir of  $(p_1, \dots, p_n)$  over  $A$  on  $N$ .

**LEMMA 2.8.** *Let  $p_1, \dots, p_n$  be pairwise distinct cuts of  $M$  and let  $q_1, \dots, q_n$  be extensions of  $p_1 \upharpoonright A, \dots, p_n \upharpoonright A$  on  $N$ , respectively, with  $q_i \notin \{\pm\infty_N\}$ . Then  $(q_1, \dots, q_n)$  is an heir of  $(p_1, \dots, p_n)$  over  $A$  if and only if the following condition holds:*

*if  $\varphi(u_1, \dots, u_n, v_1, \dots, v_n)$  is an  $\mathcal{L}$ -formula with parameters from  $A$  and  $\bar{\alpha}, \bar{\beta} \in N^n$  such that  $\alpha_1 < q_1 < \beta_1, \dots, \alpha_n < q_n < \beta_n$  and  $N \models \varphi(\bar{\alpha}, \bar{\beta})$  then there are  $\bar{a}, \bar{b} \in M^n$  such that  $a_1 < p_1 < b_1, \dots, a_n < p_n < b_n$  and  $M \models \varphi(\bar{a}, \bar{b})$*

**PROOF.** If  $(q_1, \dots, q_n)$  is an heir of  $(p_1, \dots, p_n)$  over  $A$ , then certainly the condition holds. The converse follows from a straightforward simplification of the condition in 2.6.  $\dashv$

The next theorem is the crucial starting point for the model theory of cuts in o-minimal expansions of groups.

**THEOREM 2.9** (Fixpoint criterion for heirs). *Let  $T$  be an o-minimal expansion of DOAG. Let  $A, M, N$  be models of  $T$  with  $A \prec M, N$ . Let  $q$  be a cut of  $N$  and let  $p$  be a cut of  $M$  with  $p \upharpoonright A = q \upharpoonright A$ , such that  $p^L$  is not  $A$ -definable (in particular if  $p$  is not principal). Then the following are equivalent:*

- (i)  $q$  is an heir of  $p$  over  $A$ .
- (ii) If  $\beta \in N$  is a realization of  $q \upharpoonright A$ , then  $(q \upharpoonright A \langle \beta \rangle)$  is an heir of  $p$  over  $A$ .
- (iii) For each  $A$ -definable map  $f : A \rightarrow A$  with  $f(p) = p$  we have  $f(q) = q$ .

**PROOF.** (i) $\Rightarrow$ (ii) follows directly from the definition of ‘heir’.

(ii) $\Rightarrow$ (iii). Take  $f$  as in (iii), suppose  $f(p) = p$  and assume there is some  $\beta \in N$  with  $f(q) \leq \beta < q$ . Since  $f$  is  $A$ -definable, we have  $f(p \upharpoonright A) = p \upharpoonright A$ , too. In particular the 1-type  $f(q)$  is an extension of  $p$ , so  $\beta$  is a realization of  $p \upharpoonright A$  and  $f(q \upharpoonright A \langle \beta \rangle) \leq \beta < q \upharpoonright A \langle \beta \rangle$ . Therefore we may assume that  $N = A \langle \beta \rangle$ .

By 2.5,  $f$  is strictly increasing in some open  $A$ -definable neighborhood  $I$  of  $p \upharpoonright A$ , in particular  $f(q)$  is a cut, thus  $f(q) \neq \beta$  and  $f(I_N)$  contains the realizations of  $p \upharpoonright A$  in  $N$  (again,  $I_N$  denotes the interval of  $N$  defined by the formula which

defines  $I$ ). Thus, there is some  $\tilde{\beta} \in I_N$  with  $\beta = f(\tilde{\beta})$ . Since  $f(q) < \beta = f(\tilde{\beta})$  and  $\tilde{\beta} \in I_N$  we have  $q < \tilde{\beta}$ . Together with  $\beta < q$  this means

$$(N, q^L) \models \exists x, \tilde{x} \mathcal{D}(x) \wedge \neg \mathcal{D}(\tilde{x}) \wedge x = f(\tilde{x}) \wedge x \in I.$$

Since  $N = A\langle\beta\rangle$  and  $\beta$  realizes  $p \upharpoonright A$  we may apply (ii) and we get

$$(M, p^L) \models \exists x, \tilde{x} \mathcal{D}(x) \wedge \neg \mathcal{D}(\tilde{x}) \wedge x = f(\tilde{x}) \wedge x \in I.$$

Hence there are  $\alpha, \tilde{\alpha} \in M$  with  $f(\tilde{\alpha}) = \alpha < p < \tilde{\alpha}$  and  $\alpha \in I$ . Since  $f$  is strictly increasing in  $I_M$  it follows  $f(p) < p$ , in contradiction to our assumption.

(iii) $\Rightarrow$ (i). Let  $\varphi(x, y)$  be an  $\mathcal{L}$ -formula with parameters from  $A$  and let  $\alpha, \beta \in N$  with  $\alpha < q < \beta$  and  $N \models \varphi(\alpha, \beta)$ . By 2.8 it is enough to find elements  $a, b \in M$  with  $a < p < b$  and  $M \models \varphi(a, b)$ .

CASE 1.  $\beta$  is not a realization of  $p \upharpoonright A$ . Then there is some  $c \in A$  with  $p \upharpoonright A < c \leq \beta$ . Let  $\psi(x)$  be the formula  $\exists y c \leq y \wedge \varphi(x, y)$ . Then  $N \models \psi(\alpha)$  and it is enough to find some  $a \in M$  with  $a < p$  and  $M \models \psi(a)$ .

Since  $\psi$  has parameters from  $A$ , the set  $S$  of all realizations of  $\psi$  in  $A$  is nonempty and the cut  $S^-$  of  $A$  is  $A$ -definable, hence principal, and different from  $+\infty_A$ . If  $S^- < p \upharpoonright A$ , then there is some element  $a \in S$  with  $a < p \upharpoonright A$ , so  $a < p$  and  $M \models \psi(a)$ , too.

Hence we may assume that  $q \upharpoonright A = p \upharpoonright A \leq S^-$ . Since  $\alpha < q$  and  $N \models \psi(\alpha)$  it follows that for every  $b \in A$  with  $p \upharpoonright A < b$  there is an element  $a \in A$ ,  $a < b$  such that  $A \models \psi(a)$ . Therefore,  $p \upharpoonright A = S^-$  is principal and not of the form  $d^-$  for all  $d \in A$ .

If  $S^-$  is of the form  $d^+$  for some  $d \in A$ , then there is some  $\varepsilon > 0$ ,  $\varepsilon \in A$  such that  $(d, d + \varepsilon) \subseteq S$ . Since  $p^L$  is not  $A$ -definable we have  $d_M^+ < p$  and there is some  $a \in M$ ,  $a < p$  with  $d < a < d + \varepsilon$ . Since  $(d, d + \varepsilon)_M \subseteq S_M$ ,  $a$  satisfies  $a < p$  and  $M \models \psi(a)$ . Finally if  $S^- = -\infty_A$ , then  $A \models \forall x \exists z z \leq x \wedge \psi(z)$ . Then also  $M \models \forall x \exists z z \leq x \wedge \psi(z)$  and since  $p^L \neq \emptyset$  ( $p$  is not principal), there are elements  $a \in M$  with  $a < p$  and  $M \models \psi(a)$ .

CASE 2.  $\alpha$  is not a realization of  $p \upharpoonright A$ . Analog to case 1. Observe that we do not need assumption (iii) in case 1 and 2.

CASE 3.  $\alpha$  and  $\beta$  realize  $p \upharpoonright A$ . We may assume that  $p \upharpoonright A < c$  for some  $c \in A$  (otherwise we proceed with  $-p$  and  $-q$ ). We have to find  $a, b \in M$  with  $a < p < b$  and  $M \models \varphi(a, b)$ .

Since  $\alpha < \beta < c$  we may replace  $\varphi(x, y)$  by  $\varphi(x, y) \wedge x < y < c$  and we may assume that  $A \models \forall x, y \varphi(x, y) \rightarrow x < y < c$ .

Suppose for all  $a, b \in M$  with  $a < p < b$  we have  $M \models \neg \varphi(a, b)$ . We define  $f : A \rightarrow A$  by

$$f(a) := \begin{cases} \sup\{b \in A \mid A \models \varphi(a, b)\} & \text{if } A \models \exists y \varphi(a, y), \\ a & \text{otherwise.} \end{cases}$$

By o-minimality and since  $\varphi(x, y)$  implies  $y < c$ ,  $f(a)$  is a well defined element for all  $a \in A$  with  $A \models \exists y \varphi(a, y)$ . Thus  $f$  is an  $A$ -definable map. Keep in mind that the case  $p \upharpoonright A = -\infty_A$  is not excluded in what follows. Of course  $p \neq \pm\infty_M$ , since by assumption  $p^L$  and  $p^R$  are not  $A$ -definable. We claim that  $f$  violates condition (iii). First we prove  $f(p) = p$ .

CASE 3.1.  $p = m^+$  for some  $m \in M$ .

Since  $p^L$  is not  $A$ -definable, we have  $m \notin A$ , thus  $m \models p \upharpoonright A$ . Since  $m < p$ ,  $M \not\models \varphi(m, b)$  for all  $b \in M$ ,  $p < b$  (i.e.,  $m < b$ ) and the definition of  $f$  implies  $f(m) = m$ . Since  $f$  is  $A$ -definable, this means  $f(p \upharpoonright A) = p \upharpoonright A$ , hence by 2.5,  $f$  is strictly increasing and continuous in some  $A$ -definable neighborhood  $I \subseteq A$  of  $p \upharpoonright A$ . Since  $f(m) = m$  and  $m \in I_M$ , it follows  $f(m^+) = m^+$ , thus  $f(p) = p$ .

CASE 3.2.  $p = m^-$  for some  $m \in M$ .

Again  $m \notin A$  and  $m \models p \upharpoonright A$ , since  $p^L$  is not  $A$ -definable. Since  $p = m^-$  and  $f$  is  $A$ -definable, it follows that  $f$  is continuous in some open,  $A$ -definable interval  $I \subseteq A$  with  $m \in I_M$ .

We have  $M \not\models \varphi(a, b)$  for all  $a, b \in M$  with  $a < m \leq b$  (which means  $a < p < b$ ). Therefore  $f(a) \leq m$  for all  $a \in M$ ,  $a < m$ . If  $f(a) < m$  for  $a < m$ ,  $a$  close to  $m$ , then  $f(m^-) = m^-$  and case 3.2 is proved. Otherwise there is some  $\varepsilon \in M$ ,  $\varepsilon > 0$  such that  $f(a) = m$  whenever  $a \in (m - \varepsilon, m)$ . By continuity, also  $f(m) = m$ . This again means that  $f(p \upharpoonright A) = p \upharpoonright A$ , hence by 2.5,  $f$  is strictly increasing and continuous in some  $A$ -definable neighborhood  $I \subseteq A$  of  $p \upharpoonright A$ . But  $f$  is also constant in  $(m - \varepsilon, m)$ , a contradiction.

CASE 3.3.  $p$  is not principal.

Then for each  $a \in M$  with  $a < p$  we have  $f(a) < p$  by assumption and because of  $p \neq m^-$  ( $m \in M$ ). Let  $I \subseteq A$  be an  $A$ -definable open interval with  $p \upharpoonright A \in I$ , such that  $f$  is monotonous on  $I$ . From  $a \leq f(a) < p$  ( $a < p$ ,  $a \in I_M$ ) we know that  $f$  must be strictly increasing in some neighborhood of  $p$ . Since  $p \neq m^+$  for all  $m \in M$  this shows  $f(p) = p$ .

We have proved that  $f(p) = p$ . In particular  $f(p \upharpoonright A) = p \upharpoonright A$  and by 2.5, there is some  $A$ -definable neighborhood  $I$  of  $p \upharpoonright A$ , such that  $f$  is strictly increasing in  $I$ . Since  $\alpha$  and  $\beta$  realize  $p \upharpoonright A$  we have  $\alpha, \beta \in I_N$ . As  $\beta < c$  we have  $f(\alpha) \geq \beta$  by definition of  $f$ . Since  $f$  is strictly increasing in  $I_N$  we get  $f(q) \neq q$  in contradiction to (iii).  $\dashv$

COROLLARY 2.10. *Let  $M \prec N$  be models of  $T$  and let  $p$  be a nonprincipal cut of  $M$ .*

- (i) *The least and the largest extension of  $p$  on  $N$  are heirs of  $p$ .*
- (ii) *If  $\dim N/M = 1$  then the least and the largest extension of  $p$  on  $N$  are the only heirs of  $p$  on  $N$ .*

PROOF. (i). If  $f : M \rightarrow M$  is  $M$ -definable with  $f(p) = p$  then by 2.5,  $f$  is strictly increasing in some open  $M$ -definable neighborhood containing  $p$ . Thus the least and the largest extension of  $p$  on  $N$  are fixed by  $f$  as well. By 2.9 this proves (i).

(ii). Let  $q$  be an extension of  $p$  on  $N$ , different from the least and the largest extension of  $p$  on  $N$ . Then there are realizations  $\alpha, \beta$  of  $p$  with  $\alpha < q < \beta$ . As  $\dim N/M = 1$  we have  $\beta \in M\langle\alpha\rangle$ . Since  $M\langle\alpha\rangle$  is the definable closure of  $M \cup \{\alpha\}$  there is an  $M$ -definable map  $f : M \rightarrow M$  with  $f(\alpha) = \beta$ . Since  $\alpha$  and  $\beta$  realize  $p$  we have  $f(p) = p$  and  $f$  is strictly increasing in some open  $M$ -definable neighborhood containing  $p$ . This shows  $f(q) \neq q$  and  $q$  is not an heir of  $p$ .  $\dashv$

Another Consequence of 2.9 is that the ‘‘heir’’ relation is total, i.e., if  $A \prec M, N$  are models of  $T$ ,  $q$  is a cut of  $N$  and  $p$  is a cut of  $M$  with  $p \upharpoonright A = q \upharpoonright A$ , then  $p$

is an heir of  $q$  over  $A$  or  $q$  is an heir of  $p$  over  $A$ . We don't need this and omit the proof.

**§3. The invariance group.** We start with basic facts about cuts of ordered abelian groups. Let  $M = (M, +, 0, \leq)$  be an ordered abelian group and let  $X \subseteq M$  be a set. We write  $-X := \{-x \mid x \in X\}$  and  $a + X := \{a + x \mid x \in X\}$  for  $a \in M$ .

If  $p$  is a cut of  $M$  and  $a \in M$ , then certainly  $-p := (-p^R, -p^L)$  and  $a + p := (a + p^L, a + p^R)$  are again cuts of  $M$ . Moreover we write

$$|p| := \begin{cases} p & \text{if } p > 0, \\ -p & \text{if } p < 0. \end{cases}$$

**First Warning.** If  $p$  is a cut of  $M$ , then there is no canonical way to define a cut “ $2 \cdot p$ ” of  $M$ . That is, in general the cut  $\{2 \cdot x \mid x \in p^L\}^+$  is strictly less than the cut  $\{2 \cdot x \mid x \in p^R\}^-$  (e.g., if  $M = \mathbb{Z}$  and  $p$  is different from  $\pm\infty$ ).

In this paper we are interested in cuts of expansions of divisible ordered abelian groups mainly. In this case, the couple  $n \cdot p := (\{n \cdot x \mid x \in p^L\}, \{n \cdot x \mid x \in p^R\})$  is a cut of  $M$  again for every  $n \in \mathbb{N}$ . Observe that  $n \cdot (-p) = -(n \cdot p)$  and that  $n \cdot (a \pm p) = na \pm n \cdot p$  for all  $a \in M$ ,  $n \in \mathbb{N}$ . These observations make the basic computations concerning cuts much easier in the divisible case and we'll restrict ourself to divisible groups soon.

**LEMMA AND DEFINITION 3.1.** *Let  $M$  be an abelian ordered group. Then  $M$  operates on the cuts of  $M$  via translation, i.e., via  $(a, p) \mapsto a + p$ . The stabilizer of this action at a cut  $p$  is a convex subgroup of  $M$  which is called the **invariance group** or the **width** of  $p$ :*

$$G(p) := \{a \in M \mid a + p = p\}.$$

Furthermore we define

$$\hat{p} := G(p)^+$$

and we have  $|p| \geq \hat{p}$ .

**PROOF.** Straightforward. +

For example the principal cuts  $a^+$  and  $a^-$  have invariance group  $\{0\}$ . Observe also, that if  $p - a$  is an edge of a convex subgroup for some  $a \in M$ , then this group is  $G(p)$ . Another important observation is the following lemma.

**LEMMA 3.2.** *Let  $M \subseteq N$  be ordered abelian groups, let  $a \in M$  and let  $q$  be a cut of  $N$ . Then  $(a + q) \upharpoonright M = a + (q \upharpoonright M)$ . In particular,  $\hat{q} \upharpoonright M \leq \widehat{q \upharpoonright M}$ .*

**PROOF.** If  $b \in M$ , then  $b \in (a + q)^L \iff b - a \in q^L \iff b - a \in (q \upharpoonright M)^L \iff b \in (a + (q \upharpoonright M))^L$ . This shows  $(a + q) \upharpoonright M = a + (q \upharpoonright M)$ . It follows  $G(q) \cap M \subseteq G(q \upharpoonright M)$ , in other words  $\hat{q} \upharpoonright M \leq \widehat{q \upharpoonright M}$ . +

**Second Warning.** If  $M \subseteq N$  are ordered abelian groups (divisible or not), if  $\alpha \in N \setminus M$  and if  $p$  is a cut of  $M$ , then the expression  $\alpha + p$  is **not** defined here and there is no canonical way to define it in our setup.

If  $M \subseteq N$  are ordered abelian groups and  $G$  is a convex subgroup of  $M$ , then the convex hull  $H$  of  $G$  in  $N$  is again a convex subgroup of  $N$  with  $H \cap M = G$  and  $H^+$  is the least extension of  $G^+$  on  $N$ . If  $M$  is divisible, then the largest extension of  $G^+$  on  $N$  also is an edge of a convex subgroup of  $N$ :

**LEMMA 3.3.** *Let  $M \subseteq N$  be ordered abelian groups such that  $M$  is divisible and let  $G$  be a subgroup of  $M$ . Let  $q$  be the largest extension of the cut  $G^+$  of  $M$  on  $N$ . Then  $q$  is the upper edge of a convex subgroup of  $N$ .*

**PROOF.** Let  $\alpha \in N$ ,  $0 < \alpha < q$  and suppose  $2\alpha > q$ . Then there is some  $a \in M$  with  $G < a \leq 2\alpha$ , since  $q$  is the largest extension of  $G^+$  on  $N$ . Since  $M$  is divisible, there is  $b \in M$  with  $a = 2b$  and since  $G$  is a subgroup of  $M$ , also  $G < b$ . Hence  $G < b \leq \alpha < q$ , which is impossible, since  $q$  extends  $G^+$ .  $\dashv$

**Third Warning:** Lemma 3.3 is not true in general for ordered abelian groups!

If  $K$  is an ordered field and  $p$  is a cut of  $K$  with  $p > 0$ , then  $p$  is a cut of the ordered abelian groups  $(K, +, \leq)$  and  $(K^{>0}, \cdot, \leq)$ . Hence we can associate two invariance groups with  $p$ . The group  $G(p)$  denotes the invariance group of  $p$  with respect to  $(K, +, \leq)$ .

**DEFINITION 3.4.** Let  $p$  be a cut of an ordered field  $K$ . We define the **multiplicative invariance group** of  $p$  as

$$G^*(p) = \{a \in K \mid a \cdot p = p\}.$$

Hence  $G^*(p)$  is the invariance group of  $|p|$  with respect to  $(K^{>0}, \cdot, \leq)$ . Observe that

$$2 \in G^*(p) \iff |p| = \hat{p}.$$

We recall the warning from above: if  $p$  is a cut of an ordered field,  $p > 0$ , then the expression “ $p^2$ ” is not well defined in general! Cuts of ordered fields is a subtle matter. We only need one proposition here, which shows how both invariance groups are linked if  $p$  is not an edge of a convex subgroup of  $(K, +, \leq)$ :

**PROPOSITION 3.5.** *Let  $K$  be an ordered field and let  $p$  be a cut of  $K$  with  $|p| > \hat{p}$ . There is some  $c \in K$  such that*

$$G^*(p) = c \cdot G(p) + 1 (= \{c \cdot a + 1 \mid a \in G(p)\}).$$

**PROOF.** We may assume that  $p > \hat{p}$ . Let  $H := G^*(p)$ . Since  $p > \hat{p}$  we have  $2 \notin H$ .

**CLAIM 1.**  $H - 1$  is a convex subgroup of  $(K, +, \leq)$ .

$H - 1$  is convex, since  $H$  is convex. Hence we only have to show that  $2 \cdot (H - 1) \subseteq H - 1$  and  $H - 1 = -(H - 1)$ . Let  $\varepsilon \in H - 1$ ,  $\varepsilon > 0$ . Then  $0 < 2\varepsilon < (1 + \varepsilon)^2 - 1 \in H - 1$ , hence  $2\varepsilon \in H - 1$ . Since  $2 \notin H$  we have  $\frac{\varepsilon^2}{1 - \varepsilon} < \varepsilon$ , thus  $1 < \frac{1}{1 - \varepsilon} = 1 + \varepsilon + \frac{\varepsilon^2}{1 - \varepsilon} < 1 + 2\varepsilon \in H$ . We get  $\frac{1}{1 - \varepsilon} \in H$ , therefore  $-\varepsilon \in H - 1$ .

If  $\varepsilon > 0$  with  $-\varepsilon \in H - 1$ , then  $1 < 1 + \varepsilon < \frac{1}{1 - \varepsilon} \in H$ , that is  $\varepsilon \in H - 1$ .

**CLAIM 2.**  $H - 1 = \{a \in K \mid |a| \cdot p < \hat{p}\} = \{a \in K \mid |a| \cdot p \leq \hat{p}\}$ .

The second equality holds since  $p > \hat{p}$ . To see the first equality we may assume that  $a > 0$ . If  $a \cdot p < \hat{p}$ , then easily  $(1 + a) \cdot p = p$ . Conversely take  $h \in H$  and assume  $(h - 1) \cdot p > \hat{p}$ .

First suppose  $h > 1$ . Then there is some  $0 < h_1 < p$  with  $(h - 1)h_1 \notin G(p)$ , hence there is some  $h_2 \in K$ ,  $0 < h_1 \leq h_2 < p$  with  $h_2 + (h - 1)h_1 > p$ . It follows  $p = hp > hh_2 = h_2 + (h - 1)h_2 \geq h_2 + (h - 1)h_1 > p$ , a contradiction.

This argument shows that  $h > 1$  and  $h \cdot p = p$  imply  $(h - 1) \cdot p \leq \hat{p}$ , thus  $(h - 1) \cdot p < \hat{p}$ . On the other hand, if  $0 < h < 1$  and  $h \cdot p = p$  then by Claim 1 we

have  $1 - h = -(h - 1) \in H - 1$ , whence  $(2 - h) \cdot p = p$  and  $2 - h > 1$ . By what we have just proved it follows  $(1 - h) \cdot p < \hat{p}$ .

Now we prove the Proposition. Let  $q := H^+ - 1$ . By Claim 1 it is enough to find some  $c \in K$  with  $q = c \cdot \hat{p}$ . By elementary real algebra, there is an ordered field  $L$  containing  $K$  and realizations  $\alpha, \gamma$  of  $\hat{p}$  and  $p$  respectively. By Claim 2 we know that  $\beta := \frac{\alpha}{\gamma}$  realizes  $q$ . Let  $G'$  be the convex hull of  $G(p)$  in  $K(\alpha, \gamma)$  and let  $\alpha'$  be a realization of  $G'^+$  from an ordered field extension of  $L$ . Note that  $\alpha' \leq \alpha$ . Certainly  $\frac{\alpha'}{\beta}$  is a realization of  $\frac{1}{\beta} \cdot G'^+$ , hence of  $U^+$ , where  $U := K \cap \frac{1}{\beta} \cdot G'$ . Since  $U$  is a convex subgroup of  $(K, +, \leq)$  and  $p > \hat{p}$ , the element  $\gamma = \frac{\alpha}{\beta}$  is not a realization of  $U^+$ , hence  $\frac{\alpha'}{\beta} \leq a \leq \frac{\alpha}{\beta}$  for some  $a \in K$ . As  $\alpha$  and  $\alpha'$  realize  $\hat{p}$  it follows that  $a \cdot \beta$  realizes  $\hat{p}$ . Since  $\beta$  realizes  $q$  this means  $q = \frac{1}{a} \cdot \hat{p}$ .  $\dashv$

Because of this proposition, the multiplicative invariance group of a cut of an ordered field which is not an edge of a subgroup, pops up rarely in the present paper. On the other hand if  $G$  is a (convex) subgroup of  $(K, +, \leq)$ , then the multiplicative invariance group of  $G^+$  is a significant invariant of  $G$  (cf. 5.1). This will be analyzed later in detail (cf. Section 5).

**From now on we work with an o-minimal extension  $T$  of divisible, ordered, abelian groups in the language  $\mathcal{L}$ .** Recall again, that any definably closed subset of a model of  $T$  different from  $\{0\}$  is an elementary substructure of  $M$ .

Let  $M$  be a model of  $T$ , let  $p$  be a cut of  $M$  and let  $f : M \rightarrow M$  be  $M$ -definable such that  $f$  is not constant in any neighborhood of  $p$ . Then all the operations  $a + p, |p|, \dots$  defined above, coincide with the application of the corresponding functions  $x \mapsto a + x, x \mapsto |x|, \dots$  to the 1-type  $p$ . This gives us a strong tool to analyze the movements of cuts in o-minimal structures.

First we have another consequence of the fixpoint criterion 2.9.

**COROLLARY 3.6.** *If  $p$  is a nonprincipal cut of  $M$ , then the following are equivalent:*

- (i)  $p$  has a principal heir.
- (ii) All cuts extending  $p$  on every elementary extension of  $M$  are heirs of  $p$ .
- (iii) If  $f : M \rightarrow M$  is  $M$ -definable and  $f(p) = p$ , then  $f(x) = x$  in some open interval containing  $p$ .
- (iv) If  $\alpha$  is a realization of  $p$ , then  $\alpha$  is the unique realization of  $p$  in  $M\langle\alpha\rangle$ .
- (v) If  $\alpha$  is a realization of  $p$ , then  $M$  is dense in  $M\langle\alpha\rangle$ , i.e., for all  $\beta < \gamma$  from  $M\langle\alpha\rangle$  we have  $(\beta, \gamma) \cap M \neq \emptyset$ .
- (vi)  $G(p) = 0$ .

**PROOF.** (iii) $\Rightarrow$ (ii) follows immediately from 2.9 and (ii) $\Rightarrow$ (i) is obvious.

(i) $\Rightarrow$ (iii). Let  $N \succ M$ ,  $\beta \in N$  and suppose  $\beta^+$  is an heir of  $p$ . Let  $f$  be strictly monotonous on the  $M$ -interval  $I$  and  $p \in I$ . Since  $p$  is not principal we have  $\beta \in I$ , thus  $f(\beta^+) = f(\beta)^+$  and  $\beta \models p$ . From  $f(p) = p$  and 2.9 we get  $\beta^+ = f(\beta^+) = f(\beta)^+$ , hence  $f(\beta) = \beta$  and  $f(x) = x \in p$ .

As (iv) is just a reformulation of (iii) we already know that (i)-(iv) are equivalent. Certainly (v) implies (iv).

(i) $\Rightarrow$ (v) Let  $\gamma < \beta$  be elements of  $M\langle\alpha\rangle$  and suppose  $\gamma \notin M$ . There is  $f : M \rightarrow M$ , definable in  $M$  with  $f(\alpha) = \gamma$ . The cut  $\text{tp}(\gamma/M)$  is not principal and fulfills (i), hence (iv) again. (iv) says, that  $\gamma$  is the unique realization of  $\text{tp}(\gamma/M)$  in  $M\langle\gamma\rangle = M\langle\alpha\rangle$ , so we can find an element  $m \in M$  with  $\gamma < m < \beta$ .

Clearly (iii) implies (vi).

(vi) $\Rightarrow$ (iii). Suppose  $f(p) = p$  and  $f(x) = x \notin p$ , say  $f(x) > x \in p$ . Let  $I$  be an interval with  $p \in I$  such that  $f$  is strictly increasing on  $I$ . There is a closed interval  $I' \subseteq I$  with  $p \in I'$  such that  $f(x) - x$  is monotonous or constant. In any case there is some  $a \in M$  such that  $f(x) - x > a > 0$  for all  $x \in I'$ . Certainly we have  $a + p = p$  in this case.  $\dashv$

Examples of cuts (in the theory of pure real closed fields) with the properties of 3.6 are all cuts of the field of real algebraic numbers that are realized by transcendental numbers from  $\mathbb{R}$ .

**DEFINITION 3.7.** A cut  $p$  is called **dense** if it is nonprincipal and if the equivalent conditions of 3.6 hold for  $p$ .

Let  $\mathcal{D}$  be a new unary predicate and let  $T^{\text{dense}}$  be the  $\mathcal{L}(\mathcal{D})$ -theory  $T$  together with the axiom

$\mathcal{D}$  is the set of elements on the left side of a dense cut

By 3.6(vi) this can be expressed by a single  $\forall\exists$ -sentence.

The following theorem has been proved for a special case in [MMS], Section 2.3. The model theory of dense pairs of o-minimal structures is also well understood (cf. [vdD2]). For example the theory of pairs  $(N, M)$ , where  $M \prec N$  is proper and dense in  $N$ , is a complete theory.

**THEOREM 3.8.** *Let  $T$  be an o-minimal theory containing the theory of divisible ordered abelian groups. Let  $A \prec M, N$  be models of  $T$  and let  $p, q$  be dense cuts of  $M, N$  respectively. Then  $(M, p^L) \equiv_A (N, q^L)$  if and only if  $p \upharpoonright A = q \upharpoonright A$ . Hence if  $T$  has quantifier elimination and a universal system of axioms, then  $T^{\text{dense}}$  has quantifier elimination.*

**PROOF.** Since  $T$  expands the theory DOAG,  $T$  has quantifier elimination and a universal system of axioms if we add function symbols for all 0-definable functions. Therefore, by adding names for these functions, we may assume that  $T$  has quantifier elimination and a universal system of axioms. By the Blum criterion for quantifier elimination and our assumptions on  $T$  it is enough to prove the following. Let  $(M, D)$  and  $(N, E)$  be models of  $T^{\text{dense}}$  and let  $(A, U)$  be a common  $\mathcal{L}(\mathcal{D})$ -substructure such that  $A \prec M, N$  and  $(N, E)$  is  $|A|^+$ -saturated. Let  $\alpha \in M$ . Then there is some  $\beta \in N$  such that the  $\mathcal{L}(\mathcal{D})$ -structures  $(A\langle\alpha\rangle, D \cap A\langle\alpha\rangle)$  and  $(A\langle\beta\rangle, E \cap A\langle\beta\rangle)$  are isomorphic over  $A$ . Since  $(N, E)$  is  $|A|^+$ -saturated it is enough to prove for  $A$ -definable maps  $f_1, \dots, f_k : A \rightarrow A$  the following:

There is some  $\beta \in N$  such that  $p := \text{tp}(\alpha/A) = \text{tp}(\beta/A)$  and such that

$$f_i(\alpha) \in D \iff f_i(\beta) \in E \text{ for each } i.$$

If  $f_i(\alpha) \leq a \in U$ , then  $f_i(x) \leq a \in p$  and we can cross out  $f_i$  from the list. Similarly if  $f_i(\alpha) \geq a \in A \setminus U$ . Furthermore we can suppose that  $\alpha$  realizes the cut  $U^+$  of  $A$ , that is  $p = U^+$  and  $f_i(p) = p$  for all  $i$ . In a small neighborhood of  $p$  the maps  $f_i$  are comparable and this is witnessed in the type  $p$ . Thus it remains to show:

If  $\alpha \models p = U^+$  and  $f, g : A \rightarrow A$  are  $A$ -definable maps such that  $f(p) = g(p) = p$ ,  $f(\alpha) \in D$  and  $g(\alpha) \notin D$ , then there is some  $\beta \in N$  such that  $p = \text{tp}(\beta/A)$ ,  $f(\beta) \in E$  and  $g(\beta) \notin E$ .

Because  $E^+$  is a dense cut,  $g \circ f^{-1}(x) > x \in p$  and  $g \circ f^{-1}(p) = p$ , 3.6(iii) gives some  $\gamma \in N$  such that  $\gamma \models p$ ,  $\gamma \in E$  and  $g \circ f^{-1}(\gamma) \notin E$ . We choose  $\beta = f^{-1}(\gamma) \models p$ . Then  $f(\beta) = \gamma \in E$  and  $g(\beta) = g \circ f^{-1}(\gamma) \notin E$ . Since  $\beta$  realizes  $p$ , the theorem is proved.  $\dashv$

The theorem above frequently enables us to exclude dense cuts from examinations. In particular, this is necessary if we want to apply facts about nonprincipal cuts to  $\hat{p}$ . Note that  $\hat{p}$  is the upper edge of a convex subgroup of  $M$ , so by 3.6,  $\hat{p}$  is principal if and only if  $p$  is dense or principal.

Next we introduce another basic invariant of cuts of divisible ordered abelian groups, namely the signature of a cut. We need a lemma to define it.

**LEMMA 3.9.** *Let  $M$  be a divisible ordered abelian group, let  $G, H$  be convex subgroups of  $M$  and let  $a, b \in M$ . Then  $a + G^+ \neq b + H^-$ .*

**PROOF.** Note that  $-H^+ = H^-$ . For the proof we may assume that  $a = 0$ . Suppose  $b + H^- = G^+$ . We write  $p$  for this cut. As  $M$  is divisible, the pair  $(2 \cdot p^L, 2 \cdot p^R)$  is again a cut, denoted by  $2 \cdot p$ . Since  $p = G^+$  and  $G$  is a group we have  $2 \cdot p = G^+ = b + H^-$ . Since  $p = b + H^-$  we have  $2 \cdot p = 2b + H^-$ . Hence  $2b + H^- = b + H^-$ , which implies  $b \in H$ . But this contradicts  $b + H^- = G^+$ .  $\dashv$

**DEFINITION 3.10.** Let  $M$  be a divisible ordered abelian group and let  $p$  be a cut of  $M$ . By 3.9 we may define the **signature** of  $p$  as the following integer:

$$\text{sign } p := \begin{cases} 1 & \text{if } p = a + G^+ \text{ for some } a \in M \text{ and some convex subgroup } G \text{ of } M, \\ -1 & \text{if } p = a - G^+ \text{ for some } a \in M \text{ and some convex subgroup } G \text{ of } M, \\ 0 & \text{otherwise.} \end{cases}$$

If  $M$  is a real closed field, we define the **multiplicative signature**  $\text{sign}^* p$  of  $p$  as the signature of  $|p|$  with respect to  $(M^{>0}, \cdot, \leq)$  (e.g.,  $\text{sign}^* p = 1$  means that  $p = a \cdot G^*(p)^+$  for some  $a \in M^*$ ).

Observe again that there is at most one convex subgroup  $G$  of  $M$  such that  $p = a + G^+$  or  $p = a - G^+$  for some  $a \in M$ , namely  $G = G(p)$ .

**EXAMPLES 3.11.**

A. The principal cuts  $a^+$  and  $+\infty$  have signature 1 and the principal cuts  $a^-$  and  $-\infty$  have signature  $-1$ . Moreover, dense cuts (cf. 3.7) have signature 0, since they are nonprincipal and by 3.6 they have invariance group  $\{0\}$ .

Moreover 3.6 implies that a cut  $p$  of a divisible, ordered abelian group  $M$  has invariance group  $\{0\}$  if and only if  $p$  is dense or of the form  $a^\pm$  for some  $a \in M$ .

B. Let  $p$  be a cut of a divisible, ordered abelian group  $M$ ,  $p \neq \pm\infty_M$ . Let  $\bar{M}$  be the group  $M/G(p)$ . Since  $G(p)$  is convex,  $\bar{M}$  is again a divisible ordered abelian group, so that the residue map  $\varphi : M \rightarrow \bar{M}$  is order preserving. Then, as  $G(p)$  is the invariance group of  $p$ , we have  $\varphi(g) \neq \varphi(h)$  for all  $g, h \in M$  with  $g < p < h$ . Therefore,  $\bar{p} := (\varphi(p^L), \varphi(p^R))$  is again a cut of  $\bar{M}$ . It turns out that  $\bar{p}$  has invariance group  $\{0\}$  and

- (i) for every  $a \in M$ ,  $p = a \pm \hat{p}$  if and only if  $\bar{p} = \bar{a}_M^\pm$ .
- (ii)  $\text{sign } p = 0$  if and only if  $\bar{p}$  is dense.

Conversely if  $q$  is a cut of  $\bar{M}$ , then certainly  $r := (\varphi^{-1}(q^L), \varphi^{-1}(q^R))$  is a cut of  $M$  with  $(\varphi(r^L), \varphi(r^R)) = q$  and in the case  $G(q) = \{0\}$  we get  $G(r) = G(p)$ .

C. In this example we exhibit certain cuts realized in fields of generalized power series. The definition and basic properties of these fields can be found in chapter 2 of the book [Da-Wo]. Let  $R$  be a real closed field, let  $N := R((t^{\mathbb{Q}}))$  be the field of generalized power series with coefficients in  $R$  and exponents in  $\mathbb{Q}$ ; hence the elements of  $N$  are maps  $b : \mathbb{Q} \rightarrow R$  with well-ordered support  $\text{supp}(b)$ . Recall that  $N$  is a real closed field. Let  $M$  be the real closure of  $R(t)$  in  $N$ , where  $t$  stands for  $t^1$  as usual. Every element  $a \in M$  is a Puiseux-series  $a = \sum_{n \geq n_0} a_n t^{\frac{n}{k}}$  for some  $n_0 \in \mathbb{Z}, k \in \mathbb{N}$ .

CLAIM 1. If  $G$  is a convex subgroup of  $(M, +, \leq)$ , then  $G^+$  is not realized in  $N$ .

PROOF. If  $c \in N, c > 0$  is a power series,  $c_0 \in R, c_0 > 0, \gamma \in \mathbb{Q}$  and  $c = c_0 \cdot t^\gamma +$  terms of higher order, then  $c < \frac{3}{2}c_0 t^\gamma < 2c$  and  $c_0 t^\gamma \in M$ . Hence  $c$  and  $2c$  do not realize the same cut of  $M$ . Since  $G$  is a convex subgroup of  $M$ , this shows that  $c$  can not realize  $G^+$ .  $\dashv$

Claim 1 implies that every cut of  $M$  which is realized in  $N$  has signature 0. We compute the invariance groups of these cuts now. First let  $b \in N \setminus M, b > 0$  so that the order type of  $\text{supp}(b)$  is the order type  $\omega$  of the natural numbers. Let  $p$  be the cut of  $M$  realized by  $b$ .

CLAIM 2. If  $c \in N$ , then  $c$  realizes  $p$  (in particular  $c \notin M$ ) if and only if  $b$  is a truncation of  $c$ , i.e.,  $\text{supp}(c - b) > \text{supp } b$  (hence, if the support of  $b$  is unbounded in  $\mathbb{Q}$ , then  $c = b$ ).

PROOF. We may assume that  $b < c$ . The elements in  $M$  have finite support or unbounded support of order type  $\omega$ . Therefore, a straightforward computation shows that there is no  $a \in M$  with  $b \leq a \leq c$  if and only if  $b$  is a truncation of  $c$  (note that  $b \notin M$  by assumption).  $\dashv$

Claim 2 implies that every cut of  $M$  which is realized in  $N$  can also be realized by an element of  $N$ , whose support has order type  $\omega$ . Moreover with our element  $b$  from above and the cut  $p$  of  $M$  realized by  $b$ , the claim implies  $G(p) = \{g \in M \mid \text{supp}(b) < \text{supp}(g)\}$ .

Finally we compute the multiplicative signature of  $\hat{p} = G(p)^+$  and the multiplicative invariance group  $G^*(\hat{p})$ . Let  $\xi$  be the upper edge of  $\text{supp}(b)$  in  $\mathbb{Q}$ . So  $\xi$  is a cut of  $\mathbb{Q}$  and since  $\text{supp}(b)$  has order type  $\omega$ , the cut  $\xi$  can not be of the form  $-\infty$  or  $\gamma^+$  for some  $\gamma \in \mathbb{Q}$ .

Let  $v : M^{>0} \rightarrow \mathbb{Q}$  be the map  $v(a) = \min \text{supp}(a)$ . The kernel of  $v$  is the group  $V^{*>0}$  of positive units of the valuation ring  $V := \{a \in M \mid \text{supp}(a) \geq 0\}$  of  $M$ . Moreover  $G(p)^{>0} = \{g \in M^{>0} \mid \xi < v(g)\}$ , hence the cut  $\hat{p}$  of  $M^{>0}$  is

$$\hat{p} = (v^{-1}(\xi^R), v^{-1}(\xi^L)). \quad (*)$$

CASE 1.  $\xi = +\infty$ . Then  $G(p) = \{0\}$ , hence  $G^*(\hat{p}) = M^{>0}$ .

Now let  $\xi \neq +\infty$ . Then the invariance group of  $\xi$  is a convex subgroup of  $(\mathbb{Q}, +)$  different from  $\mathbb{Q}$ , thus it is  $\{0\}$ . Therefore an element  $a \in M$  satisfies  $a \cdot \hat{p} = \hat{p}$  if and only if  $a > 0$  and  $v(a) = 0$ . In other words  $G^*(\hat{p}) = V^{*>0}$ .

CASE 2.  $\xi = \gamma^-$  for some  $\gamma \in \mathbb{Q}$ . Then  $G(p) = \{g \in M \mid \gamma \leq \text{supp}(g)\}$ , thus  $G(p) = t^\gamma \cdot V$ . Since  $V^+$  is the upper edge of the multiplicative group  $V^{*>0}$  we get  $\text{sign}^* \hat{p} = 1$ .

CASE 3.  $\zeta$  is not principal. Then  $\text{sign}^* \hat{p} = 0$ . This follows from  $G^*(\hat{p}) = V^{*>0}$ ; if  $\text{sign}^* \hat{p} = 1$ , then  $\hat{p} = a \cdot G^*(\hat{p})^+ = a \cdot V^+$  for some  $a \in M^{>0}$ , which implies  $\zeta = v(a)^-$ ; if  $\text{sign}^* \hat{p} = -1$ , then  $\hat{p} = a \cdot G^*(\hat{p})^- = a \cdot (V^{*>0})^- = a \cdot \mathfrak{m}^+$  for some  $a \in M$  (where  $\mathfrak{m}$  is the maximal ideal of  $V$ ), which implies  $\zeta = v(a)^+$ . Both conclusions contradict the assumption  $\text{sign} \zeta = 0$ .

Note that in our determination of  $G^*(\hat{p})$  and  $\text{sign}^* \hat{p}$  we used the same consideration as in example B above. Indeed, if we equip  $\mathbb{Q}$  with the reverse order, then  $v$  is an homomorphism of ordered groups. Moreover (\*) says that  $\hat{p}$  is the “preimage” of  $\zeta$  under  $v$ , so the cases 2 and 3 correspond to items (i) and (ii) of example B. This finishes example C.

More examples can be found in Section 6 below, where we introduce a method to produce cuts with prescribed signature.

Let again  $T$  denote an o-minimal expansion of the theory of divisible, ordered abelian groups.

**PROPOSITION 3.12.** *Let  $p$  be a cut of  $M \models T$ . Then*

- (i)  *$\text{sign} p \geq 0$  if and only if for all  $N \succ M$ , if  $q$  is the largest extension of  $p$  on  $N$ , then the cut  $\hat{q}$  is the largest extension of  $\hat{p}$  on  $N$ .*
- (ii)  *$\text{sign} p \leq 0$  if and only if for all  $N \succ M$ , if  $q$  is the least extension of  $p$  on  $N$ , then the cut  $\hat{q}$  is the largest extension of  $\hat{p}$  on  $N$ .*
- (iii) *If  $q$  is the least or the largest extension of  $p$  on some  $N \succ M$ , then  $\hat{q}$  is the least or the largest extension of  $\hat{p}$  on  $N$ , respectively.*
- (iv) *If  $\text{sign} p = 0$  and  $\alpha$  is a realization of  $p$  in some  $N \succ M$ , then the map  $2\alpha - x$  swaps the least extension of  $p$  on  $N$  and the largest extension of  $p$  on  $N$ .*
- (v) *If  $p$  is omitted in  $N \succ M$  then  $\text{sign} q = \text{sign} p$  for the unique extension  $q$  of  $p$  on  $N$ .*

**PROOF.** Obviously we may assume that  $T$  is the theory of divisible, ordered abelian groups.

(i). If  $\text{sign} p = -1$ ,  $p = a - \hat{p}$  with  $a \in M$ , then the largest extension  $q$  of  $p$  on  $M\langle\alpha\rangle$ ,  $\alpha \models p$ , is  $a - r$ , where  $r$  is the least extension of  $\hat{p}$  on  $M\langle\alpha\rangle$ . Therefore  $\hat{q} = r$  is not the largest extension of  $\hat{p}$  on  $M\langle\alpha\rangle$ . Conversely let  $q$  be the largest extension of  $p$  on some  $N \succ M$  and suppose  $\hat{q}$  is not the largest extension  $r$  of  $\hat{p}$  on  $N$ . Pick  $\beta \in N$  with  $\hat{q} < \beta < r$ . Since  $\hat{q} < \beta$  we have  $\beta + q > q$  and since  $q$  is the largest extension of  $p$  on  $N$  there is some  $a \in M$  with  $p < a < \beta + q$ , i.e.,  $a - \beta < q$ . Since  $\beta < r$  we have  $a - r < a - \beta$ , so  $a - r < q$  and  $(a - r) \upharpoonright M \leq q \upharpoonright M = p$ . Since  $(a - r) \upharpoonright M = a - \hat{p} \geq p$  we get  $p \leq a - \hat{p} = (a - r) \upharpoonright M \leq q \upharpoonright M = p$ , i.e.,  $p = a - \hat{p}$  has signature  $-1$ .

(ii) is (i) applied to  $-p$ .

(iii). Say  $q$  is the largest extension of  $p$ . If  $\text{sign} p \geq 0$ , then (iii) follows from (i). If  $p = a - \hat{p}$ , then  $a - q$  is the least extension of  $\hat{p}$  on  $N$ , hence this extension is  $\hat{q}$ .

(iv). Let  $q$  denote the largest extension of  $p$  on  $N$ . We claim that  $\alpha + \hat{q} = q$ . Since  $\alpha \models p$  we have  $\alpha < q$ , hence  $\alpha + \hat{q} \leq q$ . Because of  $\text{sign} p = 0$  and (i), the cut  $\hat{q}$  is the largest extension of  $\hat{p}$  on  $N$ . So if  $\beta \in N$  with  $\alpha + \hat{q} < \beta < q$ , then there is some  $a \in M$  with  $\hat{p} < a \leq \beta - \alpha$ . But then  $a + p > p$  and there is some  $b \in M$  with  $a + p > b > p$ . Since  $\alpha$  and  $\beta$  are realizations of  $p$  it follows  $a + \alpha > b > \beta$ , in contradiction to  $a \leq \beta - \alpha$ .

This shows  $\alpha + \hat{q} = q$ . The same argument applied to  $-p$  and the largest extension  $r$  of  $-p$  (note that  $\text{sign } -p = -\text{sign } p = 0$ ) shows  $r = -\alpha + \hat{r}$ . Since  $-r$  is the least extension of  $p$  and  $\text{sign } p = 0$  we can apply (i) and (ii), which gives  $\hat{r} = \hat{q}$ . Thus  $-r = \alpha - \hat{q}$  is the least extension of  $p$  on  $N$ . Certainly the map  $x \mapsto 2\alpha - x$  swaps  $\alpha + \hat{q}$  and  $\alpha - \hat{q}$ .

(v). First suppose  $\text{sign } p \neq 0$ , say  $p = a + \hat{p}$ . Let  $H$  be the convex hull of  $G(p)$  in  $N$  and let  $r := a + H^+$ . As  $a \in M$ ,  $r$  extends  $p$ , hence  $q = r$  has signature  $1 = \text{sign } p$ .

Conversely suppose  $\text{sign } q \neq 0$ , say  $q = \alpha + \hat{q}$ ,  $\alpha \in N$ . Since  $p$  is omitted in  $N$ , there is some  $a \in M$  with  $\alpha < a < p$ . Then  $a < q$ , hence  $q = a + \hat{q}$ , which implies  $p = a + \hat{p}$ .  $\dashv$

DEFINITION 3.13. Let  $p$  and  $q$  be cuts of a model  $M$  of  $T$ . We define

$$p \sim q \text{ :} \Leftrightarrow \text{there is some } M\text{-definable map } f : M \longrightarrow M \text{ with } f(p) = q.$$

It follows easily from the monotonicity theorem that  $p \sim q$  implies  $q \sim p$  (see also [Ma], Lemma (3.1)). So  $\sim$  is an equivalence relation among the cuts of  $M$ .

LEMMA 3.14. Let  $p$  be a cut and let  $q$  be a cut extending  $p$  on  $N \succ M$ . Then

- (i) If  $f : M \longrightarrow M$  is  $M$ -definable and  $q$  is an heir of  $p$ , then  $f(q)$  is an heir of  $f(p)$ .
- (ii) If  $p = a \pm \hat{p}$  for some  $a \in M$  then  $q$  is an heir of  $p$  if and only if  $q = a \pm \hat{q}$  and  $\hat{q}$  is an heir of  $\hat{p}$ .
- (iii) If  $p \not\sim \hat{p}$ , then  $q$  is an heir of  $p$  if and only if  $\hat{q}$  extends  $\hat{p}$ .

PROOF. (i) follows easily from 2.8. (ii). Say  $p = a + \hat{p}$ . If  $\hat{q}$  is an heir of  $\hat{p}$  and  $q - a = \hat{q}$ , then clearly  $q = a + \hat{q}$  is an heir of  $p = a + \hat{p}$ . Conversely, if  $q$  is an heir of  $p$ , then  $q - a$  is an heir of  $p - a = \hat{p}$ . Since  $\hat{p}$  is the upper edge of a convex subgroup it follows that  $q - a$  is the upper edge of a convex subgroup as well. Hence  $q - a = \hat{q}$ .

In order to prove (iii) it suffices to show that  $q$  is an heir of  $p$  if  $\hat{p} \not\sim p$  and  $\hat{p} \subseteq \hat{q}$ . As  $\text{sign } p = 0$  the cut  $p$  is not principal and we may use 2.9. So let  $f : M \longrightarrow M$  be  $M$ -definable with  $f(p) = p$ . We have to show that  $f(q) = q$ . From  $f(p) = p$  and 2.5 we know that  $f$  is strictly increasing on some open interval with endpoints in  $M$ , containing  $p$ . Therefore we may assume that  $f : M \longrightarrow M$  is bijective. By replacing  $f$  with  $f^{-1}$  if necessary we may furthermore assume that  $f(x) > x$  in some open  $M$ -definable interval  $I$  containing  $p$ . Let  $\alpha$  be a realization of  $q$ . Since  $f(\alpha)$  and  $\alpha$  realize  $p$  it follows from the definition of  $\hat{p}$  that  $\text{tp}(f(\alpha) - \alpha/M) \leq \hat{p}$ . Since  $p \not\sim \hat{p}$  the cut  $\hat{p}$  is not realized in  $M\langle\alpha\rangle$ , hence there is some  $a \in M$  with  $0 < f(\alpha) - \alpha < a < \hat{p}$ . By assumption  $a < \hat{q} = G(q)^+$ , thus  $a + q = q$ . From  $\alpha < f(\alpha) < a + \alpha$  we get  $f(q) = q$ .  $\dashv$

Hence in the cases  $\text{sign } p \neq 0$  or  $p \not\sim \hat{p}$  the question if  $q$  is an heir of  $p$  can be reduced to the question if  $\hat{q}$  is an heir of  $\hat{p}$  or if  $\hat{q}$  extends  $\hat{p}$ . A reformulation of the alternative “ $\text{sign } p \neq 0$  or  $p \not\sim \hat{p}$ ” is 3.15(i).

REMARK 3.15. Let  $p$  be a cut of  $M$ , let  $f : M \longrightarrow M$  be  $M$ -definable with  $f(p) = \hat{p}$  and let  $N \succ M$ . Then

- (i) If  $p$  is realized in  $N$ , then the following are equivalent:

- (a)  $\text{sign } p \neq 0$ .
- (b) If  $q$  is an heir of  $p$  on  $N$ , then  $f(q) = \hat{q}$ .
- (ii) If  $p$  is omitted in  $N$ , then  $f(q) = \hat{q}$  for the unique extension  $q$  of  $p$  on  $N$ .

PROOF. (i) (a) $\Rightarrow$ (b). Say  $p = a + \hat{p}$ . By 3.14(ii) we have  $q = a + \hat{q}$  and the cut  $\hat{q}$  is an heir of  $\hat{p}$ . Since the map  $f_1(x) := f(a + x)$  fixes the cut  $\hat{p}$  we know  $f_1(\hat{q}) = \hat{q}$ , i.e.  $f(q) = f_1(\hat{q}) = \hat{q}$ .

(i) (b) $\Rightarrow$ (a). Suppose  $\text{sign } p = 0$ . By 3.12(i),(ii) the least and the largest extension  $q_1$  and  $q_2$  of  $p$  on  $N$  have the same invariance group. On the other hand we have  $f(q_1) \neq f(q_2)$ , since  $f$  is strictly monotonous in an  $M$ -definable neighborhood of  $p$ . By (b) the cut  $f(q_i)$  is the upper edge of the invariance group of  $q_i$ , a contradiction.

(ii) holds since  $q$  is an heir of  $p$ , so  $\hat{q}$  is the unique extension of  $\hat{p}$  on  $N$ .  $\dashv$

DEFINITION 3.16. If  $p$  is a cut of  $M$  with  $\text{sign } p = 0$  and  $p$  is omitted in  $N \succ M$  then by 3.12(v), the unique extension  $q$  of  $p$  on  $N$  has again signature 0.

We say that  $p$  has the **signature alternative** if  $\text{sign } p \neq 0$  or if  $p \not\prec \hat{p}$  and  $q \not\prec \hat{q}$  where  $q$  denotes the unique extension of  $p$  on  $M\langle\alpha\rangle$  ( $\alpha \models \hat{p}$ ).

LEMMA 3.17. Let  $M \prec N$  be models of  $T$ , let  $p_1, \dots, p_n$  be cuts of  $M$  and let  $(q_1, \dots, q_n)$  be an heir of  $(p_1, \dots, p_n)$  on  $N$ . Let  $\varphi(\bar{x}, \bar{y}, \bar{z})$  be an  $\mathcal{L}(\mathcal{D}_1, \dots, \mathcal{D}_n)$ -formula, where  $\bar{x}, \bar{y}$  and  $\bar{z}$  are  $n$ -tuples of variables. If there are  $\bar{\alpha}, \bar{\beta}, \bar{\gamma} \in N^n$  with  $(N, q_1^L, \dots, q_n^L) \models \varphi(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ ,  $\hat{q}_i < \alpha_i$  and  $\beta_i < q_i < \gamma_i$  ( $1 \leq i \leq n$ ), then there are  $\bar{a}, \bar{b}, \bar{c} \in M^n$  with  $(M, p_1^L, \dots, p_n^L) \models \varphi(\bar{a}, \bar{b}, \bar{c})$ ,  $\hat{p}_i < a_i$  and  $b_i < p_i < c_i$  ( $1 \leq i \leq n$ ).

PROOF. Choose  $\delta_1, \dots, \delta_n \in N$  with  $\delta_i < q_i < \delta_i + \alpha_i$ . Since  $(q_1, \dots, q_n)$  is an heir of  $(p_1, \dots, p_n)$ , by 2.8 we get  $\bar{b}, \bar{c} \in M^n$  such that

$$(M, p_1^L, \dots, p_n^L) \models \exists \bar{u}, \bar{v} \varphi(\bar{u}, \bar{b}, \bar{c}) \wedge \bigwedge v_i \in \mathcal{D}_i < u_i + v_i \text{ and} \\ b_i < p_i < c_i \text{ (} 1 \leq i \leq n \text{)}.$$

This proves the Lemma.  $\dashv$

LEMMA 3.18. Let  $M \prec N$  be models of  $T$ , let  $p$  be a nonprincipal cut of  $M$  and let  $q$  be a cut of  $N$  extending  $p$ . Assume  $\hat{q}$  is an heir of  $\hat{p}$  and  $\alpha \in N$  is a realization of  $\hat{p}$  with  $\alpha < \hat{q}$ . Then  $G(q) \cap M\langle\alpha\rangle = G(q \upharpoonright M\langle\alpha\rangle)$ .

PROOF. By 3.2, we only have to show  $\supseteq$ . We may assume that  $p > 0$  and that  $\hat{p}$  is not principal. Let  $f(\alpha) \in M\langle\alpha\rangle$ ,  $f(\alpha) > 0$ , be such that  $f(\alpha) + q \upharpoonright M\langle\alpha\rangle = q \upharpoonright M\langle\alpha\rangle$ . If  $\text{tp}(f(\alpha)/M) > \hat{p}$  then there is some  $m \in M$  with  $f(\alpha) \geq m \notin G(p)$  and we get  $f(\alpha) + q \upharpoonright M\langle\alpha\rangle > q \upharpoonright M\langle\alpha\rangle$ . So by the choice of  $f(\alpha)$  we have  $\text{tp}(f(\alpha)/M) \leq \hat{p}$ .

If  $\text{tp}(f(\alpha)/M) < \hat{p}$  there is some  $m \in M$  with  $f(\alpha) \leq m \in G(p) \subseteq G(q)$ , hence  $f(\alpha) + q = q$ . So we may suppose that  $f(\alpha)$  realizes  $\hat{p}$ . Then  $f(\hat{p}) = \hat{p}$ , so  $f$  is strictly increasing in some open  $M$ -definable interval  $I$  containing  $\hat{p}$  (2.5). Since  $\hat{q}$  is an heir of  $\hat{p}$  we have  $f(\hat{q}) = \hat{q}$ . As  $\alpha < \hat{q}$  and  $f$  is strictly increasing on  $I$  we have  $f(\alpha) < \hat{q}$  too, i.e.,  $f(\alpha) + q = q$ .  $\dashv$

PROPOSITION 3.19. Let  $M \prec N$  be models of  $T$ , let  $p$  be a cut of  $M$  with  $\text{sign } p = 0$  and let  $q$  be a cut of  $N$  extending  $p$  such that  $\hat{q}$  is an heir of  $\hat{p}$ . Suppose  $p$  has the signature alternative. Then  $(q, \hat{q})$  is an heir of  $(p, \hat{p})$ .

PROOF. Because  $p \not\prec \hat{p}$ ,  $q$  is an heir of  $p$  by 3.14(iii). By 2.8 we have to show: if  $\alpha_1, \alpha_2, \beta_1$  and  $\beta_2$  are elements of  $N$  and  $\varphi(v_1, v_2, v_3, v_4)$  is an  $\mathcal{L}$ -formula with parameters from  $M$ , such that  $\alpha_1 < \hat{q} < \alpha_2$ ,  $\beta_1 < q < \beta_2$  and  $N \models \varphi(\alpha_1, \alpha_2, \beta_1, \beta_2)$  then there are  $a_1, a_2, b_1, b_2 \in M$  such that  $a_1 < \hat{p} < a_2$ ,  $b_1 < p < b_2$  and  $M \models \varphi(a_1, a_2, b_1, b_2)$ .

By 3.17 we may assume that  $\alpha_1$  is a realization of  $\hat{p}$ . Since  $p \not\prec \hat{p}$  the cut  $p$  has a unique extension  $q_1 = q \upharpoonright M\langle\alpha_1\rangle$  on  $M\langle\alpha_1\rangle$ . Since  $p$  has the signature alternative we get  $q_1 \not\prec \hat{q}_1$ . By 3.18 we have  $G(q) \cap M\langle\alpha_1\rangle = G(q \upharpoonright M\langle\alpha_1\rangle)$ . Hence  $\hat{q}$  extends  $\hat{q}_1$  and  $\hat{q}_1$  extends  $\hat{p}$ . By 3.14(iii)  $q$  is an heir of  $q_1$ . By 3.17 applied to  $q_1$  and  $q$ , there are  $M$ -definable maps  $f, g_1, g_2$  such that

$$\alpha_1 < \hat{q}_1 < f(\alpha_1), \quad g_1(\alpha_1) < q_1 < g_2(\alpha_1) \text{ and} \\ M\langle\alpha_1\rangle \models \varphi(\alpha_1, f(\alpha_1), g_1(\alpha_1), g_2(\alpha_1)).$$

Since  $p$  is omitted in  $M\langle\alpha_1\rangle$  there are  $c, d \in M$  with  $g_1(\alpha_1) < c < p < d < g_2(\alpha_1)$ . Let  $\psi(v_1, v_2)$  be the formula  $\exists v_3 v_4 (v_3 < c \wedge d < v_4 \wedge \varphi(v_1, \dots, v_4))$ . As  $\hat{q}_1$  is an heir of  $\hat{p}$  and  $\alpha_1 < \hat{q}_1 < f(\alpha_1)$ ,  $M\langle\alpha_1\rangle \models \psi(\alpha_1, f(\alpha_1))$  we get some  $a_1 \in M$  with

$$a_1 < \hat{p} < f(a_1) \text{ and } M \models \psi(a_1, f(a_1)).$$

Hence there are  $b_1, b_2$  with  $b_1 < c$ ,  $d < b_2$  and  $M \models \varphi(a_1, f(a_1), b_1, b_2)$ . So with  $a_2 := f(a_1)$  we found the required elements.  $\dashv$

**§4. Review of  $T$ -convex valuation rings.** For the rest of this article we work with a complete, o-minimal extension  $T$  of the theory of real closed fields in the language  $\mathcal{L}$ . Basic valuation theory for models of these theories has been developed in the paper [vdD-Lew]:

DEFINITION 4.1. Let  $M$  be a model of  $T$ . A convex valuation ring  $V$  of  $M$  is called  **$T$ -convex**, if  $V$  is closed under all  $\emptyset$ -definable, continuous maps  $M \rightarrow M$ .

For example, if  $M_0 \prec M$  and if  $V$  is the convex hull of  $M_0$  in  $M$ , then  $V$  is  $T$ -convex.

PROPOSITION 4.2. Let  $V$  be a  $T$ -convex valuation ring of a model  $M$  of  $T$  and let  $M_0 \prec M$  with  $M_0 \subseteq V$ . Then

- (i) If  $f : M^n \rightarrow M$  is a continuous,  $M_0$ -definable map, then  $f(V^n) \subseteq V$ .
- (ii) If  $\alpha \in V$  and  $\alpha > M_0$ , then  $M_0\langle\alpha\rangle \subseteq V$ .

PROOF. (i) is [vdD-Lew], (2.9) and (2.10). (ii) is [vdD-Lew], (2.13).  $\dashv$

THEOREM 4.3. Let  $V \subseteq M \models T$  be  $T$ -convex and  $M_0 \prec M$  with  $M_0 \subseteq V$ . Then  $M_0$  is maximal definably closed in  $V$  if and only if the composed map  $M_0 \rightarrow V \rightarrow \kappa_V := V/\mathfrak{m}(V)$  is onto. In this case we have:

- (i)  $V$  is the convex hull of  $M_0$  in  $M$  and  $M_0$  is **tame** in  $M$ , i.e., each  $M_0$ -bounded element in  $M$  is infinitely close to an element of  $M_0$ .
- (ii) If  $M'_0 \subseteq V$  is maximal definably closed, then there is a unique  $\mathcal{L}$ -isomorphism  $M_0 \rightarrow M'_0$ , which commutes with the residue maps.

PROOF. [vdD-Lew], (2.12) and (2.15).  $\dashv$

Thus  $M_0 \longrightarrow \kappa_V$  induces an  $\mathcal{L}$ -structure on  $\kappa_V$ , which is independent of the choice of  $M_0$ .  $\kappa_V$  is again a model of  $T$ . We always consider  $\kappa_V$  to be this expansion of the field  $\kappa_V$ . 4.3 implies

**COROLLARY 4.4.** *Let  $V \subseteq M \models T$  be  $T$ -convex and  $M \prec N$ . If  $W \subseteq N$  is a  $T$ -convex ring of  $N$  lying over  $V$ , then the canonical morphism  $\kappa_V \longrightarrow \kappa_W$  is an elementary map of models of  $T$ .*  $\dashv$

Let  $\mathcal{L}_{\text{convex}}$  be the language  $\mathcal{L}$  expanded by a new unary predicate  $\mathcal{O}$  for the  $T$ -convex valuation ring and let  $T_{\text{convex}}$  be the  $\mathcal{L}_{\text{convex}}$ -theory of pairs  $(M, V)$ , where  $M \models T$  and  $V$  is a proper  $T$ -convex valuation ring of  $M$ .

**THEOREM 4.5.**  *$T_{\text{convex}}$  is a complete  $\mathcal{L}_{\text{convex}}$ -theory.  $T_{\text{convex}}$  is weakly o-minimal (i.e., each  $M$ -definable subset of a model  $(M, V)$  of  $T_{\text{convex}}$  is a finite union of convex subsets of  $M$ ). If  $T$  has quantifier elimination and a universal system of axioms, then  $T_{\text{convex}}$  has quantifier elimination.*

**PROOF.** By [vdD-Lew], (3.10), (3.13) and (3.14).  $\dashv$

**COROLLARY 4.6.** *Let  $(M, V) \models T_{\text{convex}}$  and let  $A \subseteq M$ . Then*

- (i) *Each subset of  $M^n$ , which is  $A$ -definable in  $(M, V)$  is a Boolean combination of sets of the form  $f_i^{-1}(V)$  or  $f_i^{-1}(0)$  for a finite number of  $A$ -definable maps  $f_1, \dots, f_r : M^n \longrightarrow M$ .*
- (ii) *If  $p$  is a nonprincipal cut of  $M$  such that  $p^L$  is  $A$ -definable in  $(M, V)$  then  $f(V^+) = p$  for some  $A$ -definable homeomorphism  $f : M \longrightarrow M$ .*
- (iii) *If  $V \neq M$ ,  $N \succ M$  and  $q$  is a cut of  $N$  extending  $V^+$ , then  $q$  is an heir of  $V^+$  if and only if  $q$  is the upper edge of a  $T$ -convex valuation ring of  $N$ .*
- (iv) *If  $p$  is a cut of  $M$  with  $p \sim V^+$  and  $r$  is an heir of  $p$  on  $N \succ M$ , then  $r \sim W^+$  for some  $T$ -convex valuation ring of  $W$  of  $N$ , lying over  $V$ .*

**PROOF.** It is enough to prove the Corollary in the case  $A = \emptyset$ . We work in the expanded language  $\mathcal{L}^{df}$  with the theory  $T^{df}$  (cf. [vdD-Lew], (2.3) and (2.4)), where we have a function symbol for each  $\emptyset$ -definable function  $M^n \longrightarrow M$ . Since  $T$  has definable Skolem functions the theory  $T^{df}$  has quantifier elimination and a universal system of axioms. By 4.5 the theory  $(T^{df})_{\text{convex}}$  has quantifier elimination. This already implies Claim (i).

(ii). By (i) the set  $p^L$  is a Boolean combination of sets of the form  $f_i^{-1}(V)$  or  $f_i^{-1}(0)$  for a finite number of  $A$ -definable maps  $f_1, \dots, f_r : M^n \longrightarrow M$ . Since  $T_{\text{convex}}$  is weakly o-minimal and  $p$  is not principal, there are  $a, b \in M$ ,  $a < p < b$  and some  $i \in \{1, \dots, r\}$  such that  $f_i^{-1}(V) \cap (a, b) = p^L \cap (a, b)$  or  $f_i^{-1}(V) \cap (a, b) = p^R \cap (a, b)$ . In any case  $f_i(V^+) = p$ . Since  $f_i$  is continuous inside some  $A$ -definable neighborhood of  $p$ , we can change  $f_i$  to a  $\emptyset$ -definable homeomorphism.

(iii). Since  $(T^{df})_{\text{convex}}$  is model complete we have that  $W^+$  is an heir of  $V^+$  for each  $T$ -convex valuation ring  $W$  of  $N$ , lying over  $V$ . Conversely if  $q$  is an heir of  $p$ , then it is easy to see that  $W := q^L \cap (-q)^R$  is a  $T$ -convex valuation ring of  $N$ .

(iv). If  $f : M \longrightarrow M$  is definable with  $f(p) = V^+$ , then  $f(r)$  is an heir of  $V^+$  and by (iii) applied to  $q = f(p)$  we get (iv).  $\dashv$

**THEOREM 4.7.** *Let  $M \prec N$  be tame, let  $B$  be the convex hull of  $M$  in  $N$ . If  $X \subseteq N^n$  is definable in  $(N, B)$  with parameters from  $N$ , then  $X \cap M^n$  is definable in  $M$ .*

**PROOF.** [vdD1], Theorem (A).  $\dashv$

### §5. The invariance ring.

DEFINITION 5.1. Let  $K$  be an ordered field and let  $G$  be a convex subgroup of  $(K, +, \leq)$ . We define the **invariance ring**  $V(G)$  of  $G$  as

$$V(G) := \{a \in K \mid a \cdot G \subseteq G\}$$

As  $G$  is a convex subgroup of  $K$  it is not difficult to see that the set  $V(G)$  is a convex valuation ring of  $K$  and  $G$  is a  $V(G)$ -submodule of  $K$ . The maximal ideal of  $V(G)$  is denoted by  $\mathfrak{m}(G)$ . If  $p$  is a cut of  $K$  we define the invariance ring  $V(p)$  of  $p$  as  $V(p) := V(G(p))$ . The maximal ideal of  $V(p)$  is denoted by  $\mathfrak{m}(p)$ .

Observe that the group  $V(G)^{>0}$  of positive units of  $V(G)$  is the invariance group of the cut  $G^+$  with respect to the multiplicative, ordered group of positive elements of  $K$ . Observe also that  $V(p) = K$  if and only if  $G(p)$  is equal to  $K$  or to  $\{0\}$ . So if  $K$  is a real closed field, then  $V(p) = K$  if and only if  $\hat{p}$  is principal if and only if  $p$  is principal or dense.

**Warning.** Even in the case  $1 \notin G$ , the ring  $V(G)$  is in general not the largest convex subring  $W$  of  $K$ , such that  $G$  is contained in the maximal ideal of  $W$ .  $V(G)$  has nothing to do with  $W$ !

Let  $R \subseteq S$  be real closed fields and let  $V$  denote the intersection of all  $V(p)$ , where  $p$  runs through the cuts of  $R$ , which are realized in  $S$ . One can show that the convex hull  $W$  of  $V$  in  $S$  is the largest convex valuation ring of  $S$ , such that the residue field of  $W \cap R$  is dense in the residue field of  $W$ . Hence this valuation ring is the same as defined by Macintyre ([Mac]) and Baur ([Ba]). We don't use this and omit the proof.

We use the invariance ring of a convex subgroup in this section in order to describe heirs of the upper edge of a convex subgroup of a model  $M$  of  $T$  — provided the invariance ring of the group is definable in some  $(M, V)$ , where  $V \subseteq M$  is a  $T$ -convex valuation ring. This will lead to a model completeness result (cf. 5.6) for these groups and it serves as a preparation for the results in Section 7 on the model theory of cuts with such invariance groups.

THEOREM 5.2. *Let  $M$  be a model of  $T$ , let  $V$  be a  $T$ -convex valuation ring of  $M$  and let  $A$  be a subset of  $M$ . If  $p$  is a nonprincipal cut of  $M$ , such that  $p^L$  is  $A$ -definable in  $(M, V)$ , then there is an  $A$ -definable map  $f : M \rightarrow M$  and elements  $a, b \in \text{dcl}(A)$ ,  $b \neq 0$ , such that  $f(V^+) = V(p)^+$  and such that  $p = a + b \cdot V(p)^+$  or  $p = a + b \cdot \mathfrak{m}(p)^+$ . In particular  $\text{sign } p \neq 0$  and  $\text{sign}^* \hat{p} \neq 0$ .*

*Moreover  $V \subseteq G(p)$  or  $G(p) \subseteq \mathfrak{m}(V)$ .*

PROOF. First note that  $V$  is proper, since  $p$  is definable in  $(M, V)$  and nonprincipal.

CLAIM 1. If  $G$  is a convex subgroup of  $(M, +, \leq)$ , which is  $M$ -definable in  $(M, V)$ , then  $V \subseteq G$  or  $G \subseteq \mathfrak{m}(V)$ .

Suppose  $\mathfrak{m}(V) \subsetneq G \subsetneq V$ . Let  $M_0 \prec M$  be tame such that  $V$  is the convex hull of  $M_0$  and let  $G_0 := G \cap M_0$ . Since  $\mathfrak{m}(V) \subsetneq G \subsetneq V$ , the set  $G_0$  is a convex subgroup of  $(M_0, +, \leq)$  with  $G_0 \neq 0$  and  $G_0 \neq M_0$ .

Since there is no proper definable convex subgroup in an o-minimal expansion of a group, this contradicts 4.7.

By Claim 1 we already know the “moreover” part of the Theorem. Since  $V(p)$  is definable in  $(M, V)$  with parameters from  $A$  we can apply 4.6(ii) to  $V(p)^+$ . We get an  $A$ -definable homeomorphism  $f : M \rightarrow M$  with  $f(V^+) = V(p)^+$ .

It remains to find elements  $a, b \in \text{dcl } A$  with  $p = a + b \cdot V(p)^+$  or  $p = a + b \cdot \mathfrak{m}(p)^+$ .

**CLAIM 2.** There are elements  $a, b \in M$ ,  $b \neq 0$  with  $p = a + b \cdot V(p)^+$  or  $p = a + b \cdot \mathfrak{m}(p)^+$ .

Since  $p^L$ ,  $G(p)$  and  $V(p)$  are definable in  $(M, V)$  with parameters from  $M$  we have  $p \sim V^+$ ,  $\hat{p} \sim V^+$  and  $V(p)^+ \sim V^+$  by 4.6(ii). Let  $\alpha$  be a realization of  $p$  and let  $N := M\langle\alpha\rangle$ . Let  $V_1$  be the convex hull of  $V$  in  $N$  and let  $V_2$  be the largest convex valuation ring of  $N$  with  $V_2 \cap M = V$ . Easily we have that  $V_2^+$  is the largest extension of  $V^+$  on  $N$ , so  $V_2^+$  is an heir of  $V^+$  and by 4.6(iii),  $V_2$  is  $T$ -convex. Since  $\mathfrak{m}(V_2) \subsetneq V_1 \subsetneq V_2$ , the set  $V_1$  is not definable in  $(N, V_2)$  by Claim 1. In particular  $V_1^+ \not\sim V_2^+$ .

Let  $r_1, r_2$  be the least and the largest extension of  $p$  on  $N$ . Since  $p \sim V^+$  and  $V_1^+ \not\sim V_2^+$  we get  $r_1 \not\sim r_2$ . By 3.12(iv) we have  $\text{sign } p \neq 0$ .

The same argument applied to  $\hat{p}$  and  $(M^{>0}, \cdot, \leq)$  shows that  $\text{sign}^* \hat{p} \neq 0$ . So we know  $\text{sign } p, \text{sign}^* \hat{p} \neq 0$ , i.e., there are  $a, b \in M$  such that  $p = a + b \cdot V(p)^+$  or  $p = a + b \cdot \mathfrak{m}(p)^+$ . This shows Claim 2.

Let  $M_0$  be the definable closure of  $A$ , let  $V_0 := V \cap M_0$  and let  $p_0$  be the cut  $(p^L \cap M_0, p^R \cap M_0)$ . Since  $p^L$  is  $A$ -definable in  $(M, V)$  we know from 4.6(ii) that there is an  $A$ -definable homeomorphism  $h : M \rightarrow M$  with  $h(V^+) = p$ . Clearly  $h(V_0^+) = p_0$  and  $p_0$  is  $M_0$ -definable in  $(M_0, V_0)$ .

**CLAIM 3.** The theorem holds if  $M_0 \subseteq V$  and  $\dim M/M_0 = 1$ .

As  $\dim M/M_0 = 1$  and  $V \neq M$ , 4.2(ii) implies that the cut  $V^+$  is the least extension of  $V_0^+ = +\infty_{M_0}$  on  $M$ . Thus  $p = h(V^+)$  is the least or the largest extension of  $p_0 = h(V_0^+)$  on  $M$ . Since  $V_0^+ = +\infty_{M_0}$  is a principal cut, the cut  $p_0$  is principal as well. Since  $p$  is not principal and  $p$  is the least or the largest extension of the principal cut  $p_0$  we get  $p = \pm V^+$  if  $p_0 = \pm\infty_{M_0}$  and  $p = a \pm \mathfrak{m}(V)^+$  if  $p_0 = a_{M_0}^\pm$  ( $a \in M_0$ ). This proves Claim 3.

**CLAIM 4.** If  $M_0 \not\subseteq V$ ,  $\varepsilon \in \{\pm 1\}$  and if  $a, b \in M_0$  with  $p_0 = a + b \cdot (V(p_0)^+)^{\varepsilon}$ , then  $p = a + b \cdot (V(p)^+)^{\varepsilon}$  (recall that for a cut  $\xi$  of an ordered field,  $\xi^{\varepsilon}$  denotes the result of the application of the map  $x^{\varepsilon}$  to the cut  $\xi$ ).

This is so, since  $V_0$  is a proper  $T$ -convex valuation ring of  $M_0$ , hence  $(M_0, V_0) \prec (M, V)$  (this follows from 4.5 applied to the extension  $T^{\text{df}}$  of  $T$ , cf. [vdD-Lew](2.3)). As  $p^L$  is  $A$ -definable in  $(M, V)$  the valuation ring  $V(p)$  is  $A$ -definable in  $(M, V)$  as well, hence there is an  $A$ -definable homeomorphism  $h_1 : M \rightarrow M$  with  $h_1(V^+) = V(p)^+$ . As  $p = h(V^+)$ , the equation  $h_1(V^+) = V(p)^+$  can be expressed by an  $\mathcal{L}_{\text{convex}}$ -sentence with parameters from  $A$ . Since  $(M_0, V_0) \prec (M, V)$  we get  $h_1(V_0^+) = V(p_0)^+$ . Consequently  $h(V_0^+) = a + b \cdot h_1(V_0^+)^{\varepsilon}$  and  $(M_0, V_0) \prec (M, V)$  imply  $p = a + b \cdot (V(p)^+)^{\varepsilon}$ . This shows Claim 4.

Now we prove that  $p = a + b \cdot V(p)^+$  or  $p = a + b \cdot \mathfrak{m}(p)^+$  for some  $a, b \in M_0 = \text{dcl } A$ .

**CASE 1.**  $M_0 \not\subseteq V$ .

By Claim 2 applied to  $p_0$  and  $M_0$  there are  $a, b \in M_0$  and  $\varepsilon \in \{\pm 1\}$  with  $p_0 = a + b \cdot (V(p_0)^+)^{\varepsilon}$ . By Claim 4 we get  $p = a + b \cdot (V(p)^+)^{\varepsilon}$ .

CASE 2.  $M_0 \subseteq V$ .

Let  $\gamma \in M$ ,  $\gamma > V$ ,  $M_1 := M_0 \langle \gamma \rangle$ ,  $p_1 := (p^L \cap M_1, p^R \cap M_1)$  and  $V_1 := V \cap M_1$ . Then  $p_1^L$  is definable in  $(M_1, V_1)$ , so by Claim 3 applied to  $p_1$ , there are  $a, b \in M_0$  and  $\varepsilon \in \{\pm 1\}$  with  $p_1 = a + b \cdot (V(p_1)^+)^{\varepsilon}$ . By Claim 4 applied to  $M_1$  instead of  $A$  we get  $p = a + b \cdot (V(p)^+)^{\varepsilon}$ .  $\dashv$

REMARK 5.3. We'll apply the results of Section 3 also to the expansion of the multiplicative group of positive elements of  $M$  induced by  $M$ . This means the following.

Let  $\mathcal{L}^{>0}$  be the signature of ordered groups,  $\mathcal{L}_0 := \{+, -, <, 0\}$ , together with an  $n$ -ary predicate  $R_\varphi$  for each  $\mathcal{L}$ -formula  $\varphi$  with exactly  $n$  free variables. If  $M$  is a model of  $T$ , then  $M^{>0}$  is the  $\mathcal{L}^{>0}$ -structure with universe  $\{a \in M \mid M \models a > 0\}$  and the following interpretation of the non logical symbols: the underlying divisible, ordered group of  $M^{>0}$  is the multiplicative group of positive elements of  $M$ . If  $\varphi$  is an  $\mathcal{L}$ -formula with exactly  $n$  free variables, then  $R_\varphi^{M^{>0}} := \varphi[M^n] \cap (0, \infty)^n$ . If we relativize to  $(0, \infty)$  we can inductively assign to each  $\mathcal{L}^{>0}$ -formula a formula from  $\mathcal{L}$ . We define  $T^{>0} := Th(M^{>0})$ . As  $T$  is complete,  $T^{>0}$  does not depend on the choice of  $M$ .  $T^{>0}$  is an  $o$ -minimal extension of divisible ordered abelian groups, because  $M^{>0}$  is  $o$ -minimal. If  $N$  is a model of  $T^{>0}$ , we can construct in an obvious way a model  $M$  of  $T$  such that  $M^{>0} = N$ . The transition  $M \mapsto M^{>0}$  respects elementary extensions, as well as definable closures: since  $T$  is  $o$ -minimal, we can recover  $M$ -definable subsets of  $M^n$  from  $M^{>0}$ -definable subsets of  $(M^{>0})^n$ . If  $M$  is a model of  $T$  then the  $n$ -types of  $M$  which contain  $\bigwedge x_i > 0$  are in one-to-one correspondence with the  $n$ -types of  $M^{>0}$ . Observe that for these types the notion of an heir is not changed by switching the point of view. We leave the arguments in this way; the transition  $T \mapsto T^{>0}$  in the applications below is always straightforward. For example a consequence of 5.2 is:

COROLLARY 5.4. *If  $G$  is a convex subgroup of  $(M, +, \leq)$ , such that  $V(G)^+ \sim V^+$  for some  $T$ -convex valuation ring  $V$  of  $M$ , then the signature alternative holds for the cut  $G^+$  with respect to the structure  $M^{>0}$ . In particular  $\text{sign}^* G^+ = 0$  implies  $G^+ \not\sim V(G)^+$ .*

PROOF. Assume  $\text{sign}^* G^+ = 0$ ; there is no  $M$ -definable map  $f$  with  $f(G^+) = V(G)^+$  by 5.2. If  $\alpha \models V(G)^+$  and  $q$  denotes the unique extension of  $G^+$  on  $M \langle \alpha \rangle$ , then  $q$  is again the upper edge of a convex subgroup  $H$  of  $(M \langle \alpha \rangle, +)$  with  $\text{sign}^* H^+ = 0$  (3.12(v) applied to  $M^{>0}$  and  $G^+$ ). Furthermore  $V(H)^+$  is the largest extension of  $V(G)^+$  on  $M \langle \alpha \rangle$  (3.12(i),(ii) applied to  $M^{>0}$  and  $G^+$ ). In particular  $V(H)^+$  is an heir of  $V(G)^+$  by 2.10. By 4.6(iv),  $V(H)^+ \sim W^+$ , for some  $T$ -convex valuation ring  $W$  of  $M \langle \alpha \rangle$ . Again by 5.2 we have  $H^+ \not\sim V(H)^+$ .  $\dashv$

COROLLARY 5.5. *Let  $M \prec N$  be models of  $T$  and let  $G$  be a convex subgroup of  $(M, +, \leq)$  with  $\text{sign}^* G^+ = 0$ , such that  $f(V(G)^+) = V^+$  for some  $T$ -convex valuation ring  $V$  of  $M$  and some  $M$ -definable map  $f : M \rightarrow M$ . If  $H$  is a convex subgroup of  $(N, +, \leq)$  with  $H \cap M = G$ ,  $V(H) \cap M = V(G)$  and  $f(V(H)^+) = W^+$  for some  $T$ -convex valuation ring  $W$  of  $N$ , then  $(V(H)^+, H^+)$  is an heir of  $(V(G)^+, G^+)$ .*

PROOF. By 5.4, the signature alternative holds for the cut  $G^+$  with respect to the theory  $T^{>0}$ . Since  $W^+$  is an heir of  $V^+$  and  $f(V(H)^+) = W^+$ ,  $f(V(G)^+) = V^+$ ,

the cut  $V(H)^+$  is an heir of  $V(G)^+$ . Therefore we can apply 3.19 for  $G^+$  and the theory  $T^{>0}$ . This proves the Corollary.  $\dashv$

**COROLLARY 5.6.** *Suppose  $T$  is model complete. Let  $\mathcal{O}$  and  $\mathcal{G}$  be new unary predicates. Let  $M$  be a model of  $T$  and let  $G$  be a convex subgroup of  $(M, +, \leq)$  with  $\text{sign}^* G^+ = 0$ , such that  $f(V(G)^+, \bar{a}) = V^+$  for some  $T$ -convex valuation ring  $V$  of  $M$  and some  $\emptyset$ -definable map  $f : M^{n+1} \rightarrow M$ ,  $\bar{a} \in M^n$ . Let  $c_1, \dots, c_n$  be new constants and let  $T'$  be the  $\mathcal{L}(\mathcal{O}, \mathcal{G}, c_1, \dots, c_n)$ -theory, which says the following about a model  $(N, W, H, \bar{d})$ :*

- (a)  $W$  is a  $T$ -convex valuation ring.
- (b)  $H$  is a convex subgroup of  $(N, +, \leq)$  with  $\text{sign}^* H^+ = 0$ .
- (c)  $f(W^+, d_1, \dots, d_n) = V(G)^+$ .

Then  $T'$  is model complete.

**PROOF.** By 5.5 and Robinson's test for model completeness.  $\dashv$

**§6. A method for producing cuts with prescribed signature.** In this section we'll show that for every cut  $p$  of a real closed field  $M$ , there is a real closed field  $N$  containing  $M$  and a cut  $q$  extending  $p$ , such that  $\hat{q}$  extends  $\hat{p}$ ,  $V(q)$  lies over  $V(p)$  and such that  $\text{sign } q = \delta$ ,  $\text{sign}^* \hat{q} = \varepsilon$  for all prescribed values  $\delta, \varepsilon \in \{-1, 0, 1\}$ . The precise statement can be found in 6.6. This result serves as a source for examples as well as a main ingredient for the proof of the consistency part of 7.4 on model completeness of o-minimal structures expanded by cuts.

**LEMMA 6.1.** *Let  $M$  be a divisible ordered abelian group, let  $G$  be a convex subgroup of  $M$  and let  $(a_i)_{i \in \mathbb{N}}$  be a sequence in  $M$  with  $a_{i+1} \leq \frac{1}{2}a_i$  for all  $i \in \mathbb{N}$  such that  $G^+ = \{a_i \mid i \in \mathbb{N}\}^-$ . Let  $p$  be the cut  $\{\sum_{i=1}^n a_i \mid n \in \mathbb{N}\}^+$ . Then  $G(p) = G$  and  $\text{sign } p \leq 0$ . Moreover  $p < 3a_1 - G^+$ .*

**PROOF.** If  $g \in G$ ,  $g > 0$ , then  $g + \sum_{i=1}^n a_i \leq \sum_{i=1}^{n+1} a_i \leq p$ , hence  $g \in G(p)$  and  $G$  is contained in  $G(p)$ . Conversely, if  $g > G$  and  $k \in \mathbb{N}$  are such that  $a_k < \frac{g}{2}$ , then  $p < g + \sum_{i=1}^{k-1} a_i$ , since for each  $n > k$  we have  $\sum_{i=k}^n a_i \leq \sum_{i=k}^n (\frac{1}{2})^{i-k} a_k < 2a_k < g$ . So we know that  $G(p) = G$ .

We have  $p < 3a_1 - G^+$ , since  $p < 2a_1$  and  $G^+ < a_1$ . It remains to show that  $\text{sign } p \leq 0$ .

Suppose  $p = a + G^+$  for some  $a \in M$ . There is some  $k \in \mathbb{N}$  such that  $\sum_{i=1}^k a_i \geq a$ ,

hence  $p = a + G^+ \leq \sum_{i=1}^k a_i + G^+ < \sum_{i=1}^{k+1} a_i < p$ , a contradiction.  $\dashv$

**EXAMPLE.** In the situation of 6.1 the signature of  $p$  may be  $-1$ , for example if  $M = \mathbb{R}$ ,  $G = 0$  and  $a_i = (\frac{1}{2})^i$ . Moreover if the cut  $p$  constructed in 6.1 has signature 0 and  $\alpha$  is a realization of  $p$ , then the least extension  $q$  of  $p$  on  $M\langle\alpha\rangle$  is  $\{\sum_{i=1}^n a_i \mid n \in \mathbb{N}\}^+$  in  $M\langle\alpha\rangle$  and the upper edge of the convex hull  $H$  of  $G$  in  $M\langle\alpha\rangle$  is  $\{a_i \mid i \in \mathbb{N}\}^-$ . So  $\text{sign } q = -1$  and  $G(q) = H$ .

In particular the signature of  $p$  may be  $-1$  even if  $\mathbb{N} \cdot a_{i+1} < a_i$  for all  $i$ .

LEMMA 6.2. *Let  $M = M_0 \subseteq M_1 \subseteq \dots \subseteq N := \bigcup_{i < \omega} M_i$  be divisible, ordered, abelian groups, let  $G$  be a convex subgroup of  $M$  and let  $G_i$  be the convex hull of  $G$  in  $M_i$ . Suppose for each  $i \in \mathbb{N}$  there is a realization  $\alpha_i$  of  $G_{i-1}^+$  in  $M_i$ . Let  $q$  denote the cut  $\{\alpha_1 + \dots + \alpha_n \mid n \in \mathbb{N}\}^+$  of  $N$ . Then*

- (i)  $q \upharpoonright M_i$  is the cut  $\alpha_1 + \dots + \alpha_{i-1} + G_i^+$  of  $M_i$  for all  $i \in \mathbb{N}$ ; in particular  $q$  extends  $G^+$ .
- (ii)  $\text{sign } q = 0$  and  $G(q)$  is the convex hull of  $G$  in  $N$ .
- (iii) Let  $p$  be a cut of  $M$  realized in  $N$  by  $\beta$  such that  $G \subseteq G(p)$ . Then
  - (a) If  $\text{sign } p \geq 0$ , then  $\beta + q$  extends  $p$ .
  - (b) If  $\text{sign } p \leq 0$ , then  $\beta - q$  extends  $p$ .
  - (c) If  $G \subsetneq G(p)$ , then  $\beta + q$  and  $\beta - q$  extend  $p$ .

*In particular, there is an extension of  $p$  on  $N$  with signature 0 such that the invariance group of this extension is the convex hull of  $G$  in  $N$ .*

PROOF. From the choice of the  $\alpha_i$  we have  $\alpha_i > 2\alpha_{i+1}$  ( $i \in \mathbb{N}$ ).

(i). Let  $q_i$  denote the cut  $\alpha_1 + \dots + \alpha_{i-1} + G_i^+$  of  $M_i$ . Since  $\alpha_i \models G_{i-1}^+$  we have  $q_i \leq q \upharpoonright M_i$ . On the other hand for each  $k > i$  we have  $\sum_{j=i}^k \alpha_j \leq \sum_{j=i}^k \frac{1}{2}^{j-i} \alpha_i \leq 2\alpha_i \models G_{i-1}^+$ . Hence  $(q - \alpha_1 - \dots - \alpha_{i-1}) \upharpoonright M_i \leq G_i^+$  and  $q \upharpoonright M_i \leq \alpha_1 + \dots + \alpha_{i-1} + G_i^+ = q_i$ . This shows  $q_i = q \upharpoonright M_i$  and so (i) is proved.

(ii). Let  $H$  be the convex hull of  $G$  in  $N$ . Clearly  $H^+$  is the cut  $\{\alpha_i \mid i \in \mathbb{N}\}^-$  of  $N$ . So by 6.1 we have  $G(q) = H$  and  $\text{sign } q \leq 0$ . Suppose  $q = \alpha - H^+$  for some  $\alpha \in N$ . Let  $i \in \mathbb{N}$  be such that  $\alpha \in M_i$ . Then  $q \upharpoonright M_i = \alpha - (H \cap M_i)^+$ . So by (i) we get  $\text{sign } q_i = -1$  in contradiction to the definition of  $q_i$ .

(iii). We have  $q < 2\alpha_1$  and  $2\alpha_1 \models G^+$ . If  $\text{sign } p \geq 0$ , then  $\beta + 2\alpha_1$  realizes  $p$ , hence  $\beta + q$  extends  $p$ . Similarly if  $\text{sign } p \leq 0$ . This shows (a) and (b). Finally, if  $G \subsetneq G(p)$  then  $\beta \pm 2\alpha_1 \models p$ , which implies (c).  $\dashv$

COROLLARY 6.3. *Let  $T$  be an o-minimal expansion of DOAG, let  $\mathcal{C}$  be a set of cuts of  $M$  and let  $G^p$  be a convex subgroup of  $G(p)$  for each  $p \in \mathcal{C}$ . Then there is an elementary extension  $N \succ M$  with  $\dim N/M = \aleph_0 + |\mathcal{C}|$ , such that each  $p \in \mathcal{C}$  has an extension  $q$  on  $N$  with  $\text{sign } q = 0$  and such that  $G(q)$  is the convex hull of  $G^p$  in  $N$ .*

PROOF. Let  $M' \succ M$  be such that each  $p \in \mathcal{C}$  is realized in  $M'$  with  $\dim M'/M \leq |\mathcal{C}|$  and let  $H^p$  be the convex hull of  $G^p$  in  $M'$  ( $p \in \mathcal{C}$ ). Furthermore let  $\alpha_1^p$  be a realization of  $(H^p)^+$  ( $p \in \mathcal{C}$ ) and let  $M_1 := M' \langle \alpha_1^p \mid p \in \mathcal{C} \rangle$ . Let  $G_1^p$  be the convex hull of  $G^p$  in  $M_1$  ( $p \in \mathcal{C}$ ), let  $\alpha_2^p$  be a realization of  $(G_1^p)^+$  ( $p \in \mathcal{C}$ ) and let  $M_2 := M_1 \langle \alpha_2^p \mid p \in \mathcal{C} \rangle$ . Iterating this construction we get an elementary extension  $N := M' \langle \alpha_i^p \mid i \in \mathbb{N}, p \in \mathcal{C} \rangle$  of dimension  $\leq \aleph_0 + |\mathcal{C}|$  over  $M$ , where we can apply 6.2 to each  $p \in \mathcal{C}$ : Let  $q'$  be the cut  $\{\alpha_1^p + \dots + \alpha_n^p \mid n \in \mathbb{N}\}^+$  of  $N$ . By 6.2 we have  $\text{sign } q' = 0$  and  $G(q')$  is the convex hull of  $G^p$  in  $N$ . Since  $p$  is realized in  $N$  we get the corollary from 6.2(iii).  $\dashv$

LEMMA 6.4. *Let  $T$  be an o-minimal expansion of DOAG. Let  $I$  be a totally ordered index set, let  $M_i \prec M_j$  be models of  $T$  ( $i \leq j \in I$ ) and let  $p_i$  be a cut of  $M_i$  such that  $p_i \subseteq p_j$  whenever  $i \leq j \in I$ . Let  $M := \bigcup_i M_i \models T$ . There is a unique cut of  $M$  extending all  $p_i$ . We write  $\bigcup p_i$  for this cut. We have:*

- (i)  $G(\bigcup p_i) = \bigcup \bigcap_{i \in I, j \geq i} G(p_j) = \bigcap \bigcup_{i \in I, j \geq i} G(p_j)$ .
- (ii) If  $\hat{p}_j$  extends  $\hat{p}_i$  for all  $i < j$ , then  $\widehat{\bigcup p_i} = \bigcup_i \hat{p}_i$ .
- (iii) If  $\text{sign} \bigcup p_i \neq 0$  then there is some  $k \in I$  such that
- (a)  $\text{sign} p_j = \text{sign} \bigcup p_i$  for all  $j \geq k$  and
  - (b)  $\hat{p}_j$  extends  $\hat{p}_i$  for all  $j \geq i \geq k$ .
- In particular, if  $\text{sign} p_i = 0$  for all  $i$ , then  $\text{sign} \bigcup p_i = 0$ .
- (iv) If  $\hat{p}_j$  does not extend  $\hat{p}_i$  for all  $i < j$ , then  $G(\bigcup p_i) = \bigcap_i H_i$ , where  $H_i$  denotes the convex hull of  $G(p_i)$  in  $M$ .

PROOF. Since  $p_j$  is an extension of  $p_i$  for all  $i \leq j$ , the sets  $\bigcup_i p_i^L$  and  $\bigcup_i p_i^R$  define a partitioning of  $M$  into convex sets. Hence there is a unique cut  $p$  of  $M$  extending all the  $p_i$ .

(i). If  $a \in G(p) \cap M_i$ , then  $a + p = p$  implies  $a + p \upharpoonright M_i = p \upharpoonright M_i$ , so  $a \in G(p_i)$  which shows  $G(p) \subseteq \bigcup \bigcap_{i \in I, j \geq i} G(p_j)$ . The inclusion  $\bigcup \bigcap_{i \in I, j \geq i} G(p_j) \subseteq \bigcap \bigcup_{i \in I, j \geq i} G(p_j)$  is trivial. Finally if  $a \in \bigcap \bigcup_{i \in I, j \geq i} G(p_j)$ , then  $a + p = p$ , since for  $b \in p^L$  there is some  $i \in I$  with  $a, b \in p_i^L$ , hence  $a + p_j = p_j$  implies  $a + b \in p_j^L \subseteq p^L$  for any  $j \geq i$  with  $b \in G(p_j)$ .

(ii). Suppose  $\hat{p}_j$  extends  $\hat{p}_i$  — thus  $G(p_i) \subseteq G(p_j)$  — for all  $i < j$ . Then the first equality in (i) implies that  $\widehat{\bigcup p_i}$  is the upper edge of  $\bigcup_i G(p_i)$ , which is  $\bigcup \hat{p}_i$  by definition.

(iii). If  $p = a \pm \hat{p}$  with  $a \in M_j$ , then  $p - a$  is an edge of a convex subgroup of  $M$  and we write  $H$  for the intersection of  $M_j$  with this subgroup. Since  $a \in M_j$ ,  $p_j - a = p \upharpoonright M_j - a = (p - a) \upharpoonright M_j = H^\pm$ . This shows  $p_j = a \pm \hat{p}_j$  and  $\hat{p}_j = H^+ = \hat{p} \upharpoonright M_j$ . So if  $k \in I$  with  $a \in M_k$ , then  $k$  has properties (a) and (b).

(iv). Clearly  $G(p)$  contains  $\bigcap_i H_i$ . Conversely, let  $a \in G(p) \cap M_k$ . Then  $a \in G(p_j)$  for all  $j \geq k$ . If  $j < k$  then  $H_k \subseteq H_j$ , since  $\hat{p}_k$  does not extend  $\hat{p}_j$ . Hence  $a \in H_j$  as desired.  $\dashv$

PROPOSITION 6.5. *Let  $T$  be an o-minimal expansion of RCF, let  $M$  be a model of  $T$  and let  $G$  be a convex subgroup of  $(M, +, \leq)$ . Let  $V$  be a convex valuation ring of  $M$  with  $V \subseteq V(G)$ . Let  $\mathcal{E}$  be a set of cuts of  $M$  with  $G \subseteq G(p)$  for all  $p \in \mathcal{E}$ . Then there are an elementary extension  $N$  of  $M$  with  $\dim N/M = \aleph_0 + |\mathcal{E}|$  and a convex subgroup  $H$  of  $(N, +, \leq)$ , such that  $H \cap M = G$ ,  $\text{sign}^* H^+ = 0$ ,  $V(H)$  is the convex hull of  $V$  in  $N$  and such that for each  $p \in \mathcal{E}$  there is a cut  $q$  of  $N$  extending  $p$  with  $\text{sign} q = 0$  and  $G(q) = H$ .*

PROOF. First we reduce to the case where  $G(p) = G$  ( $p \in \mathcal{E}$ ) and  $V = V(G) \subsetneq M$ . For this we take a model  $M' \succ M$  and realizations  $\alpha_p, \beta$  of  $p \in \mathcal{E}$ ,  $G^+$  in  $M'$  respectively. Let  $V'$  be the convex hull of  $V$  in  $M'$ . Since  $V \subseteq V(G)$ , the upper edge of  $V'$  is less than or equal to the least extension of  $V(G)^+$  on  $M'$ . As  $\beta \models G^+$  and  $V(G)^+$  is the upper edge of the invariance group of  $G^+$  with respect to  $M^{>0}$ , it follows that the cut  $\beta \cdot V'^+$  extends  $G^+$ , consequently  $M \cap \beta \cdot V' = G$ . By 3.12(i) and (ii),  $\alpha_p + \beta \cdot V'^+$  or  $\alpha_p - \beta \cdot V'^+$  extends  $p$ . This argument shows that, upon replacing  $M$  by  $M'$ ,  $V$  by  $V'$ ,  $G$  by  $\beta \cdot V'$ , and  $p$  by that of the cuts above which extends it, we may assume  $V = V(G)$  and  $G = G(p)$  for all  $p \in \mathcal{E}$ .

We construct a chain  $M = M_0 \prec M_1 \prec M_2 \prec \dots$  of models of  $T$ , together with extensions  $p_i$  on  $M_i$  for each  $p \in \mathcal{E}$  such that

- (a) For all  $p, p' \in \mathcal{E}$  we have  $\hat{p}_i = \hat{p}'_i$ .
- (b)  $p_0 \subseteq p_1 \subseteq p_2 \subseteq \dots$ ,  $\hat{p}_0 \subseteq \hat{p}_1 \subseteq \hat{p}_2 \subseteq \dots$  and  $V(p_i)$  lies over  $V$  ( $i < \omega$ ,  $p \in \mathcal{E}$ ); note that  $V(p_i)$  is independent of  $p \in \mathcal{E}$  by (a).
- (c)  $\text{sign } p_{2i+1} = 0$  for all  $i$ .
- (d)  $\text{sign}^* \hat{p}_{2i+2} = 0$  and  $V(p_{2i+2})$  is the convex hull of  $V$  in  $M_{2i+2}$  for all  $i$ .

Take  $M_0 := M$  and  $p_0 := p$ . Suppose  $M_0, \dots, M_i$  and  $p_0, \dots, p_i$  are constructed, and (a)–(c) holds up to  $i$ . We define  $M_{i+1}$  and  $p_{i+1}$  in the following way:

CASE 1.  $i$  is even (the case  $i = 0$  is included since we do not need that  $\text{sign}^* \hat{p}_i = 0$  for this construction step).

By 6.3 applied to  $\{p_i \mid p \in \mathcal{E}\}$  and  $G(p_i)$  there is some  $M_{i+1} \succ M_i$  with  $\dim M_{i+1}/M_i = \aleph_0 + |\mathcal{E}|$  and cuts  $p_{i+1}$  of  $M_{i+1}$  extending  $p_i$  for all  $p \in \mathcal{E}$  such that  $\text{sign } p_{i+1} = 0$  and  $\hat{p}_{i+1}$  is the least extension of  $\hat{p}_i$  on  $M_{i+1}$ . By 3.12(iii),  $V(p_{i+1})$  lies over  $V(p_i)$ , hence over  $V$ .

CASE 2.  $i$  is odd.

Let  $\alpha_p$  be a realization of  $p_i$  ( $p \in \mathcal{E}$ ) and let  $r$  be the least extension of  $\hat{p}_i$  on  $M \langle \alpha_p \mid p \in \mathcal{E} \rangle$ . Then  $\alpha_p + r$  is an extension of  $p_i$  and  $V(r)$  lies over  $V(p_i)$  (3.12(iii)), hence over  $V$ . From 6.3 applied to  $T^{>0}$ ,  $r$ , and the convex hull of  $V$  in  $M_i \langle \alpha_p \mid p \in \mathcal{E} \rangle$  we get some  $M_{i+1} \succ M_i \langle \alpha_p \mid p \in \mathcal{E} \rangle$  with  $\dim M_{i+1}/M_i = \aleph_0 + |\mathcal{E}|$  and an extension  $r_1$  of  $r$  on  $M_{i+1}$  with multiplicative invariance group  $U$  such that  $\text{sign}^* r_1 = 0$  and such that  $U$  is the convex hull of  $V$  in  $M_{i+1}$ . Since  $2 \in U$ , we know that  $r_1$  is the upper edge of a convex subgroup of  $(M_{i+1}, +, \leq)$  (cf. remark following 3.4), hence  $U = V(r_1)$ . Now we take  $p_{i+1} = \alpha_p + r_1$  ( $p \in \mathcal{E}$ ), which completes the construction.

Let  $N = \bigcup M_i$ . By construction of the  $M_i$  we get  $\dim N/M = \aleph_0 + |\mathcal{E}|$ . Let  $H := \bigcup_{i < \omega} G(p_i)$ . By (a) the group  $H$  does not depend on  $p$ ; by (b) we have  $H = \bigcup_{i < \omega} G(p_{2i})$ . By (d) and 6.4 applied to  $T^{>0}$  and the sequence  $(\hat{p}_{2i+2})_{i < \omega}$  we have  $\text{sign}^* H^+ = 0$  (cf. 6.4(iii)) and  $V(H) = \bigcup_{i \in \mathbb{N}} V(p_{2i+2})$  (cf. 6.4(ii)). Since each  $V(p_{2i+2})$  is the convex hull of  $V$  we get that  $V(H)$  is the convex hull of  $V$  in  $N$ .

It remains to show that for each  $p \in \mathcal{E}$  there is a cut  $q$  of  $N$  with  $\text{sign } q = 0$  and  $G(q) = H$ . We take  $q = \bigcup p_i$  and apply 6.4 to the sequence  $(p_{2i+1})_{i < \omega}$ : clearly we have  $q = \bigcup p_{2i+1}$ ; by (c) and 6.4(iii) we have  $\text{sign } q = 0$ . By (b) and 6.4(ii) again, we have  $\hat{q} = \bigcup_{i \in \mathbb{N}} \hat{p}_{2i+1} = H^+$ .  $\dashv$

**PROPOSITION 6.6.** *Let  $T$  be an o-minimal expansion of RCF, let  $M$  be a model of  $T$  and let  $G$  be a convex subgroup of  $(M, +, \leq)$ . Let  $V$  be a proper convex valuation ring of  $M$  with  $V \subseteq V(G)$ . Let  $\mathcal{E}$  be a set of cuts of  $M$  with  $G \subseteq G(p)$  for all  $p \in \mathcal{E}$ . Let  $\varepsilon, \delta \in \{-1, 0, 1\}$ . Then there are an elementary extension  $N$  of  $M$  with  $\dim N/M = \aleph_0 + |\mathcal{E}|$  and a convex subgroup  $H$  of  $(N, +, \leq)$ , such that  $H \cap M = G$ ,  $\text{sign}^* H^+ = \varepsilon$ ,  $V(H)^+$  is an heir of  $V^+$  and such that for each  $p \in \mathcal{E}$  there is a cut  $q$  of  $N$  extending  $p$  with  $\text{sign } q = \delta$  and  $G(q) = H$ .*

**PROOF.** First we apply 6.5 and replace  $G$  by  $H$  and each  $p \in \mathcal{E}$  by a cut  $q$  extending  $p$  with  $\text{sign } q = 0$  and  $G(q) = H$ . Since  $V(H)$  is the convex hull of  $V$  in

$N \succ M$  and  $V \neq M$  we know from 2.10, that  $V(H)^+$  is an heir of  $V^+$ . So in order to prove the Proposition we may assume that  $\text{sign } p = \text{sign}^* \hat{p} = 0$ ,  $G = G(p)$  and  $V = V(p) \neq M$  ( $p \in \mathcal{E}$ ).

First we do the case  $\delta = 0$ . If  $\varepsilon = 0$ , the result is 6.5. So let  $\varepsilon \neq 0$ .

Let  $\beta$  be a realization of  $G^+$  and for each  $p \in \mathcal{E}$  let  $p_1$  be the largest extension of  $p$  on  $M\langle\beta\rangle$ . By 3.12(iii) applied to  $Th(M)^{>0}$ , the cut  $\hat{p}_1$  is the largest extension of  $G^+$  on  $M\langle\beta\rangle$ . Let  $G_1$  be the convex hull of  $G$  in  $M\langle\beta\rangle$ . Then  $\text{sign}^* \hat{p}_1 = 1$  and  $\text{sign}^* G_1^+ = -1$  (by 3.12(iv) and since  $\text{sign}^* G^+ = 0$ ). Moreover  $V(p_1)$  is the largest convex valuation ring of  $M\langle\beta\rangle$  lying over  $V$ , and  $V(p_1) = V(G_1)$  by 3.12(i),(ii).

Now we apply 6.3 to  $G(p_1)$  and  $p_1$  ( $p \in \mathcal{E}$ ). Hence there is some  $N \succ M$  with  $\dim N/M = \aleph_0 + |\mathcal{E}|$  and cuts  $q$  of  $N$  extending  $p_1$  ( $p \in \mathcal{E}$ ) such that  $G(q)$  is the convex hull of  $G(p_1)$  in  $N$ . Since  $\text{sign}^*(\hat{p}_1) = 1$  we have  $\text{sign}^* \hat{q} = 1$  as well and the cut  $V(q)^+$  is an heir of  $V(p_1)^+$ . Since  $\hat{q}$  does not depend on  $p \in \mathcal{E}$  we have proved the Proposition in the case  $\varepsilon = 1$  and  $\delta = 0$ .

The same argument for  $G_1$  and  $p_1$  ( $p \in \mathcal{E}$ ) yields the Proposition in the case  $\varepsilon = -1, \delta = 0$ .

Now we get the Proposition from the case  $\delta = 0$  as follows: for  $p \in \mathcal{E}$  take an extension  $q_p$  of  $p$  on  $N$ ,  $\text{sign } q_p = 0$  such that  $G(q_p) = H$ ,  $\text{sign}^* H^+ = \varepsilon$  and  $V(H)^+$  is an heir of  $V^+$ . Let  $N_1 \succ N$  be such that all  $q_p$  ( $p \in \mathcal{E}$ ) are realized in  $N_1$  with  $\dim N_1/N = \aleph_0 + |\mathcal{E}|$ , and let  $H_1$  be the largest extension of  $H^+$  on  $N_1$ . From 3.12(iii) applied to  $T^{>0}$ ,  $H^+$  and  $N_1$  we get that  $V(H_1)^+$  is the least or the largest extension of  $V(H)^+$ ; consequently  $V(H_1)^+$  is an heir of  $V(H)^+$ , hence of  $V^+$ . From 3.12(i),(ii) applied to  $T$ ,  $q_p$  and  $N_1$ , the least and largest extensions of  $q_p$  on  $N_1$  have invariance group  $H_1$ . These extensions have signature  $-1, +1$  respectively, since  $\text{sign } q_p = 0$ .  $\dashv$

**§7. Existentially closed extensions.** Again we work with an o-minimal expansion  $T$  of the theory of real closed fields in the language  $\mathcal{L}$ . If  $p$  is a cut of a model  $M$  of  $T$  and  $f : M \rightarrow M$  is  $M$ -definable, then by [vdD3] 7,(2.5) the map  $f$  is differentiable in a neighborhood  $I$  of  $p$  and there is an  $M$ -definable map  $g : M \rightarrow M$  with  $g(x) = f'(x)$  for all  $x \in I$ . The cut  $g(p)$  does not depend on the choice of  $I$  and  $g$ . We denote this cut by  $f'(p)$ .

**PROPOSITION 7.1.** *If  $p$  is a cut of  $M$  with  $\text{sign } p = 0$  and  $f : M \rightarrow M$  is  $M$ -definable with  $f(p) = \hat{p}$ , then  $f'(p) = \mathfrak{m}(p)^+$  or  $f'(p) = \mathfrak{m}(p)^-$ . In particular  $p \sim V(p)^+$ .*

*Remark.* *It is an open question if a cut  $p$  of an o-minimal expansion of a field with  $\text{sign } p = 0$  and  $p \sim \hat{p}$  exists at all. We'll use our Proposition to show that in many cases this is not the case (cf. 7.2).*

**PROOF.** First we prove  $|f'(p)| \leq \mathfrak{m}(p)^+$ .

Suppose  $0 < a < |f'(p)|$  for some  $a \in V(p)^*$ . Then  $\frac{1}{a} \cdot \hat{p} = \hat{p}$ , hence the map  $g(x) := \frac{1}{a} \cdot f(x)$  fulfills  $g(p) = \hat{p}$  and  $g'(p) \geq 1$ . Let  $b, c \in M$  such that  $b < p < c$  such that  $g$  is differentiable in  $[b, c]$ ,  $g(b) \geq 0$  and  $g'(x) \geq 1$  for all  $x \in [b, c]$ . Then  $0 \leq g(b) + x - b \leq g(x)$  for all  $x \in [b, c]$ , hence  $0 \leq g(b) + p - b \leq g(p) = \hat{p}$ . This implies  $g(b) + p - b = \hat{p}$ , thus  $\text{sign } p \neq 0$  in contradiction to our assumption.

It remains to show  $|f'(p)| \geq \mathfrak{m}(p)^+$ . We may assume  $p > 0$ . Suppose there is some  $a \in \mathfrak{m}(p)$  with  $0 < |f'(p)| < a$ .

CLAIM 1. There is some  $c \in M$  with  $1 + a \in G^*(p + c)$ .

Since  $\frac{1}{a} \notin V(p)$  there is some  $b \in G(p)$ ,  $b > 0$  with  $\frac{b}{a} \notin G(p)$ . Let  $d < p$  with  $d + \frac{b}{a} > p$ . We prove  $1 + a \in G^*(p + c)$  for  $c := \frac{b}{a} - d$ . Let  $m \in M$  with  $d < m < p$ . Since  $ap < b + ad$  we have  $(1 + a)(m + c) = m + c + a(m + \frac{b}{a} - d) = m + c + b + am - ad < m + c + b + b$ . As  $b \in G(p)$  we get  $m + 2b < p$ , hence  $m + 2b + c < p + c$ . This proves Claim 1.

With the element  $c$  from Claim 1 we replace  $f(x)$  by  $f(x - c)$  and  $p$  by  $p + c$ , hence we may assume that  $1 + a \in G^*(p)$ . By 3.5 we have  $1 - a \in G^*(p)$  as well.

CLAIM 2. There is a linear map  $g : M \rightarrow M$  with  $g(p) = p$  and a neighborhood  $(c, d)$  of  $p$  with  $g(x) \geq x + f(x)$  on  $(c, d)$ . For the proof of Claim 2 we have two cases.

CASE 1.  $f'(p) > 0$ .

There is a neighborhood  $[c, d]$  of  $p$ ,  $c > 0$ , where  $f$  is strictly increasing and differentiable, such that  $f(c) \in G(p)$  and  $f'|_{(c,d)} < a$ . Take  $g(x) = f(c) + (1+a)x$ . Then  $g(x) > x + f(x)$  on  $[c, d]$ , because  $g(c) > f(c) + c$  and  $g' = 1+a > 1+f' > 0$  on  $(c, d)$ . We have  $g(p) = f(c) + (1+a)p = p$ , since  $(1+a)p = p$  and  $f(c) \in G(p)$ .

CASE 2.  $f'(p) < 0$ .

There is some neighborhood  $[c, d]$  of  $p$ , where  $f$  is strictly decreasing and differentiable, such that  $f(d) \in G(p)$  and  $f'|_{(c,d)} > -a$ . Since  $(1+a) \cdot p = p$  we have  $a \cdot p \leq \hat{p}$ ; as  $\text{sign } p = 0$  we have  $a \cdot p < \hat{p}$ . Hence we may shrink  $d$  if necessary so that  $a \cdot d \in G(p)$ . Let  $g(x) = f(d) + ad + (1-a) \cdot x$ . Then  $g(x) \geq x + f(x)$  on  $[c, d]$ , since  $g(d) = f(d) + d$  and  $g' = 1-a < 1+f'$  on  $(c, d)$ . We have  $g(p) = f(d) + ad + (1-a) \cdot p = p$  since  $(1-a) \cdot p = p$  and  $f(d), a \cdot d \in G(p)$ .

Now Claim 2 is proved and we have a linear map  $g$  with  $g(p) = p$  and  $g(x) - x \geq f(x)$  on  $(c, d)$ . This means  $g_1(p) = \hat{p}$  if  $g_1(x)$  denotes  $g(x) - x$ . Since  $g_1$  is linear it is not possible that  $\text{sign } p = 0$ .  $\dashv$

THEOREM 7.2. Let  $p$  be a cut of a model of  $T$ , such that  $V(p)^+ \sim V^+$  for some  $T$ -convex valuation ring  $V$  of  $M$ . Then

- (i) The signature alternative holds for  $p$ . In particular  $\text{sign } p = 0$  implies  $p \not\sim \hat{p}$ .
- (ii) The signature alternative holds for  $|p|$  with respect to  $M^{>0}$ . In particular  $\text{sign}^* p = 0$  implies  $p \not\sim G^*(p)^+$ .

PROOF. (i). Suppose  $\text{sign } p = 0$ . If  $p \sim \hat{p}$ , 7.1 and the assumption give  $p \sim V(p)^+ \sim V^+$ ; in particular  $p^L$  is definable in  $(M, V)$ ; 5.2 implies, then, that  $\text{sign } p \neq 0$ . Hence,  $p \not\sim \hat{p}$ .

Let  $\alpha$  be a realization of  $\hat{p}$  and let  $q$  be the unique extension of  $p$  on  $M\langle\alpha\rangle$ . By 3.12(v) we have  $\text{sign } q = 0$  and by 3.12(i),(ii), the cut  $\hat{q}$  is the largest extension of  $\hat{p}$  on  $M\langle\alpha\rangle$ . By 3.12(iii) applied to  $\hat{p}$  and  $M^{>0}$  the cut  $V(q)^+$  is the least or the largest extension of  $V(p)^+$  on  $M\langle\alpha\rangle$ ; in particular it is an heir of  $V(p)^+$ . By 4.6(iv),  $V(q)^+ \sim W^+$  for some  $T$ -convex valuation ring  $W$  of  $M\langle\alpha\rangle$ . Since  $\text{sign } q = 0$ , we can apply the argument at the beginning of the proof to  $q$  instead of  $p$  and get  $q \not\sim \hat{q}$  as desired.

(ii). Let  $\text{sign}^* p = 0$ , hence  $p \neq a \cdot G^*(p)^\pm$  for all  $a \in M$ . We first show that  $p \not\sim G^*(p)^+$ .

CASE 1.  $\text{sign } p = 0$ .

By (i) we have  $p \not\sim \hat{p}$  and by 3.5 we have  $\hat{p} \sim G^*(p)^+$ , hence  $p \not\sim G^*(p)^+$ .

CASE 2.  $\text{sign } p \neq 0$ , so  $p = a \pm \hat{p}$  with some  $a \in M$ . We may assume that  $p > 0$ . If  $a \leq 0$ , then  $a \in G(p)$ , hence  $p = \hat{p}$  and  $p \not\sim G^*(p)^+ = V(p)^+$  follows from 5.2. So we may assume that  $a > 0$  and we may replace  $p$  by  $1 + \frac{1}{a} \cdot \hat{p}$ . If  $1 \notin \frac{1}{a} \cdot G(p)$ , then  $1 + \frac{1}{a} \cdot G(p)$  is a convex subgroup of  $(M^{>0}, \cdot, \leq)$  in contradiction to  $\text{sign}^* p = 0$ . So  $1 \in \frac{1}{a} \cdot G(p)$ , in other words  $p = \hat{p}$  and we may use 5.2 again.

So we know that  $\text{sign}^* p = 0$  implies  $p \not\sim G^*(p)^+$ . As in the proof of (i), this implies the signature alternative for  $|p|$  with respect to  $M^{>0}$ .  $\dashv$

REMARK. If  $T$  is polynomially bounded then 7.2(i) follows quickly from the valuation property (cf. [vdD-S], (9.2)). This property even implies that  $p \not\sim G^+$  for all convex subgroups  $G$  of  $(M, +, \leq)$  and all cuts  $p$  of  $M$  with  $\text{sign } p = 0$ .

NOTATION. Let  $p$  be a cut of a divisible ordered abelian group  $M$ . If  $\text{sign } p \neq 0$ , then  $p$  is an edge of the nonempty set  $Z(p) := \{a \in M \mid a + \hat{p} = p \text{ or } a - \hat{p} = p\}$  and we denote the other edge by  $z(p)$ . If  $\text{sign } p = 0$  we define  $z(p) := p$ .

So  $Z(p)$  is the set of all  $a \in M$  between  $p$  and  $z(p)$ . Note that  $Z(p)$  is 0-definable in the structure  $(M, p^L)$ .

If  $p$  is a cut of a real closed field  $M$ ,  $p > 0$ , then  $Z^*(p)$ ,  $z^*(p)$  denote  $Z(p)$ ,  $z(p)$  respectively with respect to  $(M^{>0}, <, \cdot)$  (where we consider  $z^*(p)$  as a cut of  $M$ , too). If  $G$  is a convex subgroup of a real closed field  $M$  we write  $z^*(G)$  instead of  $z^*(G^+)$ . Hence

$$z^*(G) := \begin{cases} a \cdot \mathfrak{m}(G)^+ & \text{if } a \cdot V(G) = G, a \neq 0, \\ a \cdot V(G)^+ & \text{if } a \cdot \mathfrak{m}(G) = G, a \neq 0, \\ G^+ & \text{if } \text{sign}^* G^+ = 0. \end{cases}$$

We come to our main result:

**THEOREM 7.3.** *Let  $M$  be a model of  $T$  and let  $p$  be a cut of  $M$  such that  $f(V(p)^+) = V^+$  for some  $T$ -convex valuation ring  $V$  of  $M$  and some  $M$ -definable map  $f : M \rightarrow M$ . Suppose  $\text{sign } p = 0$  and  $p$  is not dense. Let  $N$  be an elementary extension of  $M$  and let  $q$  be a cut of  $N$  extending  $p$  such that  $G(q) \cap M = G(p)$ ,  $V(q) \cap M = V(p)$  and  $f(V(q)^+) = W^+$ , where  $W$  is a  $T$ -convex valuation ring of  $N$ .*

- (i) *If  $\text{sign}^* \hat{p} = 0$  then  $(V(q)^+, \hat{q}, q)$  is an heir of  $(V(p)^+, \hat{p}, p)$ .*
- (ii) *If  $\text{sign}^* \hat{q} = \text{sign}^* \hat{p}$  and  $Z^*(\hat{q}) \cap M = Z^*(\hat{p})$ , then  $(V(q)^+, \hat{q}, z^*(\hat{q}), q)$  is an heir of  $(V(p)^+, \hat{p}, z^*(\hat{p}), p)$ .*

**PROOF.** Since  $p$  is not dense,  $V(p)$  is proper. Since  $V(q)^+$  extends  $V(p)^+$ ,  $W^+ = f(V(q)^+)$  extends  $V^+ = f(V(p)^+)$ . Since  $V(p)$  is proper,  $V$  is proper and  $W^+$  is an heir of  $V^+$ . So  $V(q)^+$  is an heir of  $V(p)^+$ .

(i). We may assume that  $p > 0$ . By 2.8 we have to show the following: If  $\varphi(v_1, \dots, v_6)$  is an  $\mathcal{L}(M)$ -formula and  $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in N$  are such that

$$\alpha_1 < q < \alpha_2, \beta_1 < \hat{q} < \beta_2, \gamma_1 < V(q)^+ < \gamma_2, N \models \varphi(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2),$$

then there are  $a_1, a_2, b_1, b_2, c_1, c_2 \in M$  such that

$$a_1 < p < a_2, b_1 < \hat{p} < b_2, c_1 < V(p)^+ < c_2, M \models \varphi(a_1, a_2, b_1, b_2, c_1, c_2).$$

First we prove that  $(q, \hat{q})$  is an heir of  $(p, \hat{p})$ . Since  $\hat{p}$  has multiplicative signature 0, using our assumptions on  $V(p)$  and  $V(q)$ , we can apply 5.5, which shows that  $\hat{q}$  is an heir of  $\hat{p}$ . Since  $\text{sign } p = 0$  and  $V(p)^+ \sim V^+$ , the signature alternative holds for  $p$  by 7.2(i). Hence we can apply 3.19 which shows that  $(q, \hat{q})$  is an heir of  $(p, \hat{p})$ .

If  $\gamma_1$  is not a realization of  $V(p)^+$ , then there are elements  $a_i, b_i, c_i$  as claimed: this follows from 3.17 applied to the theory  $T^{>0}$ , because  $(q, \hat{q})$  is an heir of  $(p, \hat{p})$ . Hence we may assume that  $\gamma_1$  is a realization of  $V(p)^+$ . Since  $\text{sign } p = \text{sign}^* \hat{p} = 0$ , neither  $p$  nor  $\hat{p}$  is of the form  $a \pm b \cdot (V(p)^+)^{\pm 1}$  with  $a, b \in M$ . Since  $V(p)$  is definable in  $(M, V)$ , 5.2 is applicable. We get that neither  $p^L$  nor  $G(p)$  are definable in  $(M, V(p))$ . Therefore, and since  $\gamma_1$  realizes  $V(p)^+$ ,  $p$  and  $\hat{p}$  must be omitted in  $M\langle\gamma_1\rangle$ .

Let  $p_1 := q \upharpoonright M\langle\gamma_1\rangle$  be the unique extension of  $p$  on  $M\langle\gamma_1\rangle$ . Then  $\hat{p}_1$  is the unique extension of  $\hat{p}$  on  $M\langle\gamma_1\rangle$ . Since  $V(q)^+$  is an heir of  $V(p)^+$  we can apply 3.18 to  $T^{>0}$ , thus  $V(q)$  lies over  $V(p_1)$ , which shows that  $f(V(p_1)^+)$  is the upper edge of the  $T$ -convex valuation ring  $W \cap M\langle\gamma_1\rangle$ . Hence the assumptions of the Theorem for  $p$  and  $q$  also hold for  $p_1$  and  $q$ . Because  $\text{sign}^* \hat{p}_1 = 0$  we can apply what we have already shown, namely:  $(q, \hat{q})$  is an heir of  $(p_1, \hat{p}_1)$ .

By 3.17 applied to  $T^{>0}$ ,  $(q, \hat{q})$  and  $(p_1, \hat{p}_1)$ , there are  $M$ -definable maps  $h, g_1, g_2, f_1, f_2$  with

$$f_1(\gamma_1) < p_1 < f_2(\gamma_1), \quad g_1(\gamma_1) < \hat{p}_1 < g_2(\gamma_1), \quad \gamma_1 < V(p_1)^+ < h(\gamma_1)$$

and

$$M\langle\gamma_1\rangle \models \varphi(f_1(\gamma_1), f_2(\gamma_1), g_1(\gamma_1), g_2(\gamma_1), \gamma_1, h(\gamma_1)).$$

Therefore we can find elements  $a_i, b_i, c_i$  in  $M$  as desired, if we know that  $(V(p_1)^+, \hat{p}_1, p_1)$  is an heir of  $(V(p)^+, \hat{p}, p)$ . To see this, note first that  $V(p_1)^+$  is an heir of  $V(p)^+$ . Then, it follows from a standard compactness argument (cf. [Poi] 11.01) that there are cuts  $r_1, r_2, r_3$  of  $M\langle\gamma_1\rangle$  with  $r_1 = V(p_1)^+$  such that  $(r_1, r_2, r_3)$  is an heir of  $(V(p)^+, \hat{p}, p)$ . Since  $p_1, \hat{p}_1$  are the unique extensions of  $p, \hat{p}$  respectively on  $M\langle\gamma_1\rangle$  we get  $p_1 = r_3$  and  $\hat{p}_1 = r_2$ . So  $(V(p_1)^+, \hat{p}_1, p_1)$  is an heir of  $(V(p)^+, \hat{p}, p)$  and this finishes the proof of (i).

(ii). If  $\text{sign}^* \hat{p} = \text{sign}^* \hat{q} = 0$ , then  $z^*(\hat{p}) = \hat{p}, z^*(\hat{q}) = \hat{q}$  and (ii) is implied by (i). So we assume that  $\varepsilon := \text{sign}^* \hat{p} = \text{sign}^* \hat{q} \neq 0$ , hence  $Z^*(\hat{q}) \cap M = Z^*(\hat{p}) \neq \emptyset$ . Let  $a \in Z^*(\hat{p})$  and let  $g(x) := a \cdot x^\varepsilon$ . Then  $g(V(p)^+) = \hat{p}$  and  $g(V(q)^+) = \hat{q}$ . Since  $V(q)^+$  is an heir of  $V(p)^+$ ,  $\hat{q}$  is an heir of  $\hat{p}$ . By 7.2(i) and 3.19 again, it follows that  $(\hat{q}, q)$  is an heir of  $(\hat{p}, p)$ .

Let  $h(x) = a \cdot x^{-\varepsilon}$ . Then  $h(V(p)^+) = z^*(\hat{p}), h(V(q)^+) = z^*(\hat{q})$  and it follows easily that  $(V(q)^+, \hat{q}, z^*(\hat{q}), q)$  is an heir of  $(V(p)^+, \hat{p}, z^*(\hat{p}), p)$ .  $\dashv$

**THEOREM 7.4.** *Let  $T$  be model complete and let  $f(y, x_1, \dots, x_n)$  be a 0-definable map. Let  $\mathcal{O}, \mathcal{G}, \mathcal{Z}, \mathcal{Z}^*$  and  $\mathcal{D}$  be new unary predicates and let  $c_1, \dots, c_n$  be new constants with respect to  $\mathcal{L}$ . Let  $\varepsilon, \delta \in \{-1, 0, 1\}$  and let  $\mathcal{L}^*$  be the language  $\mathcal{L}(\mathcal{O}, \mathcal{G}, \mathcal{Z}, \mathcal{Z}^*, \mathcal{D}, c_1, \dots, c_n)$ .*

*Let  $T^*$  be the  $\mathcal{L}^*$ -theory which extends  $T$  and which says the following about a model  $(M, V, G, Z, Z^*, D, d_1, \dots, d_n)$ :*

- (a)  $D = p^L$  for some cut  $p$  of  $M$ ,  $p$  neither dense nor principal with  $\text{sign } p = \delta$ .
- (b)  $V = V(p)$ .
- (c)  $G = G(p)$  and  $\text{sign}^* G^+ = \varepsilon$ .

- (d)  $f(V(p)^+, d_1, \dots, d_n) > 0$  is the upper edge of a  $T$ -convex valuation ring of  $M$ .  
This is axiomatizable by the axioms

$$g(f(V(p)^+, d_1, \dots, d_n)) \leq f(V(p)^+, d_1, \dots, d_n)$$

for each continuous, 0-definable map  $g$ .

- (e)  $Z = \{a \in M \mid a + G^+ = p \text{ or } a - G^+ = p\}$  and  $Z^* = \{a \in M^* \mid a \cdot V = G \text{ or } a \cdot \mathfrak{m} = G\}$ .

Then

- (i)  $T^*$  is consistent if and only if there is a model  $M$  of  $T$ , a convex valuation ring  $V$  of  $M$  and  $d_1, \dots, d_n \in M$  such that  $f(V^+, d_1, \dots, d_n)$  is the upper edge of a  $T$ -convex valuation ring.  
(ii) If  $T^*$  is consistent then  $T^*$  is model complete.

PROOF. (i). Certainly the condition is necessary for the consistency of  $T^*$ . Conversely suppose we have a model  $M$  of  $T$ , a convex valuation ring  $V$  of  $M$  and  $d_1, \dots, d_n \in M$  such that  $f(V^+, d_1, \dots, d_n)$  is the upper edge of a  $T$ -convex valuation ring. By 6.6 there is some  $N \succ M$  and an extension  $q$  of  $V^+$  on  $N$  with  $\text{sign } q = \delta$  and  $\text{sign}^* \hat{q} = \varepsilon$  such that  $V(q)^+$  is an heir of  $V^+$ . Hence  $f(V(q)^+, d_1, \dots, d_n)$  is an heir of  $f(V^+, d_1, \dots, d_n)$ , thus  $f(V(q)^+, d_1, \dots, d_n)$  is the upper edge of a  $T$ -convex valuation ring of  $N$  by 4.6(iii). This proves that  $T^*$  is consistent.

(ii). In order to prove model completeness of  $T^*$  we use Robinson's test. So let  $\mathcal{M} \subseteq \mathcal{N}$  be an extension of  $\mathcal{L}^*$ -structures, such that  $\mathcal{M}, \mathcal{N} \models T^*$ . We have to show that  $\mathcal{M}$  is existentially closed in  $\mathcal{N}$ . If  $\delta \neq 0$  then this follows easily from 5.6 and the assumption that  $p$  is not principal. If  $\delta = 0$ , then  $\mathcal{M}$  is existentially closed in  $\mathcal{N}$  by 7.3.  $\dashv$

We conclude with an open question, how non  $T$ -convex valuation rings can arise in o-minimal structures. Let  $T$  be the theory of the real exponential field  $(\mathbb{R}, \exp)$ , which is o-minimal by [Wi]. Let  $M$  be a model of  $T$  containing  $(\mathbb{R}, \exp)$  which is the definable closure of  $\mathbb{R}$  together with a single new element  $x \in M \setminus \mathbb{R}$ . Let  $W$  be a proper convex valuation ring of  $M$ . Then  $\log(W^{>0})$  is a convex subgroup of  $(M, +)$  and the invariance valuation ring  $W'$  of this convex subgroup is contained in  $W$ . Certainly  $W$  is  $T$ -convex — hence equal to the convex hull  $V$  of  $\mathbb{R}$  in  $M$  — if and only if  $W = W'$ . Is it possible that  $W \supsetneq W' \supsetneq W'' \supsetneq \dots$ ?

More generally, is it possible that  $V$  is not definable in  $(M, W)$ ? Observe that  $V$  is definable in  $(M, W)$  if  $W$  is definable in  $(M, V)$ , by 4.6(i).

#### REFERENCES

- [Ba] W. BAUR, *On the elementary theory of pairs of real closed fields. II*, this JOURNAL, vol. 47 (1982), no. 3, pp. 669–679.  
[Da-Wo] H. G. DALES and W. H. WOODIN, *Super-real fields*, London Mathematical Society Monographs. New Series, vol. 14, The Clarendon Press Oxford University Press, New York, 1996.  
[Ch-Dic] G. GHERLIN and M. DICKMANN, *Real closed rings. II. Model Theory*, *Annals of Pure and Applied Logic*, vol. 25 (1983), no. 3, pp. 213–231.  
[Mac] A. MACINTYRE, *Classifying pairs of real closed fields*, *Ph.D. thesis*, Stanford University, 1968.  
[MMS] D. MACPHERSON, D. MARKER, and C. STEINHORN, *Weakly o-minimal structures and real closed fields*, *Transactions of the American Mathematical Society*, vol. 352 (2000), no. 12, pp. 5435–5483.  
[Ma] D. MARKER, *Omitting types in o-minimal theories*, this JOURNAL, vol. 51 (1986), no. 1, pp. 63–74.

- [Poi] B. POIZAT, *Cours de théorie des modèles*, Villeurbanne: Nur al-Mantiq wal-Ma'rifah, 1985.
- [T1] M. TRESSL, *Dedekind cuts in polynomially bounded o-minimal expansions of real closed fields*, Dissertation, Regensburg 1996.
- [T2] ———, *The elementary theory of Dedekind cuts in polynomially bounded structures*, submitted.
- [vdD1] L. VAN DEN DRIES, *T-convexity and tame extensions. II*, this JOURNAL, vol. 62 (1997), no. 1, pp. 14–34.
- [vdD2] ———, *Dense pairs of o-minimal structures*, *Fundamenta Mathematicae*, vol. 157 (1998), no. 1, pp. 61–78.
- [vdD3] ———, *Tame topology and o-minimal structures*, London Mathematical Society Lecture Note Series, vol. 248, Cambridge University Press, Cambridge, 1998.
- [vdD-Lew] L. VAN DEN DRIES and A. H. LEWENBERG, *T-convexity and tame extensions*, this JOURNAL, vol. 60 (1995), no. 1, pp. 74–102.
- [vdD-S] L. VAN DEN DRIES and P. SPEISSEGER, *The field of reals with multisummable series and the exponential function*, *Proceedings of the London Mathematical Society. Third Series*, vol. 81 (2000), no. 3, pp. 513–565.
- [Wi] A. J. WILKIE, *Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function*, *Journal of the American Mathematical Society*, vol. 9 (1996), no. 4, pp. 1051–1094.

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