LATTICE ORDERED ABELIAN GROUPS AND THEIR MODEL THEORY

MARCUS TRESSL

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Contents

Some standard sources: For lattice ordered Abelian groups (aka Abelian ℓ-groups) our principal reference is [Darnel1995]. A shorter text is [AndFei1988]. A much cited book is [BiKeWo1977]. A more recent book is [KopMed1994] and a book also addressing fields, rings and modules is [Steinb2010]. For vector lattices, see the two volumes on Riesz spaces [LuxZaa1971, Zaanen1983]. A book on advanced techniques is [GlaHol1989], which also contains some model theory mentioned in the overview [Weispf1989].

1. Overview

1.1. **An ad hoc definition** Let X be a set and for each $x \in X$ let $\Lambda_x = (\Lambda_x, +, \leq)$ be a TOAG (Totally Ordered Abelian Group). Let

$$\Lambda = \prod_{x \in X} \Lambda_x$$

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be the product structure, i.e. $\Lambda = (\Lambda, +, \leq)$, where $(f+g)(x) = f(x) + g(x)$ and $f \leq g \iff \forall x \in X : f(x) \leq g(x)$. Then $(\Lambda, +)$ is an abelian group and $(\Lambda, \leq)$ is a poset (=partially ordered set) compatible with $+$, i.e.

$$f \leq g \Rightarrow f + h \leq g + h \text{ for all } f, g, h \in \Lambda.$$

Furthermore, for all $f, g \in \Lambda$ the supremum $f \vee g = \sup\{f, g\}$ of $\{f, g\}$ in Λ exists and is given by

$$(f \vee g)(x) = \max\{f(x), g(x)\} \ (x \in X).$$

Similarly, the infimum $f \wedge g = \inf\{f, g\}$ of $\{f, g\}$ in Λ exists and is given by

$$(f \wedge g)(x) = \min\{f(x), g(x)\} \ (x \in X).$$

An ℓ-group in this course is (up to isomorphism) a subgroup of some such Λ that is closed under $\wedge$ and $\vee$. We usually consider G as a structure in the language $\{+, -, 0, \wedge, \vee\}$.

1.2. **Examples - crucial for the development of the theory**

(0) Products of TOAGs (or in fact of ℓ-groups). These have to be named here but do not really vindicate the systematic study of a new algebraic structure.

(1) *Continuous functions.* Let X be a topological space and let Γ be a TOAG equipped with a topology making subtraction and $\max, \min : \Gamma \times \Gamma \longrightarrow \Gamma$ continuous (e.g. the order topology on Γ or the discrete topology on Γ).

Let $G = C(X, \Gamma)$ be the set of continuous functions $X \longrightarrow \Gamma$. Then G is a subgroup of $\Lambda = \Gamma^X$ closed under $\wedge$ and $\vee$, hence G is an ℓ-group.

There are variations of this example:

(a) Let M be an expansion of a TOAG and let $X \subseteq M^n$ be definable and endowed with the product of the order topology. Then the set of (continuous) definable functions $X \longrightarrow M$ is an ℓ-group.

(b) Let K be an ordered field. Then $C(X, K)$ is even a ring, the ring of continuous functions $X \longrightarrow K$. There are two ℓ-groups associated to this ring: $(C(X, K), +, \leq)$ and $(C(X, K)^{\times, >0}, \cdot, \leq)$ (positive units of the ring $C(X, K)$)

In each of the examples here we also have the ℓ-group of bounded functions: Those functions f for which there is some $\gamma \in \Gamma$ with $|f(x)| \leq \gamma$ $(x \in X)$.

(2) *Value groups of Bézout domains.* A Bézout domain is a commutative domain R such that all finitely generated ideals are principal (e.g: $\mathbb{Z}$, or valuation rings of fields). Let R be a Bézout domain with fraction field K. Let G be the group $K^\times/R^\times$, partially ordered by $aR^\times \leq bR^\times \iff b/a \in R$. Then G is an ℓ-group as one verifies easily - once we have the correct algebraic definition. This group is not given in the style of example (1) or the definition 1.1 and needs justification. This example is crucial because by the Jaffard-Kaplansky-Ohm theorem, every ℓ-group comes from a Bézout domain in this way.

We will talk about this in the course.

2. Partially ordered groups and ℓ-groups

2.1. **Definition.**
A **po-group** is a (not necessarily abelian) group $G = (G, +)$ together with a partial order $\leq$ such that for all $a, b, c \in G$ we have

$$a \leq b \Rightarrow a + c \leq b + c, c + a \leq c + b.$$

If in addition $(G, \leq)$ is a lattice, then G is a **lattice ordered group** or an **ℓ-group**[1].

In these notes all po-groups and all ℓ-groups will be assumed to be abelian; hence we just write po-group/ℓ-group to mean "abelian po-group/abelian ℓ-group"

Thus an ℓ-group here is an abelian group G together with a partial order $\leq$ such that $(G, \leq)$ is a lattice satisfying $a \leq b \Rightarrow a + c \leq b + c$ for all $a, b, c \in G$.

The main example of po-groups are subgroups of powers of a totally ordered abelian group together with the induced order. However, in general po-group are not even torsion free (think of the trivial poset).

2.2**. Basic Distributivity laws in po-groups** Let G be a po-group.

(i) By definition 2.1, every $c \in G$ gives rise to a monotone map $f_c : G \longrightarrow G,\ f_c(a) = a + c$. For $a \in G$ we then have $f_c \circ f_{-c}(a) = f_c(a - c) = a$, hence f_{-c} is the compositional inverse of f_c. It follows that f_c is an automorphism of $(G, \leq)$.

(ii) The map $G \longrightarrow G,\ a \mapsto -a$ is an isomorphism $(G, \leq) \longrightarrow (G, \geq)$, because $a \leq b$ implies $-b = a + (-a - b) \leq b + (-a - b) = -a$.
Hence if $H \subseteq G$, then $\sup H$ exists if and only if $\inf(-H)$ exists and in this case $\inf(-H) = -\sup H$.

(iii) If $g \in G$ and $H \subseteq G$, then $\sup H$ exists if and only if $\sup(g + H)$ exists and in this case $g + \sup H = \sup(g + H)$. This follows from (i), which says that the map $G \longrightarrow G,\ x \mapsto g + x$ is an automorphism of $(G, \leq)$. Similarly, if $\inf H$ exists, then $\sup(-H)$ exists and $\inf H = -\sup(-H)$ using (ii). It follows $g + \inf(H) = \inf(g + H)$ because $-(g + \inf H) = -g + \sup(-H) = \sup(-g - H) = -\inf(g + H)$.
Assuming finite suprema and infima exists, this means

$$(\vee) \qquad g + (h_1 \vee h_2) = (g + h_1) \vee (g + h_2) \text{ and}$$
$$(\wedge) \qquad g + (h_1 \wedge h_2) = (g + h_1) \wedge (g + h_2).$$

Warning: Notice that $\wedge, \vee$ do not distribute over $+$, e.g. if $a \geq 0$ and $a \neq 0$, then $a \wedge (a + a) = a \not\geq 2a = a \wedge a + a \wedge a$ and $0 \wedge (a - a) = 0 \not\leq -a = 0 \wedge a + 0 \wedge -a$.

On the other hand, (iii) implies distributivity of $\wedge$ and $\vee$ (if they exist) and a modularity law, as we see now in (iv) and below in 2.3.

(iv) If $g, h \in G$ and $g \vee h, g \wedge h$ exist in G, then

$$g + h = (g \vee h) + (g \wedge h),$$

[1] In [BiKeWo1977, 1.2.1, p. 16] ℓ-groups are called **groupe réticulé** ('reticular group')

since

$$g + h - (g \wedge h) = g + (-g \vee -h) + h, \text{ by (ii)}$$
$$= (0 \vee (g - h)) + h, \text{ by (iii)}$$
$$= h \vee g, \text{ by (iii)}.$$

(v) The posets on an abelian group that make G into a po-group are in bijection with the subsets P of G that have the property $P+P \subseteq P$ and $P \cap (-P) = \{0\}$.

(vi) Every partial order of a torsion-free abelian group G can be extended to a total order. This is important in the proof of the Jaffard-Ohm-Kaplansky Theorem; see 4.1 for a proof.

2.3. **Proposition.** *Let G be an ℓ-group. If $S \subseteq G$ is any set such that* $\sup(S)$ *exists, then for every $f \in G$ also* $\sup(f \wedge S)$ *exists and* $f \wedge \sup(S) = \sup(f \wedge S)$.[2] *Similarly, if* $\inf(S)$ *exists, then* $\inf(f \wedge S)$ *exists and* $f \vee \inf(S) = \inf(f \vee S)$.

In particular G is distributive.

Proof. Let $t = \sup(S)$. Then for $s \in S$ we have $0 \leq t - s$, hence $f \wedge t \leq (f + (t - s)) \wedge t = t - s + (f \wedge s)$ and therefore

$$(*) \qquad 0 \leq (f \wedge t) - (f \wedge s) \leq t - s.$$

Now

$$\begin{aligned} 0 &= t - \sup(S) \\ &= t + \inf(-S) \text{ and } \inf(-S) \text{ exists by 2.2(ii)} \\ &= \inf(t - S) \text{ and } \inf(t - S) \text{ exists by 2.2(iii)} \\ &= \inf\{(f \wedge t) - (f \wedge s) \mid s \in S\} \text{ and the infimum exists by } (*) \\ &= (f \wedge t) + \inf\{-(f \wedge s) \mid s \in S\} \text{ and the infimum exists by 2.2(iii)} \\ &= (f \wedge t) - \sup(f \wedge S) \text{ and } \sup(f \wedge S) \text{ exists by 2.2(ii)}. \end{aligned}$$

Thus $\sup(f \wedge S)$ exists and is equal to $f \wedge t = f \wedge \sup(S)$. Applying the automorphism $G \longrightarrow G,\ g \mapsto -g$ of $(G, \leq)$ and using (iii) we get the statement for the infimum: $f \vee \inf(S) = -(-f \wedge -\inf(S)) = -(-f \wedge \sup(S)) = -\sup((-f) \wedge S) = \inf(f \vee S)$. □

2.4**. Decomposition of elements**

Let G be an ℓ-group.

(i) Let $f \in G$.

(a) $f = (f \vee 0) + (f \wedge 0)$, by 2.2(iv) applied to f and 0. Hence f is the difference of the two elements

$$f^+ := f \vee 0, \text{ the } \textbf{positive part of } f \text{ and}$$
$$f^- := -(f \wedge 0) = (-f) \vee 0, \text{ the } \textbf{negative part of } f,$$

and $f^+, f^- \geq 0$.

[2]In this property of a lattice is called *Brouwerian*, which is different from saying that a lattice is Heyting.

(b) $(f \vee 0) \wedge -(f \wedge 0) = 0$, because

$$\begin{aligned} 0 &\leq (f \vee 0) \wedge -(f \wedge 0), \text{ as } f \wedge 0 \leq 0 \leq f \vee 0 \\ &= (f \vee 0) \wedge ((f \vee 0) - f) \text{ by (a)} \\ &= (f \vee 0) + (0 \wedge -f) \text{ by 2.2(iii)}(\wedge) \\ &= (f \vee 0) - (f \vee 0) \text{ by 2.2(ii)} \\ &= 0. \end{aligned}$$

(c) $(f \vee 0) - (f \wedge 0) = (f \vee 0) \vee -(f \wedge 0) = f \vee -f$, in particular the **absolute value**

$$|f| := f \vee -f$$

satisfies $|f| \geq 0$.
Proof. The identity $(f \vee 0) - (f \wedge 0) = (f \vee 0) \vee -(f \wedge 0)$ follows from 2.2(iv) and (b). For the second identity we have $(f \vee 0) \vee -(f \wedge 0) = (f \vee 0) \vee (-f) \vee 0$ and it suffices to show that $f \vee -f \geq 0$.
From $f \vee -f \geq f$ we get $2(f \vee -f) \geq (f \vee -f) + f = (2f) \vee 0 \geq 0$. Hence it suffices to show that for any $g \in G$ with $2g \geq 0$ we have $g \geq 0$.
This is because $2(g \wedge 0) = (g \wedge 0) + (g \wedge 0) = ((g \wedge 0) + g) \wedge (g \wedge 0) = (2g) \wedge g \wedge 0 = g \wedge 0$; thus $2(g \wedge 0) = g \wedge 0$ and so $g \wedge 0 = 0$, i.e. $g \geq 0$. ⋄

(d) **The triangle inequality**: If $g \in G$, then $|f + g| \leq |f| + |g|$.[3]
Proof. $f + g = (f \vee 0) + (f \wedge 0) + (g \vee 0) + (g \wedge 0) \leq (f \vee 0) + (g \vee 0)$, because $(f \wedge 0) + (g \wedge 0) \leq 0$. Hence $(f + g) \vee 0 \leq (f \vee 0) + (g \vee 0)$. But then also $-((f + g) \wedge 0) = (-f - g) \vee 0 \leq ((-f) \vee 0) + ((-g) \vee 0) = -(f \wedge 0) - (g \wedge 0)$. It follows that $|f + g| = ((f + g) \vee 0) - ((f + g) \wedge 0) \leq (f \vee 0) + (g \vee 0) - (f \wedge 0) - (g \wedge 0) = |f| + |g|$ by (c). ⋄

(e)
- $f \wedge -(f \wedge 0) = f \wedge 0$, because $f \wedge -(f \wedge 0) = f \wedge ((-f) \vee 0) \overset{2.3}{=\!=} (f \wedge (-f)) \vee (f \wedge 0) = (-|f|) \vee (f \wedge 0) = f \wedge 0$, since $-|f| \leq f \wedge 0$ by (c).
- $f \vee -(f \wedge 0) \overset{2.3}{=\!=} f \vee (-f) \vee 0 = |f|$, by (c).

(ii) For every $f \in G$ there are uniquely determined $g, h \in G$ with $g \wedge h = 0$ and $f = g - h$, namely $g = f \vee 0$ and $h = -(f \wedge 0) = (-f) \vee 0$.
Proof. That $f \vee 0$ and $-(f \wedge 0)$ have these property has been shown in (i).
Take $g, h \in G$ with $f = g - h$ and $g \wedge h = 0$. We show that $g = f \vee 0$ and $h = -(f \wedge 0)$. We have $f \wedge 0 = (g - h) \wedge 0 = (g \wedge h) - h$ by 2.2(iii)($\wedge$). By assumption $g \wedge h = 0$, thus $f \wedge 0 = -h$. Since $f \vee 0 = f - (f \wedge 0)$ by (i)(a) we obtain $f \vee 0 = g - h + h = g$ as required. ⋄

(iii) Every term t in the language $\{+, -, 0, \wedge, \vee\}$ is a composition of terms in the language $\{+, -, 0\}$ extended by a unary function $x \vee 0$. Similarly, t is a composition of terms in the language $\{+, -, 0\}$ extended by a unary function $x \wedge 0$.
Proof. Since $f \wedge g = -((-f) \vee (-g))$ it suffices to show that $f \vee g$ is a composition of $+, -$ and the function $x \vee 0$. This is shown in the following

[3] This property necessarily needs commutativity of G and in fact characterizes commutativity, see [Darnel1995, Prop. 4.15, p. 20] .

identities.

$$\begin{aligned} f\vee g &= \Big((f\vee 0)-(0\vee -f)\Big)\vee\Big((g\vee 0)-(0\vee -g)\Big) \text{ by (i)(a)}\\ &= \Big((f\vee 0)-(0\vee -f)-(g\vee 0)+(0\vee -g)\Big)\vee 0\\ &\quad +(g\vee 0)-(0\vee -g) \text{ by (iii)}(\vee). \end{aligned}$$

◇

(iv) Let $g, h_1, h_2 \in G$. Then
   (a) $(g\wedge h_1)+(g\wedge h_2)\leq (2g)\wedge(h_1+h_2)$ and

Polar inequality $g, h_1, h_2 \geq 0 \Longrightarrow g\wedge(h_1+h_2)\leq (g\wedge h_1)+(g\wedge h_2)$

Proof. Using distributivity of $+$ over $\wedge$ we have

$$\begin{aligned} (g\wedge h_1)+(g\wedge h_2) &= ((g\wedge h_1)+g)\wedge((g\wedge h_1)+h_2)\\ &= (g+g)\wedge(h_1+g)\wedge(g+h_2)\wedge(h_1+h_2)\\ &\begin{cases}\leq (2g)\wedge(h_1+h_2)\\ \geq g\wedge(h_1+h_2) & \text{if } g,h_1,h_2\geq 0.\end{cases} \end{aligned}$$

◇

   (b) $(2g)\vee(h_1+h_2)\leq(g\vee h_1)+(g\vee h_2)$, because by distributivity of $+$ over $\vee$ we have

$$\begin{aligned} (g\vee h_1)+(g\vee h_2) &= ((g\vee h_1)+g)\vee((g\vee h_1)+h_2)\\ &= (g+g)\vee(h_1+g)\vee(g+h_2)\vee(h_1+h_2)\\ &\geq (2g)\vee(h_1+h_2). \end{aligned}$$

(v) **Riesz decomposition** If $f, g_1, \ldots, g_n \in G^{\geq 0}$ with $f \leq g_1+\ldots+g_n$, then there are $h_i \leq g_i$ with $h_i \geq 0$ such that $f = h_1+\ldots+h_n$.
Proof. It suffices to do the case $n=2$, because then $f\leq g_1+\ldots+g_{n+1}$ gives $0\leq h_0\leq g_1+\ldots+g_n$ and $0\leq h_{n+1}\leq g_{n+1}$ with $f=h_0+h_{n+1}$ and by induction we get the assertion.

Hence assume $f\leq g_1+g_2$. Using the polar inequality 2.2(iv)(a) we know $f=f\wedge(g_1+g_2)\leq(f\wedge g_1)+(f\wedge g_2)$. Then $0\leq f-(f\wedge g_1)\leq f\wedge g_2\leq g_2$. Hence we may take $h_1=f\wedge g_1$ and $h_2=f-(f\wedge g_1)$. ◇

2.5. **ℓ-group homomorphisms and ℓ-subgroups** Let G, H be ℓ-groups. A map $\varphi: G\longrightarrow H$ is an ℓ-group **homomorphism** if and only if φ is a group homomorphism preserving $\wedge$ and $\vee$. Obviously φ then also preserves the partial order. If G is a subgroup of H and φ is the inclusion map, then G is called an **ℓ-subgroup** of H.

(i) Let $\varphi: G\longrightarrow H$ be a group homomorphism. The following are equivalent.
   (a) φ is an ℓ-group homomorphism.
   (b) φ preserves $\wedge$.
   (c) φ preserves $\vee$.
   (d) For all $f\in G$ we have $\varphi(f\wedge 0)=\varphi(f)\wedge 0$.
   (e) For all $f\in G$ we have $\varphi(f\vee 0)=\varphi(f)\vee 0$.
   (f) φ is monotone and preserves disjointness, i.e. $f\wedge g=0$ implies $\varphi(f)\wedge\varphi(g)=0$.

Proof. (a) $\iff$ (b) $\iff$ (c) follows from the identity $x \wedge y = -(-x \vee -y)$, valid in all ℓ-groups by 2.2(ii). For the same reason, (d) is equivalent to (e). Item (c) trivially implies (e). Condition (e) implies (a) by 2.2(iii). Hence (a)–(e) are equivalent. Obviously (a) implies (f).

(f)$\Rightarrow$(d),(e). Since f is monotone we have $\varphi(f \vee 0), \varphi(-(f \wedge 0)) \geq 0$. Since $(f \vee 0) \wedge -(f \wedge 0) = 0$ (by 2.4(i)) and f preserves disjointness we have $\varphi(f \vee 0) \wedge \varphi(-(f \wedge 0)) = 0$. Since $\varphi(f) = \varphi((f \vee 0) + (f \wedge 0)) = \varphi(f \vee 0) + \varphi(f \wedge 0)$ we see from 2.4(ii) that $\varphi(f \vee 0) = \varphi(f) \vee 0$ and $\varphi(f \wedge 0) = \varphi(f) \wedge 0$. ⋄

(ii) If U is a subgroup of H, then U is an ℓ-subgroup of H if and only if $U \wedge 0 \subseteq U$ if and only if $U \vee 0 \subseteq U$. This follows again from 2.2(iii).

2.6. **Corollary.** *(Weinberg)*
Let U be a subgroup of an ℓ-group G. The lattice ordered group G_U generated by U in G is

$$G_U = \{\sup_{i \in I} \inf_{j \in J} u_{ij} \mid (u_{ij})_{i \in I, j \in J} \subseteq U,\ I, J \text{ finite and nonempty}\}.$$

Proof. We only need to show that G_U is an ℓ-subgroup of G. That G_U is a sublattice of G follows from distributivity, see 2.3. Further $-G_U = G_U$ follows from 2.2(ii) and distributivity. Finally $G_U + G_U \subseteq G_U$ follows from 2.2(iii). □

2.7. **Definition.** Let K be an ordered field (we will mostly choose $K = \mathbb{Q}$ or $K = \mathbb{R}$). A K**-vector lattice** is an ℓ-group G and a K-vector space such that for all $r \in K^{\geq 0}$ and all $g \in G^{\geq 0}$ we have $r \cdot g \geq 0$; see . If $K = \mathbb{R}$, then vector lattices are also called **Riesz spaces**, see [LuxZaa1971, Zaanen1983].

2.8. **Corollary.** *Let X be a set and let V be the K-vector lattice of all functions $X \longrightarrow K$, for some ordered field K. If $U \subseteq V$ is a subgroup, then the K-vector lattice generated by U in V is*

$$K \cdot G_U = G_{K \cdot U},$$

in the notation of 2.6.

Proof. It is clear that $G_{K \cdot U} \subseteq K \cdot G_U$ and that $K \cdot G_U$ is contained in the K-vector lattice generated by U in V. Hence it is enough to show that $G_{K \cdot U}$ is closed under scalar multiplication. But this is obvious from the description of G_U in 2.6, since $r \cdot (f \vee g) = (rf \vee rg)$, when $r \geq 0$, $r \cdot (f \vee g) = (rf \wedge rg)$ if $r \leq 0$ and because G_U is distributive. □

2.9. **Proposition.** *For $n \in \mathbb{N}$ the map $\varphi : G \longrightarrow G$, $\varphi(x) = n \cdot x$ is an ℓ-group embedding, in particular G is torsion free and satisfies $(ng) \vee (nh) = n(g \vee h)$, $(ng) \wedge (nh) = n(g \wedge h)$ and $ng \leq nh \iff g \leq h$.*[4]

Proof. Since G is abelian, φ is a group homomorphism. By induction on n we see that φ is monotone: $g \leq h \Rightarrow (n+1)g = ng + g \leq nh + g \leq nh + h = (n+1)h$.

In order to show that φ is an ℓ-group homomorphism we use 2.5(i) and we only need to show that $\varphi(g) \vee 0 = \varphi(g \vee 0)$, i.e. $(ng) \vee 0 = n(g \vee 0)$.
Claim. For all $f, g \in G$ with $f \wedge g = 0$ and each $k \in \mathbb{N}$ we have $f \wedge kg = 0$.

[4]By [AndFei1988, Theorem 4.1.1], a not necessarily abelian ℓ-group is *representable* (in the sense that it is isomorphic to an ℓ-subgroup of a product of totally ordered groups) if and only if $(2g) \wedge (2h) = 2(g \wedge h)$ for all g, h.

Proof. Since $g \geq 0$ we have $kg \leq (k+1)g$. Hence $f \wedge kg = 0$ follows from $f \wedge 2^k g = 0$ (and $f \wedge g = 0$). Thus by induction it suffices to show $f \wedge 2g = 0$. But this holds by 2.2(iv)(a): Since $f, g \geq 0$ we have $f \wedge 2g = f \wedge (g+g) \leq f \wedge g + f \wedge g = 0$. $\diamond$

Now take $f \in G$. Write $g = f \vee 0$ and $h = -(f \wedge 0)$. Then $g, h \geq 0$ and by 2.4(ii) we know that $g \wedge h = 0$. By the claim we get $g \wedge nh = 0$ and then by the claim again also $(ng) \wedge (nh) = 0$. Since $f = g - h$ by 2.4(ii), we have $nf = ng - nh$. But now the uniqueness in 2.4(ii) for nf and $(ng) \wedge (nh) = 0$ shows that $ng = (nf) \vee 0$, i.e. $n(f \vee 0) = (nf) \vee 0$.

Hence φ is an ℓ-group homomorphism. φ is injective, because $ng = 0$ implies $0 = (ng) \vee 0 = \varphi(g) \vee 0 = \varphi(g \vee 0) = n(g \vee 0)$, and from $0 \leq g \vee 0 \leq n(g \vee 0)$ we obtain $g \vee 0 = 0$; similarly $g \wedge 0 = 0$ and so $g = g \vee 0 + g \wedge 0 = 0$ (using 2.4(ii)). $\square$

2.10. **Corollary.** *Let G be an ℓ-group and let H be the divisible hull of G. Then there is a unique ℓ-group structure on H such that G is an ℓ-subgroup of H. We have $\frac{f}{k} \leq \frac{g}{n} \iff nf \leq kg$, $\frac{f}{k} \vee \frac{g}{n} = \frac{nf \vee kg}{kn}$ and $\frac{f}{k} \wedge \frac{g}{n} = \frac{nf \wedge kg}{kn}$ for $f, g \in G$ and $k, n \in \mathbb{N}$.*

Proof. Straightforward using 2.9. $\square$

2.11. **ℓ-ideals**

Let G be an ℓ-group. Convex ℓ-subgroups of G are called **ℓ-ideals** of G.[5] A set S here is **convex** if $x \leq y \leq z$ and $x, z \in S$ implies $y \in S$.

(i) The ℓ-ideals of G are precisely the kernels of ℓ-group homomorphisms $G \longrightarrow H$ to some ℓ-group H. If I is an ℓ-ideal of G, then G/I is ordered by $f + I \leq g + I \iff \exists h \in I : h \leq g - f$; in particular $g + I \geq 0$ in G/I just if $h \leq g$ for some $h \in I$.

(ii) If $S \subseteq G$ is nonempty, then the ℓ-ideal generated by S in G is
$$\ell(S) = \{g \in G \mid \exists s_1, \ldots, s_n \in S : |g| \leq |s_1| + \ldots + |s_n|\}.$$

(iii) If $f \in G$, then
$$\ell(f) = \{g \in G \mid \exists n \in \mathbb{N} : |g| \leq n \cdot |f|\}.$$

(iv) A subset S of G is an ℓ-ideal of G just if $S \neq \emptyset$, $0 \leq s_1, s_2 \in S$ implies $s_1 + s_2 \in S$ and $|g| \leq |s|$ and $s \in S$ implies $g \in S$.

(v) If I, J are ℓ-ideals of G, then their sum $I + J = \{g + h \mid g \in I, h \in J\}$ is again an ℓ-ideal.

(vi) If $f, g \in G$, then
$$\ell(f) + \ell(g) = \ell(|f| \vee |g|) = \ell(|f| + |g|)$$
$$\ell(f) \cap \ell(g) = \ell(|f| \wedge |g|).$$

In particular every ℓ-ideal that is finitely generated as an ℓ-ideal is generated by a single element and the finitely generated ℓ-ideals form a lattice with $+$ as supremum and $\cap$ as infimum.

(vii) If I is an ℓ-ideal of G and $\pi : G \longrightarrow G/I$ is the residue map, then taking preimages under π is an inclusion preserving bijection between the ℓ-ideals of G/I and the ℓ-ideals of G containing I.

[5]Notice that in the non-commutative case one also requires that I is a normal subgroup of G.

Proof. (i) It is clear that the kernel of an ℓ-group homomorphisms $G \longrightarrow H$ is a convex ℓ-subgroup. Conversely let I be a convex ℓ-subgroup. We write $\bar{f} = f + I \in G/I$ for $f \in G$. We define a relation $\leq$ on G/I by $\bar{f} \leq \bar{g} \iff \exists h \in I : h \leq f - g$, which is clearly well defined. Since I is convex we see that $\bar{f} = \bar{g} \iff \bar{f} \leq \bar{g} \leq \bar{f}$. It follows easily that $\bar{f} \leq \bar{g} \iff \exists f', g' \in G : f' \leq g'$ & $f - f', g - g' \in I$ and that $\leq$ is a partial order on G/I. If $f, g \in G$, then $\overline{f \vee g}$ is the supremum of $\bar{f}$ and $\bar{g}$ in G/I: Assume $\bar{f}, \bar{g} \leq \bar{h}$ and take $a, b \in I$ with $a \leq h - f$ and $b \leq h - g$. Then $f \leq h - a \leq h - (a \wedge b)$ and $g \leq h - b \leq h - (a \wedge b)$ and so $f \vee g \leq h - (a \wedge b)$. Since I is an ℓ-subgroup we know that $a \wedge b \in I$, hence $\overline{f \vee g} \leq \bar{h}$. This implies that $(G/I, \leq)$ is an ℓ-group and the residue map $G \longrightarrow G/I$ is an ℓ-group homomorphism with kernel I.

(ii) Now let $S \subseteq G$ be nonempty. Recall from 2.4(i)(c) that $|f|$ – defined as $f \vee -f$ – satisfies $|f| = (f \vee 0) \vee -(f \wedge 0) = (f \vee 0) - (f \wedge 0) \geq 0$. In particular $0 \leq f \vee 0, -(f \wedge 0) \leq |f|$, which implies that $\ell(S)$ is contained in the ℓ-ideal generated by S using $f = (f \vee 0) + (f \wedge 0)$ (see 2.4(i)(a)). In order to see that $\ell(S)$ is an ℓ-ideal we need to show that $\ell(S)$ is convex, $-\ell(S) \subseteq \ell(S)$ and $\ell(S) + \ell(S) \subseteq \ell(S)$.

$-\ell(S) \subseteq \ell(S)$ is clear. If $g_1, g_2 \in \ell(S)$ and $g_1 \leq f \leq g_2$, then $-|g_1| \leq -f$, hence $|f| \leq |g_1| \vee |g_2| \leq |g_1| + |g_2|$, which implies $f \in \ell(S)$. Now take $g_1, g_2 \in \ell(S)$. then $|g_1 + g_2| \leq |g_1| + |g_2|$ by 2.4(i)(d), which implies $g_1 + g_2 \in \ell(S)$.

(iii) is an instance of (ii) and (iv) is an easy consequence of (ii).

(v) We verify (iv) for $I + J$. Since $I + J$ is closed under addition we only need to show that for $f \in G$, $g \in I$ and $h \in J$ with $|f| \leq |g + h|$, we have $f \in I + J$. By the triangle inequality 2.4(i)(d) we have $|g + h| \leq |g| + |h|$ and as $|g| \in I, |h| \in J$ we may replace g, h by $|g|$, $|h|$ and assume that $|f| \leq g + h$. Since $0 \leq f \vee 0, -(f \wedge 0) \leq |f|$ and $f = (f \vee 0) + (f \wedge 0)$ by 2.4(i)(a), we may assume that $f \geq 0$.

Hence $0 \leq f, g, h \leq g + h$. By Riesz decomposition 2.2(v) there are $g_0 \leq g, h_0 \leq h$ with $g_0, h_0 \geq 0$ and $f = g_0 + h_0$. But then $g_0 \in I$, $h_0 \in J$ and $f = g_0 + h_0 \in I + J$.

(vi) Since $\ell(f) + \ell(g)$ is an ℓ-ideal by (v) we get $\ell(f) + \ell(g) = \ell(|f| \vee |g|) = \ell(|f| + |g|)$. The inclusion $\ell(f) \cap \ell(g) \supseteq \ell(|f| \wedge |g|)$ follows from (iii) and for $\ell(f) \cap \ell(g) \subseteq \ell(|f| \wedge |g|)$ take $h \in \ell(f) \cap \ell(g)$. By (iii) there are $m, n \in \mathbb{N}$ with $|h| \leq m|f|, n|g|$. Then $|h| \leq (m + n)|f|, (m + n)|g|$ and so $|h| \leq (m + n)|f| \wedge (m + n)|g|$, which is equal to $(m + n)(|f| \wedge |g|)$ by 2.9. Hence $h \in \ell(|f| \wedge |g|)$.

(vii). Taking preimages under π is an inclusion preserving bijection between the subgroups of G/I and the subgroups of G containing I. Hence For a subgroup J of G/I we only need to show that J is an ℓ-ideal of G/I if and only if $\pi^{-1}(J)$ is an ℓ-ideal of G. This is straightforward using the characterization in (iv). □

2.12. *Example.* Convex subgroups of TOAGs are obviously ℓ-ideals. However, convex subgroups of an ℓ-group are in general not ℓ-subgroups. For example consider the identity function f in the ℓ-group G of all maps $\mathbb{Q} \longrightarrow \mathbb{Q}$. Then the subgroup $\mathbb{Z} \cdot f$ generated by f in G is convex. The reason is that distinct elements of $\mathbb{Z} \cdot f$ are incomparable: If $k, n \in \mathbb{Z}$ and $k \cdot f \leq n \cdot f$, then $k \cdot f(1) \leq n \cdot f(1)$ and $k \cdot f(-1) \leq n \cdot f(-1)$; but $f(1) = 1$ and $f(-1) = -1$, hence $k = n$.

2.13**. Definition of (prime) ideals and filters in posets** Let P be a partially ordered set.

(a) A subset S of P is called a **down-set** of P if $a \leq b \in S$ implies $a \in S$. The down-set generated by S in P is written as $S^{\downarrow} = \{p \in P \mid \exists s \in S : p \leq s\}$.

Dually, S is called an **up-set** of P if $a \geq b \in S$ implies $a \in S$. The up-set generated by S in P is written as $S^{\uparrow} = \{p \in P \mid \exists s \in S : p \geq s\}$.

(b) An **ideal** of a poset P is a nonempty down-set I of P that is **up-directed** (i.e. $a, b \in I \Rightarrow \exists c \in I : a, b \leq c$). If P is a lattice, this is equivalent to saying that $I \neq \emptyset$ and I satisfies $a, b \in I \iff a \vee b \in I$.

(c) A **filter** of a poset P is a nonempty up-set F of P that is **down-directed** (i.e. $a, b \in F \Rightarrow \exists c \in F : a, b \geq c$). If P is a lattice, this is equivalent to saying that $F \neq \emptyset$ and F satisfies $a, b \in F \iff a \wedge b \in F$.

(d) A **prime ideal** of P is an ideal of P whose complement is a filter. A **prime filter** of P is a filter of P whose complement is an ideal. Notice that prime ideals and prime filters are nonempty and proper subsets of P.

If P is a lattice, then an ideal I is prime just if it is proper and it satisfies $a \wedge b \in I \Rightarrow a \in I$ or $b \in I$; a filter F is prime just if it is proper and it satisfies $a \vee b \in I \Rightarrow a \in I$ or $b \in I$.

2.14. *Remark.* Let I be an ℓ-ideal of G. Then $I \cap G^{\geq 0}$ is an ideal of the lattice $G^{\geq 0}$ by 2.11(iv). However, I is an ideal of the lattice G if and only if $I = G$: If $g \in G \setminus I$, then $-|g| \notin I$, but $-|g| \leq 0 \in I$.

2.15. **Prime ℓ-ideals**

Let I be an ℓ-ideal of an ℓ-group G. The following are equivalent.

(i) G/I is totally ordered.
(ii) For all $f, g \in G$ with $f \wedge g \in I$ we have $f \in I$ or $g \in I$.
(iii) $I = G$ or $I \cap G^{\geq 0}$ is a prime ideal of the lattice $(G^{\geq 0}, \leq)$.
(iv) For all $f, g \in G$ with $f \wedge g = 0$ we have $f \in I$ or $g \in I$.
(v) The ℓ-ideals of G containing I are totally ordered by inclusion.[6]
(vi) For all ℓ-ideals J, K we have $J \cap K \subseteq I \Rightarrow J \subseteq I$ or $K \subseteq I$.
(vii) For all $f, g \in G$ with $\ell(f) \cap \ell(g) = (0)$ we have $f \in I$ or $g \in I$.

If these conditions hold, then I is called a **prime ℓ-ideal**. If G is a TOAG, then obviously the prime ℓ-ideals of G are just the convex subgroups of G.

Proof. We write $\bar{f}$ for the residue of $f \in G$ in G/I.

(i)$\Rightarrow$(ii). If $f \wedge g \in I$, then $\overline{f} \wedge \overline{g} = 0$ in G/I. Since G/I is totally ordered we must have $\overline{f} = 0$ or $\overline{g} = 0$, i.e. $f \in I$ or $g \in I$.

(ii)$\Rightarrow$(iii). Since I is convex, the set $I \cap G^{\geq 0}$ is a down-set in $G^{\geq 0}$ and as I is an ℓ-subgroup of G, $I \cap G^{\geq 0}$ is an ideal of $G^{\geq 0}$. By (ii), this ideal is prime, unless it is equal to $G^{\geq 0}$ and in that case $I = G$.

(iii)$\Rightarrow$(iv) is clear.

(iv)$\Rightarrow$(i) In order to show that the order of G/I is total it suffices to show that for each $f \in G$ the element $\overline{f}$ is comparable with $\overline{0}$. By 2.4(i)(b) we know that $(f \vee 0) \wedge -(f \wedge 0) = 0$. By (iv) we get $f \vee 0 \in I$ or $f \wedge 0 \in I$. Since the residue map $G \longrightarrow G/I$ is an ℓ-group homomorphism, this is the same as saying $\bar{f} = \bar{f} \wedge 0 \leq \overline{0}$ or $\bar{f} = \bar{f} \vee 0 \geq \overline{0}$.

Hence (i),(ii),(iii) and (iv) are equivalent. Before proving the remaining equivalences we need a

Claim. If $f, g \in G$ with $f \wedge g = 0$, then $f \wedge (n{\cdot}g) = 0$.

[6]In view of 2.15(v) one might wonder if the prime ideals of the lattice G containing a given prime ℓ-ideal form a chain. This fails in general, see 10.7.

Proof. By 2.9 we know that $(nf) \wedge (ng) = n(f \wedge g)$. Hence $(nf) \wedge (ng) = 0$ and from $f \geq 0$ we get $f \wedge ng \leq (nf) \wedge (ng) = 0$. ⋄

(i)⇒(v) By 2.11(vii), the ℓ-ideals of G containing I are in inclusion preserving bijection with the ℓ-ideals of G/I. Since convex and symmetric subsets of totally ordered sets are a chain for inclusion, we see that (i) implies (v).

(v)⇒(ii). Let $f, g \in G$ with $f \wedge g \in I$. Using 2.11(vii) the assumption in (v) implies that $\ell(\bar{f})$ and $\ell(\bar{g})$ are comparable, say $\ell(\bar{f}) \subseteq \ell(\bar{g})$. By 2.11(iii) there is some $n \in \mathbb{N}$ with $|\bar{f}| \leq n{\cdot}|\bar{g}|$. Since $\bar{f}, \bar{g} \geq 0$ this means $0 \leq \bar{f} \leq n\bar{g}$.

On the other hand $f \wedge g \in I$ says $\bar{f} \wedge \bar{g} = 0$, hence by the claim applied to G/I also $\bar{f} \wedge (n{\cdot}\bar{g}) = 0$. But then $\bar{f} = \bar{f} \wedge n|\bar{g}| = 0$ and so $f \in I$.

(iii)⇒(vi). If $f \in J \setminus I$ and $g \in K \setminus I$, then $|f| \in J \setminus I$ and $|g| \in K \setminus I$ because I, J, K are ℓ-ideals. Then $|f| \wedge |g| \in J \cap K$ and by (iii) we have $|f| \wedge |g| \notin I$.

(vi)⇒(vii) is a weakening.

(vii)⇒(iv). If $f \wedge g = 0$, then by the claim and 2.11(iii) we see that $\ell(f) \cap \ell(g) = (0)$. Hence (vii) implies $f \in I$ or $g \in I$. □

2.16. **Proposition.** *If $F \subseteq G$ is a filter and $I \subseteq G$ is an ℓ-ideal, maximal with the property $I \cap F = \emptyset$, then I is a proper prime ℓ-ideal.*[7]

Proof. Since $F \neq \emptyset$ we have $I \neq G$. Suppose I is not a prime ℓ-ideal. By 2.15 there are $g, h \geq 0$ with $g \wedge h \in I$, $g \notin I$ and $h \notin I$. By 2.11(v), the ℓ-ideal generated by I and g is $I + \ell(g)$ and by maximality of I, this ideal hits F. By 2.11(iii) there are $x \in I$ and $g_0 \in G$ with $|g_0| \leq ng$ for some $n \in N$ such that $x + g_0 \in F$. As I is an ℓ-ideal we know that $|x| \in I$. Since $x + g_0 \leq |x + g_0| \leq |x| + |g_0|$ by 2.4(i)(d), we get $|x| + |g_0| \in F$. Hence we may replace x, g_0 by $|x|, |g_0|$ and assume that $x, g_0 \geq 0$. We set $f = x + g_0$.

Similarly there are $y \in I$ and $h_0 \in G$ with $y \geq 0$ and $0 \leq h_0 \leq mh$ for some $m \in \mathbb{N}$ such that $y + h_0 =: f' \in F$. As $g, h \geq 0$ we may assume that $m = n$ by possibly enlarging either m or n. Since $g \wedge h \in I$, also $ng \wedge nh \in I$ (see 2.9) and consequently $g_0 \wedge h_0 \in I$. But then F contains

$$f \wedge f' = (x + g_0) \wedge (y + h_0) \leq x \wedge y + x \wedge h_0 + g_0 \wedge y + g_0 \wedge h_0 \in I$$

(using 2.2(iv)), a contradiction to $F \cap I = \emptyset$. □

2.17. **Abstract Nullstellensatz for ℓ-groups** If G is an ℓ-group and $S \subseteq G$, then the ℓ-ideal generated by S is

$$\ell(S) = \bigcap\{\mathfrak{p} \mid \mathfrak{p} \text{ a prime } \ell\text{-ideal of } G \text{ with } S \subseteq \mathfrak{p}\}.$$

Proof. The inclusion $\subseteq$ is clear. For the converse take $g \in G \setminus \ell(S)$ and let $F = \{h \in G \mid |g| \leq h\}$ be the up-set generated by $|g|$, hence F is a filter, too. Then $F \cap \ell(S) = \emptyset$ because $|g| \leq h \in \ell(S)$ implies $g \in \ell(S)$ by definition of "ℓ-ideal". By 2.16 there is a prime ℓ-ideal of G containing $\ell(S)$ and disjoint from F. This proves that $g \notin \bigcap\{\mathfrak{p} \mid \mathfrak{p} \text{ a proper prime } \ell\text{-ideal with } S \subseteq \mathfrak{p}\}$. □

[7]Notice that $F \cap I = \emptyset$ is equivalent to $F \cap G^{\geq 0} \cap I = \emptyset$ for any ℓ-ideal I.

3. The spectrum of an ℓ-group

3.1. **Definition.** Let G be an ℓ-group. The **ℓ-spectrum** of G is the following topological space, denoted by ℓ-Spec(G). As a set ℓ-Spec(G) consists of the prime ℓ-ideals of G, hence this includes the ℓ-ideal G. On ℓ-Spec(G) a topology is defined, namely the topology generated by the sets

$$D(f)=\{\mathfrak{p}\in \ell\text{-Spec}(G)\mid f\notin \mathfrak{p}\}$$

where $f\in G$.

Notice that if G is a TOAG, then ℓ-Spec G consists of the convex subgroups of G. For example $\ell\text{-Spec}(\mathbb{Z})=\{(0),\mathbb{Z}\}$ and the open sets are $\emptyset,\ell\text{-Spec}(\mathbb{Z})$ and the singleton set $\{(0)\}=D(1)$.

We will have a closer look at the topology soon. First we use the set ℓ-Spec(G) as an index set for a representation of G as ℓ-subgroup of a product of TOAGs. This will also give a first hint why we define the topology on ℓ-Spec(G) as above. By 2.15, for each $\mathfrak{p}\in\ell$-Spec(G) the ℓ-group $G/\mathfrak{p}$ is a TOAG. The idea now is to consider an element $f\in G$ as a function on ℓ-Spec(G), where each point $\mathfrak{p}\in\ell$-Spec(G) has its own co-domain, namely $G/\mathfrak{p}$ and the value of f at $\mathfrak{p}$ is $f/\mathfrak{p}$; so we are really thinking of $f/\mathfrak{p}$ as "$f(\mathfrak{p})$".

3.2**. Representation of ℓ-groups, I** The function

$$\begin{aligned}\rho: G&\longrightarrow \prod_{\mathfrak{p}\in\ell\text{-Spec}(G)} G/\mathfrak{p}\\ f&\longmapsto (\mathfrak{p}\mapsto f(\mathfrak{p}):=f/\mathfrak{p}).\end{aligned}$$

is an embedding of ℓ-groups.[8] In particular, every ℓ-group is isomorphic to a **subdirect product** of TOAGs (i.e. for each factor the projection of G for this factor is surjective).

Proof. Since all components of ρ are ℓ-group homomorphism also ρ is an ℓ-group homomorphism. Hence we only need to show that ρ is injective. However $\rho(f)=0$ means $f\in\bigcap_{\mathfrak{p}\in\ell\text{-Spec}(G)}\mathfrak{p}$ and this ℓ-ideal is $\{0\}$ by the Nullstellensatz 2.17. Thus $f=0$. □

We now have a closer look at the topology of ℓ-Spec(G). Viewing G as an ℓ-subgroup of functions on ℓ-Spec(G) as in 3.2 we may associate natural sets to elements $f\in G$. Most prominently,

$$V(f):=\{\mathfrak{p}\in\ell\text{-Spec}(G)\mid f(\mathfrak{p})=0\}=\{\mathfrak{p}\in\ell\text{-Spec}(G)\mid f\in\mathfrak{p}\}.$$

is the **zero set of f in** ℓ-Spec(G). If we want to view this set as a closed set of a topology it is natural to define the topology of ℓ-Spec(G) as in 3.1: Notice that $D(f)$ is the complement of $V(f)$ in ℓ-Spec(G). The topology of ℓ-Spec(G) codes a lot of information of the ℓ-group G, which is much tighter than in the case of rings. This is made explicit in 3.5 below and will be applied many times later on.

[8]Obviously we can drop the element G from ℓ-Spec(G) in the index set and still get an embedding. In fact it suffices to take the set of minimal prime ℓ-ideals as an index set. The reason for including all points of ℓ-Spec(G) is that when we consider morphisms between ℓ-groups and we want to compare the presentations in a natural way, then we need to consider all elements of ℓ-Spec(G), see 3.6.

3.3. **Observation.** *For all $f, g \in G$ we have*

$$V(f) \cap V(g) = V(|f| \vee |g|) \text{ and } V(f) \cup V(g) = V(|f| \wedge |g|),$$

The first equality follows immediately from the definition of ℓ-ideal and the second one follows from 2.15(ii). Also compare with 2.11(vi).

It follows that

(i) *the nonempty and closed subset of $\ell\text{-Spec}(G)$ are precisely the intersections of sets of the form $V(f)$.*
(ii) *the set $\overline{\mathcal{K}}(\ell\text{-Spec}(G)) = \{\emptyset\} \cup \{V(f) \mid f \in G\}$ is a bounded sublattice of the powerset of $\ell\text{-Spec}(G)$ and a basis of closed sets for the topology of $\ell\text{-Spec}(G)$.*
(iii) *the set $\mathring{\mathcal{K}}(\ell\text{-Spec}(G)) = \{\ell\text{-Spec}(G)\} \cup \{D(f) \mid f \in G\}$ is a bounded sublattice of the powerset of $\ell\text{-Spec}(G)$ and a basis of open sets for the topology of $\ell\text{-Spec}(G)$.*
(iv) *If $\mathfrak{p}, \mathfrak{q} \in \ell\text{-Spec}(G)$, then*

$$\begin{aligned} \mathfrak{p} \subseteq \mathfrak{q} &\iff \forall f \in G : \mathfrak{p} \in V(f) \Rightarrow \mathfrak{q} \in V(f) \\ &\iff \forall f \in G : \mathfrak{q} \in D(f) \Rightarrow \mathfrak{p} \in D(f) \\ &\iff \mathfrak{q} \in \overline{\{\mathfrak{p}\}}. \end{aligned}$$

If this is the case we say that $\mathfrak{q}$ is a **specialization** *of $\mathfrak{p}$ and also write $\mathfrak{p} \rightsquigarrow \mathfrak{q}$.*[9]

3.4. **The space ℓ-Spec$^*(G)$ of proper prime ℓ-ideals**
In the literature, $\ell\text{-Spec}(G)$ is often defined as the subspace of our space, where the point G is removed. We write

$$\ell\text{-Spec}^*(G) = \{\mathfrak{p} \subseteq G \mid \mathfrak{p} \text{ proper prime } \ell\text{-ideal}\}$$

for this subspace of $\ell\text{-Spec}(G)$. Since G is a closed point of $\ell\text{-Spec}(G)$, $\ell\text{-Spec}^*(G)$ is an open subspace of $\ell\text{-Spec}(G)$.

One reason for including G as a point in the spectrum is that $\ell\text{-Spec}^*(G)$ is in general not quasi-compact and $\ell\text{-Spec}(G)$ is the Non-Hausdorff Alexandroff extension of $\ell\text{-Spec}^*(G)$. Another reason is that $\ell\text{-Spec}^*$ is not functorial, see 3.6.

Notice that intersection with $\ell\text{-Spec}^*(G)$ is a poset isomorphism between the nonempty closed subsets of $\ell\text{-Spec}(G)$ and the closed subsets of $\ell\text{-Spec}^*(G)$. Similarly, intersection with $\ell\text{-Spec}^*(G)$ is a poset isomorphism between the proper open subsets of $\ell\text{-Spec}(G)$ and the open subsets of $\ell\text{-Spec}^*(G)$.

3.5. **The Galois connection for ℓ-groups** We define maps

$$\begin{aligned} V : \mathcal{P}(G) &\longrightarrow \mathcal{P}(\ell\text{-Spec}(G)) \\ S &\longmapsto V(S) = \{\mathfrak{p} \in \ell\text{-Spec}(G) \mid S \subseteq \mathfrak{p}\} \end{aligned}$$

and

$$\begin{aligned} I : \mathcal{P}(\ell\text{-Spec}(G)) &\longrightarrow \mathcal{P}(G) \\ Z &\longmapsto I(Z) = \bigcap_{\mathfrak{p} \in I} \mathfrak{p}. \end{aligned}$$

[9]Specialization is a relation in any topological space X and is defined as $x \rightsquigarrow y \iff y \in \overline{\{x\}}$. The space X is T_0 just if $\rightsquigarrow$ is a partial order.

(i) The maps V and I form an antitone Galois connection, aka **polarity**, between the posets $(\mathcal{P}(G),\subseteq)$ and $(\mathcal{P}(\ell\text{-Spec}(G)),\subseteq)$. Explicitly this means that V and I are order reversing and for all $S\subseteq G,\ Z\subseteq \ell\text{-Spec}(G)$ we have

$$Z\subseteq V(S)\iff S\subseteq I(Z).$$

Proof. That V and I are order reversing is obvious. The equivalence follows immediately from the definitions of V and $I$and does not require any of the theory of ℓ-groups developed so far:

$\Rightarrow$. If $f\in S$ and $\mathfrak{p}\in Z\subseteq V(S)$, then $f\in S\subseteq \mathfrak{p}$. This shows $S\subseteq I(Z)$.

$\Leftarrow$. If $\mathfrak{p}\in Z$ and $s\in S\subseteq I(Z)$, then $s\in\mathfrak{p}$. This shows $Z\subseteq V(S)$. ◇

(ii) If $Z\subseteq \ell\text{-Spec}(G)$ is nonempty, then $\overline{Z}=V(I(Z))$, because

$$\overline{Z}\overset{\text{using 3.3}}{=\!=}\bigcap\{V(f)\mid Z\subseteq V(f)\}\overset{\text{by (i)}}{=\!=}\bigcap\{V(f)\mid f\in I(Z)\}=V(I(Z)).$$

(iii) If $S\subseteq G$, then $\ell(S)=I(V(S))$, which is precisely the statement of the Nullstellensatz 2.17.

(iv) The image of V is the set of nonempty closed subsets of $\ell\text{-Spec}(G)$ and the image of I is the set of ℓ-ideals of G.
Proof. This follows easily from (iii) and (iv). ◇

(v) V induces an antitone isomorphism between the poset of ℓ-ideals, ordered by $\subseteq$ and the nonempty and closed subsets of $\ell\text{-Spec}(G)$ ordered by $\subseteq$. Its compositional inverse is the restriction of I.

(vi) The isomorphism from (v) restricts to an inclusion reversing bijection between $\ell\text{-Spec}(G)$ and the nonempty, closed and irreducible subsets of $\ell\text{-Spec}(G)$. Explicitly, $\mathfrak{p}\in\ell\text{-Spec}(G)$ is mapped to $\overline{\{\mathfrak{p}\}}$ and the compositional inverse maps A to its generic point.
Proof. Let I be an ℓ-ideal. Then

$V(I)$ is irreducible

$\iff \forall f,g\in G: V(I)\cap D(f),V(I)\cap D(g)\neq\emptyset\Rightarrow V(I)\cap D(f)\cap D(g)\neq\emptyset,$
by 3.3

$\iff \forall f,g\in G: V(I)\subseteq V(f)\cup V(g)\Rightarrow V(I)\subseteq V(f)$ or $V(I)\subseteq V(g)$

$\iff \forall f,g\in G: V(I)\subseteq V(|f|\wedge|g|)\Rightarrow V(I)\subseteq V(f)$ or $V(I)\subseteq V(g)$,
because $V(f)\cup V(g)=V(|f|\wedge|g|)$ by 3.3

$\iff \forall f,g\in G: |f|\wedge|g|\in I\Rightarrow f\in I$ or $g\in I$,
by (iii)

$\iff$ I is a prime ℓ-ideal, using 2.15 (and 2.11). ◇

(vii) The isomorphism from (v) restricts to an inclusion reversing bijection between the poset of finitely generated ℓ-ideals and the zero sets $V(f)$ of elements of G. By taking complements, the poset of finitely generated ℓ-ideals is isomorphic to the proper open and quasi-compact subsets of X. Another way of saying this is:

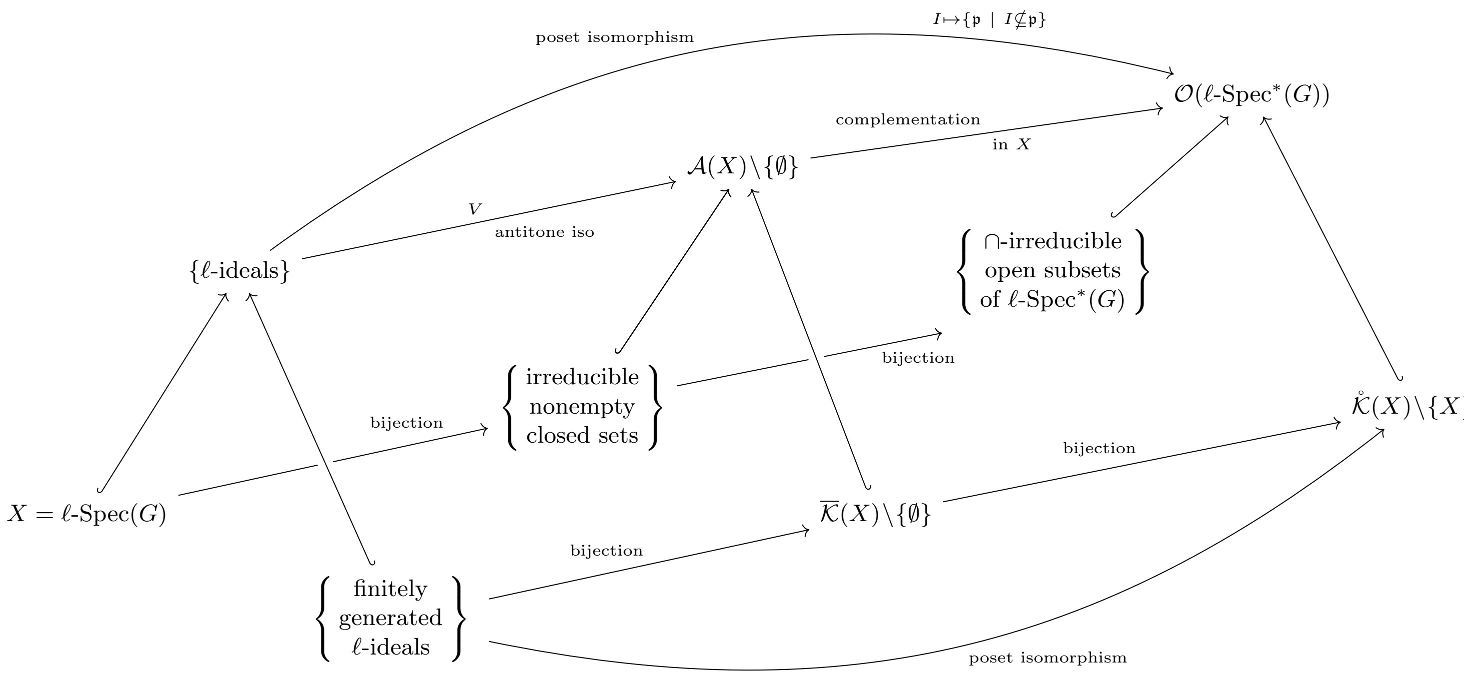
The poset of finitely generated ℓ-ideals is isomorphic to the open and quasi-compact subsets U of $\ell\text{-Spec}^*(G)$.
Proof. In view of 3.3 we only need to show that for each $f\in G$ the set $D(f)$ is a quasi-compact subset of $\ell\text{-Spec}^*(G)$. Let $g_\lambda\in G$ for λ in some index

set Λ and assume that $D(f) \cup \{V(g_\lambda) \mid \lambda \in \Lambda\}$ has the FIP. Let I be the ℓ-ideal generated by $\{g_\lambda \mid \lambda \in \Lambda\}$. We need to show that $D(f) \cap_{\lambda\in\Lambda} V(g_\lambda) \neq \emptyset$. Otherwise $V(I) \subseteq V(f)$ and so $f \in I$ by the Nullstellensatz 2.17. By 2.11(ii) there are $\lambda_1, \ldots, \lambda_n \in \Lambda$ with $|f| \leq |g_{\lambda_1}| + \ldots + |g_{\lambda_1}|$. But then $D(f) \cap V(g_{\lambda_1} \cap \ldots \cap V(g_{\lambda_n})) = \emptyset$, contradicting the FIP. ◇

(viii) ℓ-Spec$^*(G)$ is quasi-compact if and only if G has a **strong order unit**, i.e. an element u with $G = \ell(u)$.
Proof. Notice that $\ell\text{-Spec}^*(G) = \bigcup_{f\in G} D(f)$. If $\ell(u) = G$, then $\ell\text{-Spec}^*(G) = D(u)$ is quasi-compact by (vii). Conversely, if ℓ-Spec$^*(G)$ is quasi-compact, then ℓ-Spec$^*(G)$ is a finite union of $D(f)$ and by 3.3 even $\ell\text{-Spec}^*(G) = D(u)$ for some $u \in G$. Hence $V(u) = \{G\}$, which means $G = \ell(u)$ by the Nullstellensatz 2.17. ◇

(ix) The space ℓ-Spec(G) is **spectral** (see [DiScTr2019, 1.1.5] for a definition). This follows from the items above and will be discussed later in more detail.

Here is a commutative diagram summarizing what is said in 3.5.



Notice that the map $\mathcal{O}(X) \setminus \{X\} \longrightarrow \mathcal{O}(\ell\text{-Spec}^*(G)),\ U \mapsto U \setminus \{G\}$ is a poset isomorphism.

3.6. **Functoriality of** ℓ-Spec
Let $\varphi : G \longrightarrow H$ be an ℓ-group homomorphism. We define a map $\ell\text{-Spec}(\varphi) : \ell\text{-Spec}(H) \longrightarrow \ell\text{-Spec}(G)$ by

$$(\ell\text{-Spec}(\varphi))(\mathfrak{q}) = \varphi^{-1}(\mathfrak{q}).$$

Then

(i) ℓ-Spec(φ) is a continuous map and for each $f \in G$ we have

$$\ell\text{-Spec}(\varphi)^{-1}(V(f)) = V(\varphi(f)).$$

(ii) If φ is surjective, then ℓ-Spec(φ) maps $\ell\text{-Spec}^*(H)$ homeomorphically onto $V(\mathrm{Ker}(\varphi)) \setminus \{G\}$.

Proof. (i) It is clear that $\varphi^{-1}(\mathfrak{q}) \in \ell\text{-Spec}(G)$ as one checks the characterization 2.15(ii) easily. Continuity follows from $\ell\text{-Spec}(\varphi)^{-1}(V(f)) = V(\varphi(f))$. To see this equation, take $\mathfrak{q} \in \ell\text{-Spec}(H)$. Then

$$\begin{aligned} \mathfrak{q} \in V(\varphi(f)) &\iff \varphi(f) \in \mathfrak{q} \\ &\iff f \in \varphi^{-1}(\mathfrak{q}) = \ell\text{-Spec}(\varphi)(\mathfrak{q}) \\ &\iff \ell\text{-Spec}(\varphi)(\mathfrak{q}) \in V(f) \\ &\iff \mathfrak{q} \in \ell\text{-Spec}(\varphi)^{-1}(V(f)). \end{aligned}$$

(ii) follows from 2.11(vii) and (i). □

3.7. *Remark.* Observe that the restriction of ℓ-Spec(φ) to $\ell\text{-Spec}^*(H)$ does not have image in $\ell\text{-Spec}^*(G)$. For example when G is a proper prime ℓ-ideal of H. One of the two reasons why we include G in the spectrum of G is the functoriality in 3.6. The other reason is that the inclusion of G guarantees that ℓ-Spec(G) is a spectral space, i.e. a well understood topological theory is available to be used.

3.8. **Proposition.** *Let G be an ℓ-group and let H be its divisible hull, which we consider as an ℓ-supergroup of G, see 2.10. Then the map $\ell\text{-Spec}(H) \longrightarrow \ell\text{-Spec}(G)$ induced by the inclusion $G \hookrightarrow H$ is a homeomorphism.*

Proof. All ℓ-ideals of H are divisible, because for $f \in G$ and $n \in \mathbb{N}$ we have $|\frac{f}{n}| \le |f|$. Now if $\mathfrak{q}_1, \mathfrak{q}_2 \in \ell\text{-Spec}(H)$ and $f \in G, n \in \mathbb{N}$ with $\frac{f}{n} \in \mathfrak{q}_1 \setminus \mathfrak{q}_2$, then $f \in \mathfrak{q}_1 \cap G \setminus \mathfrak{q}_2 \cap G$. Hence the map is injective. For surjectivity, pick $\mathfrak{p} \in \ell\text{-Spec}(G)$ and let M be the divisible hull of the ℓ-group $G/\mathfrak{p}$. Since $H = \mathbb{Q} \otimes G$ as abelian group, there is a unique group homomorphism $\psi : H \longrightarrow M$ extending the residue map $G \longrightarrow G/\mathfrak{p}$. Straightforward checking shows that ψ is an ℓ-group homomorphism whose kernel intersects G in $\mathfrak{p}$.

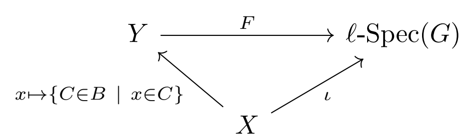
Hence $\ell\text{-Spec}(H) \longrightarrow \ell\text{-Spec}(G)$ is bijective. It is a homeomorphism because $V(\frac{f}{n}) = V(f)$ in ℓ-Spec(H) for $f \in G, n \in \mathbb{N}$ as follows from the divisibility of ℓ-ideals of H. □

3.9. **ℓ-groups of functions** Often, ℓ-groups are given in the context of functions. For example consider the ℓ-group of all (continuous/bounded/piecewise polynomial) functions $X \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ that are definable in some structure expanding the TOAG $(\mathbb{R}, +, \leq)$, or, the ℓ-group of all functions $X \longrightarrow \mathbb{R}$ that have finite image. For such groups we are looking at their ℓ-spectrum and how it relates to X.

Let X be a set and let T be a TOAG. Let $G \subseteq T^X$ be an ℓ-subgroup. For $x \in X$, let $\mathrm{ev}_x : \Gamma \longrightarrow T,\ \mathrm{ev}_x(f) = f(x)$ be the evaluation map. This is obviously an ℓ-group homomorphism and as $\Gamma/\operatorname{Ker}(\mathrm{ev}_x)$ is a totally ordered subgroup of T, we have an ℓ-prime ideal $\hat{x} := \operatorname{Ker}(\mathrm{ev}_x) = \{f \in G \mid f(x) = 0\} \in \ell\text{-Spec}(G)$. This defines a map $\iota : X \longrightarrow \ell\text{-Spec}(G),\ \iota(x) = \hat{x}$.[10]

Let
$$\mathrm{Coz}_G(X) = \{X\} \cup \{\text{cozero sets of } f \in G\},$$
where the cozero set of $f \in G$ is $\{f \neq 0\}_X = \{x \in X \mid f(x) \neq 0\}$.

Let B be any Boolean algebra of subsets of X containing $\mathrm{Coz}_G(X)$ and let Y be the space of ultrafilters of B (the Stone dual of B). We obtain a continuous map
$$F : Y \longrightarrow \ell\text{-Spec}(G),\ F(\mathfrak{u}) = \{f \in G \mid \{f = 0\}_X \in \mathfrak{u}\}$$
making the diagram



commutative and that has the following properties.

(i) F is injective if and only if every element of B is a Boolean combination of cozero sets of G. This for example is the case in the following situation:
   - G is the ℓ-group of all (continuous) and definable functions $X \longrightarrow M$, where M is an o-minimal expansion of a DOAG and $X \subseteq M^n$ is definable in M[11],
   - B is the Boolean algebra of definable subsets of X. Hence in this case Y is the space of all n-types of M containing X.

(ii) The point G of $\ell\text{-Spec}(G)$ is in the image of F if and only if there is some ultrafilter $\mathfrak{u}$ of B with $\{f = 0\}_X \in \mathfrak{u}$ for all $f \in G$, equivalently: every $f \in G$ has a zero in X.

   $\ell\text{-Spec}^*(G)$ is in the image of F if and only if for all $f, g \in G$ with $\{f = 0\}_X \subseteq \{g = 0\}_X$ we have $g \in \ell(f)$. This means that a concrete Nullstellensatz is satisfied for G. Notice that this condition implies that every function $f \in G$ without a zero in G is a strong order unit.

However, notice that even if F is bijective, it is not a homeomorphism, because Y is Hausdorff and $\ell\text{-Spec}(G)$ generally is not.

Here is an example where F is a bijection onto $\ell\text{-Spec}^*(G)$. Let $X = [0, 1]^n \subseteq \mathbb{Q}^n$, let G be the ℓ-group of continuous functions $X \longrightarrow \mathbb{Q}$ that are definable in the TOAG $\mathbb{Q}$ and let B be the Boolean algebra of definable subsets of X; i.e. $Y =$

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[10]One may think of ι as being injective, because otherwise one can think G as an ℓ-group of functions $Y \longrightarrow T$, where Y is the set of equivalence classes of X modulo the equivalence relation $\hat{x}_1 = \hat{x}_2$.

[11]By cell decomposition, X is a Boolean combination of closed and definable sets and these sets are zero sets of (continuous) definable functions

$\{p \in S_n(\mathbb{Q}) \mid X \in p\}$[12]. Then F is injective by (i) and is surjective, because the Nullstellensatz holds for this group by [Baker1968, 3.3]. Notice that every $f \in G$ without a zero is a strong order unit. For an illustrations of the specialization relation of ℓ-Spec(G) in this example, take $\mathfrak{p}, \mathfrak{q}$, corresponding to n-types p, q via F we have $\mathfrak{q} \in \overline{\{\mathfrak{p}\}}$ (i.e. $\mathfrak{p} \subseteq \mathfrak{q}$) if and only if for every definable set $C \in p$ its closure $\overline{C}$ in $\mathbb{Q}^n$ is in q.

If $n = 1$ and q is the isolated type realised by $r \in (0, 1)$ then there are exactly 3 types p whose corresponding prime ideal $\mathfrak{p}$ specializes to $\mathfrak{q}$. One is q and the other ones are the types corresponding to the cuts r^- and r^+ of $[0, 1]$.

Now let $X = \mathbb{Q}^n$ and let G be the ℓ-group of continuous functions $X \longrightarrow \mathbb{Q}$ that are definable in the TOAG $\mathbb{Q}$. Then F misses points of ℓ-Spec$^*(G)$: Let I be the ℓ-ideal of all bounded functions (so these functions do not need to have any zeroes). Then $V(I) = V(1)$ (the constant function of value 1) and the image of F is $D(1)$: Obviously F misses all points in $V(1)$; if $1 \notin \mathfrak{p}$, then all $f \in \mathfrak{p}$ must have a zero and one can construct a preimage of $\mathfrak{p}$.

Note that there are many points in $V(1)$: for each "unbounded" $p \in S_n(\mathbb{Q})$ (i.e. p does not contain a bounded set), we have $\mathfrak{q} = \{f \in G \mid \exists n \in \mathbb{N},\ S \in p : f|_S \text{ is bounded}\} \in V(I)$ and $\mathfrak{q}$ is a proper specialization of the prime ℓ-ideal corresponding to p. If $\tilde{p}$ is another unbounded type and we form $\tilde{\mathfrak{q}}$ for $\tilde{p}$ correspondingly then $\mathfrak{q}$ and $\tilde{\mathfrak{q}}$ are in general incomparable; to see a concrete example let $Q = \{x_1 > 0, \ldots, x_n > 0\} \subseteq \mathbb{Q}^n$ and choose $p, \tilde{p}$ unbounded, $Q \in p$ and $-Q \in \tilde{p}$. Then the function

$$f(x_1, \ldots, x_n) = (x_1 \vee 0) + \ldots + (x_n \vee 0)$$

is in $\tilde{\mathfrak{q}} \setminus \mathfrak{q}$ and $f(-x_1, \ldots, -x_n) \in \mathfrak{q} \setminus \tilde{\mathfrak{q}}$.

[12]Here $S_n(\mathbb{Q})$ is the space of n-types of the TOAG $\mathbb{Q}$ over the parameter set $\mathbb{Q}$. In other words, $S_n(\mathbb{Q})$ is the space of ultrafilters of B; see [StoneBoolAlg, section 3.1] for a definition of the ultrafilter space

4. The Jaffard-Ohm-Kaplansky Theorem

4.1. **Proposition.** [Darnel1995, Thm 3.7,p.11]
Let $G = (G, +, \leq)$ be a po-group and assume that G is torsion free. This for example is the case when G is an abelian ℓ-group (cf. 2.9). Then there is a total order $\sqsubseteq$ of G containing $\leq$ such that $(G, +, \sqsubseteq)$ is a po-group (and thus is lattice ordered as well).

Proof. We may assume that $G \neq \{0\}$. Let $T = G^{\geq 0} \setminus \{0\}$, or, if G is trivially ordered choose $g \in G \neq 0$ and define $T = \mathbb{N}\cdot g$. In either case T is a nonempty sub-semigroup of G containing $G^{\geq 0} \setminus \{0\}$ with $T \cap -T = \emptyset$. By Zorn's lemma, there is a maximal nonempty sub-semigroup T of G with $T \cap -T = \emptyset$ containing $G^{\geq 0} \setminus \{0\}$ and it suffices to show that $T \cup -T = G \setminus \{0\}$. Suppose there is some $g \in G \setminus (T \cup -T)$ with $g \neq 0$. Let R be the sub-semigroup of G generated by $T \cup \{g\}$. Then $R = (T + \mathbb{N}_0\cdot g) \cup \mathbb{N}\cdot g$ properly contains T and by maximality of T we have $R \cap -R \neq \emptyset$. Since G is torsion free, there are no $k, k' > 0$ with $kg = -k'g$. But then there are $t, t' \in T$ and $k, k' \geq 0$ with $t + kg = -t' - k'g$, or, $t + kg = -k'g$, or $kg = -t' - k'g'$. In either case we see that for some $t \in T$ and some $k \geq 0$ we have $t = -k\cdot g$. Since $0 \notin T$ we get $k > 0$. The same argument for $-g$ shows that $t' = k'\cdot g$ for some $t' \in T, k' > 0$. Then $k't + kt' \in T$. However

$$k't + kt' = -k'kg + kk'g = 0,$$

in contradiction to $T \cap -T = \emptyset$. □

A **Bézout domain** is a commutative domain R such that all finitely generated ideals are principal (e.g: $\mathbb{Z}$, or valuation rings of fields). Let R be a Bézout domain with fraction field K. Let G be the group $K^\times/R^\times$, partially ordered by $aR^\times \leq bR^\times \iff b/a \in R$. Then G is an ℓ-group as one verifies easily. G is called the value group of R.

4.2**. Jaffard-Ohm-Kaplansky Theorem** [AndFei1988, Thm. 11.2]
Every ℓ-group $G = (G, +, \leq)$ is the group of divisibility of a Bézout domain

Proof. (Sketch.) Let $K[G]$ be the group ring over some field K. Then $K[G]$ is a domain.
Proof. Choose a total order $\sqsubseteq$ containing $\leq$ such that $(G, +, \sqsubseteq)$ is a po-group as in 4.1. Using totality of $\sqsubseteq$ it is easy to see that $K[G]$ is a domain. ◇
Let F be the fraction field of $K[G]$. Define a map $v : K[G] \setminus \{0\} \longrightarrow G$ by $v(\sum a_g X^g) = \inf\{g \mid a_g \neq 0\}$, taken in the lattice $(G, \leq)$. Its image contains $G^{\geq 0}$ hence its extension to $F^\times$ is onto G.

We claim that v is a group homomorphism $F^\times \longrightarrow G$.
Proof. It suffices to show that $v_\mathfrak{p} := \pi_\mathfrak{p} \circ v$ is a group hom. $F^\times \longrightarrow G/\mathfrak{p}$ for all $\mathfrak{p} \in \ell\text{-Spec}(G)$. Using totality of the order of $G/\mathfrak{p}$ the definition of v shows $v_\mathfrak{p}(\sum a_g X^g) = \min\{g \bmod \mathfrak{p} \mid a_g \notin \mathfrak{p}\}$. From here one can show that $v_\mathfrak{p}$ is a homomorphism. ◇
Now define $R = \{x \in F \mid v(x) \geq 0\}$ and verify that R is a Bézout domain using the following observation: If $\gamma, \delta \in \Gamma$, then $x^\gamma + x^\delta = x^{\gamma\wedge\delta}\cdot(x^{\gamma-\gamma\wedge\delta} + x^{\delta-\gamma\wedge\delta})$, and $x^{\gamma-\gamma\wedge\delta} + x^{\delta-\gamma\wedge\delta}$ is a unit of R, because $v(x^{\gamma-\gamma\wedge\delta} + x^{\delta-\gamma\wedge\delta}) = (\gamma-\gamma\wedge\delta)\wedge(\delta-\gamma\wedge\delta) = \gamma\wedge\delta - \gamma\wedge\delta = 0$.

Finally one shows that G is the ℓ-group of divisibility of R by using that v is a homomorphism. □

5. Conrad-Harvey-Holland Representation

5.1**. The Hahn group on a poset** Let Γ be a poset and for each $\gamma \in \Gamma$ let G_γ be a TOAG. We define

$$\mathbb{H} = \mathbb{H}(\Gamma, (G_\gamma)_{\gamma\in\Gamma}) = \{a \in \prod_{\gamma\in\Gamma} G_\gamma \mid \operatorname{supp}(a) \text{ has the ACC}\}.$$

Here $\operatorname{supp}(a) = \{\gamma \in \Gamma \mid a_\gamma \neq 0\}$ denotes the **support** of a and **ACC** stands for the ascending chain condition saying that there is no infinite sequence $\gamma_1 < \gamma_2 < \gamma_3 \ldots$. If $G_\gamma = H$ is independent of γ we just write $\mathbb{H}(\Gamma, H)$.
Elements of $\mathbb{H}$ are written as

$$\sum_{\gamma\in\Gamma} a_\gamma x^\gamma.$$

Then $\mathbb{H}$ is an abelian subgroup of the product $\prod_{\gamma\in\Gamma} G_\gamma$ because $\operatorname{supp}(-a) = \operatorname{supp}(a)$ and $\operatorname{supp}(a+b) \subseteq \operatorname{supp}(a) \cup \operatorname{supp}(b)$ and clearly the union of two subsets of Γ has the ACC just if each of the two sets have the ACC.

On $\mathbb{H}$ we define a relation $\leq$ by

$$a \leq b \iff \text{for all maximal elements } \gamma \text{ of } \operatorname{supp}(b-a) \text{ we have } a_\gamma < b_\gamma.$$

This is a partial order on $\mathbb{H}$ and $(\mathbb{H}, +, \leq)$ is a po-group, called the **Hahn group on** Γ (if all G_γ are equal to $\mathbb{R}$).

Proof. Firstly, if $a \leq b$ and $c \in \mathbb{H}$, then clearly $a+c \leq b+c$. Hence in order to show transitivity it suffices to prove that $0 \leq a \leq b$ implies $0 \leq b$. Let γ be a maximal element in $\operatorname{supp}(b)$, i.e. $\gamma^\uparrow \cap \operatorname{supp}(b) = \{\gamma\}$. It follows that $\gamma^\uparrow \cap \operatorname{supp}(b-a) \subseteq \{\gamma\} \cup (\gamma^\uparrow \cap \operatorname{supp}(a))$.

If $\gamma^\uparrow \cap \operatorname{supp}(a) = \emptyset$, then $\gamma^\uparrow \cap \operatorname{supp}(b-a) = \gamma^\uparrow \cap \operatorname{supp}(b) = \{\gamma\}$ and therefore $0 = a_\gamma < b_\gamma$ as required.

Hence we may assume that $\gamma^\uparrow \cap \operatorname{supp}(a) \neq \emptyset$. Let δ be a maximal element in $\gamma^\uparrow \cap \operatorname{supp}(a)$. If $\delta > \gamma$, then $0 < a_\delta$ and $b_\delta = 0$. But this contradicts $a \leq b$. It follows that $\gamma^\uparrow \cap \operatorname{supp}(a) = \{\gamma\}$.

Hence if $a_\gamma \neq b_\gamma$, then γ is a maximal element in $\operatorname{supp}(b-a)$ and we get $0 < a_\gamma < b_\gamma$ as required. On the other hand, if $a_\gamma = b_\gamma$, then $\gamma^\uparrow \cap \operatorname{supp}(b-a) = \emptyset$ and $\gamma^\uparrow \cap \operatorname{supp}(a) = \gamma^\uparrow \cap \operatorname{supp}(b) = \{\gamma\}$, thus $b_\gamma = a_\gamma > 0$. □

5.2**. How to calculate in a Hahn group?** In the setup of 5.1 the following properties hold.

(i) Let $\Delta \subseteq \Gamma$. Then the inclusion map $\mathbb{H}(\Delta, (G_\gamma)_{\gamma\in\Delta}) \hookrightarrow \mathbb{H}(\Gamma, (G_\gamma)_{\gamma\in\Gamma})$ is obviously an embedding of po-groups.

If $\Delta = \Delta^\uparrow$ is an up-set of Γ, then the truncation map

$$\mathbb{H}(\Gamma, (G_\gamma)_{\gamma\in\Gamma}) \twoheadrightarrow \mathbb{H}(\Delta, (G_\gamma)_{\gamma\in\Delta})$$
$$a \longmapsto a \restriction \Delta := \sum_{\gamma\in\Delta} a_\gamma x^\gamma$$

is a poset homomorphism and a retraction of the inclusion map. This follows immediately from the definition of the partial order on the groups.

(ii) For $a, b \in \mathbb{H}$ we have

$$a \leq b \iff \forall \gamma \in \Gamma : a \restriction \gamma^\uparrow \leq b \restriction \gamma^\uparrow,$$

because obviously $a \geq 0$ if and only if $a \restriction \gamma^{\uparrow} \geq 0$ for all $\gamma \in \Gamma$ and therefore $a \leq b \iff b - a \geq 0 \iff \forall\gamma \in \Gamma : b \restriction \gamma^{\uparrow} - a \restriction \gamma^{\uparrow} = (b - a) \restriction \gamma^{\uparrow} \geq 0$.

(iii) Let I be a index set and let $a^i \in \mathbb{H}$ such that for all $i \neq j$ we have $\operatorname{supp}(a^i)^{\downarrow} \cap \operatorname{supp}(a^j)^{\downarrow} = \emptyset$. Then the element $b \in \mathbb{H}$ defined by

$$b_{\gamma} = \begin{cases} a_{\gamma} & \text{if } a^i_{\gamma} \neq 0 \text{ form some } i \in I, \\ 0 & \text{otherwise,} \end{cases}$$

is obviously a well defined element of $\mathbb{H}$, which we denote by $\sum_{i \in I} a^i$.

Now assume that Γ is a **root system**.[13] A **component** of $a \in \mathbb{H}$, is any element $b \in \mathbb{H}$ of the form $\sum_{\gamma \leq \delta} a_{\gamma} x^{\gamma}$, where $\delta \in \operatorname{supp}(a)^{\max}$.

Now let $a \in \mathbb{H}$. Then

(iv) For every $\gamma \in \Gamma$ there is at most one component b of a such that $\gamma \in \operatorname{supp}(b)^{\downarrow}$ (equivalently $\operatorname{supp}(b) \cap \gamma^{\uparrow} \neq \emptyset$).

Consequently a is the sum of its components in the sense of (iii).

(v) Every component of a is comparable with 0 and $a \geq 0$ if and only if each of its components is ≥ 0.

(vi) If c is a component of $a \in \mathbb{H}$ and $\gamma \in \Gamma$, then

$$c \restriction \gamma^{\uparrow} = \begin{cases} a \restriction \gamma^{\uparrow} & \text{if } \gamma \in \operatorname{supp}(c)^{\downarrow}, \\ 0 & \text{otherwise.} \end{cases}$$

(vii) If $b \in \mathbb{H}$ with $b \geq 0$, then $a \leq b$ if and only if for every component $c > 0$ of a there is a component d of b with $c \leq d$.

Proof. $\Rightarrow$. Let $c > 0$ be a component of a and $\gamma = \max(\operatorname{supp}(c))$. Since $0 < c$ and $a \leq b$ we have $\gamma^{\uparrow} \cap \operatorname{supp}(b) \neq \emptyset$ and there is a component d of b with $\gamma \in \operatorname{supp}(d)$ and $d > 0$. By (ii) it suffices to check that $c \restriction \delta^{\uparrow} \leq d \restriction \delta^{\uparrow}$ for all $\delta \in \Gamma$. Since $d \geq 0$ we may assume that $c \restriction \delta^{\uparrow} \neq 0$, hence by (vi) we know $\delta \in \operatorname{supp}(c)^{\downarrow} = \gamma^{\downarrow}$ and $c \restriction \delta^{\uparrow} = a \restriction \delta^{\uparrow}$. But then $\delta \in \operatorname{supp}(d)^{\downarrow}$ and by (vi) we get $d \restriction \delta^{\uparrow} = b \restriction \delta^{\uparrow} \geq a \restriction \delta^{\uparrow} = c \restriction \delta^{\uparrow}$.

$\Leftarrow$. By (ii) it suffices to show that $a \restriction \gamma^{\uparrow} \leq b \restriction \gamma^{\uparrow}$ for all $\gamma \in \Gamma$. Since $b \geq 0$ we may assume that $\gamma \in \operatorname{supp}(c)^{\downarrow}$ for some component $c > 0$ of a. Take a component d of b with $c \leq d$. Then

$$\begin{aligned} a \restriction \gamma^{\uparrow} &= c \restriction \gamma^{\uparrow} \text{ by (vi)} \\ &\leq d \restriction \gamma^{\uparrow} \text{ since } c \leq d \\ &= b \restriction \gamma^{\uparrow} \text{ by (vi) as } c \leq d \text{ implies } \gamma \in \operatorname{supp}(d)^{\downarrow}. \end{aligned}$$ ◇

5.3. **Corollary.** *If Γ is a root system then $\mathbb{H}$ is an ℓ-group*[14] *and for $a \in \mathbb{H}$ we have*

$$(a \vee 0)_{\gamma} = \begin{cases} a_{\gamma} & \textit{if there is } \delta \in \gamma^{\uparrow} \cap \operatorname{supp}(a)^{\max} \textit{ with } a_{\delta} > 0, \\ 0 & \textit{otherwise.} \end{cases}$$

In the terminology of 5.2, the element $a \vee 0$ is the sum of the positive components of a and $a \vee 0$ is also the supremum of the set of positive components of a (in $\mathbb{H}^{\geq 0}$). In particular,

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[13] Γ is a **root system** if for all $\gamma \in \Gamma$ the up-set $\gamma^{\uparrow}$ is a chain.

[14] If all G_{γ} are divisible, then $\mathbb{H}$ is an ℓ-group if and only if Γ is a root system. See [Darnel1995, Theorem 51.3, p.334].

(i) $a \wedge 0 = -((-a) \vee 0)$ *is the infimum of the set of all negative components of* a.

(ii) $|a| = (a \vee 0) \vee -(a \wedge 0)$ *is the sum and also the supremum of the set*

$$\{c, -d \mid c \text{ positive component of } a,\ d \text{ negative component of } a\}.$$

(iii) $\mathrm{supp}(a \vee 0)$ *is an up-set of* $\mathrm{supp}(a)$ *and so* $\mathrm{supp}(a \wedge 0) = \mathrm{supp}((-a) \vee 0)$ *is an up-set of* $\mathrm{supp}(a)$ *as well.*

(iv) $\mathrm{supp}(|a|) = \mathrm{supp}(a)$ *by (ii) using 5.2(iv).*

Proof. Let b be the sum of the positive components of a. Obviously $b \geq 0$. Further, $b \geq a$, because $a - b$ is the sum of all negative components of A and so $a - b \leq 0$. Now if $c \geq 0$ and $c \geq a$, then every positive component of a is less or equal to some component of c. Hence by 5.2(vii) we see that $b \leq c$. This shows that $b = a \vee 0$ is the supremum of the set of positive components of a. Obviously $(a \vee 0)_\gamma = b_\gamma$ satisfies the assertion. □

From now on Γ will always be a root system.

5.4. **Corollary.** *Let* $a, b \in \mathbb{H}$. *Then*

$$a \wedge b = 0 \iff a, b \geq 0 \text{ and } \mathrm{supp}(a)^{\downarrow} \cap \mathrm{supp}(b)^{\downarrow} = \emptyset.$$

Proof. $\Rightarrow$. Suppose there is some $\varepsilon \in \mathrm{supp}(a)^{\downarrow} \cap \mathrm{supp}(b)^{\downarrow}$. Then there are components c of a and d of b with $\varepsilon \in \gamma^{\downarrow} \cap \delta^{\downarrow}$, where $\gamma = \max \mathrm{supp}(c)$ and $\delta = \max \mathrm{supp}(d)$. Since Γ is a root system we may assume that $\gamma \leq \delta$.

If $\gamma < \delta$ or $c_\gamma \neq d_\delta$ then c, d are comparable and from $c \leq a, d \leq b$ we get $0 < c \leq a, b$ or $0 < d \leq a, b$, in each case a contradiction to $a \wedge b = 0$. Hence $\gamma = \delta$ and $c_\gamma = d_\delta$. But then the element $u = c_\gamma x^\gamma + (c - c_\gamma x^\gamma) \wedge (d - c_\gamma x^\gamma)$ satisfies $0 < u \leq c, d$ again contradicting $a \wedge b = 0$.

$\Leftarrow$. If $a \wedge b > 0$ then take a component c of $a \wedge b$ and set $\gamma = \max \mathrm{supp}(c)$. Then $\gamma \in \mathrm{supp}(c)^{\downarrow} \cap \mathrm{supp}(d)^{\downarrow} \subseteq \mathrm{supp}(a)^{\downarrow} \cap \mathrm{supp}(b)^{\downarrow} = \emptyset$, which is impossible. □

5.5. **The Hahn sum** In the setup of 5.1 we define the **Hahn sum** on Γ as

$$\mathscr{S} = \Sigma(\Gamma, (G_\gamma)_{\gamma \in \Gamma}) = \{a \in \prod_{\gamma \in \Gamma} G_\gamma \mid \mathrm{supp}(a) \text{ is finite}\}.$$

By 5.3, $\mathscr{S}$ is an ℓ-subgroup of $\mathbb{H}$ (use 2.5(ii)).

One can form other ℓ-subgroups of $\mathbb{H}$. For example

$$G_1 = \{a \in \mathbb{H} \mid \mathrm{supp}(a) \text{ has the DCC}\},$$
$$G_0 = \{a \in \mathbb{H} \mid C \cap \mathrm{supp}(a) \text{ is finite for every graph component of } \Gamma\}.$$

G_0 and G_1 are easily seen to be ℓ-subgroups of $\mathbb{H}$. We get ℓ-subgroups

$$\mathscr{S} \subsetneq G_0 \subsetneq G_1 \subsetneq \mathbb{H}.$$

Note that elements $a \in \mathbb{H}$ with the property that all components have finite support do not form a subgroup, because Γ could be $D + \{\infty\}$, where D is discrete and infinite, a could have support D and b could have support $\{\infty\}$.

The Conrad-Harvey-Holland Representation theorem says that every ℓ-group is an ℓ-subgroup of $\mathbb{H}(\Gamma, \mathbb{R})$ for some appropriate choice of Γ. We show how to choose the data now but leave the proof to the literature.

5.6. **The archimedean property** Let G be an Abelian ℓ-group.

(i) G is called **archimedean** if for all $g,h\in G^{\geq 0}$ with $g\neq 0$ there is some $n\in\mathbb{N}$ with $n{\cdot}g\not\leq h$; In other words: if $0\leq ng\leq h$ for all $n\in\mathbb{N}$, then $g=0$. In this sense, G does not contain infinitesimal elements. [15]

(ii) If G is a TOAG then clearly G is archimedean just if it has no proper nontrivial convex subgroup. If G is any ℓ-group and $\mathfrak{p}\in\ell\text{-Spec}^*(G)$, then $\mathfrak{p}$ is maximal in $\ell\text{-Spec}^*(G)$ if and only if $G/\mathfrak{p}$ is archimedean. In this sense the archimedean TOAGs play a similar role for ℓ-groups as fields do for commutative rings.

Recall **Hölder's theorem**, saying that the archimedean TOAGs are exactly the ℓ-subgroups of $(\mathbb{R},+,\leq)$.

Proof. This is clear in one direction. If G is an ℓ-group, then clearly its divisible hull is archimedean as well and we may assume that G is archimedean. Then G has $(\mathbb{Q},+,\leq)$ as an ordered subgroup and an embedding $\varphi: G\longrightarrow\mathbb{R}$ can be defined as follows: Take $g\in G\setminus\mathbb{Q}$ and let $P=\{q\in\mathbb{Q}\mid q<g\}$. Since G is archimedean, P is nonempty, bounded from above and its supremum r in $\mathbb{R}$ is not in $\mathbb{Q}$. Now define $\varphi(g)=r$. Straightforward checking shows that φ is an embedding of TOAG's. $\diamond$

(iii) G is called **hyper-archimedean** if every homomorphic image is archimedean.

Claim. G is hyper-archimedean if and only if $\dim(\ell\text{-Spec}(G))=1$ or $G=0$.

Proof. Since G is the unique maximal point of $\ell\text{-Spec}(G)$, the dimension of $\ell\text{-Spec}(G)$ can only be zero if $G=0$. If G is hyper-archimedean and $\mathfrak{p}\in\ell\text{-Spec}(G)$, then $G/\mathfrak{p}$ is archimedean. But this is a TOAG, hence it has no nontrivial convex subgroups. Hence $\mathfrak{p}$ has at most one specialization, showing that $\dim(\ell\text{-Spec}(G))\leq 1$.

Conversely assume $\dim(\ell\text{-Spec}(G))\leq 1$, let $\pi: G\twoheadrightarrow H$ be surjective ℓ-group homomorphism. Then $\ell\text{-Spec}(\pi):\ell\text{-Spec}(H)\longrightarrow\ell\text{-Spec}(G)$ is a homeomorphism onto $\mathbb{H}(\mathrm{Ker}(\psi))$, which implies that $\ell\text{-Spec}(H)$ has dimension ≤ 1. Hence it suffices to show that G is archimedean. Take $g,h\in G$ with $g\neq 0$. Then there is some $\mathfrak{p}\in\ell\text{-Spec}(G)$ with $g\notin\mathfrak{p}$ Since $\dim(\ell\text{-Spec}(G))\leq 1$ there is some $n\in\mathbb{N}$ such that $ng\not\leq h$ in $G/\mathfrak{p}$. But then $ng\not\leq h$ in G as well.$\diamond$

5.7. **Values and regular groups** Let G be an ℓ-group and let $\mathfrak{p}$ be an ℓ-ideal of G. The following are equivalent.

(i) There is some $f\in G$ such that $\mathfrak{p}$ is maximal among ℓ-ideals not containing f.

(ii) $\mathfrak{p}$ is a locally closed point of $\ell\text{-Spec}^*(G)$, i.e. $\mathfrak{p}$ is a proper prime ℓ-ideal and the set $\{\mathfrak{p}\}$ is a locally closed subset[16] of the space $\ell\text{-Spec}^*(G)$.

If these conditions hold, then $\mathfrak{p}$ is called a **regular subgroup** of G.

If $f\in G$, then $\mathfrak{p}$ is maximal among ℓ-ideal not containing f if and only if $\mathfrak{p}$ is a closed point of the space $D(f)$. In this case $\mathfrak{p}$ is called a **value of** f. If f has exactly one value, then f is called **special**.

Proof. This is an easy exercise in translating (i) into topology, which is expressed by (ii). Also see [DiScTr2019, 4.5.10]. □

[15]We work here with abelian ℓ-groups throughout, but it should be mentioned that Archimedean ℓ-groups are automatically abelian, see [Darnel1995, Thm. 53.3, p. 360]

[16]A subset S of a topological space is called **locally closed** if it is locally a closed set, i.e. if for each $s\in S$ there is an open neighborhood U of s such that $S\cap U$ is closed in U. This is easily seen to be equivalent to the property that S is the intersection of an open and a closed set.

5.8**. Properties of locally closed points** If G is an ℓ-group we set

$$\Gamma(G) = \{\mathfrak{p} \in \ell\text{-Spec}^*(G) \mid \mathfrak{p} \text{ is locally closed}\}.$$

(i) $\Gamma(G)$ is a root system for specialization in ℓ-Spec$^*(G)$ (i.e. for inclusion) and is a root system, because ℓ-Spec(G) is already a root system, cf. 2.15(v).

(ii) If $\mathfrak{p} \in \Gamma(G)$ then there is an immediate specialization $\mathfrak{p}^+$ of $\mathfrak{p}$ in ℓ-Spec(G), i.e. $\mathfrak{p} \subsetneq \mathfrak{p}^+$ and every $\mathfrak{q} \in \ell$-Spec(G) with $\mathfrak{p} \subseteq \mathfrak{q} \subseteq \mathfrak{p}^+$ satisfies $\mathfrak{p} = \mathfrak{q}$ or $\mathfrak{q} = \mathfrak{p}^+$. So see this, take $f \in G$ that has $\mathfrak{p}$ as a value. Then one checks that $\mathfrak{p}^+ = \mathfrak{p} + \ell(f)$ using 2.15.

Consequently, $\mathfrak{p}^+/\mathfrak{p}$ has no proper nontrivial convex subgroups and is thus archimedean.

(iii) $\Gamma(G)$ is a subspace of ℓ-Spec(G) and one can show that it is the smallest topological space that has ℓ-Spec$^*(G)$ as its sobrification. "Smallest" here means that it embeds into every space whose sobrification is isomorphic to ℓ-Spec$^*(G)$. We will not need this later on and refer to see [DiScTr2019, 4.5.21] instead; notice that ℓ-Spec$^*(G)$ is sober.

5.9**. Locally closed points in** $\mathbb{H}$ Let $\mathbb{H} = \mathbb{H}(\Gamma, (G_\gamma)_{\gamma\in\Gamma})$ be the Hahn group on the root system Γ as in 5.1. For $\Delta \subseteq \Gamma$ we define

$$I_\Delta := \{a \in \mathbb{H} \mid \operatorname{supp}(a) \cap \Delta = \emptyset\}.$$

Then

(i) $I_\Delta = \mathbb{H}(\Gamma \setminus \Delta)$ is an ℓ-subgroup of $\mathbb{H}$.

(ii) If Δ is an up-set of Γ, then I_Δ is an ℓ-ideal of $\mathbb{H}$. If all G_γ are nonzero, then I_Δ is an ℓ-ideal of $\mathbb{H}$ if and only if Δ is an up-set of Γ.

Now let Δ be an up-set of Γ.

(iii) If Δ is a chain, then I_Δ is an ℓ-prime ideal of $\mathbb{H}$. If all G_γ are nonzero, then Δ is a chain if and only if I_Δ is an ℓ-prime ideal of $\mathbb{H}$.

(iv) For $\gamma \in \Gamma$ the sets

$$\mathbb{H}_\gamma := \{a \in \mathbb{H} \mid \operatorname{supp}(a) \cap \gamma^\uparrow = \emptyset\}, \text{ and}$$
$$\mathbb{H}_{\gamma^+} := \{a \in \mathbb{H} \mid \operatorname{supp}(a) \cap \gamma^\uparrow \subseteq \{\gamma\}\}$$

are ℓ-prime ideals of $\mathbb{H}$. The ℓ-group $\mathbb{H}_\gamma$ is an ℓ-prime ideal of $\mathbb{H}_{\gamma^+}$, $\mathbb{H}_{\gamma^+} = \mathbb{H}_\gamma \oplus G_\gamma{\cdot}x^\gamma$ and the truncation map $\mathbb{H}_{\gamma^+} \longrightarrow G_\gamma{\cdot}x^\gamma$ is an ℓ-group homomorphism that is induces an ℓ-group isomorphism $\mathbb{H}_{\gamma^+}/\mathbb{H}_\gamma \longrightarrow G_\gamma$.

(v) $\mathbb{H}_\gamma$ is a locally closed point of $\mathbb{H}$ if and only if G_γ has a smallest nontrivial convex subgroup. This is the case in particular when G_γ is archimedean.

If G_γ is archimedean, then $\mathbb{H}_{\gamma^+} = \mathbb{H}_\gamma^+$ in the terminology of 5.8 and $\mathbb{H}_\gamma$ is a value of $g \in \mathbb{H}$ if an only if γ is a maximal element of $\operatorname{supp}(g)$.

Proof. (i) holds by 5.2(i).

(ii) By (i) we know that I_Δ is closed under addition. If $0 \le a \le b$, then $\operatorname{supp}(a) \subseteq \operatorname{supp}(b)^\downarrow$ as follows from 5.2(vii). Furthermore $\operatorname{supp}(a) = \operatorname{supp}(|a|)$ for all $a \in \mathbb{H}$ by 5.3.

Hence if Δ is an up-set of Γ, then the characterizations 2.11(iv) holds true and I_Δ is an ℓ-ideal of $\mathbb{H}$.

Conversely assume Δ is not an up-set and all G_γ are nonzero. Take $\delta \in \Delta$ and $\gamma \in \Gamma \setminus \Delta$ with $\delta < \gamma$. Choose $a_\delta \in G_\delta$ with $a_\delta > 0$ and $a_\gamma \in G_\gamma$ with $a_\gamma > 0$. Then $0 \le a_\delta x^\delta < a_\gamma x^\gamma \in I_\Delta$, but $a_\delta x^\delta \notin I_\Delta$, hence I_Δ is not convex.

(iii) Now assume that Δ is a chain. In order to check primality we use 5.4: If $a \wedge b = 0$, then $\mathrm{supp}(a)^{\downarrow} \cap \mathrm{supp}(b)^{\downarrow} = \emptyset$. Suppose there are $\delta_a \in \Delta \cap \mathrm{supp}(a)^{\downarrow}$ and $\delta_b \in \Delta \cap \mathrm{supp}(b)^{\downarrow}$. Since Δ is a chain we may assume $\delta_a \leq \delta_b$; but then $\delta_a \in \mathrm{supp}(a)^{\downarrow} \cap \mathrm{supp}(b)^{\downarrow}$, a contradiction. Hence we may assume that $\mathrm{supp}(a)^{\downarrow} \cap \Delta = \emptyset$ and so $a \in I_\Delta$.

Conversely assume Δ is not a chain and all G_γ are nonzero. Take incomparable $\delta_1, \delta_2 \in \Delta$. In the root system Γ this means $\delta_1^{\downarrow} \cap \delta_2^{\downarrow} = \emptyset$. Take $a_i \in G_{\delta_i}$, $a_i > 0$. Then $a_1 x^{\delta_1}, a_2 x^{\delta_2} \notin I_\Delta$, but $a_1 x^{\delta_1} \wedge a_2 x^{\delta_2} = 0$ (by 5.4), showing that I_Δ is not prime.

(iv) Since $\mathbb{H}_\gamma = I_{\gamma^\uparrow}$ and $\mathbb{H}_{\gamma^+} = I_{\gamma^\uparrow \setminus \{\gamma\}}$ we know from (iii) that $\mathbb{H}_\gamma$ and $\mathbb{H}_{\gamma^+}$ are prime ℓ-ideals. The remaining statements follow from 5.2(i).

(v) $\mathbb{H}_\gamma$ is a locally closed point if and only if it is isolated in its closure in ℓ-Spec(G). This is equivalent to saying that there is a smallest nontrivial convex subgroup in $\mathbb{H}_{\gamma^+}/\mathbb{H}_\gamma \cong G_\gamma$ (by (iv)). This shows the first equivalence.

Now assume G_γ is archimedean. If $g \in \mathbb{H}$, then γ is is a maximal element of $\mathrm{supp}(g)$ if and only if $g \in \mathbb{H}_{\gamma^+} \setminus \mathbb{H}_\gamma$ and these are precisely the elements that have H_γ as value. □

5.10**. Conrad-Harvey-Holland Theorem** Let G be an ℓ-group. Then there is an ℓ-group embedding

$$\varphi : G \longrightarrow \mathbb{H}(\Gamma(G), (\mathbb{Q} \otimes (\mathfrak{p}^+/\mathfrak{p}))_{\mathfrak{p} \in \Gamma(G)}).$$

such that for each $\mathfrak{p} \in \Gamma(G)$ and every $g \in \mathfrak{p}^+$ we have $\varphi(g)_{\mathfrak{p}} = g \bmod \mathfrak{p}$.

Since all groups $\mathfrak{p}^+/\mathfrak{p}$ are archimedean, they can be embedded into $(\mathbb{R}, +, \leq)$ (Hölder) and therefore there is also an ℓ-group embedding

$$G \longrightarrow \mathbb{H}(\Gamma(G), \mathbb{R}).$$

Proof. See [AndFei1988, Theorem 3.2], or [Darnel1995, Theorem 51.7, p.338]. .

The strategy of the proof is as follows: For $\mathfrak{p} \in \Gamma(G)$ we write $\pi_{\mathfrak{p}} : \mathfrak{p}^+ \longrightarrow \mathfrak{p}^+/\mathfrak{p}$ be the residue map. Let $H \subseteq G$ and let $\varphi : H \longrightarrow \mathbb{H} = \mathbb{H}(\Gamma(G))$ be a homomorphism of groups. The we say that φ preserves coefficients if for all $\mathfrak{p} \in \Gamma(G)$ and every $g \in \mathfrak{p}^+ \cap H$ we have $\varphi(g)_{\mathfrak{p}} = \pi_{\mathfrak{p}}(g)$. Then one shows:

Claim 1. If $\varphi : G \longrightarrow \mathbb{H}$ is a group homomorphism that preserves coefficients, then φ is an ℓ-group embedding.

Claim 2. Every group homomorphism $\varphi : H \longrightarrow \mathbb{H}$ that preserves coefficients, defined on a proper subgroup H of G, can be extended to a group homomorphism that preserves coefficients on a properly larger subgroup of G.

Starting with $H = \{0\}$, claim 2 and the lemma of Zorn then gives a group homomorphism $\varphi : G \longrightarrow \mathbb{H}$ that preserves coefficients and by claim 1 φ is an ℓ-group embedding. □

Model theory of abelian ℓ-groups

The model theoretic analysis of an abelian ℓ-group G is, roughly speaking, centered around two invariants of G. The first one is the common theory of the TOAGs $G/\mathfrak{p}$, where $\mathfrak{p} \in \ell\text{-Spec}(G)$; the second one is the topological type of ℓ-Spec(G). Recall that G is naturally embedded into $\prod_{\mathfrak{p}\in\ell\text{-Spec}(G)} G/\mathfrak{p}$. "Topological type" here refers to the homeomorphism type of the space ℓ-Spec(G). In order to say something about that, one may apply Stone duality, which implies that ℓ-Spec(G) is completely determined by its lattice L of quasi-compact open subsets. This lattice is a first order structure, hence we can attach model theoretic invariants to it; most importantly the theory of this lattice.

In this course we focus on two cases.

(I) *Divisible ℓ-groups.* Here the first invariant above, the common theory of the TOAGs $G/\mathfrak{p}$, is the easiest such theory and we will see that the model theory of G can essentially be reduced to the model theory of L. Caveat: L is not always interpretable in G, however we will still obtain interesting applications.

(II) *Projectable ℓ-groups.* This is kind of opposite to (I). Here a principal restriction on ℓ-Spec(G) is made, namely it is assumed that ℓ-Spec(G) is *stranded* (the graph components of the specialization relation are chains) and the subspace of minimal points is Boolean.

6. Model theory of divisible ℓ-groups

This is in large parts from [SheWei1987b].

6.1. **Standard structures** If G is an ℓ-subgroup of functions $X \longrightarrow M$ for some TOAG M, then we say **standard structure** (for $G \subseteq M^X$) for the triple (G, L, P), where L is the lattice of zero sets of functions $f \in G$ and $P : G \longrightarrow L$ is defined by $P(f) = \{f \geq 0\} = \{x \in X \mid f(x) \geq 0\}$. We will also use the map $Z : G \longrightarrow L$, $Z(f) = \{f = 0\} = P(f) \cap P(-f)$. Notice that $P(f) = Z(f \wedge 0)$.

Notice that every totally ordered abelian group G will be considered as a standard structure, by setting X a singleton set and $M = G$. In fact recall that abelian ℓ-groups are representable, i.e. each such group G is isomorphic to an ℓ-subgroup of a product of totally ordered abelian groups. Amalgamating these totally ordered groups into a divisible ordered abelian group Γ shows that G is isomorphic to some ℓ-subgroup of Γ^X for some set X. Hence every abelian ℓ-group can be expanded to a standard structure with $M = (\Gamma, +, \leq)$ and $X = \ell\text{-Spec}(G)$ (or $\ell\text{-Spec}^*(G)$).

Observe that in general Γ cannot be chosen to be $\mathbb{R}$, because if $G \subseteq \mathbb{R}^X$, then for $x \in X$ the prime ℓ-ideal $\mathfrak{p}_x = \{f \in G \mid f(x) = 0\}$ of G has the property that $G/\mathfrak{p}_x$ is archimedean. Now if G is a totally ordered abelian group with no proper maximal convex subgroup, then this is only possible if $G \subseteq \mathfrak{p}_x$. But then $G = 0$.

6.2. **Languages** We will work with the following first order-languages

$$\begin{aligned}
\mathscr{L}^{gr} &= \{+, -, 0, \leq\} \text{ the language of po-groups} \\
\mathscr{L}^{l-gr} &= \mathscr{L}^{gr}(\wedge, \vee) \text{ the language of } \ell\text{-groups} \\
\mathscr{L}^{po} &= \{\sqsubseteq, \top\} \text{ the language of posets} \\
\mathscr{L}^{lat} &= \mathscr{L}^{po}(\sqcap, \sqcup) \text{ the language of lattices.}
\end{aligned}$$

Consider the following two-sorted language $\mathscr{L}^{gr,po}$. In the first sort (called the **group sort**) we take $\mathscr{L}^{gr}$, in the second sort (called the **space sort**) we take $\mathscr{L}^{po}$. In addition there is a unary function symbol P from the group sort to the space sort.

If (G, L, P) is a standard structure for $G \subseteq M^X$, then we view (G, L, P) as $\mathscr{L}^{gr,po}$-structure whose group sort is interpreted as G in the language $\mathscr{L}^{gr}$, whose space sort is interpreted as the lattice of G-zero sets in the language $\mathscr{L}^{po}$ and where the function $G \longrightarrow L$ interprets the function symbol P. We will also write $Z(f)$ for the formula $P(f) \sqcap P(-f)$.

Let T be the common theory of all standard structures in the language $\mathscr{L}^{gr,po}$.

We are also considering the extensions $\mathscr{L}^{gr,lat}$, $\mathscr{L}^{l-gr,po}$, $\mathscr{L}^{l-gr,lat}$ of the language $\mathscr{L}^{gr,po}$, where the superscripts indicate the first order languages that are put on the two sorts. It is clear how each standard structures can be definably expanded to these languages. The common theory of all standard structures in any of the two sorted languages is a an extension of T by definitions. The corresponding theory is denoted by putting the appropriate superscripts to T.

For the remaining ingredients of an elimination procedure we need an assumption on the standard structures:

6.3. **Definition.** [SheWei1987b, page 5]
A standard structure $G = (G, L, P)$ for $G \subseteq X^M$ is said to be **closed under patching** if for all $f, g \in G$ and all G-zero sets $A, B \in L$ with $A \cap B \subseteq \{f = g\}$

there is some $h \in G$ with $A \subseteq \{h = f\}$ and $B \subseteq \{h = g\}$. Notice that this is an elementary condition of the two sorted structure (G, L, P), expressed by

$$\forall f, g, f_1, g_1 \Big(Z(f_1) \sqcap Z(g_1) \sqsubseteq Z(f-g) \to \exists h (Z(f_1) \sqsubseteq Z(h-f) \,\&\, Z(g_1) \sqsubseteq Z(h-g)) \Big),$$

where all variables are from the group sort.

6.4. *Remark.* If (G, L, P) is a model of the theory T, then it is isomorphic to a standard structure: P is a lattice homomorphism and we may take X to be the image of the map $\mathrm{PrimI}(L) \longrightarrow \ell\text{-Spec}(G)$. Notice that T "knows" that G is a subgroup of some M^X: $f < g$ means $P(g-f) = \top$ and $P(f-g) \neq \top$, hence there is a prime ℓ-ideal $\mathfrak{p}$ with $g > f$ in $G/\mathfrak{p}$.

It follows that $T \cup \{\forall f, g (f < g \iff P(g-f) = \top \,\&\, P(f-g) \neq \top)\}$ axiomatizes standard structures. Using the sentence in 6.3 we see that standard structures closed under patching are first order axiomatisable as well.

6.5. *Examples.*

(1) Many examples of standard structures that are closed under patching stem from a context where some form of Tietze extension property holds for functions from G. For example if X is a metric space and G is the ℓ-group of continuous functions $X \longrightarrow \mathbb{R}$. The the classical Tietze extension theorem says that for every closed subset A of X, every continuous functions $A \longrightarrow \mathbb{R}$ can a continuous extension $X \longrightarrow \mathbb{R}$. Hence the standard structure given by $G \subseteq M^X$ is closed under patching.

Another example comes from o-minimality. Suppose M is (an o-minimal expansion of) a real closed field and X could be a locally closed and definable subset of some M^n. Let G be the ℓ-group of continuous and definable functions $X \longrightarrow M$. Then the Tietze extension property of M (see. [vdDrie1998, Chapter 8, (3.10)]) implies that the standard structure given by $G \subseteq M^X$ is closed under patching.

Proof. Take $f, g \in G$ and G-zero sets A, B with $A \cap B \subseteq \{f = g\}$ Then the map $h_0 : A \cup B \longrightarrow M$ defined by

$$h_0(p) = \begin{cases} f(p) & \text{if } p \in A, \\ g(p) & \text{if } p \in B. \end{cases}$$

is continuous (and definable if we are in the o-minimal context). By the Tietze extension property, h_0 has an extension to some $h \in G$. This h has the required properties. ⋄

One possible axiomatic approach to the Tietze extension property in a standard structure may be found in [Tressl2016, Definition 2.1].

(2) For any ordered field K the ℓ-group G of semi-linear continuous functions $K^2 \longrightarrow K$ fails to have the Tietze extension property and in fact G is also not closed under patching. In fact, G has a lattice ordered subgroup with the same zero sets as G, but which is not an elementary substructure of G. For details see [Tressl2016, 4.7]

6.6. **Lemma.** *If the image of $X \longrightarrow \ell\text{-Spec}^*(G), x \mapsto \{f \in G \mid f(x) = 0\}$ is patch dense (meaning: for all $a, b \in G$ with $V(a) \cap D(b) \neq \emptyset$ there is some $x \in X$ with $a(x) = 0 \neq b(x)$), then (G, L, P) is closed under patching.*

In particular, every ℓ-group G can be expanded to a standard structure that is closed under patching: Take $X = \ell\text{-Spec}(G)$, $L = \mathring{\mathcal{K}}(\ell\text{-Spec}(G))$ and $P(f) = D(f \wedge 0)$.

Proof. Take $a, b, f, g \in G$ and assume $\{a = 0\}_X \cap \{b = 0\}_X \subseteq \{f = g\}_X$. By patch density this implies $V(a) \cap V(b) \subseteq V(f - g)$. Then $V(|a| \vee |b|) = V(a) \cap V(b) \subseteq V(f - g)$ and so $f - g \in \ell(|a| \vee |b|)$. Now by 2.11(vi) we have $\ell(|a| \vee |b|) = \ell(a) + \ell(b)$. Take $u \in \ell(a)$, $v \in \ell(b)$ with $f - g = u + v$ and define $h = f - u = g + v$. Then $V(a) \subseteq V(u) = V(h - f)$ and $V(b) \subseteq V(v) = V(h - g)$ as required. □

6.7. **Theorem.** [SheWei1987b, Theorem 2.1] *Let φ be an $\mathscr{L}^{l-gr,lat}$-formula. Then φ is equivalent in every divisible standard structure (i.e. the underlying ℓ-group is divisible) that is closed under patching, to a formula (in the same variables as φ) of the form*

$$(*) \qquad \exists \xi_1 \dots \xi_k \Big(\gamma \ \& \ \bigwedge_{i=1}^{k} \xi_i = P(t_i) \Big),$$

where γ is an $\mathscr{L}^{lat}$-formula, the ξ_i are distinct variables of space sort and t_i are $\mathscr{L}^{gr}$-terms.

6.8. **Corollary.** *In the situation of 6.7, L is stably embedded in (G, L, P).*

Proof. Let $Z \subseteq L^n$ be parametrically definable in (G, L, P). Then there is an $\mathscr{L}^{l-gr,lat}$-formula $\varphi(\bar{x}, \bar{\eta}, \bar{\tau})$, where $\bar{x}$ are variables of group sort and $\bar{\eta}, \bar{\tau}$ are variables of space sort and $\bar{f} \subseteq G^{\bar{x}}$, $\bar{A} \in L^{\bar{\tau}}$ such that Z is defined in $P : G \longrightarrow L$ by $\varphi(\bar{f}, \bar{\eta}, \bar{A})$. Putting φ into the form stated in 6.7 it is clear that Z is parametrically definable in L. □

6.9. **Corollary.** *In the situation of 6.7, if the lattice L is decidable, then so is the standard structure (G, L, P) as well as the ℓ-group G.*

Proof. Every sentence of the form $(*)$ in 6.7 is a sentence of the lattice L. The recursive list of sentences true in L will thus give a recursive list of all sentences true in (G, L, P), because standard structures closed under patching are first order axiomatisable by 6.4. □

Applications

There are two issues with the applicability of 6.7. Firstly, we need to check when the standard structure is closed under patching. Secondly, the lattice L might not be interpretable in the ℓ-group G. Hence if we are interested in G alone, the introduction of L might not be appropriate. Notice that the lattice L often interprets true arithmetic, i.e it interprets $(\mathbb{N}, +, \cdot)$, see [Tressl2017].

The two issues are connected: Recall from 6.6 that every ℓ-group G can be expanded to a standard structure. In order to interpret the lattice $\mathring{\mathcal{K}}(\ell\text{-Spec}(G))$ naturally, one would ideally want to define the binary relation $\ell(a) = \ell(b)$ of G.

6.10**. Defining L in G** In order to interpret L in G naturally we need to check that the binary relation $\{f = 0\} \subseteq \{g = 0\}$ of G is definable. Assuming that there is some $g \in G$ without zeroes, this is equivalent to showing that the unary relation $\{f = 0\} = \perp$ is definable in G:

For $f, g \in G$ we have

$$\begin{aligned} \{f \neq 0\} \cap \{g \neq 0\} &= \{|f| \wedge |g| \neq 0\} \text{ and} \\ \{f \neq 0\} \cup \{g \neq 0\} &= \{|f| \vee |g| \neq 0\}. \end{aligned}$$

Hence

$$\begin{aligned} \{f = 0\} \cup \{g = 0\} &= \{|f| \wedge |g| = 0\} \text{ and} \\ \{f = 0\} \cap \{g = 0\} &= \{|f| \vee |g| = 0\}. \end{aligned}$$

In particular

(i) $\{f = 0\} \supseteq \{g \neq 0\} \iff \{f \neq 0\} \cap \{g \neq 0\} = \emptyset \iff |f| \wedge |g| = 0$, hence the relation $\{f = 0\} \supseteq \{g \neq 0\}$ is 0-definable in G, and,

(ii) $\{f = 0\} \subseteq \{g \neq 0\} \iff \{f \neq 0\} \cup \{g \neq 0\} = K^n \iff \{|f| \vee |g| = 0\} = \emptyset$, hence the relation $\{f = 0\} \subseteq \{g \neq 0\}$ is 0-definable in G, provided the set of functions without zeroes is 0-definable in G. Notice that if we can define $\{f = 0\} \subseteq \{g \neq 0\}$, then we can also define $\{f = 0\} \subseteq \{h = 0\}$ by saying $\forall g(\{h = 0\} \subseteq \{g \neq 0\} \to \{f = 0\} \subseteq \{g \neq 0\})$: If $h(x) \neq 0$ then take g with $\{h = 0\} \subseteq \{g \neq 0\}$ and $g(x) = 0$; if the condition holds, we must have $f(x) \neq 0$.

Notice that in the undecidability proof of $FA\ell(n)$ for $n \geq 3$ in [GlMaPo2005] the main work is indeed to show that the set of all functions with no zeroes is definable. Then as $n \geq 3$ they invoke Grzegorczyk's "Undecidability of some topological theories" result on Heyting algebras.

Exercise: For an ordered field K, the ℓ-group of all continuous semi-linear functions $K \longrightarrow K$, defines L, so here L is the lattice of subsets of K that are finite unions of intervals of the form $[a, b]$, $(-\infty, b]$ or $[a, +\infty)$. Hint: First express, in the ℓ-group G the property of $f \in G$ that its zero set has empty interior. Then express the property of f that its cozero set is definably connected.

7. The positive cone

Again all ℓ-groups here are Abelian.

7.1**. Ideals of up-sets** Let P be a poset and let $B \subseteq P$ be a nonempty up-set.

(i) The map

$$\Theta_0 : \{I \mid I \text{ is an ideal of } P \text{ with } B\cap I\neq\emptyset\} \longrightarrow \{J \mid J \text{ is an ideal of } B\}$$
$$I \longmapsto I\cap B$$

is bijective and its compositional inverse sends J to $J^{\downarrow}$ (in P).

(ii) Now let P be a distributive lattice and let B also be closed under meets. Then B is itself a distributive lattice and the bijection Θ_0 in (i) restricts to a bijection

$$\{I \mid I \text{ is a prime ideal of } P \text{ with } B\cap I\neq\emptyset\} \longrightarrow \{J \mid J \text{ is a prime ideal of } B\}$$
$$I \longmapsto I\cap B.$$

If B is the principal up-set generated by $b\in P$, then the left hand side is equal to $V(b)\subseteq \mathrm{PrimI}(P)$.

Proof. (i) It is clear that Θ_0 is well defined and for an ideal J of B, the set $J^{\downarrow}$ is an ideal of P with $\Theta_0(J^{\downarrow})=J$. Now take an ideal I of P with $I\cap B\neq\emptyset$, say $b\in B\cap I$. Then $I=(I\cap B)^{\downarrow}$, because the inclusion $\supseteq$ is obvious and for $a\in I$, there is some $c\in I$ with $a,b\leq c$; hence $c\in I\cap B$ and $a\in (I\cap B)^{\downarrow}$.

(ii) If I is a prime ideal of P and $b,b'\in B$ with $b\wedge b'\in P\cap B$, then b or b' is in $P\cap B$, hence $P\cap B$ is a prime ideal of B. Conversely, let J be a prime ideal of B. By (i) it remains to show that $J^{\downarrow}$ is a prime ideal of P. Take $p,p'\in P$ with $p\wedge p'\in J^{\downarrow}$ and take $b\in J$ with $p\wedge p'\leq b$. Then $b=b\vee(p\wedge p')=(b\vee p)\wedge(b\vee p')$ and $b\vee p, b\vee p'\in B$ as B is an up-set of P. Since J is a prime ideal of B we get $b\vee p\in J$ or $b\vee p'\in J$, say $b\vee p\in J$. It follows $p\in J^{\downarrow}$, as required. □

7.2. *Remark.* For any distributive lattice L, we extend the poset L by a new element ∞ and get a new poset $L_\infty = L\,\dot{\cup}\{\infty\}$ by declaring $a<\infty$ for all $a\in L$. It is straightforward to see that L_∞ is a again a distributive lattice (just check $x\wedge(y\vee z)\leq (x\wedge y)\vee(x\wedge z)$) and L is a sublattice of L_∞. Further, any morphism between distributive lattices $\varphi: L\longrightarrow M$ extends uniquely to a homomorphism $\varphi_\infty : L_\infty\longrightarrow M_\infty$ that respects the top elements.

Now assume L has a bottom element. Then L_∞ is a bounded distributive lattice and we define

$$\mathrm{PrimI}_\infty(L)=\{I\cap L \mid I\in \mathrm{PrimI}(L_\infty)\}$$

together with the induced topology given by the bijection that sends I to $I\cap L$ (observe that L is a prime ideal of L_∞ and so $L\in \mathrm{PrimI}_\infty(L)$).

7.3. **Definition.** Let G be an ℓ-group. We define $T(G)$ to be the set

$$T(G)=\{S\subseteq G^{\geq 0} \mid S+S\subseteq S \text{ and } S \text{ is an ideal of the lattice } G^{\geq 0}\}.$$

Observe that every down-set of $G^{\geq 0}$ that is closed under addition, is already an ideal of $G^{\geq 0}$, because $f\vee g\leq f+g$ for $f,g\in G^{\geq 0}$.

We equip $T(G)$ with the topology that has the sets

$$D(f)=\{S\in T(G)\mid f\notin S\}$$

as an open subbasis.

7.4. *Remark.* Consider the algebraic structure $T = (T(G) \cup \{+\infty\}, \wedge, +, +\infty, 0)$. Then T is a commutative semiring with addition $\wedge$ and multiplication $+$. The neutral element for addition in T is $+\infty$ and the neutral element for multiplication in T is 0. It is distributive because $x + (y \wedge z) = (x + y) \wedge (x + z)$ holds universally by 2.2(iii)($\wedge$). Hence $T(G)$ is a "Tropical semiring", which explains the use of the letter 'T'.

A standard argument in spectral spaces shows that $T(G)$ is a spectral subspace of the spectral space of all ideals of $G^{\geq 0}$ (which is equal to the spectral space of all proper ideals of $(G^{\geq 0})_\infty$, see 7.2 and notice that $(G^{\geq 0})_\infty$ is a closed and constructible point in the space of ideals of $(G^{\geq 0})_\infty$).

Similarly, we know that the set

$$\ell\text{-}Id(G) := \{\ell\text{-ideals of } G\}$$

carries a spectral space of which the $D(f) = \{I \in \ell\text{-}Id(G) \mid f \notin I\}$ form a subbasis. (As we use the same notation $D(f)$ in $\ell\text{-}Id(G)$ and in $T(G)$ for $f \geq 0$ we will mention the ambient space if necessary.)

7.5. **Proposition.** *Let G be an ℓ-group.*

(i) The map

$$\Theta : \ell - Id(G) \longrightarrow T(G);\ \Theta(I) = I \cap G^{\geq 0}$$

is a homeomorphism.

(ii) The inverse of Θ maps $S \in T(G)$ to $S - S$, which is equal to the convex hull of $S \cup -S$.

(iii) The map Θ restricts to a homeomorphism

$$\ell\text{-Spec}(G) \longrightarrow \mathrm{PrimI}_\infty(G^{\geq 0}) \cap T(G).$$

(Notice that $\mathrm{PrimI}_\infty(G^{\geq 0}) \cap T(G)$ is a spectral subspace of $\mathrm{PrimI}_\infty(G^{\geq 0})$.)

(iv) The minimal prime ideals of the lattice $G^{\geq 0}$ are in $T(G)$, hence the map Θ restricts to a homeomorphism

$$(\ell\text{-Spec}(G))^{\min} \longrightarrow \mathrm{PrimI}_\infty(G^{\geq 0})^{\min}.$$

Proof. First we show that Θ is well defined. If $I \in \ell\text{-Spec}(G)$ and $S = I \cap G^{\geq 0}$, then obviously $S + S \subseteq S$. Since I is convex, S is a down-set of $G^{\geq 0}$. Since I is a sublattice of H, S is an ideal of the lattice $G^{\geq 0}$.

Now let $S \subseteq G^{\geq 0}$ be an ideal of the lattice $G^{\geq 0}$ with $S + S \subseteq S$. We show that $I := S - S$ is an ℓ-ideal of G and prove (ii). As $S + S \subseteq S$ it is clear that $I - I \subseteq I$, hence I is a subgroup. I is the convex hull of $S \cup -S$ in G: If $s, t \in S$, then $-t \leq s - t \leq s$. On the other hand for any $g \in G$ in the convex hull of $S \cup -S$ there are $s, t \in S$ with $-t \leq g \leq s$; then $0 \leq g \vee 0 \leq s, -t \leq g \wedge 0 \leq 0$ and so $g = (g \vee 0) - (-(g \wedge 0)) \in S - S = I$.

It follows that I is convex in G, but it also follows that I is a sublattice of G: If $g, g' \in I$, the pick $s, s', t, t' \in S$ with $-t \leq g \leq s$ and $-t' \leq g' \leq s'$. Then $-(t \vee t') = (-t) \wedge (-t') \leq g \wedge g' \leq g \vee g' \leq s \vee s'$ and since S is an ideal of $G^{\geq 0}$, we see that I is a sublattice and thus I is an ℓ-ideal.

As each $S \in T(G)$ is a down set of $G^{\geq 0}$ we also know $\Theta(S - S) = (S - S) \cap G^{\geq 0} = S$. Hence the bijectivity of Θ and item (ii) are shown if we confirm that $\Theta(I) - \Theta(I) = I$ for any ℓ-ideal of G. However, this follows from $g = (g \vee 0) - (-(g \wedge 0))$ and $g \vee 0, g \wedge 0 \in \Theta(I)$ for all $g \in I$.

It remains to show that Θ is a homeomorphism. As we already know that Θ is bijective, it suffices to remark that the sets $D(f)$ with $f \geq 0$ form a subbasis on both sides (for ℓ-$Id(G)$ observe that $V(g) = V(g \vee 0) \cap V(-(g \wedge 0)))$, and obviously Θ restricts to a bijection between these sets.

(iii) We need to show that an ℓ-ideal I of G is prime if and only if $I \cap G^{\geq 0}$ is prime in the lattice $G^{\geq 0}$. But this in fact could be taken as the definition of primality, or it is a well known characterization of prime ℓ-ideals (see [Darnel1995, Thm. 9.1, p. 49]).

(iv) (This also follows from 7.6). Let $\mathfrak{p}$ be a prime ideal of the lattice $G^{\geq 0}$. We just need to find an element in $\bigcap_{f \in G^{\geq 0} \setminus \mathfrak{p}} D(f) \subseteq \ell\text{-Spec}(G^{\geq 0})$. By compactness, it suffices to show that for all $f_1, \ldots, f_n \in G^{\geq 0} \setminus \mathfrak{p}$, the subset $D(f_1) \cap \ldots \cap D(f_n)$ of ℓ-Spec$(G^{\geq 0})$ is not empty. If it were empty, then $f_1 \wedge \ldots \wedge f_n = 0$, which implies $f_1 \wedge \ldots \wedge f_n = 0 \in \mathfrak{p}$. But then $f_i \in \mathfrak{p}$ as $\mathfrak{p}$ is prime, a contradiction. □

7.6. **Every lattice ideal contains a largest additive ideal** Let $I \subseteq G^{\geq 0}$ be an ideal of the lattice $G^{\geq 0}$.

(i) The set
$$I^\bullet = \{f \in G^{\geq 0} \mid \mathbb{N}\cdot f \subseteq I\}$$
is in $T(G)$ and is thus the largest element of $T(G)$ contained in I.

(ii) If I is a prime ideal of the lattice $G^{\geq 0}$, then $I^\bullet$ is a prime ideal of the lattice $G^{\geq 0}$. Hence if I is a minimal prime ideal of the lattice $G^{\geq 0}$, then $I = I^\bullet \in T(G)$.

(iii) If $F \subseteq G^{\geq 0}$ is a filter and $I \in T(G)$ is an ℓ-ideal, maximal with the property $I \cap F = \emptyset$, then I is a prime ideal.

Consequently – using 7.5 – if $F \subseteq G^{\geq 0}$ is a filter and $I \subseteq G$ is an ℓ-ideal, maximal with the property $I \cap F = \emptyset$, then I is a prime ℓ-ideal. It follows that every ℓ-ideal I of G is the intersection of all the prime ℓ-ideals containing I: For $f \in G \setminus I$ we may assume that $f \vee 0 \notin I$ and we get a prime ℓ-ideal $\mathfrak{p}$ of G containing I and not containing $f \vee 0$, thus $f \notin \mathfrak{p}$ either.

Proof. (i). $I^\bullet$ is a down-set of $G^{\geq 0}$ because for $0 \leq g \leq f \in I^\bullet$ we have $\mathbb{N}\cdot g \subseteq (\mathbb{N}\cdot f)^\downarrow \subseteq I$. $I^\bullet$ is closed under $\vee$ because if $f, g \in I^\bullet$, then $n(f \vee g) = nf \vee ng \in I$ for all $n \in \mathbb{N}$, as I is closed under $\vee$.

$I^\bullet$ is closed under addition: If $\mathbb{N}\cdot f, \mathbb{N}\cdot g \subseteq I$, then $n\cdot(f + g) \leq 2n\cdot(f \vee g)$ for all $n \in \mathbb{N}$. Since $f \vee g \in I^\bullet$ we get $f + g \in I^\bullet$.

(ii) Take $f, g \in G^{\geq 0}$ and assume that $f \wedge g \in I^\bullet$. We may assume that $f \notin I^\bullet$. Hence for some $k \in \mathbb{N}$ we have $k\cdot f \notin I$. But then $n\cdot f \notin I$ for all $n \geq k$. Now for $n \geq k$ we have $n\cdot(f \wedge g) = (nf) \wedge (ng) \in I$ and primality of I entails $ng \in I$. This shows that $\mathbb{N}\cdot g \subseteq I$, hence $g \in I^\bullet$ as required.

(iii) By [DiScTr2019, 3.2.1] there is a prime ideal $J \subseteq G^{\geq 0}$ of the lattice $G^{\geq 0}$ with $I \subseteq J$ and $J \cap F = \emptyset$. Since $I \in T(G)$ we have $I \subseteq J^\bullet$. By maximality of I we have $I = J^\bullet$ and from (ii) we know that $J^\bullet$ is a prime ideal. □

7.7. **Principal down-sets in bounded distributive lattices** (this can also be done via [DiScTr2019, Theorem 5.4.10])
Let L be a bounded distributive lattice and let $a \in L$. Let $\pi : L \longrightarrow a^\downarrow$, $\pi(x) = x \wedge a$.

(i) π is a surjective homomorphism of bounded distributive lattices and a retraction of the poset embedding $a^{\downarrow} \hookrightarrow L$ (which itself is obviously not an embedding of bounded distributive lattices, unless $a = \top$).
(ii) $\mathrm{PrimI}(\pi)$ maps $\mathrm{PrimI}(a^{\downarrow})$ homeomorphically onto $D(a)$.

Proof. We work via Stone duality. Let $s : D(a) \hookrightarrow X = \mathrm{PrimI}(L)$ be the inclusion map. Then $\mathring{\mathcal{K}}(s)$ is isomorphic to π, because $\mathring{\mathcal{K}}(s)$ maps $D(x)$ to $s^{-1}(D(x)) = D(x) \cap D(a) = D(a \wedge x)$. □

7.8. **Intervals in ℓ-groups** We spell out 7.7 in the context of ℓ-groups: Let G be an ℓ-group and let $f \in G^{\geq 0}$. Let $\pi : G^{\geq 0} \cup \{\infty\} \longrightarrow [0, f], \pi(x) = x \wedge f$.

(i) π is a surjective homomorphism of bounded distributive lattices and π is a section of the poset embedding $[0, f] \hookrightarrow G^{\geq 0} \cup \{\infty\}$.
(ii) $\mathrm{PrimI}(\pi)$ maps $\mathrm{PrimI}([0, f])$ homeomorphically onto $D(f)$ (which here has to be understood as $\{\mathfrak{p} \in \mathrm{PrimI}(G^{\geq} \cup \{\infty\}) \mid f \notin \mathfrak{p}\}$).

8. Pseudo-complementation and regularization in topological form

8.1. Review of the spectrum of a bounded distributive lattice. Let $L = (L, \leq, \wedge, \vee, \top, \bot)$ be a bounded distributive lattice. We define the (prime ideal) spectrum $\mathrm{PrimI}(L)$ of L as follows.

As a set, $\mathrm{PrimI}(L)$ consist of all prime ideals of L (cf. 2.13). If L is a Boolean algebra, then prime ideals are precisely the prime ideals of the commutative ring that has $\wedge$ as multiplication and $(x \wedge \neg y) \vee (y \wedge \neg x)$ ('symmetric difference') as addition; furthermore the prime ideals of a Boolean algebras are precisely the complements of ultrafilters and the ultrafilter space is homeomorphic to the prime spectrum of the algebra, viewed as a commutative ring; cf. [StoneBoolAlg, 3.2]

On $\mathrm{PrimI}(L)$ a topology is defined, namely the topology generated by the sets

$$D(a) = \{\mathfrak{p} \in \mathrm{PrimI}(L) \mid a \notin \mathfrak{p}\}$$

where $a \in L$. We write $V(a) = \{\mathfrak{p} \mid a \in \mathfrak{p}\}$ for the complement of $D(a)$ in $\mathrm{PrimI}(L)$.

Warning: $\mathrm{PrimI}(L)$ is **not** Hausdorff, unless L is a Boolean algebra.

Now write X for the space $\mathrm{PrimI}(L)$.

(1) A subset of X is of the form $D(a)$ for some $a \in L$ if and only if it is open and at the same time quasi-compact (i.e., it has the Heine-Borel cover property). We write

$$\mathring{\mathcal{K}}(X) = \{U \subseteq X \mid U \text{ is open and quasi-compact}\} = \{D(a) \mid a \in L\},$$
$$\overline{\mathcal{K}}(X) = \{A \subseteq X \mid X \setminus A \text{ is open and quasi-compact}\} = \{V(a) \mid a \in L\}.$$

(2) (Stone representation) $\mathring{\mathcal{K}}(X)$ is a bounded sublattice of the powerset of X, in particular X itself is quasi-compact, and the map

$$L \longrightarrow \mathring{\mathcal{K}}(X),\ a \mapsto D(a)$$

is an isomorphism of bounded lattices.

(3) Every nonempty, closed and irreducible subset A of X has a unique generic point, i.e. a point p satisfying $A = \overline{\{p\}}$. In particular X satisfies the **$\mathbf{T}_0$-separation axiom** saying that distinct points can be separated by open sets.

(4) X is a **spectral space** and every spectral space is of this form (so one might read the definition of $\mathrm{PrimI}(L)$ as a definition of spectral spaces). For a topology-intrinsic definition of spectral spaces see [DiScTr2019, section 1].

(5) The **patch topology** (aka **constructible topology**) on X is the topology that has the set $\{D(a) \cap V(b) \mid a, b \in L\}$ as a basis of open sets. X_{con} denotes the set X with the patch topology. Theorem: X_{con} is compact and totally disconnected, it is a Boolean space.

(6) The **inverse topology** on X is the topology that has the set $\{V(a) \mid a \in L\}$ as a basis of *open* sets. X_{inv} denotes the set X with the inverse topology and X_{inv} is naturally homeomorphic to $\mathrm{PrimI}(L_{\mathrm{inv}})$, which is (as a set) equal to the set of prime filters of L.

8.2. *Examples.*

(i) If L is a chain with smallest and largest element, then $X = \mathrm{PrimI}(L)$ consists of the proper nonempty down-sets of L, i.e. the Dedekind cuts of L. The topology is generated by the sets $D(a) = \{\text{all cuts below } a\}$. The space is irreducible with generic point $\{\perp\}$. For more about this example see [DiScTr2019, Example 1.6A and section 3.6]

(ii) Let G is an ℓ-group. Recall from 3.5(vii) that $\mathring{\mathcal{K}}(\ell\text{-Spec}(G))$ is poset isomorphic to the lattice $M = \{G\} \cup \{\text{principal } \ell\text{-ideals}\}$. It follows that $\ell\text{-Spec}(G) \cong \mathrm{PrimI}(M)$.

8.3. **The space of minimal points** *Let $X = \mathrm{PrimI}(L)$ as in 8.1. We write $X^{\min}$ for the subspace of minimal prime ideals of L, endowed with the topology induced by X. Let Z be the closure of $X^{\min}$ in the patch topology.*

(i) A point $x \in X$ is in $X^{\min}$ if and only if $\forall y \in X : x \in \overline{\{y\}} \Rightarrow x = y$ (because $x \in \overline{\{y\}}$ means $y \subseteq x$), if and only if for every $V \in \overline{\mathcal{K}}(X)$ with $x \in V$ we have $x \in \mathrm{int}(V)$.

(ii) $X^{\min}$ (viewed as a subspace of X) is compact if and only if it is Boolean, if and only if it is a patch closed subset of X, if and only if it is a spectral space and the inclusion map $X^{\min} \hookrightarrow X$ is spectral.

(iii)

8.4. **Open and closed regularizations** [DiScTr2019, 4.4.21] Let X be any topological space and let $S \subseteq X$. Then S is called **regular open** if $S = \mathrm{int}(\overline{S})$ and **regular closed** if $S = \overline{\mathrm{int}(S)}$.

(i) For each $O \in \mathcal{O}(X)$ the set $N(O) = \mathrm{int}(\overline{O})$ is the **open regularization** of O, the smallest regular open subset containing O, [Koppel1989, p. 25]. Similarly, for $A \in \mathcal{A}(X)$, the set $\overline{N}(A) = \overline{\mathrm{int}(A)}$ is the **closed regularization** of A. This is the largest regular closed set contained in A. For each $O \in \mathcal{O}(X)$ we have $X \setminus N(O) = \overline{N}(X \setminus O)$.

(ii) The subset $RO(X) \subseteq \mathcal{O}(X)$ of regular open sets is a complete Boolean algebra, [Koppel1989, Theorem 1.37]. Similarly, $RC(X) \subseteq \mathcal{A}(X)$ the set of regular closed sets, is a complete Boolean algebra. Via complementation the Boolean algebras $RO(X)$ and $RC(X)$ are anti-isomorphic to each other. However, note that they are not sublattices of $\mathcal{O}(X)$ and $\mathcal{A}(X)$, respectively. The Boolean operations in $RO(X)$ are described in [Koppel1989, Theorem 1.37].

(iii) The maps $N : \mathcal{O}(X) \to RO(X)$ and $\overline{N} : \mathcal{A}(X) \to RC(X)$ are homomorphisms of bounded lattices. But if they are considered as maps into $\mathcal{O}(X)$ and $\mathcal{A}(X)$ then they are <u>not</u> lattice homomorphisms. For example one checks for $U, V \in \mathcal{O}(X)$ that

$$N(U \cap V) = N(U) \cap N(V) \qquad \overline{N}(U \cup V) = \overline{N}(U) \cup N(V)$$
$$N(U \cup V) = \mathrm{int}(\,\overline{N(U) \cup N(V)}\,) \qquad \overline{N}(U \cap V) = \overline{\mathrm{int}(\overline{N}(U) \cap \overline{N}(V))}.$$

(In particular, infima and suprema in $RO(X)$ are given by $U \wedge V = U \cap V$ and $U \vee V = \mathrm{int}(\,\overline{U \cup V}\,)$.)

Now we assume that X is a spectral space and we consider the restrictions $N : \mathring{\mathcal{K}}(X) \to RO(X)$ and $\overline{N} : \overline{\mathcal{K}}(X) \to RC(X)$ of N and $\overline{N}$, which are homomorphisms of bounded lattices. Their images are denoted by L and $\overline{L}$. These are bounded sublattices of $RO(X)$ and $RC(X)$, but not of $\mathcal{O}(X)$ and $\mathcal{A}(X)$.

Let $Z \subseteq X$ be the patch closure of $X^{\min}$ and $i \colon Z \to X$ the inclusion.

(iv) The restriction maps

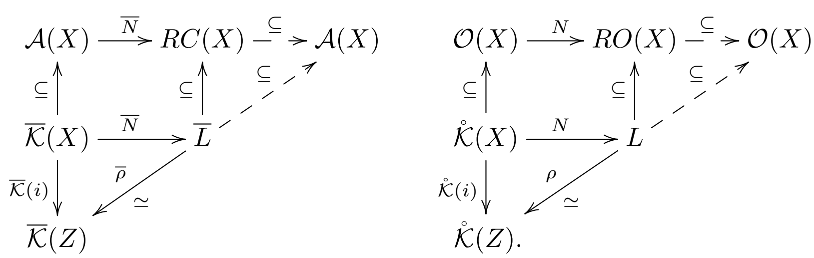
$$\rho\colon L \longrightarrow \mathring{\mathcal{K}}(Z)\colon O \mapsto O \cap Z$$
$$\overline{\rho}\colon \overline{L} \longrightarrow \overline{\mathcal{K}}(Z), B \mapsto B \cap Z$$

are poset isomorphisms.

Proof. We prove the assertion for $\overline{\rho}$. The assertion for ρ then follows by taking complements.

If $A \in \overline{\mathcal{K}}(X)$ then $A \cap Z \in \overline{\mathcal{K}}(Z)$, and $\overline{N}(A) = \overline{A \cap Z} = \text{Spez}(A \cap Z)$. It follows that $\overline{N}(A) \cap Z = A \cap Z$. We claim that the restriction map $\overline{\rho}\colon \overline{L} \to \overline{\mathcal{K}}(Z), B \mapsto B \cap Z$ is an isomorphism. It is clear that $\overline{\rho}$ is a homomorphism of bounded lattices. For injectivity, consider $B, B' \in \overline{L}$ with $B \cap Z = \overline{\rho}(B) = \overline{\rho}(B') = B' \cap Z$. But then $B = \text{Spez}(B \cap Z) = \text{Spez}(B' \cap Z) = B'$. For surjective, pick $C \in \overline{\mathcal{K}}(Z)$ and $A \in \overline{\mathcal{K}}(X)$ with $C = A \cap Z$. Then $C = \overline{N}(A) \cap Z = \overline{\rho}(\overline{N}(A))$. $\diamond$

(v) The following diagrams exhibit the various lattices and maps discussed here. The solid arrows are lattice homomorphisms, whereas the dashed arrows are poset homomorphisms:



Note that L is a Boolean algebra if and only if $\mathring{\mathcal{K}}(Z)$ is a Boolean algebra, if and only if Z is Boolean (i.e., $Z = X^{\min}$ and $X^{\min}$ is proconstructible), if and only if $\overline{\mathcal{K}}(Z)$ is a Boolean algebra, if and only if $\overline{L}$ is a Boolean algebra. Then the Boolean algebras $\mathring{\mathcal{K}}(Z)$ and $\overline{\mathcal{K}}(Z)$ coincide.

(vi) In conclusion, the commutative triangles of bounded lattice homomorphisms at the bottom of the two diagrams in (v) show that the regularization maps $N : \mathring{\mathcal{K}}(X) \to RO(X)$ and $\overline{N} : \overline{\mathcal{K}}(X) \to RC(X)$ are dual to the inclusion map $i : Z \longrightarrow X$.

8.5. **Definition.** [BalDwi1974, Chapter VIII] Let L be a bounded lattice and $a \in L$. If there is a largest $c \in L$ with $a \wedge c = \perp$, then c is called a **pseudo-complement** of a and is denoted by $\sim a$. If all elements of L have a pseudo-complement, then L is called **pseudo-complemented**. For example topologies are pseudo-complemented: if O is open, then $\sim O$ is the interior of the complement of O. Notice that $\sim\sim O$ is the open regularization of O, denoted by $N(O)$ in 8.4.

A prominent class of pseudo-complemented lattices are **Heyting algebras**. These are bounded distributive lattices L such that for all $a, b \in L$ there is a largest $c \in L$ with $a \wedge c \leq b$. Equivalently: For each $b \in L$ the interval $[b, \top]$ of L is pseudo-complemented. An example of a Heyting algebra is the lattice of open semi-algebraic subsets of $\mathbb{R}^n$.

8.6. **Topological characterization of pseudo-complemented lattices** [DiScTr2019, Thm. 8.3.9] *Let L be a bounded distributive lattice and let $X = \text{PrimI}(L)$ be its Stone dual. The following are equivalent.*

(i) *L is pseudo-complemented*

(ii) *For every $U \in \mathring{\mathcal{K}}(X)$ the closure $\overline{U}$ of U in X is constructible (equivalently: $\overline{U} \in \overline{\mathcal{K}}(X)$).*

(iii) *The following two conditions hold.*

  (a) *For $U \in \mathring{\mathcal{K}}(X)$ the open regularization $N(U) = \operatorname{int}(\overline{U})$ belongs to $\mathring{\mathcal{K}}(X)$, and*

  (b) *$X^{\min}$ is compact.*

9. ℓ-groups acting on ℓ-groups

Literature: There doesn't seem to be anything in the books [AndFei1988], [Steinb2010], [Darnel1995] or [KopMed1994].

A reference is [Glass1981].

If G is an ℓ-group, then the group $\mathrm{Aut}(G)$ of lattice automorphisms of G is again an ℓ-group (in general not commutative). An action of an ℓ-group G on an ℓ-group H is an ℓ-group homomorphism $\sigma : G \longrightarrow \mathrm{Aut}(H)$, $g \mapsto \sigma_g$. If G is abelian, then the image of this action is obviously again abelian.

We fix such an action now, where G and H are abelian and write $g+h$ for $\sigma_g(h)$. More generally for $C \subseteq G$ and $D \subseteq H$ we write

$$C + D = \{\sigma_g(h) \mid g \in C, h \in D\}.$$

In particular $G + h$ is the orbit of h under the given action.

If D is an ideal of H then obviously $g + D$ is again an ideal of H and similar for prime ideals (as $h \mapsto g+h$ is an automorphisms of the poset H). Hence G operates on the (prime) ideals of the lattice H.

9.1**. The invariance ideal of an ideal** Let D be an ideal of the lattice H. Define

$$\mathcal{I}(D) = \{g \in G \mid g + D = D\}.$$

Then

$$(+) \qquad \mathcal{I}(D) \cap G^{\geq 0} = \{g \in G^{\geq 0} \mid g + D \subseteq D\},$$

and $\mathcal{I}(D)$ is an ℓ-ideal of G called the invariance ideal of D.

Proof. To see (+) take $g \in G^{\geq 0}$. Since D is a down-set we know that $D \subseteq g + D$. Hence $g + D = D \iff g + D \subseteq D$, showing (+).

In order to verify that $\mathcal{I}(D)$ is an ℓ-ideal we use 2.11(iv) and we need to show that $\mathcal{I}(D) + \mathcal{I}(D) \subseteq \mathcal{I}(D)$, which is clear, and that $|g_0| \leq |g|$ and $g \in \mathcal{I}(D)$ implies $g_0 \in \mathcal{I}(D)$. Towards this end, first note that $g \in \mathcal{I}(D)$ obviously implies $-g \in \mathcal{I}(D)$. Now if $d \in D$, then as D is an ideal of H we know $\sigma_g(d) \vee \sigma_{-g}(d) = (g + d) \vee (-g + d) \in D$. Then

$$\begin{aligned}\sigma_{|g|}(d) &= \sigma_{g\vee -g}(d) \text{ by definition of } |g| \\ &= (\sigma_g \vee \sigma_{-g})(d), \text{ as } \sigma \text{ is a lattice homomorphism} \\ &= \sigma_g(d) \vee \sigma_{-g}(d), \text{ by definition of the } \ell\text{-group } \mathrm{Aut}(H) \\ &\in D.\end{aligned}$$

Hence $|g| + D \subseteq D$. Since $|g| \geq 0$ and D is a down-set of H we also know $(-|g|) + D \subseteq D$, hence $D \subseteq |g| + D$. This shows that $|g| + D = D$. But then $|g_0| \leq |g|$ easily implies $g_0 \vee 0, -(g_0 \wedge 0) \in \mathcal{I}(D)$ and so $g_0 = (g_0 \vee 0) + (g_0 \wedge 0) \in \mathcal{I}(D)$ as well. □

9.2. *Example.* If $\varphi : G \longrightarrow H$ is an ℓ-group homomorphism, then the map $\sigma : G \longrightarrow \mathrm{Aut}(H),\ g \mapsto (h \mapsto \varphi(g) + h)$ defines an ℓ-group action.

9.3. **Proposition.** *Let D be an ideal of H.*

(i) $\mathcal{I}(D)$ is a prime ℓ-ideal if and only if for all $g, g' \in G$ with $g \wedge g' = 0$ and each $d \in D$ we have $g + d \in D$ or $g' + d \in D$.

(ii) If D is a prime ideal of the lattice H, then $\mathcal{I}(D)$ is a prime ℓ-ideal.
(iii) If $H = G$ and $\sigma_g(h) = g + h$ (addition of G), then D is a prime ideal of the lattice H if and only if $\mathcal{I}(D)$ is a prime ℓ-ideal.

Proof. (i) If $\mathcal{I}(D)$ is a prime ℓ-ideal and $g \wedge g' = 0$, then $g \in \mathcal{I}(D)$ or $g' \in \mathcal{I}(D)$ showing that $g + d \in D$ or $g' + d \in D$ for all $d \in D$.

Conversely, assume $\mathcal{I}(D)$ is not a prime ℓ-ideal. By 2.15 there are $g_1, g_2 \in G \setminus \mathcal{I}(D)$ with $g_1 \wedge g_2 = 0$. Then $g_1, g_2 \geq 0$ and by 9.1(+) there are $d_i \in D$ with $g_i + d_i \notin D$. Let $d = d_1 \vee d_2 \in D$. By assumption we may assume that $g_1 + d \in D$. But from $g_1 + d_1 \leq g_1 + d$ we get $g_1 + d_1 \in D$, a contradiction.

(ii) If $g \wedge g' = 0$ in G and $d \in D$, then $(g + d) \wedge (g' + d) = \sigma_g(d) \wedge \sigma_{g'}(d) = (\sigma_g \wedge \sigma_{g'})(d) = \sigma_{g \wedge g'}(d) = \sigma_0(d) = \mathrm{id}_H(d) = d \in D$, thus $g + d \in D$ or $g' + d \in D$. By (i) we see that $\mathcal{I}(D)$ is a prime ℓ-ideal.

(iii) By (ii) we only need to show that D is a prime ideal of the lattice H assuming $\mathcal{I}(D)$ is a prime ℓ-ideal. We know that $G/\mathcal{I}(D)$ is a chain. Let $\pi : G \longrightarrow G/\mathcal{I}(D)$ be the residue map. Then $\pi^{-1}(\pi(D)) = D$ because for $g \in G$ and $d \in D$ with $\pi(g) = \varphi(d)$ we have $g - d \in \mathcal{I}(D)$ and so $g \in d + \mathcal{I}(D) \subseteq D$.

As $\pi(D)$ is prime as an ideal of a chain, also D is prime. □

9.4. **Corollary.** *The set $\mathfrak{I}$ of all ideals D of the lattice H with the property that $\mathcal{I}(D)$ is a prime ℓ-ideal is a spectral subspace of the space of all ideals of the lattice H.*

If $G = H$ and $\sigma_g(h)$ is addition in G, then $\mathfrak{I}$ is a spectral subspace of $\mathrm{PrimI}(G)$

Proof. By 9.3(i), we know that

$$\mathfrak{I} = \bigcap \{D(h) \cup V(g + h) \cup V(g' + h) \mid h \in H, g, g' \in G, g \wedge g' = 0\},$$

where $V(h)$ denotes the set of all ideals of the lattice H containing h and $D(h)$ is its complement. Hence $\mathfrak{I}$ is a spectral subspace of the space of all ideals of the lattice H.

The second assertion then follows from 9.3(iii) □

9.5. **Observation.** *Here we work with $H = G$ and the natural operation of G on itself via translation.*

Let U be an ℓ-ideal of G and let $\pi : G \longrightarrow G/U$ be the residue map. Then the map

$$\begin{aligned} \{\textit{Ideals of the lattice } G/U\} &\longrightarrow \{D \subseteq G \mid D \textit{ ideal of the lattice } G \textit{ with } U \subseteq \mathcal{I}(D)\} \\ J &\longmapsto \pi^{-1}(J) \end{aligned}$$

is a poset isomorphism. Its inverse sends D to $\pi(D)$. Consequently this map restricts to a poset isomorphism between the corresponding sets of prime ideals.

Proof. The map is obviously well defined and for each ideal J we have $\pi(\pi^{-1}(J)) = J$. It remains to show that $\pi^{-1}(\pi(D)) = D$ for each ideal D of the lattice G with $U \subseteq \mathcal{I}(D)$. The inclusion $\supseteq$ is clear. Then take $g \in \pi^{-1}(\pi(D))$ and $d \in D$ with $\pi(g) = \pi(d)$. This means $g - d \in U \subseteq \mathcal{I}(D)$ and so $g \in d + \mathcal{I}(D) \subseteq D$ as required. □

9.6. *Remark.* Let L be a bounded distributive lattice and let $\mathrm{Idl}(L)$ be the set of ideals of L. Then $\mathrm{Idl}(L)\times\mathrm{Idl}(L)$ is isomorphic as a poset to $\mathrm{Idl}(L\times L)$, an isomorphism is given by sending (I,J) to $I\times J$, the inverse sends K to $(\pi_1(K),\pi_2(K))$. This is straightforward to check.

In the context of topology this reads as follows. Let X,Y be any spaces, assuming $X\cap Y=\emptyset$. Then the map $\mathcal{O}(X\oplus Y)\longrightarrow\mathcal{O}(X)\times\mathcal{O}(Y), O\mapsto(O\cap X,O\cap Y)$ is a obviously poset isomorphism whose inverse sends (U,V) to $U\cup V$.

In order to connect the two assertion let $X=Y$ be a spectral space, and $L=\mathring{\mathcal{K}}(X)$. Then $X\oplus X$ is again spectral and $\mathring{\mathcal{K}}(X\oplus X)\cong L\times L$. Now recall that $\mathrm{Idl}(L)$ as a poset is a frame that is isomorphic to $\mathcal{O}(X)$, given by $I\mapsto\bigcup_{a\in L}D(a)$ with inverse $O\mapsto\{a\in L\mid D(a)\subseteq O\}$. In conclusion

$$\mathrm{Idl}(L\times L)\cong\mathcal{O}(\mathrm{PrimI}(L\times L))\cong\mathcal{O}(X\oplus X)\cong\mathcal{O}(X)\times\mathcal{O}(X)\cong\mathrm{Idl}(L)\times\mathrm{Idl}(L).$$

10. Polars and projectable ℓ-groups

10.1. **Colon Ideals and Polars** *Let I be an ℓ-ideal of an ℓ-group G and let $S \subseteq G$. We define the* **colon ideal**

$$I : S = \{f \in G \mid \forall s \in S : |f| \wedge |s| \in I\}.$$

Then $I : S$ is an ℓ-ideal and

$$(*)\qquad \begin{aligned} V(I : S) &= \{G\} \cup \overline{V(I) \setminus V(S)}, \text{ and} \\ V(I : S) \cap \ell\text{-Spec}^*(G) &= \overline{V(I) \setminus V(S)}^{\ell\text{-Spec}^*(G)}. \end{aligned}$$

If $I = (0)$ then

$$S^{\perp} := \{f \in G \mid \forall s \in S : |s| \wedge |f| = 0\} \ (= (0) : S)$$

is called the **polar of** *S, hence*

$$(\dagger)\qquad \begin{aligned} V(S^{\perp}) &= \{G\} \cup \overline{\ell\text{-Spec}(G) \setminus V(S)} \text{ and} \\ V(S^{\perp}) \cap \ell\text{-Spec}^*(G) &= \overline{\ell\text{-Spec}^*(G) \setminus V(S)}^{\ell\text{-Spec}^*(G)}. \end{aligned}$$

If $S = \{f\}$ is a singleton we just write $I : f$ and $f^{\perp}$. The ℓ-ideals of the form $f^{\perp\perp}$ are called **principal polars**. *. Explicitly:*

$$f^{\perp\perp} = \{g \in G \mid \forall h \in G : (|h| \wedge |f| = 0 \Rightarrow |h| \wedge |g| = 0)\}.$$

Notice that

$$\begin{aligned} V(I : S) &= \overline{V(I) \setminus V(S)}, \text{ if } V(I) \not\subseteq V(S), \text{ i.e. } S \not\subseteq I, \\ V(S^{\perp}) &= \overline{\ell\text{-Spec}(G) \setminus V(S)}, \text{ if } S \text{ contains a nonzero element, and} \\ V(f^{\perp}) &= \overline{D(f)}, \text{ if } f \neq 0. \end{aligned}$$

Proof. To see that $I : S$ is an ℓ-ideal we use 2.11(iv) and we need to show that $I : S \neq \emptyset$, $|g| \leq |f|$ and $f \in I : S$ implies $g \in I : S$ and that $0 \leq f_1, f_2 \in I : S$ implies $f_1 + f_2 \in I : S$. The first two properties are clear. If $0 \leq f_1, f_2 \in I : S$ and $s \in S$, then $|f_1 + f_2| \wedge |s| = |s| \wedge (f_1 + f_2) \leq |s| \wedge f_1 + |s| \wedge f_2$ by the polar inequality 2.2(iv)(a). Hence $f_1 + f_2 \in I : S$.

Now for the proof of $(*)$. The second equation follows from the first (notice that $G \in V(S)$, hence $V(I) \setminus V(S) \subseteq \ell\text{-Spec}^*(G)$). If $V(I) \subseteq V(S)$, then $S \subseteq I$ and $I : S = G$, showing $(*)$. Hence we may assume that $V(I) \not\subseteq V(S)$. Then $G \in \overline{V(I) \setminus V(S)}$ and $(*)$ says that $V(I : S) = \overline{V(I) \setminus V(S)}$. By taking complements this is equivalent to saying that $V(S) \cup (\ell\text{-Spec}(G) \setminus V(I))$ has interior $\ell\text{-Spec}(G) \setminus V(I : S)$.

Since $V(I) \not\subseteq V(S)$ we have $V(S) \cup (\ell\text{-Spec}(G) \setminus V(I)) \neq \ell\text{-Spec}(G)$ and by 3.3(iii) it suffices to show that for each $f \in G$ we have $D(f) \cap V(I : S) = \emptyset \iff$

$D(f) \subseteq V(S) \cup (\ell\text{-Spec}(G) \setminus V(I))$:

$$\begin{aligned} D(f) \cap V(I:S) = \emptyset &\iff V(I:S) \subseteq V(f) \\ &\iff f \in I:S, \text{ since } I:S \text{ is an } \ell\text{-ideal} \\ &\iff \forall s \in S : |f| \wedge |s| \in I \\ &\iff \forall s \in S : V(I) \subseteq V(|f| \wedge |s|) = V(f) \cup V(s) \\ &\iff \forall s \in S : D(f) \cap V(I) \subseteq V(s) \\ &\iff D(f) \cap V(I) \subseteq V(S), \text{ since } V(S) = \bigcap_{s \in S} V(s) \\ &\iff D(f) \subseteq V(S) \cup (\ell\text{-Spec}(G) \setminus V(I)). \end{aligned}$$ □

10.2. **Basic properties of polars** Let G be an ℓ-group.

(i) If $(S_i)_{i\in I}$ is any collection of subsets of G, then obviously $(\bigcup_{i\in I} S_i)^\perp = \bigcap_{i\in I} S_i^\perp$.

(ii) If $S \subseteq G$ then

$$V(S^{\perp\perp}) = \{G\} \cup \overline{\text{int}(V(S))}, \text{ and}$$
$$V(S^{\perp\perp}) \cap \ell\text{-Spec}^*(G) = \overline{\text{int}(V(S) \cap \ell\text{-Spec}^*(G))}^{\ell\text{-Spec}^*(G)}$$

Proof.

$$\begin{aligned} V(S^{\perp\perp}) &= \{G\} \cup \overline{\ell\text{-Spec}(G) \setminus V(S^\perp)}, \text{ by } (\dagger) \text{ in } 10.1 \\ &= \{G\} \cup \overline{\ell\text{-Spec}(G) \setminus (\{G\} \cup \overline{\ell\text{-Spec}(G) \setminus V(S)})} \\ &= \begin{cases} \{G\} \cup \overline{\ell\text{-Spec}(G) \setminus (\overline{\ell\text{-Spec}(G) \setminus V(S)})} & \text{if } S \not\subseteq \{0\}, \\ \{G\} \cup \overline{\ell\text{-Spec}(G) \setminus \{G\}} & \text{if } S \subseteq \{0\}. \end{cases} \\ &= \begin{cases} \{G\} \cup \overline{\text{int}(V(S))} & \text{if } S \not\subseteq \{0\}, \text{ by} \\ \ell\text{-Spec}(G) & \text{if } S \subseteq \{0\} \end{cases} \end{aligned}$$ ◇

10.3. **Cardinal Sums and Summands** Let G be an ℓ-group and let $H \subseteq G$ be an ℓ-subgroup. The following are equivalent.

(i) There is an ℓ-group H' and an ℓ-group isomorphism $G \longrightarrow H \times H'$ mapping H onto $H \times \{0\}$.

(ii) H is an ℓ-ideal and the residue map $\pi : G \longrightarrow G/H$ splits, i.e. there is some ℓ-group homomorphism $\iota : G/H \longrightarrow G$ such that $\pi \circ \iota = \text{id}_{G/H}$.

(iii) H is an ℓ-ideal and $G = H + H^\perp$.

(iv) H is an ℓ-ideal and $H^\perp$ is the unique convex subgroup of G with the property $G = H \oplus H^\perp$.

(v) H is an ℓ-ideal and $V(H) \setminus \{G\}$ is an open subset of $\ell\text{-Spec}^*(G)$ (in other words: $\ell\text{-Spec}(G) \setminus V(H)$ is a closed subset of $\ell\text{-Spec}^*(G)$).

(vi) H is an ℓ-ideal and $V(H) \setminus \{G\}$ is a clopen subset of $\ell\text{-Spec}^*(G)$ and its complement in $\ell\text{-Spec}^*(G)$ is $V(H^\perp) \setminus \{G\}$.

(vii) H is an ℓ-ideal and for every $g \in G^{\geq 0}$ there is a largest element $h \in H$ with $0 \leq h \leq g$.

If G has a strong order unit and H is an ℓ-ideal, then these conditions are equivalent to

(viii) There is some $f \in G$ with $H = \ell(f)$ and $G = \ell(f) + f^\perp$.

Proof. (i)⇒(iv). In the product $H \times H'$ of the two ℓ-groups H, H', all terms in the language of ℓ-groups are evaluated component wise. This implies easily that $H \times \{0\}$ and $\{0\} \times H'$ are ℓ-ideals and that $(H \times \{0\})^{\perp} = (\{0\} \times H')$ in $H \times H'$. The isomorphism in (i) then transfers this fact and entails (iv) .

(iv)⇒(iii) is a weakening.

(iii)⇒(ii). Since $H \cap H^{\perp} = \{0\}$ the restriction of π to $H^{\perp}$ is injective. As $G = H + H^{\perp}$, this restriction is also surjective. Hence we may take ι to be the compositional inverse of the restriction.

(ii)⇒(i). Let $\varphi : H \times G/H \longrightarrow G$ be the map $i + \iota$, where $i : H \hookrightarrow G$ is the inclusion. Clearly φ is an isomorphism.

(iv)⇒(vi) If $\mathfrak{p}$ is a prime ℓ-ideal, then $H \subseteq \mathfrak{p}$ or $H^{\perp} \subseteq \mathfrak{p}$ because if there is some $h \in H \setminus \mathfrak{p}$ then $|h| \wedge |h'| = 0 \in \mathfrak{p}$ implies $h' \in \mathfrak{p}$ for all $h' \in H^{\perp}$.

On the other hand If $H, H' \in \mathfrak{p}$, then (iv) implies $G \subseteq \mathfrak{p}$ and $\mathfrak{p}$ is not in ℓ-Spec$^*(G)$. This shows that The complement of $V(H) \setminus \{G\}$ in ℓ-Spec$^*(G)$ is $V(H^{\perp}) \setminus \{G\}$ which proves (vi).

(vi)⇒(v) is a weakening.

(v)⇒(i) We may assume that $H \neq \{0\}$. Then by 10.1 we know that $V(H^{\perp}) = \ell\text{-Spec}(G) \setminus V(H)$. By (v) we know that ℓ-Spec$(G) \setminus V(H)$ is a closed subset of ℓ-Spec$^*(G)$. It follows that $V(H^{\perp}) = \{G\} \cup (\ell\text{-Spec}(G) \setminus V(H))$. But then $G = H + H^{\perp}$, because an element not in $H + H^{\perp}$ would be avoided by a prime ℓ-ideal containing H and $H^{\perp}$. Since we obviously have $H \cap H^{\perp} = \{0\}$ this shows $G = H \oplus H^{\perp}$ as a group. It follows easily that $G = H \oplus H^{\perp}$ as an ℓ-group.

(i)⇒(vii) We may assume that $G = H \times H'$. By the definition of the partial order of $H \times H'$ if $g = (h, h') \in H \times H'$ and $f \in H$, then $(0, 0) \leq (f, 0) \leq (h, h')$ if and only if $f \leq h$. This shows (vii)

(vii)⇒(v) Let $\mathfrak{p} \in V(H) \setminus \{G\}$. We need to show that $\mathfrak{p} \in D(f) \subseteq V(H)$ for some $f \in G$. Take $g \in G^{\geq 0}$ with $g \notin \mathfrak{p}$. By (vii) there is a largest element h of H with $0 \leq h \leq g$. Since $h \in H \subseteq \mathfrak{p}$ we have $g - h \notin \mathfrak{p}$, i.e. $\mathfrak{p} \in D(g - h)$, and it remains to show that $D(g - h) \subseteq V(H)$. This follows if we show that $(g - h) \wedge h' = 0$ for all $h' \in G^{\geq 0}$: We have $h \leq h + ((g - h) \wedge h') = g \wedge (h + h') \leq g$ and so by choice of h we get $h = h + ((g - h) \wedge h')$, i.e. $(g - h) \wedge h' = 0$.

This shows the first equivalences. Now assume that G has a strong order unit, i.e. ℓ-Spec$^*(G)$ is quasi-compact. If (viii) holds, then clearly (i) holds. Conversely, if (v) holds, then as ℓ-Spec$^*(G)$ is quasi-compact, also ℓ-Spec$(G) \setminus V(H)$ is quasi-compact. Hence there is some $f \in G$ with ℓ-Spec$(G) \setminus V(H) = D(f)$, i.e. $V(H) = V(f)$ and $H = \ell(f)$. □

10.4. **Corollary.** *Let G be an ℓ-group and let $f \in G$. The following are equivalent.*

(i) $G = f^{\perp} + f^{\perp\perp}$.

(ii) The closure of $D(f)$ in ℓ-Spec$^*(G)$ is open (and thus clopen).

(iii) For every $g \in G$ with $g \geq |f|$, $|f|$ has a pseudo-complement in the lattice $[0, g]$, hence there is a largest element h in $[0, g]$ with $h \wedge |f| = 0$.

Proof. (i)⇔(ii). By (†) in 10.1, $V(f^{\perp}) \setminus \{H\}$ is the closure of $D(f) = \ell\text{-Spec}^*(G) \setminus V(f)$ in ℓ-Spec$^*(G)$. By 10.3(iii)⇔(v), $G = f^{\perp} + f^{\perp\perp}$ just if $V(f^{\perp}) \setminus \{H\}$ is open in ℓ-Spec$^*(G)$. Hence $G = f^{\perp} + f^{\perp\perp}$ if and only if the closure of $D(f)$ in ℓ-Spec$^*(G)$ is open (and thus clopen).

(i)⇒(iii) By 10.3(iii)⇒(vii) applied to $H = f^\perp$, there is a largest element h with $h \wedge |f| = 0$ such that $0 \leq h \leq g$.

(iii)⇒(i) We may replace f by $|f|$ and assume that $f \geq 0$. By 10.3(vii)⇒(iii) applied to $H = f^\perp$, it suffices to show that for each $g \in G^{\geq 0}$ there is a largest $h \in f^\perp$ with $0 \leq h \leq g$; in other words that there is a largest h with $h \wedge f = 0$ such that $0 \leq h \leq g$. By (iii) there is a largest $h \in [0, g \vee f]$ with $h \wedge f = 0$. But this h satisfies $h = h \wedge (g \vee f) = (h \wedge g) \vee (h \wedge f) = h \wedge g$, i.e. $h \leq g$. □

10.5. **Definition.** An ℓ-group G is called **projectable** if every principal polar is a summand of G, i.e. for all $f \in G$ we have $G = f^\perp + f^{\perp\perp}$. An ℓ-group G is called **strongly projectable** if every polar is a summand of G.

10.6. *Remark.* Recall from [Blyth2005, section 7.2] that a **Stone lattice** is a pseudo-complemented bounded distributive lattice L satisfying $\sim a \vee \sim\sim a = \top$ for all $a \in L$. In other words: every pseudo-complement has a complement.[17] Using 10.1 we see that G is strongly projectable just if the lattice of open subsets $\ell\text{-Spec}^*(G)$ is a Stone lattice.

10.7. **Proposition.** *The following conditions are equivalent for every ℓ-group G.*

(i) $\ell\text{-Spec}^*(G)$ *is stranded.*

(ii) For every $\mathfrak{p} \in \ell\text{-Spec}(G)$ the prime ideals of the lattice G containing $\mathfrak{p}$ are totally ordered by inclusion. Equivalently: $\mathrm{PrimI}(G^{\geq 0}_\infty)$ *is a spectral root system.*

(iii) $\mathrm{PrimI}(G^{\geq 0})$ *is stranded.*

Proof. (i)⇒(ii) Let $\mathfrak{p} \in \ell\text{-Spec}^*(G)$ and let $\mathfrak{a}, \mathfrak{b}$ be prime ideals of the lattice G with $\mathfrak{p} \subseteq \mathfrak{a}, \mathfrak{b}$. By 9.3(ii), the invariance ideals $I(\mathfrak{a}), I(\mathfrak{b})$ of $\mathfrak{a}, \mathfrak{b}$ respectively are prime ℓ-ideals. Since $\mathfrak{p} + I(\mathfrak{a}) \subseteq \mathfrak{a}$ by definition of $I(\mathfrak{a})$ and $\ell\text{-Spec}^*(G)$ is stranded we either have $\mathfrak{p} + I(\mathfrak{a}) = G$, hence $\mathfrak{a} = G$ contains $\mathfrak{b}$ and we are done, or, $\mathfrak{p}$ and $I(\mathfrak{a})$ are comparable. Similarly, $\mathfrak{p}$ and $I(\mathfrak{b})$ are comparable. Since $\ell\text{-Spec}^*(G)$ is stranded also $I(\mathfrak{a})$ and $I(\mathfrak{b})$ are comparable, say $\mathfrak{q} := I(\mathfrak{a}) \subseteq I(\mathfrak{b})$. Let $\pi : G \longrightarrow G/\mathfrak{q}$ be the residue map. As $\mathfrak{q}$ is a prime ℓ-ideal, $\pi(\mathfrak{a}), \pi(\mathfrak{b})$ are comparable in $G/\mathfrak{q}$. Since $\mathfrak{q} \subseteq I(\mathfrak{a}), I(\mathfrak{b})$ we get $\mathfrak{a} = \pi^{-1}(\pi(\mathfrak{a})), \mathfrak{b} = \pi^{-1}(\pi(\mathfrak{b}))$. Hence also $\mathfrak{a}$ and $\mathfrak{b}$ are comparable.

(ii)⇒(i) Suppose $\ell\text{-Spec}^*(G)$ is not stranded. Then there are prime ℓ-ideals $\mathfrak{p}_0, \mathfrak{q}_0 \subseteq \mathfrak{r} \neq G$ such that $\mathfrak{p}_0, \mathfrak{q}_0$ are incomparable. Then take $a \in G \setminus \mathfrak{r}$, $a \geq 0$ and define $\mathfrak{p} = a + \mathfrak{p}_0^\downarrow$ and $\mathfrak{q} = a + \mathfrak{q}_0^\downarrow$, prime ideals of the lattice G (see 7.1). Now $\mathfrak{r} \subseteq a + \mathfrak{p}_0^\downarrow$ by 9.5 applied to $U = \mathfrak{p}_0$. Similarly $\mathfrak{r} \subseteq a + \mathfrak{q}_0^\downarrow$. However $\mathfrak{p}$ and $\mathfrak{q}$ are not comparable as one sees easily using the assumption that $\mathfrak{p}_0$ and $\mathfrak{q}_0$ are not comparable. If $\mathfrak{p} \subseteq \mathfrak{q}$, then for all $p \in \mathfrak{p}_0$ there are $y \in G$ and $q \in \mathfrak{q}_0$ with $a + p = a + y$ and $y \leq q$. But then $p = y \leq \mathfrak{q}_0$ and $p \in \mathfrak{q}_0$ - this is impossible because $\mathfrak{p}_0$ is not contained in $\mathfrak{q}_0$.

Hence the prime ideals of the lattice G containing $\mathfrak{r}$ do not form a chain for inclusion and neither form the prime ideals of $G^{\geq 0}$ containing $\mathfrak{r} \cap G^{\geq 0}$ a chain.

(iii)⇒(ii) is a weakening.

(i),(ii)⇒(iii). By (ii) it suffices to show that for prime ideals $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ of the lattice $G^{\geq 0}$ with $\mathfrak{a}, \mathfrak{b} \subseteq \mathfrak{c}$, the sets $\mathfrak{a}, \mathfrak{b}$ are comparable for inclusion. Notice that $\mathfrak{c} \neq G^{\geq 0}$. Let $\mathfrak{p}, \mathfrak{q}$ be minimal prime ℓ-ideals with $\mathfrak{p} \cap G^{\geq 0} \subseteq \mathfrak{a}$ and $\mathfrak{q} \cap G^{\geq 0} \subseteq \mathfrak{b}$. If $\mathfrak{p} = \mathfrak{q}$ then by (i) we know that $\mathfrak{a}$ and $\mathfrak{b}$ are comparable. So assume $\mathfrak{p} \neq \mathfrak{q}$. Let $\mathfrak{r}$ be a minimal prime ℓ-ideal of G contained in $I(\mathfrak{c})$. Then either $\mathfrak{p} \neq \mathfrak{r}$ or $\mathfrak{q} \neq \mathfrak{r}$, say $\mathfrak{p} \neq \mathfrak{r}$.

[17]A **Stone algebra** is a Stone lattice expanded by the pseudo-complementation map.

By minimality $\mathfrak{p}$ and $\mathfrak{r}$ are incomparable and by (i) we know that $G \subseteq \mathfrak{p} + \mathfrak{r}$ and so also $G \subseteq \mathfrak{p} + I(\mathfrak{c})$. On the other hand, $\mathfrak{p} \subseteq \mathfrak{a} \subseteq \mathfrak{c}$ implies $(\mathfrak{p} + I(\mathfrak{c})) \cap G^{\geq 0} \subseteq \mathfrak{c}$ by definition of the invariance ideal $I(\mathfrak{c})$. But then $G^{\geq 0} \subseteq \mathfrak{c}$ in contradiction to the choice of $\mathfrak{c}$. □

10.8. **z-ideals** An ℓ-ideal I is called a **z-ideal** if for all $f \in I$ we have $f^{\perp\perp} \subseteq I$. As noted in the footnote of , there is no good reason for this name and in fact it contradicts the notion of a z-ideal in rings (of continuous functions) as we see now:

Let I be an ℓ-ideal of an ℓ-group G. By 10.2(ii), we know that $V(f^{\perp\perp}) = \{G\} \cup \overline{\mathrm{int}(V(f))}$ for every $f \in G$. Hence I is a z-ideal if and only if the following condition holds

$$(*) \qquad \forall f \in G: \left(V(I) \subseteq V(f) \implies V(I) \subseteq \{G\} \cup \overline{\mathrm{int}(V(f))}\right).$$

A prime ℓ-ideal is a z-ideal if and only if $\mathfrak{p} = G$ or $\mathfrak{p} \neq G$ and $\mathfrak{p}$ is in the patch closure of $(\ell\text{-Spec}(G))^{\min}$. Hence prime ideals of rings of continuous functions that are z-deals in the sense of rings of continuous functions are in general not z-ideals in the sense here.

Proof. We may assume that $\mathfrak{p} \neq G$. First assume $\mathfrak{p}$ is a z-ideal. By [DiScTr2019, Prop. 4.4.10(i)] it suffices to show that every constructible subset C of ℓ-Spec(G) with $\mathfrak{p} \in C$ has nonempty interior (for the spectral topology of ℓ-Spec(G)). We may assume that $C = V(f) \cap D(g)$ for some $f, g \in G$. Since $\mathfrak{p}$ is a proper z-ideal and $\mathfrak{p} \in V(f)$, we know that $\mathfrak{p} \in \overline{\mathrm{int}(V(f))} \cap D(g)$. In particular, $D(g)$ hits $\overline{\mathrm{int}(V(f))}$ and therefore $\mathrm{int}(V(f)) \cap D(g) \neq \emptyset$. Thus C has nonempty interior.

Conversely assume $\mathfrak{p}$ is in the patch closure of $(\ell\text{-Spec}(G))^{\min}$ and $f \in G$ with $V(\mathfrak{p}) \subseteq V(f)$. Suppose $\mathfrak{p} \notin \{G\} \cup \overline{\mathrm{int}(V(f))}$. Then $\mathfrak{p} \notin \overline{\mathrm{int}(V(f))}$ and there is some $g \in G$ with $\mathfrak{p} \in D(g)$ and $D(g) \cap \mathrm{int}(V(f)) = \emptyset$. But then $\mathfrak{p}$ is in the constructible set $D(g) \cap V(f)$ and this set has empty interior. This contradicts [DiScTr2019, Prop. 4.4.10(i)]. ◇

10.9. **Theorem.** *The following are equivalent for every ℓ-group G.*

(i) G is projectable.
(ii) ℓ-Spec$^*(G)$ is stranded and ℓ-Spec$(G)^{\min} \cup \{G\}$ is a patch closed subset of ℓ-Spec(G).
(iii) Every proper prime ℓ-ideal contains exactly one prime z-ideal.
(iv) Every bounded interval of G is a Stone lattice.
(v) For every $f \in G^{\geq 0}$ the interval $[0, f]$ is a pseudo-complemented lattice.

If G has a strong order unit u, then these conditions are equivalent to each of the following.

(vi) $\mathring{\mathcal{K}}(\ell\text{-Spec}^*(G))$ is a Stone lattice.
(vii) $[0, |u|]$ is a pseudo-complemented lattice.

Warning: *One may ask whether the conditions above are already implied under the assumption that $\mathring{\mathcal{K}}(\ell\text{-Spec}^*(G))$ is pseudo-complemented (and G has a strong order unit). However this fails badly in general. For example if G is the ℓ-group of continuous semi-linear functions $[0, 1] \subseteq \mathbb{Q} \longrightarrow \mathbb{Q}$. Then G has a strong order unit, namely the constant function of value 1. Then $\mathring{\mathcal{K}}(\ell\text{-Spec}^*(G))$ is even a Heyting algebra, but ℓ-Spec$^*(G)$ is connected, so it is far away from being a Stone lattice.*

Similarly, if A is the ring of continuous semi-algebraic functions $[0,1]\subseteq\mathbb{R}\longrightarrow\mathbb{R}$. Then $G=(A,+,\leq)$ has a strong order unit, namely the constant function of value 1 but $\operatorname{Spec}(A)$ *is a connected Heyting space.*

Proof. (i)⇒(ii) To see that $\ell\text{-Spec}^*(G)$ is stranded take distinct $\mathfrak{p},\mathfrak{q}\in\ell\text{-Spec}(G)^{\min}$. We now work in the spectral space $\ell\text{-Spec}(G)$ and show that $\mathfrak{p}+\mathfrak{q}=G$. Suppose $\mathfrak{p}+\mathfrak{q}\neq G$. Take $f\in G$ with $\mathfrak{p}\in D(f)$ and $\mathfrak{q}\notin D(f)$. By 10.4 and (i) the closure A of $D(f)$ in $\ell\text{-Spec}^*(G)$ is open. Since $D(f)$ is a patch closed subset of $\ell\text{-Spec}(G)$, A is contained in the specializations of $D(f)$. Since $\mathfrak{p}+\mathfrak{q}\neq G$ we know $\mathfrak{p}+\mathfrak{q}\in A$. As A is open we get $\mathfrak{q}\in A\cap\ell\text{-Spec}(G)^{\min}$. However $A\cap\ell\text{-Spec}(G)^{\min}=D(f)$ because the specializations of $D(q)$ are the specializations of $\ell\text{-Spec}(G)^{\min}\cap D(f)$. This contradicts $\mathfrak{q}\notin D(f)$.

Now we show that $\ell\text{-Spec}(G)^{\min}\cup\{G\}$ is a patch closed subset of $\ell\text{-Spec}(G)$. Take $\mathfrak{p}\in\ell\text{-Spec}(G)\setminus\ell\text{-Spec}(G)^{\min}$ with $\mathfrak{p}\neq G$ and choose some $\mathfrak{q}\in\ell\text{-Spec}(G)^{\min}$ with $\mathfrak{q}\subsetneq\mathfrak{p}$. Take $f\in\mathfrak{p}\setminus\mathfrak{q}$. Then $\mathfrak{p}\in U:=V(f)\cap(\overline{D(f)}\setminus\{G\})$. Since $\overline{D(f)}=\operatorname{Spez}(D(f))$ we know that $U\cap(\ell\text{-Spec}(G)^{\min}\cup\{G\})=\emptyset$.

By (i), using 10.4 we know that $\overline{D(f)}\setminus\{G\}$ is open. Hence U is a constructibly open neighborhood of $\mathfrak{p}$ that is disjoint from $\ell\text{-Spec}(G)^{\min}\cup\{G\}$.

(ii)⇒(i). By 10.6 we need to show that for each $f\in G$ the set $\overline{D(f)}\cap\ell\text{-Spec}^*(G)$ is open. Now, $\overline{D(f)}=\operatorname{Spez}(D(f))=\operatorname{Spez}(D(f)\cap\ell\text{-Spec}(G)^{\min})$. Since $\ell\text{-Spec}^*(G)$ is stranded by (ii), we see that

$\ell\text{-Spec}(G)\setminus\overline{D(f)}=\operatorname{Spez}(\ell\text{-Spec}(G)^{\min}\cap V(f))\setminus\{G\}$, which is equal to

$$\operatorname{Spez}((\ell\text{-Spec}(G)^{\min}\cup\{G\})\cap V(f))\setminus\{G\}=\ell\text{-Spec}^*(G)\setminus\overline{D(f)}.$$

Since $\ell\text{-Spec}(G)^{\min}\cup\{G\}$ is patch closed by (ii) we see that $\operatorname{Spez}((\ell\text{-Spec}(G)^{\min}\cup\{G\})\cap V(f))$ is closed. But this set intersects $\ell\text{-Spec}^*(G)$ in $\ell\text{-Spec}^*(G)\setminus\overline{D(f)}$, which shows that $\ell\text{-Spec}^*(G)\setminus\overline{D(f)}$ is closed in $\ell\text{-Spec}^*(G)$ as required.

(ii)⇒(iii). By 10.8, the proper prime z-ideals are exactly those that are in the patch closure of $\ell\text{-Spec}(G)^{\min}$. Since this patch closure is $\ell\text{-Spec}(G)^{\min}\cup\{G\}$ by (ii) and $\ell\text{-Spec}(G)^{\min}$ is stranded there is exactly one prime z-ideal contained in $\mathfrak{p}$, namely the unique minimal prime ℓ-ideal contained in $\mathfrak{p}$.

(iii)⇒(ii). Let Z be the patch closure of $\ell\text{-Spec}(G)^{\min}\cup\{G\}$. Using 10.8 we see that (iii) implies that there are no specializations in $Z\cap\ell\text{-Spec}^*(G)$. But this is only possible if $Z=\ell\text{-Spec}(G)^{\min}\cup\{G\}$. We see that (iii) then implies that $\ell\text{-Spec}^*(G)$ is stranded.

Hence we know that (i),(ii) and (iii) are equivalent. For the rest of the proof recall from 7.8 that the spectrum of the bounded distributive lattice $[0,f]$ is naturally homeomorphic to $D(f)$ (formed in $\operatorname{PrimI}(G^{\geq 0}\cup\{\infty\})$ and therefore $[0,f]\cong\mathring{\mathcal{K}}(D(f))$.

(ii)⇒(iv). Since $\ell\text{-Spec}^*(G)$ is stranded, also $\operatorname{PrimI}(G^{\geq 0})$ is stranded by 10.7. By 7.5, $\ell\text{-Spec}(G)^{\min}\cup\{G\}$ and $\operatorname{PrimI}(G^{\geq 0})^{\min}\cup\{G\}$ are homeomorphic.

These two properties are then inherited on all open and quasi-compact subspaces S of $\operatorname{PrimI}(G^{\geq 0})$, i.e. S is stranded and $S^{\min}$ is patch closed (notice that the point G gets removed if S is proper and $S\cap(\ell\text{-Spec}(G)^{\min}\cup\{G\})=S^{\min}$). Hence for all $f\in G^{\geq 0}$, $D(f)$, taken in $\operatorname{PrimI}(G^{\geq}0)$, is stranded with patch closed space of minimal points. By `\ref{QCOPStoneLattice} in PseudoComplemented.tex` this means that $\mathring{\mathcal{K}}(D(f))$ is a Stone lattice, in particular $\mathring{\mathcal{K}}(D(f))$ is pseudo complemented.

But $\mathring{\mathcal{K}}(D(f))$ is isomorphic to $[0,f]$. Since all bounded intervals of G are isomorphic as a lattice to an interval $[0,f]$ with $f \geq 0$ we get (iv).

(iv)⇒(v) is a weakening and (v)⇒(i) holds by 10.4.

This shows the first equivalences. Now assume that G has a strong order unit u, i.e. $\ell\text{-Spec}^*(G) = D(u) = D(|u|)$ is a spectral space and G is a constructible point of ℓ-Spec(G).

(ii)⇔(vi) holds by `\ref{QCOPStoneLattice} in PseudoComplemented.tex` applied to $\ell\text{-Spec}^*(G)$.

(v)⇒(vii) is a weakening.

(vii)⇒(v) For $f \in G^{\geq 0}$ we have $[0,f] \cong \mathring{\mathcal{K}}(D(f))$, where $D(f)$ is formed in $\mathrm{PrimI}(G^{\geq 0} \cup \{\infty\})$. As $D(f) \subseteq D(u)$ we see that $D(f)$ is semi-Heyting (see [DiScTr2019, 8.3.3(i)]). □

10.10. **Corollary.** *Let T be a Tychonoff space and let G be the ℓ-group of continuous functions. Then G is projectable if and only if* $(\operatorname{Spec} C(X))^{\min}$ *is compact and* $\operatorname{Spec} C(X)$ *is stranded.*

Proof. Recall that $\operatorname{Spec}(C(X))$ is a spectral subspace of $\ell - \operatorname{Spec}(G)$ and $(\operatorname{Spec} C(X))^{\min} = (\ell - \operatorname{Spec}(G))^{\min}$. Hence if G is projectable, then by 10.9 we get that $(\operatorname{Spec} C(X))^{\min}$ is compact and $\operatorname{Spec} C(X)$ is stranded. Conversely if these conditions hold, then by 10.9 it remains to show that $\ell\text{-Spec}^*(G)$ is stranded. Suppose this is not the case. Then there are distinct $\mathfrak{p}, \mathfrak{q} \in (\ell - \operatorname{Spec}(G))^{\min}$ such that $\mathfrak{p} + \mathfrak{q} \neq G$. But then $\mathfrak{p}, \mathfrak{q}$ are prime ideals of $C(X)$ and so also $\mathfrak{p} + \mathfrak{q}$ is a prime ideal of $C(X)$, a contradiction. □

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The University of Manchester, Department of Mathematics, Oxford Road, Manchester M13 9PL, UK
Homepage: http://personalpages.manchester.ac.uk/staff/Marcus.Tressl/
Email address: marcus.tressl@manchester.ac.uk