C^*-ALGEBRAS

MARCUS TRESSL

Course Notes

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1. Revision of Banach spaces

Throughout, $\mathbb{K}$ denotes the field of real or complex numbers. Recall that a **normed space** $(V, |\,.\,|)$ over $\mathbb{K}$ is a $\mathbb{K}$-vector space V together with a function $|\,.\,| : V \longrightarrow \mathbb{R}^{\geq 0}$ (called the norm) satisfying $|a+b| \leq |a| + |b|$, $|\lambda{\cdot}a| = |\lambda|{\cdot}|a|$ and $|a| = 0 \iff a = 0$ for all $a, b \in V$, $\lambda \in \mathbb{K}$. Here $|\lambda| \in \mathbb{R}$ also denotes the modulus of $\lambda \in \mathbb{K}$.

The norm induces a metric on V via $d(a,b) = |a-b|$ and therefore any normed vector space carries a topology induced by this metric, called the **norm-topology**. We write $\mathring{B}_\rho(x)$ and $\bar{B}_\rho(x)$ for the open ball and closed ball of radius $\rho \in \mathbb{R}$ around x in a normed space.

The normed space $(V, |\,.\,|)$ is called a **Banach space** if V with the induced metric is complete, i.e. every Cauchy-sequence converges. The standard example here is the finite dimensional Banach space $\mathbb{K}^n$ with the euclidean norm: $|x| = \sqrt{x_1\bar{x}_1 + ... + x_n\bar{x}_n}$ (here $\bar{z}$ is complex conjugation).

1.1. *Example.* (Continuous functions)
Let X be a topological Hausdorff space and let

- $C(X, \mathbb{K})$ denote the set of $\mathbb{K}$-valued continuous functions on X.
- $C_b(X, \mathbb{K})$ denote the set of bounded $\mathbb{K}$-valued continuous functions on X. On $C_b(X, \mathbb{K})$ we define $|f| := \sup_{x \in X} |f(x)|$.
- $C_c(X, \mathbb{K})$ denote the set of $\mathbb{K}$-valued continuous functions f with compact support (where $\mathrm{supp}(f) = \overline{\{f \neq 0\}}$).

Then

(i) $C(X, \mathbb{K})$ together with pointwise addition and multiplication is a commutative $\mathbb{K}$-algebra, where $a \in \mathbb{K}$ is mapped to the constant function with value a.
(ii) $C_b(X, \mathbb{K})$ and $C_c(X, \mathbb{K})$ are linear subspaces of $C(X, \mathbb{K})$, in fact $C_b(X, \mathbb{K})$ is a subalgebra of $C(X, \mathbb{K})$ and $C_c(X, \mathbb{K})$ is an ideal of both rings.
(iii) $C_b(X, \mathbb{K})$ together with $|\,.\,|$ is a Banach space (easy exercise) and $C_c(X, \mathbb{K})$ together with $|\,.\,|$ is a normed vector space. However, $C_c(X, \mathbb{K})$ is not a Banach space in general.

1.2. **Fact.** *Let $(V, |\,.\,|)$ be a normed space over $\mathbb{K}$.*

(i) Addition $V \times V \longrightarrow V$ and scalar multiplication $\mathbb{K} \times V \longrightarrow V$ are continuous maps.
(ii) The completion B of the metric space V is a Banach space. Recall that B is the set of all Cauchy-sequences of V modulo null-sequences. Addition and scalar multiplication are defined pointwise. The norm of B is the unique continuous extension of the norm of V. Recall that V is dense in B.
(iii) If W is a subspace (i.e. a sub vector space), then the closure $\overline{W}$ of W is again a subspace of V. If $(V, |\,.\,|)$ is a Banach space then $\overline{W}$ (together with the induced norm) is again a Banach space
(iv) If W is a closed subspace of V, then the quotient space V/W is a normed space, the norm is given by

$$|x + W| = \inf_{w \in W} |x - w|$$

and the residue map $\pi : V \longrightarrow V/W$ is continuous and open; in fact π maps the open ball $\mathring{B}_\rho(x)$ onto the open ball $\mathring{B}_\rho(x+W)$.

(v) If W is a closed subspace of V and V is a Banach space, then $(V/W, |\,.\,|)$ is again a Banach space.

Proof. (i)-(iii) are straightforward.

(iv). To see the triangle inequality $|x+y+W| \leq |x+W| + |y+W|$ pick $\varepsilon > 0$ and $w_x, w_y \in W$ with $|x - w_x| < |x+W| + \varepsilon$ and $|y - w_y| < |y+W| + \varepsilon$. Then $|x+y+W| \leq |x - w_x + y - w_y| < |x+W| + |y+W| + 2\varepsilon$ and as $\varepsilon > 0$ was arbitrary the triangle inequality follows. Since W is closed, every element of norm 0 is 0, hence $|\,.\,|$ on V/W is indeed a norm.

A straightforward calculation shows

$$\pi^{-1}(\mathring{B}_\rho(x+W)) = \bigcup_{w\in W} \mathring{B}_\rho(x+w) = \mathring{B}_\rho(x) + W.$$

It is then clear that π is continuous and

$$\pi(\mathring{B}_\rho(x)) = \pi(\mathring{B}_\rho(x) + W) = \mathring{B}_\rho(x+W).$$

(v). Let $(x_n + W)$ be a Cauchy sequence in V/W. Let (x_{n_k}) be a subsequence of (x_n) such that

$$|x_{n_l} - x_{n_k} + W| < \frac{1}{2^k} \text{ for all } l \geq k.$$

We define $y_k := x_{n_{k+1}} - x_{n_k}$ and show that $\sum_k (y_k + W)$ converges in V/W (this simply means that $(x_{n_k} + W)$ converges in V/W and we are done, as (x_n) is Cauchy).

Choose $w_n \in W$ with

$$|y_k - w_k| < |y_k + W| + \frac{1}{2^k}.$$

We claim that with $z_k = y_k - w_k$, the series $\sum_{k=1}^\infty z_k$ converges in V: Since V is a Banach space we only need to show that the sequence (s_k) of partial sums of (z_k) is Cauchy: We have

$$\begin{aligned}
|s_l - s_k| &\overset{\text{say } l>k}{=} |z_l + \dots + z_{k+1}| = |y_l - w_l + \dots + y_{k+1} - w_{k+1}| \leq \\
&\leq |y_l - w_l| + \dots + |y_{k+1} - w_{k+1}| < \\
&< |y_l + W| + \dots + |y_{k+1} + W| + \frac{1}{2^{k-1}} = \\
&= |x_{n_{l+1}} - x_{n_l} + W| + \dots + |x_{n_{k+2}} - x_{n_{k+1}} + W| + \frac{1}{2^{k-1}} < \\
&< \frac{1}{2^l} + \dots + \frac{1}{2^{k+1}} + \frac{1}{2^{k-1}} \to 0 \text{ as } k, l \to \infty.
\end{aligned}$$

Hence $\sum_{k=1}^\infty z_k$ converges to some $z \in V$ and it remains to show that $\sum_k (y_k + W)$ converges in V/W to $z + W$. However, $y_k + W = z_k + W$ and the continuity of $V \twoheadrightarrow V/W$ gives the result. □

1.3. *Example.* (Continuous functions continued)
The closure of the subspace $C_c(X, \mathbb{K})$ of the Banach space $C_b(X, \mathbb{K})$ (together with the supremum norm) is the subspace $C_v(X, \mathbb{K})$ of all $\mathbb{K}$-valued continuous functions f on X that **vanish at** ∞, i.e. for all $\varepsilon > 0$ the set $\{|f| \geq \varepsilon\}$ is compact (exercise).

It is worth mentioning (and helps appreciating what is developed later) that our intuition of finite dimensional Banach spaces does not catch the general situation. Here two prominent examples:

(1) The closed unit ball in a Banach space is not compact, unless the space is finite dimensional. (Proof upon request). In our example: let $X = \mathbb{R}$ and let f_n be a continuous function with $|f_n| = 1$ and support $[n, n+1]$ (a bump function). Then no subsequence of (f_n) converges.

(2) Given a closed subspace W of a Banach space V and some $v \in V \setminus W$, there is in general no unique distance point of v to W: This already happens in finite dimensions, e.g. If $V = \mathbb{K}^2$ mit the maximum norm. More generally, the distance of $1 \in C_b(X, \mathbb{K})$ to the closed subspace $C_v(X, \mathbb{K})$ is attained at many points. The distance here is

$$\inf_{f \in C_v(X,\mathbb{K})} |1 - f|,$$

which is 1. However, this infimum is attained at <u>all</u> real valued functions g with the property $0 \leq g \leq 1$.

In general the situation is worse. Here is an example of a proper closed subspace W of a Banach space V such that <u>no</u> point $v \in V \setminus W$ has any distance point to W (cf. [Alt, p.126, U2.3]):

We work with $C([0,1]) = C([0,1], \mathbb{R})$. Let $V = \{f \in C([0,1]) \mid f(0) = 0\}$. V itself is a closed subspace of $C([0,1])$, hence V is a Banach space. We let $W := \{g \in V \mid \int g \ \mathrm{d}t = 0\}$. It is clear that W is a closed subspace of V.

Take $f \in V \setminus W$ and let $A = \int f \ \mathrm{d}t$. Then, for each $g \in W$ we have

$$(*) \qquad |A| \overset{\text{as } \int g \mathrm{d}t = 0}{=} |\int (f-g) \ \mathrm{d}t| \overset{\text{as } f-g \neq 0 \text{ and (f-g)(0)=0}}{<} |f - g|.$$

On the other hand for $n \in \mathbb{N}$, the function $h_n := (1 + \frac{1}{n}) \cdot t^{\frac{1}{n}}$ has $\int h \ \mathrm{d}t = 1$. Hence $g_n := f - A \cdot h_n \in W$ and

$$|f - g_n| = |A| \cdot |h_n| = |A| \cdot (1 + \frac{1}{n}).$$

This, together with $(*)$ shows that the distance of f to W is $|A|$, but this distance is never attained in W.

2. Definition of C^*-algebras

2.1. **Definition.** A **normed $\mathbb{K}$-algebra** is a normed space $(V, |\,.\,|)$ over $\mathbb{K}$ together with an operation $\cdot : V \times V \longrightarrow V$ such that $(V, +, \cdot)$ is a ring (possibly without unit) satisfying

- $\lambda\cdot(a\cdot b) = (\lambda\cdot a)\cdot b = a\cdot(\lambda\cdot b)$ for all $a, b \in V$, $\lambda \in \mathbb{K}$, and
- $|a\cdot b| \le |a|\cdot|b|$ for all $a, b \in V$.

If $a\cdot b = b\cdot a$ for all $a, b \in V$, then V is called **commutative**. If V has a (necessarily unique) neutral element w.r.t. multiplication, then V is called **unital** and the multiplicative neutral element is denoted by $\mathbb{1}$.

A **Banach algebra over** $\mathbb{K}$ is a normed $\mathbb{K}$-algebra such that the underlying normed space is a Banach space.

A finite dimensional example of a unital Banach algebra over $\mathbb{K}$ is given by the matrix ring $M_n(\mathbb{K})$, where the norm of a matrix A is defined by

$$|A| = \max\{|Ax| \mid x \in \mathbb{K}^n,\ |x| \le 1\}$$

and the norm on $\mathbb{K}^n$ is the euclidean norm. Note that in this example in general we have $|A\cdot A| < |A|^2$, $|A\cdot B| \ne |B\cdot A|$ and $|A\cdot B| < |A|\cdot|B|$: take $n = 2$ and

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Another example is $C_b(X, \mathbb{K})$, which is a unital, commutative Banach algebra where multiplication is given pointwise. $C_c(X, \mathbb{K})$ is a normed subalgebra of $C_b(X, \mathbb{K})$ without unit. Since the norm here is the supremum norm we have $|f\cdot g| \le |f|\cdot|g|$ for all f, g; however, $f\cdot g$ might be zero, whereas both f and g are not (identically) zero.

2.2. **Definition.** A **Banach $*$-algebra** is a Banach algebra A over $\mathbb{C}$ together with a map $* : A \longrightarrow A$ such that

$*1$: The map $*$ is **conjugate linear**, i.e. $(a + b)^* = a^* + b^*$ and $(\lambda a)^* = \bar{\lambda} a^*$ for all $\lambda \in \mathbb{C}$, $a \in A$.

$*2$: For all $a \in A$ we have $a^{**} = a$.

$*3$: For all $a, b \in A$ we have $(a\cdot b)^* = b^*\cdot a^*$.

Maps satisfying $*1$, $*2$ and $*3$ are called **involutions** (of the normed algebra A).

$*4$: For all $a \in A$ we have $|a^*| = |a|$.

If in addition A satisfies

C^*: For all $a \in A$ we have $|a^* a| = |a|^2$,

then A is called a **C^*-algebra**.

Note that condition C^* supersedes condition $*4$: since $|a|^2 = |a^* a| \le |a^*|\cdot|a|$ (from the Banach algebra axioms) we get $|a| \le |a^*|$; now the idempotency of the involution gives $|a^*| \le |a^{**}| = |a|$.

Going back to our example of continuous functions we see that $C_b(X, \mathbb{C})$ is a C^*-algebra when choosing $f^*(x) := \overline{f(x)}$ (complex conjugation).

Our first goal in the course is to show the

2.3. **Gelfand-Naimark theorem**
Every unital and commutative C^*-algebra is of the form $C(X, \mathbb{C})$ for a compact space X.

It should be mentioned that there is also a characterisation of real unital commutative Banach algebras of the form $C(X, \mathbb{R})$, where X is compact, namely:

2.4. **Theorem.** *A real unital commutative Banach algebra A is isometrical isomorphic to $C(X, \mathbb{R})$ in the supremum norm, where X is compact if and only if $|\mathbb{1}| = 1$ and for all elements a, b in the algebra we have*

$$|a^2 - b^2| \leq |a^2 + b^2|.$$

Proof. This follows easily from 2.3 applied to $A \oplus i{\cdot}A$ The condition on the norm, guarantees that $A \oplus i{\cdot}A$ together with $(f + ig)^* = f - ig$ and $|f + ig| = \sqrt{|f^2 + g^2|}$ is a C^*-algebra (and A is the subalgebra of self-adjoint elements of $A \oplus i{\cdot}A$, i.e. of those h, satisfying $h^* = h$).

For a direct proof see [AlbKal, Theorem 4.2.5]. □

The goal in the non-commutative case.

Let us now look at non-commutative examples. First recall that a Hilbert-space over $\mathbb{K}$ is a $\mathbb{K}$-vector space H together with a positive definite, hermitian form $\langle , \rangle$ which is complete under the norm $|x| := \sqrt{\langle x, x\rangle}$:

$\langle , \rangle$ has the following properties:

2.5.

(1) $\langle , \rangle$ is a **scalar product** (or **sesquilinear form**) of V: i.e. $\langle x, y\rangle$ is linear in x and **anti-linear** in y, so $\langle x, \lambda y\rangle = \bar{\lambda}\langle x, y\rangle$ for $\lambda \in \mathbb{C}$. If $\mathbb{K} = \mathbb{R}$, this means that $\langle , \rangle$ is a bilinear form

(2) $\langle , \rangle$ is **hermitian**, i.e. $\langle y, x\rangle = \overline{\langle x, y\rangle}$. If $\mathbb{K} = \mathbb{R}$ this means $\langle , \rangle$ is symmetric.
Hence if $\langle , \rangle$ is hermitian, then $\langle x, x\rangle$ is a real number in any case.

(3) The hermitian form $\langle , \rangle$ is called **positive**, if $\langle x, x\rangle \geq 0$ for all x and **positive definite**, if $\langle x, x\rangle > 0$ for all $x \neq 0$.

Note: In Hilbert spaces the issue with distance points is gone. One can show that for each point $x \in H$ and all closed subspaces W of H, there is a unique distance point $y \in H$ of x to W.

If H is a Hilbert space we may define a Banach-algebra $\mathcal{B}(H)$ as follows:

- $\mathcal{B}(H)$ is the set of bounded operators on H, where a **bounded operator** on H is a linear map $\varphi : H \longrightarrow H$ such that the norm
$$|\varphi| = \sup\{|\varphi(x)| \mid |x| \leq 1\} \text{ is finite.}$$
With this norm, $\mathcal{B}(H)$ is a Banach space.
- Operators in $\mathcal{B}(H)$ can be composed and with this operation, $\mathcal{B}(H)$ is a Banach algebra.
- If $\varphi \in \mathcal{B}(H)$, then φ has an **adjoint** $\varphi^* \in \mathcal{B}(H)$ uniquely defined via
$$\langle \varphi(x), y\rangle = \langle x, \varphi^*(y)\rangle \ (x, y \in H).$$
(This is due to Riesz' representation theorem)
It turns out that $\varphi \mapsto \varphi^*$ is an involution on $\mathcal{B}(H)$ and $\mathcal{B}(H)$ together with this involution is a C^*-algebra.

Note that if $H = \mathbb{K}^n$, then $\mathcal{B}(H)$ simply is the matrix algebra $M_n(\mathbb{K})$ and the adjoint of $A \in \mathcal{B}(H)$ is the conjugate transpose $\bar{A}^T$ as defined in linear algebra.

Warning. There is a clash of notations in the literature: some authors call the adjoint of a square matrix $A \in M_n(\mathbb{K})$ the $n \times n$ matrix $B = (b_{ij})$ defined by
$$b_{ij} = (-1)^{i+j} \cdot \det(C_{ji})$$
where C_{ji} is the matrix obtained from A by removing the j^{th} row and the i^{th} column. This is **not** the adjoint of A in our sense.

The overall goal of the course is to show the following

2.6. **Gelfand-Naimark-Segal theorem**
Any C^*-algebra is isometrically $*$-isomorphic to a C^*-algebra of operators on a Hilbert space.

The exact statement gives a much better description of the representation, but also needs more terminology. In the proof of the GNS-theorem (Gelfand-Naimark-Segal) certain Hilbert spaces are constructed and this construction is referred to as the **GNS-construction**.

The most important instance of an Hilbert space for us, which is also useful for understanding the GNS-construction and operators on Hilbert spaces is the following:

Let $(X, \mathfrak{A}, \mu)$ be a measure space (so $\mu : \mathfrak{A} \longrightarrow [0, \infty]$) and let $\mathscr{L}^2(\mu)$ be the $\mathbb{K}$-vector space of all measurable functions $f : X \longrightarrow \mathbb{K}$ such that $f \cdot \bar{f}$ (defined pointwise) is integrable.

Let $\mathscr{N}(\mu)$ be the subspace of all $f \in \mathscr{L}^2(\mu)$ which are 0, μ-almost everywhere. Then

$$L^2(\mu) := \mathscr{L}^2(\mu)/N$$

is a Hilbert space, where the sesquilinear form is given by

$$\langle f, g \rangle = \int_X f \cdot \bar{g} \, \mathrm{d}\mu.$$

(It is common to denote the equivalence class of $f \in \mathscr{L}^2(\mu)$ in $L^2(\mu)$ by f again.)

The most prominent example of measure spaces occurring here will be regular Borel measures on topological spaces.

Returning to the commutative case, the GNS-construction applied to the C^*-algebra $C([0,1])$ represents $C([0,1])$ as an algebra of operators on $L^2([0,1])$ (w.r.t. Lebesgue measure): An element $f \in C([0,1])$ is mapped onto the operator $b_f : L^2([0,1]) \longrightarrow L^2([0,1]);\ b_f(g) = f \cdot g$.

3. Bounded operators and functionals on normed spaces

Firstly, recall that two norms $|\ |_1$ and $|\ |_2$ on a $\mathbb{K}$-vector space V are called **equivalent** if there are $c, C > 0$ such that for all $x \in V$ we have

$$c \cdot |x|_1 \leq |x|_2 \leq C \cdot |x|_1.$$

Also recall that all norms on a finite dimensional $\mathbb{K}$-vector space are equivalent.

3.1**. Finite products of normed spaces**

Let $V_1, ..., V_n$ be normed spaces over $\mathbb{K}$. Then the product space $V := V_1 \times ... \times V_n$ is again a normed space with either the norm

$$|x| = \max\{|x_1|, ..., |x_n|\}, \text{ or any of the norms}$$

$$|x| = \sqrt[p]{\max\{|x_1|^p, ..., |x_n|^p\}}, \text{ where } 1 \leq p < \infty$$

All these norms are equivalent and induce the product topology on V. If each V_i is a Banach space, then also V is a Banach space.

Proof. This is clear using the inequalities proving the equivalence of these norms on finite dimensional spaces. □

3.2**. Basic facts about bounded linear maps**

A liner map $\lambda : V \longrightarrow W$ between normed spaces over $\mathbb{K}$ is continuous (at 0) if and only if λ is **bounded**, i.e. there is some $M \in \mathbb{R}$, $M \geq 0$ such that for all $x \in V$ we have

$$|\lambda(x)| \leq M \cdot |x|.$$

If this is the case, then the infimum of all these M's is attained and called the **norm of** λ, denoted by $|\lambda|$. We have

$$|\lambda| = \sup\{|\lambda(x)| \mid |x| \leq 1\} \overset{\text{if } V \neq 0}{=} \sup\{|\lambda(x)| \mid |x| = 1\}.$$

The set of bounded linear maps $V \longrightarrow W$, denoted by $L(V, W)$ is a normed space over $\mathbb{K}$ w.r.t the norm defined above and we have

(i) If Z is another normed space over $\mathbb{K}$ and $\lambda \in L(V, W)$, $\mu \in L(W, Z)$, then $\mu \circ \lambda \in L(V, Z)$ with $|\mu \circ \lambda| \leq |\mu| \cdot |\lambda|$.

(ii) If W is a Banach space, then also $L(V, W)$ is a Banach space.

Proof. If λ is bounded then λ is clearly continuous at 0 as $\bar{B}_1(0) \subseteq \lambda^{-1}(\bar{B}_M(0))$ and so by linearity, λ is continuous everywhere. Note that the infimum m of all the bounds M is also a bound, since this can be verified at each $x \in V$.

Conversely, suppose λ is continuous at 0. Take $\rho > 0$ with $\bar{B}_\rho(0) \subseteq \lambda^{-1}(\bar{B}_1(0))$. Thus with $M = \frac{1}{\rho}$ we have $|\lambda(x)| \leq M \cdot |x|$ for all $x \in V$ with $|x| \leq 1$, so by linearity, λ is bounded.

If λ is bounded, then using linearity of λ it is clear that $M \in \mathbb{R}$ is a bound for λ if and only if

$$|\lambda(x)| \leq M$$

for all $x \in V$ with $|x| = 1$. This indeed proves

$$|\lambda| = \sup\{|\lambda(x)| \mid |x| \leq 1\} = \sup\{|\lambda(x)| \mid |x| = 1\}.$$

Next we need to show that $L(V,W)$ is a vector space and $|\ |$ is a norm on $L(V,W)$ and the only thing that needs to be shown is $|\lambda+\tau| \leq |\lambda|+|\tau|$ for all $\lambda, \tau \in L(V,W)$. However, using the representation above, this is clear.

(i) holds since for $x \in V$ we have

$$|\mu \circ \lambda(x)| \leq |\mu|\cdot|\lambda(x)| \leq |\mu|\cdot|\lambda|\cdot|x|$$

To see (ii), take a Cauchy sequence (λ_n) in $L(V,W)$. Then for each $x \in V$, $(\lambda_n(x))$ is Cauchy in W and converges to some $\lambda(x)$. It is straightforward to see that λ is the limit of the λ_n in $L(V,W)$.

□

3.3. *Remark.* A linear map $\lambda : V \longrightarrow W$ between normed spaces that is bounded on some open ball $\mathring{B}_\varepsilon(v)$ of V by $M > 0$ is bounded with $|\lambda| \leq \frac{2M}{\varepsilon}$.

Proof. Take $x \in V$ with $|x| \leq \varepsilon$. Then

$$|\lambda(x)| \leq |\lambda(v+x)| + |\lambda(v)| \leq 2M$$

□

3.4. *Remark.* Recall that a function $f : V \longrightarrow W$ between normed spaces over $\mathbb{K}$ is **differentiable** at $x_0 \in V$ if there is some $T \in L(V,W)$ and some map $\psi : V \longrightarrow W$ with $\lim_{h\to 0} \psi(h) = 0$ such that

$$f(x) = f(x_0) + T(x - x_0) + |x - x_0|\psi(x - x_0) \text{ for all } x \in V.$$

In this case T is called the derivative of f at x_0. Clearly if f is already in $L(V,W)$, then $f'(x_0) = f \in L(V,W)$ is independent of x_0, hence f' is constant, equal to f.

A **functional** of a vector space V is a linear map $V \longrightarrow \mathbb{K}$. For a normed space V, the normed space of bounded functionals is called the **dual space of** V and written as

$$V' := L(V, \mathbb{K}).$$

To see an example of a bounded functional on a normed space, let $\lambda : C([a,b], \mathbb{K}) \longrightarrow \mathbb{K}$ be integration w.r.t Lebesgue measure: So $\lambda(f) = \int_a^b f\mathrm{d}t$. Then $|\lambda(f)| \leq |f|\cdot(b-a)$. Hence λ is bounded and $|\lambda|$ is the measure of $[a,b]$. In fact this example is typical and a version of Riesz representation theorem says that for every locally compact space, every bounded functional of $C_b(X,\mathbb{K})$ can be obtained from integration w.r.t a (signed) Borel measure on X.

To see a (natural) example of an unbounded functional on a normed space, let $\lambda : C_c(\mathbb{R}, \mathbb{K}) \longrightarrow \mathbb{K}$ be integration w.r.t Lebesgue measure: So $\lambda(f) = \int_{-\infty}^{+\infty} f\mathrm{d}t$. Now if $M > 0$, it is clear that there is a continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ with compact support and of norm 1 such that $\lambda(f) > M = M\cdot|f|$. Hence λ is not bounded.

If $\lambda \in V'$, then clearly the kernel $\mathrm{Ker}(\lambda) := \lambda^{-1}(0)$ is a closed subspace of V of codimension 1. This property of λ characterises membership in V', i.e. if W is a closed subspace of V of codimension 1, then the induced linear map $\lambda : V \longrightarrow \mathbb{K}$ is continuous (cf. 1.2(iv), where we also use that all norms on $\mathbb{K}$ are equivalent). Moreover by 1.2(iv), every $\lambda \in V'$ is open.

A **bounded operator of a normed space** V is a bounded linear map $V \longrightarrow V$. We define

$$\mathcal{B}(V) = L(V,V).$$

By 3.2, $\mathcal{B}(V)$ is a normed $\mathbb{K}$-algebra and a Banach algebra if V is a Banach space.

3.5. **Hahn-Banach**
Let V be an $\mathbb{R}$-vector space and let $p : V \longrightarrow \mathbb{R}$ be **sublinear**, i.e.

$$p(\alpha x) = \alpha p(x) \text{ and } p(x+y) \leq p(x) + p(y) \ (x, y \in V, \alpha \in \mathbb{R}, \alpha \geq 0)$$

Let W be a subspace of V. Then every functional $\lambda : W \longrightarrow \mathbb{R}$ with $\lambda \leq p$ can be extended to a functional $\bar{\lambda} : V \longrightarrow \mathbb{R}$ with $\bar{\lambda} \leq p$.

Proof. By Zorn it suffices to extend λ to a subspace of dimension 1 over W. So we may assume that $V = W + z{\cdot}\mathbb{R}$ with $z \notin W$. Now the extensions of λ to V are given by choosing $c \in \mathbb{R}$ and defining

$$\lambda_c(x + z\alpha) = \lambda(x) + c{\cdot}\alpha.$$

So the only thing we need to do is to choose c such that $\lambda_c \leq p$. This condition reads as

$$(*) \qquad \lambda(x) + c{\cdot}\alpha \leq p(x + z{\cdot}\alpha) \ (x \in W, \alpha \in \mathbb{R}).$$

For $\alpha = 0$ this is true for every choice of c by assumption. For $\alpha > 0$, $(*)$ reads as

$$(+) \qquad c \leq p(\frac{x}{\alpha} + z) - \lambda(\frac{x}{\alpha}).$$

For $\alpha < 0$, $(*)$ reads as

$$(-) \qquad c \geq \lambda(-\frac{x}{\alpha}) - p(-\frac{x}{\alpha} - z).$$

It is therefore enough to choose $c \in \mathbb{R}$ such that

$$\sup_{x \in W} \{\lambda(x) - p(x - z)\} \leq c \leq \inf_{x \in W} \{p(x + z) - \lambda(x)\}$$

However, if $x, x' \in W$, then $\lambda(x) - p(x - z) \leq p(x' + z) - \lambda(x')$ because $\lambda(x) + \lambda(x') = \lambda(x + x') \leq p(x + x') = p(x' + z + x - z) \leq p(x' + z) + p(x - z)$. Hence we can choose c as desired. □

As a corollary

3.6. **Hahn-Banach for bounded functionals**
Let W be an arbitrary subspace of a normed space V over $\mathbb{K}$. Every $\lambda \in W'$ can be extended to some $\tilde{\lambda} \in V'$ with $|\tilde{\lambda}| = |\lambda|$.

Proof. If $\mathbb{K} = \mathbb{R}$, then define $p : V \longrightarrow \mathbb{R}$ by $p(x) := |\lambda|{\cdot}|x|$ $(x \in V)$. p is of course sublinear and for $y \in W$ we have $\lambda(y) \leq |\lambda|{\cdot}|y| = p(y)$. By 3.5, there is a linear functional $\tilde{\lambda}$ of V extending λ with $\tilde{\lambda}(x) \leq p(x) = |\lambda|{\cdot}|x|$ for all $x \in V$. It follows $|\tilde{\lambda}| \leq |\lambda|$. However $|\tilde{\lambda}| \geq |\lambda|$ holds for any (bounded) extension of λ.

If $\mathbb{K} = \mathbb{C}$, then apply the following recipe:

(a) By scalar restriction, consider V, W as normed spaces over $\mathbb{R}$
(b) Consider $\lambda_r : Re \circ \lambda : W \longrightarrow \mathbb{R}$. Then $|\lambda_r| \leq |\lambda|$ and from the real case we get an extension $\mu : V \longrightarrow \mathbb{R}$ of λ_r with $|\mu| = |\lambda_r|$.
(c) Define

$$\tilde{\lambda}(x) = \mu(x) - i{\cdot}\mu(i{\cdot}x) \ (x \in V)$$

and verify that $\tilde{\lambda}$ is a linear functional $V \longrightarrow \mathbb{C}$.
(d) Verify that $|\tilde{\lambda}| = |\lambda|$.

Only the inequality $|\tilde{\lambda}| \leq |\lambda|$ in (d) needs a proof: Take $x \in V$ and write $\tilde{\lambda}(x) = r{\cdot}e^{i\Theta}$. Then

$$|\tilde{\lambda}(x)| = r = Re(e^{i\Theta}\tilde{\lambda}(x)) = Re(\tilde{\lambda}(e^{i\Theta}x)) = \mu(e^{i\Theta}x) \leq |\mu|{\cdot}|x| \leq |\lambda||x|$$

as desired. □

In particular for every nonzero $x \in V$, there are bounded functionals λ of V with $|\lambda| = 1$ and $|\lambda(x)| = |x|$: Apply, 3.6 with $W = x{\cdot}\mathbb{K}$ and the bounded linear functional $x{\cdot}\alpha \mapsto |x|{\cdot}\alpha$.

From this we get

3.7. **Corollary.** *If V is a normed space over $\mathbb{K}$, then the evaluation map*

$$\varepsilon : V \longrightarrow V'',$$

which sends $x \in V$ to the evaluation map $\varepsilon(x) : V' \longrightarrow \mathbb{K}$ at x (so $\varepsilon(x)(\lambda) = \lambda(x)$) is an isometric embedding of normed spaces (so $|x| = |\varepsilon(x)|$ for all $x \in V$).

Proof. We have

$$|\varepsilon(x)| = \sup\{|\lambda(x)| \mid \lambda \in V',\ |\lambda| = 1\} \leq |x|$$

and the other inequality holds by the remark preceding the corollary. □

3.8**. Banach-Steinhaus (aka uniform boundedness principle)**
Let V, W be normed space over $\mathbb{K}$ and assume V is a Banach space. Let $F \subseteq L(V,W)$. If for every $v \in V$, the set

$$F(v) := \{f(v) \mid f \in F\} \text{ is bounded in } W,$$

then F is bounded in $L(V,W)$.

Proof. We use completeness of V by utilising the Baire category theorem. Note that this theorem is applicable in every complete metric space. Let

$$V_n := \{v \in V \mid F(v) \subseteq \bar{B}_n(0)\}.$$

Then by assumption, $V = \bigcup_n V_n$. Now $V_n = \bigcap_{f\in F} f^{-1}(\bar{B}_n(0))$ is closed and so by the Baire category theorem, one of the V_n must have nonempty interior. Say

$$(*) \qquad B_\varepsilon(v) \subseteq V_n.$$

Now take $f \in F$. Then $(*)$ says that f is bounded on $B_\varepsilon(v)$ by n. Then by 3.3, f is bounded by $\frac{2n}{\varepsilon}$. □

3.9. **Definition.** The **weak*-topology** on the dual space V' of a normed space V over $\mathbb{K}$ is defined as the weakest topology on V' making all evaluation maps $V' \longrightarrow \mathbb{K}$ continuous.

Another way of seeing the weak*-topology is the following. Let $\mathbb{K}^V$ be the $\mathbb{K}$-vector space of all functions $V \longrightarrow \mathbb{K}$. Then $\mathbb{K}^V$ carries the product topology which has an open basis of neighborhoods of 0, given by

$$(+) \qquad \prod_{x\in F} \mathring{B}_\varepsilon(0) \times \prod_{x \in V\setminus F} V,$$

where F ranges over finite subsets of V and $\varepsilon > 0$. Translating these sets to $\mu \in \mathbb{K}^V$ gives an open neighborhood basis of μ. It is clear that addition and scalar multiplication of $\mathbb{K}^V$ are continuous. Now observe that V' is a subvector space of $\mathbb{K}^V$ and the weak*-topology on V' is the induced relative topology:

The weak*-topology by definition has an open neighborhood basis of $0 \in V$ for the weak*-topology given by the sets

$$U_{F,\varepsilon} = \{\lambda \in V' \mid |\lambda(F)| < \varepsilon\} = \bigcap_{x \in F} \varepsilon(x)^{-1}(\mathring{B}_\varepsilon(0)),$$

where F ranges over finite subsets of V and $\varepsilon > 0$. However, $U_{F,\varepsilon}$ is the intersection of V' with the set displayed in (+). Translating these sets to $\mu \in V'$ gives an open neighborhood basis of μ.

3.10**. Banach-Alaoglu** Let V be a normed space and let E be the unit ball in V'. Then E is compact in the weak*-topology

Proof. Let

$$C := \prod_{x \in V} \bar{B}_{|x|}(0) \subseteq \mathbb{K}^V.$$

Then C with the product topology is a topological subspace of $\mathbb{K}^V$ and C is compact by Tychonoff. Let $W \subseteq \mathbb{K}^V$ be the (not necessarily bounded) functionals of V. Then for $\lambda \in W$ we have

$$\lambda \in C \iff \forall x \in V \ |\lambda(x)| \leq |x| \iff |\lambda| \leq 1.$$

This means

$$E = C \cap W.$$

Hence if we show that W is closed in $\mathbb{K}^V$, then E is a closed subset of the compact space C and therefore E is compact.

To see that W is closed in $\mathbb{K}^V$, take $f \in \mathbb{K}^V$ in the closure of W. We must show that f is linear. Take $x, y \in V$ and let $\varepsilon > 0$. By definition of the topology of $\mathbb{K}^V$, there is some $\lambda \in W$ such that in the coordinates x, y and $x + y$ we have

$$|\lambda(x) - f(x)| < \varepsilon, \ |\lambda(y) - f(y)| < \varepsilon \text{ and } |\lambda(x + y) - f(x + y)| < \varepsilon.$$

Hence

$$|f(x + y) - f(x) - f(y)| < |f(x + y) - \lambda(x + y) + \lambda(x) - f(x) + \lambda(y) - f(y)| < 3\varepsilon$$

and so $f(x + y) = f(x) + f(y)$. A similar argument shows that $f(\alpha x) = \alpha f(x)$ and so f is linear as desired. □

Warning. The unit sphere $S = \{\lambda \in V' \mid |\lambda| = 1\}$ of V' is weak*-dense in the unit ball $B = \{\lambda \in V' \mid |\lambda| \leq 1\}$, unless V is finite dimensional. In particular V is finite dimensional if and only if S is weak*-compact.

Proof. Let $\lambda \in B$ and consider a basic open neighborhood

$$U = \{\mu \in V' \mid |\mu(F)| < \varepsilon\}$$

of 0 for some finite subset F of V and $\varepsilon > 0$. We need to show that $S \cap (\lambda + U) \neq \emptyset$. In fact we show that there is some $\mu \in V'$ with $\mu(x) = 0$ for all $x \in F$ such that $|\lambda + \mu| = 1$:

Let $W \subseteq V'$ be the intersection of the kernels of the evaluation maps $\varepsilon(x) : V' \longrightarrow \mathbb{K}$, where $x \in F$. Define $p : W \longrightarrow \mathbb{R}$ by $p(\mu) = |\lambda + \mu|$. As V' is not finite dimensional, $K \neq \{0\}$ and therefore p attains arbitrary large real values. Since p is continuous and $p(0) = |\lambda| \leq 1$, there must be some $\mu \in K$ with $|\lambda + \mu| = 1$. This μ has the required properties. □

For a normed space V over $\mathbb{K}$ and a subset X of V', let

$$(X, \tau^*)$$

denote the topological space X in the weak*-topology.

By 3.7, evaluation induces an isometric embedding $\varepsilon : V \longrightarrow V''$. For $x \in V$, the evaluation map $\varepsilon(x) : V' \longrightarrow \mathbb{K}$ is by definition weak*-continuous, hence $\varepsilon(V) \subseteq C((V', \tau^*), \mathbb{K})$. By linearity of the $\varepsilon(x)$ it is clear that the composition Φ given by

$$V \xrightarrow{\varepsilon} \varepsilon(V) \hookrightarrow C((V', \tau^*), \mathbb{K}) \xrightarrow{\text{restriction}} C((\bar{B}_1(0), \tau^*), \mathbb{K})$$

is an embedding of $\mathbb{K}$-vector spaces (where $\bar{B}_1(0)$ is the closed unit ball in V').

By 3.10, $(\bar{B}_1(0), \tau^*)$ is compact, hence $C((\bar{B}_1(0), \tau^*), \mathbb{K})$ is a Banach algebra over $\mathbb{K}$ with the supremum norm. The norm of $\varepsilon(x)$ in this Banach algebra is by definition, the norm of $\varepsilon(x)$ in V''. Thus, Φ is an isometric embedding

$$V \longrightarrow C((\bar{B}_1(0), \tau^*), \mathbb{K}).$$

We have shown the following.

3.11. **Corollary.** *Let V be a normed space over $\mathbb{K}$. Then there is a compact space X such that V is isometrical isomorphic to a subspace of $C(X, \mathbb{K})$ (observe that this subspace is necessarily closed if V is a Banach space).*

The "natural" choice of X is the unit ball of V' in the weak-topology.* □

3.12. *Remark.* If we start with a completely regular space X and let $V = C_b(X, \mathbb{K})$, then we can find X as a subspace of the unit sphere $\{\lambda \in V' \mid |\lambda| = 1\}$, topologised with the weak*-topology:

For each $x \in X$, let $\Lambda_x : V \longrightarrow \mathbb{K}$ be the evaluation map $\Lambda_x(f) = f(x)$. Clearly Λ_x is linear and bounded of norm

$$|\Lambda_x| = \sup\{|f(x)| \mid f \in V, |f| \leq 1\} = 1.$$

Since X is completely regular, V separates points of X and therefore we obtain an injective map

$$\Lambda : X \longrightarrow V'; x \mapsto \Lambda_x$$

from X into the unit sphere of V'. We claim that Λ is a homeomorphism onto its image if we equip V' with the weak*-topology: To see this, take a sub-basic open neighborhood of 0, $U = \{\lambda \in V' \mid |\lambda(f)| < \varepsilon\}$ of V' in the weak*-topology. Then for $x \in X$ we have

$$x \in \Lambda^{-1}(U) \iff |\lambda_x(f)| < \varepsilon \iff |f(x)| < \varepsilon.$$

Hence $\Lambda^{-1}(U) = \{|f| < \varepsilon\}$, which is open as f is continuous.

To show that Λ is an homeomorphism it remains to show for a closed subset A of X that the complement of $\Lambda(A)$ in $\Lambda(X)$ is open in $\Lambda(X)$: Take $y \in X \setminus A$. Since X is completely regular there is some $f \in V$ with $f(y) = 1$, which vanishes on A. Let

$$U = \{\lambda \in V' \mid |(\lambda - \lambda_y)(f)| < 1\}$$

Then U is an open neighborhood of λ_x in V' w.r.t. the weak*-topology. As we have seen above,

$$\Lambda^{-1}(U) = \{x \in X \mid |f(x) - f(y)| < 1\}.$$

Since f vanishes on A, $\Lambda^{-1}(U)$ is disjoint from A and therefore U is disjoint from $\Lambda(A)$. This shows that Λ is a homeomorphism onto its image.

Now look at the restriction $\rho : V' \longrightarrow W'$, where $W = C_c(X, \mathbb{K})$ equipped with the weak*-topology. The restriction is weak*-continuous since already the restriction $\mathbb{K}^V \longrightarrow \mathbb{K}^W$ in the product topologies is continuous. Then we compose Λ with this restriction and - provided X is locally compact - the same proof as above shows that $\rho \circ \Lambda$ is a homeomorphism onto its image; the local compactness is used when we construct some $f \in W$ with $f(y) = 1$ that vanishes on A.

4. The spectrum of an element in a Banach algebra

Throughout, A denotes a normed algebra over $\mathbb{K}$. If A is unital, the neutral element w.r.t. multiplication is denoted by $\mathbb{1}$. Observe that $|\mathbb{1}| = |\mathbb{1}\cdot\mathbb{1}| \leq |\mathbb{1}|\cdot|\mathbb{1}|$ and so $1 \leq |\mathbb{1}|$ (but in general $1 < |\mathbb{1}|$ because if we replace $|\ .\ |$ by $2\cdot|\ .\ |$ we obtain again a unital Banach algebra over $\mathbb{K}$). Also recall that the rule $|xy| \leq |x||y|$ implies that multiplication is jointly continuous.

If A is unital, then A embeds into the operator algebra $\mathcal{B}(A)$ via

$$x \mapsto L_x,$$

where $L_x(y) = x\cdot y$. Moreover the rule $|xy| \leq |x||y|$ exactly says $|L_x| \leq |x|$ and we have

$$|L_x| = \sup\{|xy| \mid |y| \leq 1\} \geq |x\cdot\frac{\mathbb{1}}{|\mathbb{1}|}| = \frac{|x|}{|\mathbb{1}|}.$$

Thus

$$\frac{|x|}{|\mathbb{1}|} \leq |L_x| \leq |x|$$

for all $x \in A$. Hence the original norm on A and the norm induced by $\mathcal{B}(A)$ are equivalent. Further, observe that $|L_{\mathbb{1}}| = 1$ since $|L_{\mathbb{1}}(\mathbb{1})| = |\mathbb{1}|$.

It should also be mentioned that we can always adjoin a unit to A: Define $A_{\mathbb{1}} = A \oplus \mathbb{K}$ and extend multiplication so that A is a (two-sided) ideal of $A_{\mathbb{1}}$ (this indicates how to define multiplication: $(x \oplus \alpha)\cdot(y \oplus \beta) = (xy + \alpha y + \beta x \ \oplus \alpha\beta)$) and define $|x \oplus \alpha| = |x| + |\alpha|$. Then $0 \oplus 1$ is the unit of $A_{\mathbb{1}}$ and has norm 1. It is straightforward to check that $A_{\mathbb{1}}$ is a normed $\mathbb{K}$-algebra and A is isometrical isomorphic to the closed subspace $A \oplus 0$ of $A_{\mathbb{1}}$.

Now we look at quotients of A. By an **ideal of** A we mean a subset I of A with the property $I - I \subseteq I$ such that $ax, xb \in I$ for all $a, b \in A$, $x \in I$. Recall that for any ideal I, A/I is again a ring with multiplication $(x+I)\cdot(y+I) = xy + I$. Moreover, the closure of I in A is again an ideal of A since multiplication is continuous. However, I might not be proper if I is proper. An example is $A = \mathbb{K}[T]$ viewed as a normed subalgebra of $C([0,1], \mathbb{K})$ and $I = (1 - \frac{1}{2}T)$. I is proper but the closure of I contains $\mathbb{1}$: By Stone-Weierstrass, there is a sequence of polynomials (P_n) that converges to $\frac{1}{1-\frac{1}{2}T}$ (or choose directly P_n as the n^{th} partial sum of $\sum_k (\frac{T}{2})^k$). Hence $P_n\cdot(1 - \frac{1}{2}T) \in I$ converges to $\mathbb{1}$.

4.1. **Proposition.** *Let A be a normed $\mathbb{K}$-algebra and let I be a closed subspace and an ideal of A. Then the normed space A/I over $\mathbb{K}$ (as defined in 1.2(iv)) is again a normed $\mathbb{K}$-algebra. If A is unital with $|\mathbb{1}| = 1$ and I is proper, then also A/I is unital with unit $\mathbb{1} + I$ and $|\mathbb{1} + I| = 1$.*

Proof. We have to show that $|(x+I)(y+I)| \leq |x+I|\cdot|y+I|$:

$$\begin{aligned} |x+I|\cdot|y+I| &= (\inf_{a\in I}|x+a|)\cdot(\inf_{b\in I}|y+b|) = \inf_{a,b\in I}(|x+a|\cdot|y+b|) \geq \\ &\geq \inf_{a,b\in I}(|(x+a)\cdot(y+b)|) = \inf_{a,b\in I}|xy+\underbrace{ay+xb+ab}_{\in I}| \geq \\ &\geq \inf_{c\in I}|xy+c| = |xy+I| = |(x+I)(y+I)|. \end{aligned}$$

Hence A/I is a normed $\mathbb{K}$-algebra. If A is unital and $I \neq A$, then $|\mathbb{1}+I| \leq |\mathbb{1}| = 1$ by assumption. However $1 \leq |\mathbb{1}+I|$ is true anyway. □

A **unit** (or **invertible element**) of A is an element $u \in A$ that has a (necessarily unique) two-sided multiplicative inverse in A, denoted by u^{-1}. The set of units is denoted by $\mathcal{G}(A)$. Obviously, $\mathcal{G}(A)$ is a subgroup of the multiplicative monoid A. Note that $1 \leq |\mathbb{1}| = |u\cdot u^{-1}|$ implies $|u^{-1}| \geq |u|^{-1}$.

For the rest of this section A is a unital Banach algebra over $\mathbb{K}$

4.2. **Lemma.** *Every $x \in A$ with $|\mathbb{1}-x| < 1$ is invertible with inverse*

$$x^{-1} = \sum_{n=0}^{\infty} x^n.$$

Proof. We have $|x^n| \leq |x|^n$ and since A is a Banach space, $z = \sum_{n=0}^{\infty} x^n$ exists in A. We have $(\mathbb{1}-x)\cdot z = \mathbb{1}$, because multiplication is continuous and so

$$\mathbb{1} - x^{k+1} = (\mathbb{1}-x)\cdot\sum_{n=0}^{k} x^n$$

converges to $\mathbb{1}$. Similarly $z\cdot(\mathbb{1}-x) = \mathbb{1}$. □

4.3. **Proposition.** *$\mathcal{G}(A)$ is open in A and the map $\mathcal{G}(A) \longrightarrow \mathcal{G}(A)$ that sends x to x^{-1} is continuous.*

Proof. By 4.2 we have $\mathring{B}_1(\mathbb{1}) \subseteq \mathcal{G}(A)$. Now if $x \in \mathcal{G}(A)$, then multiplication with x to the left is an homeomorphism $l_x : A \longrightarrow A$ that fixes $\mathcal{G}(A)$ setwise. Hence $l_x(\mathring{B}_1(\mathbb{1}))$ is an open neighborhood of x in A, contained in $\mathcal{G}(A)$. Thus $\mathcal{G}(A)$ is open.

To see that $x \mapsto x^{-1}$ is continuous we fix $x_0 \in \mathcal{G}(A)$ and show that $i(x) := x^{-1}$ is continuous at x_0. Since $x^{-1} = (x_0^{-1}\cdot x)^{-1}\cdot x_0^{-1}$, we see that $i = r \circ i \circ l$, where l is multiplication to the left with x_0^{-1} and r is multiplication to the right with x_0^{-1}. As $i \circ l(x_0) = \mathbb{1}$ it suffices to show that i is continuous at $\mathbb{1}$: We have

$$|x^{-1} - \mathbb{1}| \leq |x^{-1}|\cdot|\mathbb{1}-x|,$$

so we only need to show that $|x^{-1}|$ is bounded for x in some neighborhood of $\mathbb{1}$. If $|\mathbb{1}-x| < \frac{1}{2}$, then

$$|x^{-1}| = |\sum_{n=0}^{\infty}(\mathbb{1}-x)^n| \leq \sum_{n=0}^{\infty}|(\mathbb{1}-x)^n| \leq \sum_{n=0}^{\infty}|\mathbb{1}-x|^n \leq 2$$

as required. □

Remark: If A is just a normed algebra, then $\mathcal{G}(A)$ is in general not open in A: an example is $\mathbb{K}[T]$ viewed as a normed subalgebra of $C([0,1],\mathbb{K})$). Clearly $\mathcal{G}(A) = \mathbb{K}\setminus\{0\}$, but $\mathbb{1} \in \mathcal{G}(A)$ is not in the interior of $\mathcal{G}(A)$, since the polynomial $p(x) = 1 - \frac{1}{n}T$ is not in $\mathcal{G}(A)$ and satisfies $|\mathbb{1} - p| = |\frac{1}{n}T| = \frac{1}{n}$.

4.4. **Corollary.** *If I is a proper ideal of A, then also the closure of I is a proper ideal of A.*

Proof. Since I is proper we have $I \subseteq A \setminus \mathcal{G}(A)$, which is closed by 4.3. Hence the closure of I is also contained in $A \setminus \mathcal{G}(A)$. □

4.5. **Definition.** The **spectrum of** $x \in A$ is defined as

$$\sigma_A(x) := \{\lambda \in \mathbb{K} \mid x - \lambda{\cdot}\mathbb{1} \notin \mathcal{G}(A)\} \subseteq \mathbb{K}.$$

Note that $\sigma_A(0) = \{0\}$. For matrix algebras, $\sigma_A(x)$ is the set of eigenvalues of A. If $A = C(X,\mathbb{K})$ for a compact space X, then for $f \in A$, $\lambda \in \sigma_A(f)$ if and only if $f - \lambda$ is not a unit in A, i.e. $f - \lambda$ has a zero in $\mathbb{K}$. Thus $\sigma_A(f)$ is the image of f

4.6. **Theorem.** *If A is a unital Banach algebra over $\mathbb{K}$ and $x \in A$, then $\sigma_A(x)$ is a compact subset of $\mathbb{K}$ contained in the ball of radius $|x|$ around 0.*

If $\mathbb{K} = \mathbb{C}$, then $\sigma_A(x)$ is non-empty. Observe that $\sigma_A(x)$ can be empty if $\mathbb{K} = \mathbb{R}$, e.g, if $x \in A = M_n(\mathbb{R})$ does not have (real) Eigenvalues.

Proof. If $|\lambda| > |x|$, then $|\mathbb{1} - (\mathbb{1} - \frac{x}{\lambda})| < 1$, hence $\mathbb{1} - \frac{x}{\lambda} \in \mathcal{G}(A)$ and $x - \lambda\mathbb{1} = -\lambda{\cdot}(\mathbb{1} - \frac{x}{\lambda}) \in \mathcal{G}(A)$, too. Hence $\sigma_A(x)$ is contained in the ball of radius $|x|$ around 0.

To show that $\sigma_A(x)$ is compact it remains to show that $\mathbb{K} \setminus \sigma_A(x)$ is open: We have $\lambda \in \mathbb{K} \setminus \sigma_A(x) \iff x - \lambda{\cdot}\mathbb{1} \in \mathcal{G}(A)$ and we know from 4.2 that $\mathcal{G}(A)$ is open in A. Now the function $\Lambda : \mathbb{K} \longrightarrow A$; $\Lambda(\lambda) = x - \lambda{\cdot}\mathbb{1}$ is continuous, and so

$$\mathbb{K} \setminus \sigma_A(x) = \Lambda^{-1}(\mathcal{G}(A))$$

is open.

It remains to show that $\sigma_A(x)$ is non-empty, provided $\mathcal{K} = \mathbb{C}$: Otherwise consider the function $f : \mathbb{C} \longrightarrow A$,

$$f(\lambda) := (x - \lambda{\cdot}\mathbb{1})^{-1}.$$

Claim. f is differentiable with derivative

$$f'(\lambda) = (x - \lambda\mathbb{1})^{-2}$$

(to be precise, $f'(\lambda)$ is the element of $L(\mathbb{C}, A)$ that maps 1 to $(x - \lambda\mathbb{1})^{-2}$).

Proof. For $h \in \mathbb{C}$, $h \neq 0$ we have

$$\begin{aligned}
\frac{f(\lambda+h) - f(\lambda+h)}{h} &= \frac{(x - (\lambda+h){\cdot}\mathbb{1})^{-1} - (x - \lambda{\cdot}\mathbb{1})^{-1}}{h} = \\
\frac{(x - (\lambda+h){\cdot}\mathbb{1})^{-1}{\cdot}\,(x - \lambda{\cdot}\mathbb{1} - (x - (\lambda+h){\cdot}\mathbb{1}))\,{\cdot}(x - \lambda{\cdot}\mathbb{1})^{-1}}{h} &= \\
\frac{(x - (\lambda+h){\cdot}\mathbb{1})^{-1}{\cdot}h{\cdot}(x - \lambda{\cdot}\mathbb{1})^{-1}}{h} &= \\
(x - (\lambda+h){\cdot}\mathbb{1})^{-1}{\cdot}(x - \lambda{\cdot}\mathbb{1})^{-1} &\to (x - \lambda\mathbb{1})^{-2}
\end{aligned}$$

when $h \to 0$. □

From the claim we get a contradiction as follows: Pick $\varphi \in A'$. Then also $g := \varphi \circ f : \mathbb{C} \longrightarrow \mathbb{C}$ is differentiable, hence an entire function. Now

$$(*) \qquad f(\lambda) = (x - \lambda{\cdot}\mathbb{1})^{-1} = (-\lambda)^{-1}(\mathbb{1} - \lambda^{-1}x)^{-1} = (-\lambda)^{-1}{\cdot}\sum(\lambda^{-1}x)^n \to 0$$

as $|\lambda| \to \infty$. It follows that f is bounded and therefore also g is bounded. By Liouville's theorem, g is constant. But $(*)$ then shows that $g = 0$.

So we have shown that $\varphi \circ f = 0$ for all $\varphi \in A'$ which implies that $f(\lambda) = (x - \lambda{\cdot}\mathbb{1})^{-1}$ is 0, a contradiction. □

4.7. **Corollary.** *If A is a unital Banach algebra over $\mathbb{K}$, then for every nonzero $\mathbb{K}$-algebra homomorphism $\varphi : A \longrightarrow \mathbb{K}$ we have*

(i) $\varphi(\mathbb{1}) = 1$.
(ii) $\varphi(x) \in \sigma_A(x)$ *for all* $x \in A$.
(iii) φ *is bounded of norm* ≤ 1.

Proof. (i) As φ is nonzero there is some $x \in A$ with $\varphi(x) \neq 0$. Then $\varphi(x) = \varphi(x{\cdot}\mathbb{1}) = \varphi(x){\cdot}\varphi(\mathbb{1})$, thus $\varphi(\mathbb{1}) = 1$.
(ii) Let $y := x - \varphi(x){\cdot}\mathbb{1}$. Clearly $\varphi(y) = 0$. Hence if y were invertible then

$$1 = \varphi(\mathbb{1}) = \varphi(y){\cdot}\varphi(y^{-1}) = 0,$$

a contradiction. This shows (ii).
(iii) By 4.6, $\sigma_A(x)$ is contained in the ball of radius $|x|$ around 0. From (ii) we get $|\varphi| \leq 1$. □

The proof shows that actually only item (iii) of 4.7 is a consequence of 4.6. However 4.7 is true for $\mathbb{K} = \mathbb{C}$ and for $\mathbb{K} = \mathbb{R}$. In both cases, 4.7 and additional assumptions on A will allow us to construct non-zero ring homomorphisms $A \longrightarrow \mathbb{K}$ and therefore also elements in $\sigma_A(x)$ for all $x \in A$. Commutativity is one such assumption and methods from commutative algebra can be used conveniently. Commutativity is not enough though (in the real case), since not every unital commutative Banach algebra A over $\mathbb{R}$ possesses a non-trivial ring homomorphism $A \longrightarrow \mathbb{R}$: think of $A = \mathbb{C}$.

Another consequence of 4.6 is:

4.8**. Gel'fand-Mazur I**
If A is a complex, unital Banach algebra and every nonzero element of A is invertible, then $A = \mathbb{C}$.

Proof. If $x \in A$ then as $\sigma_A(x) \neq \emptyset$, we know that $x - \lambda\mathbb{1}$ is not invertible in A. By assumption, $x - \lambda\mathbb{1} = 0$, i.e. $x = \lambda{\cdot}\mathbb{1}$. □

5. The spectrum of a Banach algebra

A **character of a unital Banach** A (or **multiplicative functional**) over $\mathbb{K}$ is a non-zero $\mathbb{K}$-algebra homomorphism $A \longrightarrow \mathbb{K}$. (Note that in the case $\mathbb{K} = \mathbb{R}$, every (nontrivial) ringhomomorphism $\varphi : A \longrightarrow \mathbb{R}$ is automatically an $\mathbb{R}$-algebra homomorphism, since $\varphi(\mathbb{1}) = 1$ as in the proof of 4.7(i) and from this one sees that φ is the identity on $\mathbb{Q}\cdot\mathbb{1}$. Now it just remains to show that φ is continuous on $\mathbb{R}\cdot\mathbb{1}$. This follows essentially from the fact that φ maps positive elements of $\mathbb{R}\cdot\mathbb{1}$ to positive elements of $\mathbb{R}$)

5.1. **Definition.** The **spectrum of a unital Banach** algebra A is defined as

$$\mathrm{Sp}(A) = \{\text{all characters of A}\}.$$

Recall from 4.7, that $\mathrm{Sp}(A) \subseteq A'$.

5.2. **Proposition.** $\mathrm{Sp}(A)$ *is a weak***-compact subset of* A'.

Proof. By 4.7 we know that, that $\mathrm{Sp}(A)$ is contained in the (norm-)unit Ball of V' and by Banach-Alaoglu, this ball is weak*-compact. Hence it is enough to show that $\mathrm{Sp}(A)$ is weak*-closed. This goes exactly as in the proof of 3.10, where we showed that the set W of linear maps $A \longrightarrow \mathbb{K}$ is closed in $\mathbb{K}^A$; Take $f \in \mathbb{K}^A$ in the closure of $\mathrm{Sp}(A)$. By Banach-Alaoglu we already know that $f \in V'$ is of norm ≤ 1. We must show that f is multiplicative (we know already that $f \neq 0$, since $\varphi(\mathbb{1}) = \mathbb{1}$ for all $\varphi \in \mathrm{Sp}(A)$). Take $x, y \in A$ and let $\varepsilon > 0$. By definition of the topology of $\mathbb{K}^V$, there is some $\varphi \in A$ such that in the coordinates x, y and $x + y$ we have

$$|\varphi(x) - f(x)| < \varepsilon,\ |\varphi(y) - f(y)| < \varepsilon \text{ and } |\varphi(x\cdot y) - f(x\cdot y)| < \varepsilon.$$

Hence

$$\begin{aligned}
|f(xy) - f(x)f(y)| &= |f(xy) - \varphi(xy) + \varphi(x)\varphi(y) - f(x)f(y)| = \\
&= |f(xy) - \varphi(xy) + \varphi(x)\varphi(y) - \varphi(x)f(y) + \varphi(x)f(y) - f(x)f(y)| \leq \\
&\leq |f(xy) - \varphi(xy)| + |\varphi(x)|\cdot|\varphi(y) - f(y)| + |f(y)|\cdot|\varphi(x) - f(x)| < \\
&\overset{\text{as } |\varphi|,|f|\leq 1}{<} \varepsilon + |x|\cdot\varepsilon + |y|\cdot\varepsilon \to 0 \text{ as } \varepsilon \to 0.
\end{aligned}$$

Thus $f(xy) = f(x)f(y)$ as desired. $\square$

We now define the **Gel'fand transform** for unital Banach algebras as the composition of the maps

$$A \xrightarrow{\varepsilon} \varepsilon(A) \hookrightarrow C((A', \tau^*), \mathbb{K}) \xrightarrow{\text{restriction}} C((\mathrm{Sp}(A), \tau^*), \mathbb{K})$$

which is explicitly written as $x \mapsto \hat{x}$, where $\hat{x} : \mathrm{Sp}(A) \longrightarrow \mathbb{K}$ is defined as

$$\hat{x}(\varphi) = \varphi(x)$$

for any character of φ of A. Observe that

$$\widehat{x\cdot y} = \hat{x}\cdot\hat{y} \text{ for all } x, y \in A$$

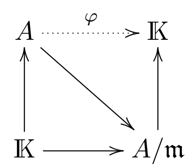
and

$$|\hat{x}| \leq |x| \text{ for all } x \in A.$$

(Recall the discussion before 3.11). Hence $\hat{}: A \longrightarrow C(\operatorname{Sp} A, \mathbb{C})$ is a continuous algebra homomorphism.

Recall that a **maximal ideal** of a ring A is a proper ideal of A which is not properly contained in any other proper ideal of A. If A is unital, then every proper ideal I of A is contained in a maximal ideal of A: By Zorn, there are maximal elements $\mathfrak{m}$ in the set of all ideals J of A with $I \subseteq J \not\ni \mathbb{1}$.

5.3. **Observation.** *If A is a unital Banach-algebra over $\mathbb{K}$ and $\varphi : A \longrightarrow \mathbb{K}$ is a character, then $\operatorname{Ker}\varphi$ is a maximal ideal of A and φ can be reconstructed from $\operatorname{Ker}\varphi$ via the commutative diagram*



because φ is the unique $\mathbb{K}$-algebra homomorphism which makes this diagram commutative.

5.4. *Example.* For $n \geq 2$, the only proper two-sided ideal of $M_n(\mathbb{K})$ is the zero ideal. (Recall that $M_n(\mathbb{C})$ is a C^*-algebra). In particular, $\operatorname{Sp}(M_n(\mathbb{K})) = \emptyset$.

Proof. This of course works over all fields. Let $P \in M_n(\mathbb{K})$ and suppose the ij-th entry of P is nonzero, call it α. For all $k \in \{1, ..., n\}$ we can swap rows and columns of P and find $E_k, F_k \in M_n(\mathbb{K})$ such that $E_k P F_k$ has the element α in its k-th diagonal position. Then let $T_k \in M_n(\mathbb{K})$ be the matrix which has exactly one non zero entry namely $\frac{1}{\alpha}$ in the k-th diagonal position. Then

$$\mathbb{1} = \sum_{k=1}^{n} E_k P F_k T_k$$

is in the (two-sided) ideal generated by P. □

This is the point were commutativity comes into the place:

If A is a commutative ring and 0 is the only non-trivial ideal of A, then every non-zero element of A is invertible (so A is a field).

Let $\operatorname{Max} A$ denote the set of maximal ideals of an arbitrary ring A.

5.5. **Gel'fand-Mazur II**

If A is a complex unital <u>commutative</u> Banach-algebra, then the map

$$\begin{aligned} \operatorname{Ker} : \operatorname{Sp}(A) &\longrightarrow \operatorname{Max}(A) \\ \varphi &\longmapsto \operatorname{Ker}(\varphi) \end{aligned}$$

is bijective. Since $\operatorname{Max} A \neq \emptyset$, it follows that $\operatorname{Sp}(A) \neq \emptyset$.

Proof. We construct the inverse: Let $\mathfrak{m}$ be a maximal ideal of A. Then $\mathfrak{m}$ is a proper ideal and so also the closure of $\mathfrak{m}$ in A is is a proper ideal by 4.4. By 4.1, $A/\mathfrak{m}$ is again a unital Banach-algebra, which clearly inherits commutativity. Moreover, the only ideal of $A/\mathfrak{m}$ is the zero ideal (it is straightforward to see that the preimage of any ideal of $A/\mathfrak{m}$ under $A \twoheadrightarrow A/\mathfrak{m}$ is again an ideal). Now by commutativity, $A/\mathfrak{m}$ is a field. However, we want that $A/\mathfrak{m}$ is isomorphic to $\mathbb{C}$: By Gel'fand-Mazur I (see 4.8, this is not available in the real case), we know this, or better: the map $\mathbb{C} \longrightarrow A/\mathfrak{m}; \lambda \mapsto \lambda{\cdot}\mathbb{1}$ is an isomorphism. Hence we obtain a character $\varphi : A \longrightarrow A/\mathfrak{m} \longrightarrow \mathbb{C}$.

This character is the unique (by the isomorphism theorem from commutative algebra) preimage of $\mathfrak{m}$ under our map Ker. □

If $\mathbb{K} = \mathbb{R}$, then 5.5 fails: For example if $A = \mathbb{C}$: Then $\operatorname{Sp}(A) = \emptyset$, but $\operatorname{Max} A = \{(0)\}$.

5.6. **Corollary.** *If A is a complex unital commutative Banach-algebra, then the Gel'fand transform of $x \in A$ is a continuous and surjective map*

$$\hat{x} : \operatorname{Sp} A \twoheadrightarrow \sigma_A(x).$$

It follows that

$$|\hat{x}| = \sup\{|\hat{x}(\varphi)| \mid \varphi \in \operatorname{Sp}(A)\} = \sup\{|\lambda| \mid \lambda \in \sigma_A(x)\}.$$

The number on the right hand side is called the spectral radius of x and we will see shortly that we have to compute this number.

Proof. We already know that $\hat{x}$ is continuous. Let $\lambda \in \sigma_A(x)$, i.e. $x - \lambda{\cdot}\mathbb{1}$ is not invertible. Since A is commutative, $x - \lambda{\cdot}\mathbb{1}$ is contained in a maximal ideal of A. The corresponding character $\varphi \in \operatorname{Sp}(A)$ then annihilates $x - \lambda{\cdot}\mathbb{1}$, in other words $\hat{x}(\varphi) = \varphi(x) = \lambda$. □

We turn to C^*-algebras (see 2.2) and show that the Gel'fand transform in the commutative case is a C^*-algebra isomorphism $\hat{\ } : A \longrightarrow C(\operatorname{Sp} A, \mathbb{C})$.

5.7. **Definition.** An element x of a C^*-algebra A is called **self-adjoint** (or **hermitian**), if $x^* = x$.

For example $x^*{\cdot}x$ is self-adjoint for all $x \in A$. The importance of self-adjoint elements comes from the observation that every $x \in A$ can be uniquely written in the form $a + i{\cdot}b$, where $a, b \in A$ are self-adjoint (a, b are called the **real and the imaginary part** of x). Explicitly, take

$$a = \frac{1}{2}(x + x^*) \text{ and } b = \frac{1}{2i}(x - x^*).$$

Now if $x = h + ik$ with self adjoint $h, k \in A$ then $x^* = h - ik$ so that $h = \frac{1}{2}(x + x^*)$ and $k = \frac{1}{2i}(x - x^*)$.

5.8. **Proposition.** *If A is a C^*-algebra and $h \in A$ is self-adjoint, then*

$$\sigma_A(x) \subseteq \mathbb{R}.$$

This generalises the linear algebra statement "hermitian matrices have real Eigenvalues"

Proof. Since h is self-adjoint, the $\mathbb{C}$-subalgebra $\mathbb{C}[h]$ generated by h in A is a $*$-closed subalgebra and it is clear that $\mathbb{C}[h]$ is commutative.

Let B be the closure of $\mathbb{C}[h]$ in A. Since multiplication and the involution are continuous, B is a commutative C^*-subalgebra of A (and so B is also a C^*-algebra). By definition we have $\sigma_A(x) \subseteq \sigma_B(x)$ and therefore it suffices to prove that $\sigma_B(x) \subseteq \mathbb{R}$. We may therefore replace A by B and assume that A is commutative.

Remark: This reduction to the commutative case would also work if h were only **normal**, i.e. $h^*h = hh^*$. The self-adjointness property of h will be used in a crucial way below.

Since A is commutative we know from 5.6 that $\hat{h} : \operatorname{Sp} A \twoheadrightarrow \sigma_A(x)$ is surjective and therefore it suffices to show that $\varphi(h) \in \mathbb{R}$ for every character φ of A. We do this by showing

$$|\exp(i \cdot t \cdot \varphi(h))| \leq 1 \text{ for all } t \in \mathbb{R}.$$

Fix $t \in \mathbb{R}$ and write $u = e^{i \cdot t \cdot h}$ defined as

$$u = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} h^n.$$

Since φ is continuous and a $\mathbb{C}$-algebra homomorphism we have

$$\varphi(u) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \varphi(h)^n = \exp(i \cdot t \cdot \varphi(h)).$$

Since $\varphi \in \operatorname{Sp} A$ has norm ≤ 1 it suffices to show that

$$|u| \leq 1.$$

Here we will use that h is self-adjoint and that A satisfies $|x^*x| = |x|^2$ for all x:

Firstly we have

$$u^* = \left(\sum_{n=0}^{\infty} \frac{(it)^n}{n!} h^n\right)^* \overset{\text{as } * \text{ is continuous}}{=} \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} (h^*)^n \overset{\text{as } h \text{ is self-adjoint}}{=}$$
$$= \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} h^n.$$

This shows $u^* \cdot u = \mathbb{1}$ (exercise!). Now by the C^*-rule in C^*-algebras we get $|\mathbb{1}| = |u^* \cdot u| = |u|^2$ and therefore $|u| = 1$ as required. □

5.9. **Corollary.** *Every character of a C^*-algebra respects the involution.*

Proof. Let $\varphi : A \longrightarrow \mathbb{C}$ be our character. Write $x = a + ib$ with self-adjoint elements $a, b \in A$. Then

$$\varphi(x^*) = \varphi(a - ib) = \varphi(a) - i\varphi(b) = \overline{\varphi(x)}$$

since $\varphi(a), \varphi(b) \in \mathbb{R}$ by 5.8 (and 4.7). □

6. Gel'fand-Naimark: Representation of commutative C^*-algebras

Here is another application of the commutative case to arbitrary complex, unital Banach algebras:

6.0. **Observation.** *Let A be a unital Banach algebra over $\mathbb{K}$ and let C be a (normed) $\mathbb{K}$-subalgebra of A.*

(i) If $\mathcal{G}(A) \cap C \subseteq \mathcal{G}(C)$, then $\sigma_C(x) = \sigma_A(x)$ for all $x \in C$.

For $u \in \mathcal{G}(A) \cap C$ we have the following.

(ii) If C is commutative then $C[u^{-1}]$ is again commutative.

(iii) If $* : A \longrightarrow A$ is an involution (so here $\mathbb{K} = \mathbb{C}$) and C is commutative and closed under this involution, then also $C[u^{-1}, (u^{-1})^*]$ is commutative and closed under the involution.

Proof. (i). It is trivial that $\sigma_A(x) \subseteq \sigma_C(x)$. Conversely, if $\lambda \in \mathbb{K} \setminus \sigma_A(x)$, then $u := x - \lambda \cdot \mathbb{1} \in C$ is invertible in A. So by assumption, u is invertible in C and therefore $\lambda \notin \sigma_C(x)$.

(ii). As $u \in C$ commutes with all elements y of C, also u^{-1} commutes with every element y of C:

$$uy = yu \Rightarrow yu^{-1} = u^{-1}uyu^{-1} = u^{-1}yuu^{-1} = u^{-1}y.$$

Consequently, $C[u^{-1}]$ is commutative.

(iii) Since $(u^{-1})^* = (u^*)^{-1}$ and $u^* \in C$ we can apply (i) for $C[u^{-1}]$. Hence $C[u^{-1}, (u^{-1})^*]$ is commutative and it is clear that this ring is closed under the involution. □

6.1. **Corollary.** *Let A be a unital Banach algebra over $\mathbb{K}$ and let C be a commutative (normed) $\mathbb{K}$-subalgebra of A. Then*

(i) C is contained in a maximal commutative (normed) $\mathbb{K}$-subalgebra B of A and each such algebra B is a unital Banach algebra with $\mathcal{G}(A) \cap B = \mathcal{G}(B)$.

(ii) If A is a unital C^*-algebra and C is closed under $*$, then C is contained in a maximal commutative $\mathbb{C}$-subalgebra B of A closed under $*$ and each such algebra B is a unital C^*-subalgebra with $\mathcal{G}(A) \cap B = \mathcal{G}(B)$.

In both situations we have $\sigma_B(x) = \sigma_A(x)$ for all $x \in B$.

Proof. (i) By Zorn, it is clear that B exists. Since B is commutative, also the closure of B in A is commutative, hence by maximality, B is already closed.

By 6.0(i), the maximality of B implies $\mathcal{G}(A) \cap B \subseteq \mathcal{G}(B)$.

(ii) As in (i) it is again clear that B is a commutative $\mathbb{C}^*$-subalgebra of A. This time, the maximality of B and 6.0(ii) imply that $\mathcal{G}(A) \cap B \subseteq \mathcal{G}(B)$.

So in both cases we have $\mathcal{G}(A) \cap B = \mathcal{G}(B)$. Hence by 6.0(iii), we have $\sigma_B(x) = \sigma_A(x)$ for all $x \in B$. □

6.2. **Corollary.** *Let A be a complex, unital Banach algebra and let $x \in A$.*

(i) If x is invertible, then $\sigma_A(x^{-1}) = \sigma_A(x)^{-1}$.

(ii) For every polynomial $P \in \mathbb{C}[T]$ we have

$$P(\sigma_A(x)) = \sigma_A(P(x)).$$

Proof. (i) If x is invertible, then for $\lambda \in \mathbb{C}$ we have

$$\lambda \in \sigma_A(x^{-1}) \iff x^{-1} - \lambda \cdot \mathbb{1} \notin \mathcal{G}(A) \iff -\lambda^{-1} \cdot \mathbb{1} + x \notin \mathcal{G}(A) \iff \lambda^{-1} \in \sigma_A(x)$$

as required.

(ii). By 6.1 applied to the commutative $\mathbb{C}$-algebra $\mathbb{C}[x]$ we may assume that A is commutative. In this case, we know from 5.6, that $\sigma_A(y) = \hat{y}(\mathrm{Sp}(A))$ for all $y \in A$. Hence

$$\begin{aligned} P(\sigma_A(x)) = P(\hat{x}(\mathrm{Sp}(A))) = P(\hat{x})(\mathrm{Sp}(A)) &\overset{\text{as } \hat{} \text{ is a } \mathbb{C}\text{-algebra hom.}}{=} \\ = \widehat{P(x)}(\mathrm{Sp}(A)) = \sigma_A(P(x)). \end{aligned}$$

□

Let A be a unital, complex Banach algebra and let $x \in A$. The **spectral radius** $r(x)$ of x is the supremum (maximum) of all "eigenvalues" of x:

$$r_A(x) = \sup\{|\lambda| \mid \lambda \in \sigma_A(x)\}.$$

We have seen in 5.6, that for commutative A, we actually have $r_A(x) = |\hat{x}|$, hence $r(x)$ is the norm of the Gel'fand transform of x. In general, the spectral radius can be computed as follows:

6.3**. Spectral radius formula** Let A be a unital, complex Banach algebra and let $x \in A$. Then

$$r_A(x) = \lim_{n\to\infty} |x^n|^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} |x^n|^{\frac{1}{n}}$$

Observe that the right hand side here does not depend on A, in the sense that it does not change if we replace A by a larger Banach algebra. Whereas the $\sigma_A(x)$ will change in general under this replacement. The formula implies that the spectral radius does not change.

Proof. Write

$$\gamma = \inf_{n\in\mathbb{N}} |x^n|^{\frac{1}{n}}.$$

We first show that

$$\lim_{n\to\infty} |x^n|^{\frac{1}{n}} = \gamma.$$

Take $\varepsilon > 0$ and fix $k \in \mathbb{N}$ with $|x^k|^{\frac{1}{k}} < \gamma + \varepsilon$. For $n \in \mathbb{N}$, write

$$n = q_n \cdot k + r_n$$

with $q_n \in \mathbb{N}_0$ and $0 \le r_n < k$. Then

$$|x^n|^{\frac{1}{n}} = |(x^k)^{q_n} \cdot x^{r_n}|^{\frac{1}{n}} \le |x^k|^{\frac{q_n}{n}} \cdot |x|^{\frac{r_n}{n}}.$$

Now $\frac{q_n}{n} = \frac{q_n}{q_n k + r_n} \to \frac{1}{k}$ as $n \to \infty$ and so $|x^k|^{\frac{q_n}{n}} \cdot |x|^{\frac{r_n}{n}} \to |x^k|^{\frac{1}{k}}$ as $n \to \infty$. It follows that $|x^n|^{\frac{1}{n}} < \gamma + \varepsilon$ for all but finitely many n.

This shows $\lim_{n\to\infty} |x^n|^{\frac{1}{n}} = \gamma$.

Now we show that $r_A(x) \le \gamma$. By 6.2, we know that $r_A(x)^n = r_A(x^n)$ and by 4.6 we know that $r_A(x^n) \le |x^n|$. Thus

$$r_A(x) \le |x^n|^{\frac{1}{n}} \text{ for all } n \in \mathbb{N}$$

and so $r_A(x) \le \gamma$.

Finally we show $r_A(x) \ge \gamma$.

Claim. For every $\varphi \in A'$ and all $\lambda \in \mathbb{C}$ with $|\lambda| > r_A(x)$, the sequence

$$(\varphi(\frac{x^n}{\lambda^{n+1}}))_{n\in\mathbb{N}} \subseteq \mathbb{C} \text{ is bounded.}$$

To see this, define

$$\begin{aligned} g : \mathbb{C} \setminus \sigma_A(x) &\longrightarrow \mathbb{C} \\ \lambda &\longmapsto \varphi((x - \lambda\cdot\mathbb{1})^{-1}). \end{aligned}$$

g is differentiable and so holomorphic (the proof is identical to the one given in 4.6).

On the other hand for all $\lambda > |x|$, we know that

$$(*) \qquad g(\lambda) = \sum_{n=0}^{\infty} \frac{\varphi(x^n)}{\lambda^{n+1}}$$

From function theory (see [Knopp, §24, Satz 1] or [Werner2, Theorem II 3.3]) we then know that the series in $(*)$ is also convergent for all $\lambda \in \mathbb{C}$ with $|\lambda| > r_A(x)$. This proves the claim.

Now fix $\lambda \in \mathbb{R}$ with $\lambda > r_A(x)$ and let $F \subseteq A''$ be the set of all evaluation maps given by $\frac{x^n}{\lambda^{n+1}}$, $n \in \mathbb{N}$. The claim says that $F(\varphi)$ is a bounded subset of $\mathbb{C}$ for every $\varphi \in A'$. By Banach-Steinhaus (see 3.8) we then know that $F \subseteq L(A', \mathbb{C}) = A''$ is bounded. Since evaluation is an isometric embedding, it follows that also $(\frac{x^n}{\lambda^{n+1}})_{n\in\mathbb{N}}$, is bounded. Pick $M > 0$ with

$$|\frac{x^n}{\lambda^{n+1}}| \leq M \text{ for all } n \in \mathbb{N}.$$

It follows $|x^n| \leq M\cdot\lambda\cdot\lambda^n$ and so

$$|x^n|^{\frac{1}{n}} \leq (M\cdot\lambda)^{\frac{1}{n}}\cdot\lambda \text{ for all } n.$$

Hence

$$\gamma = \lim_{n\to\infty} |x^n|^{\frac{1}{n}} \leq \lambda$$

for all $\lambda \in \mathbb{R}$ with $\lambda > r_A(x)$ as desired. □

6.4**. Gel'fand-Naimark**

Let A be a unital commutative C^*-algebra. Then the Gel'fand transform

$$\hat{} : A \longrightarrow C(\operatorname{Sp} A, \mathbb{C})$$

is an isometric $*$-isomorphism.

Proof. We first show that $\widehat{x^*} = (\hat{x})^*$ for every $x \in A$. Take $\varphi \in \operatorname{Sp}(A)$. Then

$$\widehat{x^*}(\varphi) = \varphi(x^*) \overset{\text{by } 5.9}{=} \overline{\varphi(x)} = \overline{\hat{x}(\varphi)} = (\hat{x})^*(\varphi).$$

Suppose we know already that $\hat{}$ is isometrical. Then it is of course injective and an isomorphism onto a closed C^* subalgebra of $C(\operatorname{Sp}(A))$. Since the image separates points of $\operatorname{Sp}(A)$ by definition, we can use the complex version of the Stone-Weierstrass theorem (see [Lang, III,§2, Theorem 1.4]; observe that we need that the image of $\hat{}$ is closed under conjugation, which is assured by what we have shown above) to see that the image is indeed all of $C(\operatorname{Sp}(A))$.

Hence it remains to show that $|\hat{x}| = |x|$ for all $x \in A$. Suppose we knew this for self-adjoint elements. Then

$$\begin{aligned}|\hat{x}|^2 &= \text{by definition} = |\bar{\hat{x}}\hat{x}| = \text{as } \hat{} \text{ preserves the involution}\\ &= |\widehat{x^*}\hat{x}| = \text{as } \hat{} \text{ preserves multiplication}\\ &= |\widehat{x^*x}| = \text{as } x^*x \text{ is self-adjoint and we are assuming this}\\ &= |x^*x| = |x|^2 \text{ by the } C^*\text{-algebra rule}\end{aligned}$$

and therefore $|\hat{x}| = |x|$.

So the only thing that remains to show is

$$|\hat{h}| = |h| \text{ for every self-adjoint } h \in A.$$

By 5.6 we know $r_A(h) = |\hat{r}|$. By the C^*-property we know $|h^2| = |h|^2$ and so by induction $|h^{2^n}| = |h|^{2^n}$ for all $n \in \mathbb{N}$. But now the formula for the spectral radius tells us

$$r_A(h) = \lim_{n\to\infty} |h^n|^{\frac{1}{n}},$$

and so indeed $r_A(h) = |h|$. □

6.5. **Corollary.** *The norm of a unital (not necessarily commutative) C^*-algebra A is uniquely determined by the $\mathbb{C}$-algebra and the $*$-algebra structure of A. For $x \in A$ we have*

$$|x| = \sqrt{\sup\{|\lambda| \mid x^*x - \lambda{\cdot}\mathbb{1} \textit{ is not invertible in } A\}}$$

Proof. Since $|x|^2 = |x^*x|$ it suffices to show that

$$|x^*x| = \sup\{|\lambda| \mid x^*x - \lambda{\cdot}\mathbb{1} \text{ is not invertible in } A\}$$

Let B be a maximal commutative C^*-subalgebra of A containing x^*x (not that $\mathbb{C}[x^*x]$ is closed under $*$, since x^*x is self-adjoint). By 6.1, B is a (commutative) C^*-subalgebra of A and so by 6.4,

$$|x^*x| = |\widehat{x^*x}|_{\mathrm{Sp}(B)} \stackrel{5.6}{=} \sup\{|\lambda| \mid \lambda \in \sigma_B(x^*x)\}.$$

By 6.1, $\sigma_B(x^*x) = \sigma_A(x^*x)$, and so the result follows. □

Later we will see that also the involution of a C^*-algebra is uniquely determined.

6.6. **Observation.** *Let A be a C^*-algebra generated by a single element x of A (hence A is the closure of the $*$-subalgebra $\mathbb{C}[x, x^*]$ generated by x and x^*). Then the Gel'fand transform $\hat{x} : \mathrm{Sp}(A) \longrightarrow \sigma_A(x)$ is injective.*

If x is normal (i.e. $x^*x = xx^*$), then $\hat{x} : \mathrm{Sp}(A) \longrightarrow \sigma_A(x)$ is an homeomorphism.

Proof. This follows from 5.9, which says that every character of A respects the involution. Hence if $\varphi, \psi \in \mathrm{Sp}(A)$ with $\hat{x}(\varphi) = \hat{x}(\psi)$, then $\varphi(x) = \psi(x)$, so $\varphi(x^*) = \psi(x^*)$, which means that φ and ψ agree on $\mathbb{C}[x, x*]$. Since $\mathbb{C}[x, x*]$ is dense in A, φ must be equal to ψ.

If x is normal, then $\mathbb{C}[x, x^*]$ is commutative, and therefore also A is commutative. By 5.6 we then know that $\hat{x}$ is bijective. Since $\mathrm{Sp}(A)$ is compact (by 5.2) and $\sigma_A(x) \subseteq \mathbb{C}$ is Hausdorff we get the assertion. □

6.7. **Corollary.** *Let A be a C^*-algebra generated by a single normal element x of A Then the map*

$$\begin{array}{rcl} \Phi : A & \longrightarrow & C(\sigma_A(x), \mathbb{C}) \\ a & \longmapsto & \hat{a} \circ \hat{x}^{-1} \end{array}$$

is an isometric $*$-isomorphism which maps x to the inclusion $\sigma_A(x) \hookrightarrow \mathbb{C}$.

Proof. Since x is normal, A is commutative and our map is the composition of the Gel'fand transform

$$\hat{} : A \longrightarrow C(\operatorname{Sp} A, \mathbb{C})$$

with the isometric $*$-isomorphism

$$\begin{array}{rcl} C(\operatorname{Sp} A, \mathbb{C}) & \longrightarrow & C(\sigma_A(x), \mathbb{C}) \\ f & \longmapsto & f \circ \hat{x}^{-1} \end{array}$$

induced by the homeomorphism $\hat{x}^{-1} : \sigma_A(x) \longrightarrow \operatorname{Sp} A$ (see 6.6). □

6.8. **Proposition.** *Let A be a C^*-algebra and let B be a C^*-subalgebra of A. Then $\mathcal{G}(A) \cap B = \mathcal{G}(B)$. In particular $\sigma_B(x) = \sigma_A(x)$ for all $x \in B$ (see 6.0(i)).*

Proof. It suffices to show for $u \in \mathcal{G}(A)$ that

$$u^{-1} \in \overline{\mathbb{C}[u, u^*]}.$$

We first assume that $u = u^*$ is self-adjoint. Then $C := \overline{\mathbb{C}[u]}$ and $B := \overline{\mathbb{C}[u, u^{-1}]}$ are commutative C^*-subalgebras of A. We obtain a commutative diagram of isometric C^*-embeddings

$$\begin{array}{ccc} B & \xrightarrow{\cong} & C(\operatorname{Sp}(B)) \\ \uparrow & & \uparrow e \\ C & \xrightarrow{\cong} & C(\operatorname{Sp}(C)) \end{array}$$

and in order to confirm $C = B$ is suffices to see that $C(\operatorname{Sp}(C))$ separates points of $C(\operatorname{Sp}(B))$ (then use the complex Stone-Weierstrass): Note that e is given by $e(f)(\varphi) = f(\varphi \restriction C)$ for $\varphi \in \operatorname{Sp}(B), f \in C(\operatorname{Sp}(C))$. Then, if $\varphi \neq \psi$ are form $\operatorname{Sp}(B)$ we must have $\varphi(u) \neq \psi(u)$ (otherwise $\varphi(u^{-1}) = \psi(u^{-1})$ and then $\varphi = \psi$). Thus $e(\hat{u})(\varphi) = \hat{u}(\varphi \restriction C) = \varphi(u) \neq \psi(u) = e(\hat{u})(\psi)$. Thus $e(\hat{u})$ separates φ from ψ.

This shows the proposition in the case when u is self-adjoint. In general, we then know that $(u^*u)^{-1} \in \mathbb{C}[u^*u] \subseteq \mathbb{C}[u, u^*]$ and so

$$u^{-1} = (u^*u)^{-1} \cdot u^* \in \mathbb{C}[u, u^*].$$

□

7. Positivity in C^*-algebras

7.1. **Definition.** An element x of a C^*-algebra A is called **positive** if $x = h^2$ for some self-adjoint $h \in A$. We write $x \leq y$ if $y - x$ is positive.

In the case $A = C(X)$ we see that $f \in A$ is positive if and only if f is real valued (i.e. f is self-adjoint) and $f \geq 0$ everywhere.

Example. If $A = M_2(\mathbb{C})$, then it is clear that the self-adjoint elements of A are precisely the matrices of the form

$$x = \begin{pmatrix} r & z \\ \bar{z} & s \end{pmatrix} \text{ with } r, s \in \mathbb{R} \text{ and } z \in \mathbb{C}.$$

Hence the positive elements of A are precisely the matrices of the form

$$(\dagger) \qquad x^2 = \begin{pmatrix} r^2 + |z|^2 & (r+s)z \\ (r+s)\bar{z} & s^2 + |z|^2 \end{pmatrix} \text{ with } r, s \in \mathbb{R} \text{ and } z \in \mathbb{C}.$$

For example

$$\begin{pmatrix} 1 & i \\ -i & 0 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$$

are self-adjoint, and their squares

$$\begin{pmatrix} 2 & i \\ -i & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

are positive. On the other hand the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is self-adjoint but not positive in this sense (as can be seen by looking at (†))

Positivity in this sense should not be confused with the notion of a 'positive matrix', see [Schaefer, Chapter I, §2].

7.2. **Proposition.** *We have $x \geq 0 \iff x$ is self-adjoint and $\sigma_A(x) \subseteq [0, \infty)$.*

If this is the case, then x has a positive square root in A.

Proof. If $x \geq 0$, then clearly $x = h^2$ with $h = h^*$ is self-adjoint. Now we may switch to the commutative $A = \overline{\mathbb{C}[h]}$ (by 6.8) and see that $\sigma_A(x) = \hat{x}(\operatorname{Sp} A) = \hat{h}^2(\operatorname{Sp} A) \subseteq [0, \infty)$.

Conversely suppose $x = x^*$ and $\sigma_A(x) \subseteq [0, \infty)$. We may again switch to $A = \overline{\mathbb{C}[x]}$. By 6.7 we know that

$$\begin{aligned} \Phi : A &\longrightarrow C(\sigma_A(x), \mathbb{C}) \\ a &\longmapsto \hat{a} \circ \hat{x}^{-1} \end{aligned}$$

is an isometric $*$-isomorphism and x is mapped to the inclusion $\varepsilon : \sigma_A(x) \hookrightarrow \mathbb{C}$. Since $\sigma_A(x) \subseteq [0, \infty)$ we can take the (pointwise positive) square root f of this inclusion, which is a positive element of $C(\sigma_A(x), \mathbb{C})$ satisfying $\varepsilon = f^2$. Hence $\varepsilon \geq 0$ in $C(\sigma_A(x), \mathbb{C})$ is the square of a positive element and therefore $x \geq 0$ is the square of a positive element in A. □

7.3. **Proposition.** *(Real algebraic properties of C^*-algebras I)*
Let A be a unital C^*-algebra.

(i) If $x \in A$ with $0 \leq x$ and $x \leq 0$, then $x = 0$
(ii) If $x \in A$ is positive, then there is a unique positive $s \in A$ with $s^2 = x$. We write $x^{\frac{1}{2}}$ or $\sqrt{x}$ for s. If x is invertible and positive, then also s is invertible.
(iii) If $x, y \in A$ are positive, then also $x + y$ is positive.
(iv) If $x, y, z \in A$ with $x \leq y \leq z$ then $x \leq z$.
(v) If $a, b \in A$ are self-adjoint with $-b \leq a \leq b$, then $|a| \leq |b|$.
(vi) If $x, y \in A$ with $0 \leq x \leq y$ and x is invertible, then also y is invertible (and $0 \leq |y^{-1}| \leq |x^{-1}|$)

Proof. (i) By 7.2 we have $\sigma_A(x) \subseteq [0, \infty)$ and $\sigma_A(-x) \subseteq [0, \infty)$, which implies $\sigma_A(x) = \{0\}$. By 6.7 this means $x = 0$.

(ii). By 7.2 we know that $x = s^2$ for some $s \geq 0$ from $\overline{\mathbb{C}[x]}$. Now if $t \in A$ is positive with $t^2 = x$, then $s, x \in \overline{\mathbb{C}[t]}$. However $\overline{\mathbb{C}[t]}$ is commutative and for rings of continuous functions the statement is certainly true. Thus $s = t$.

If x is positive and invertible, then again working in the context of continuous functions, it is obvious that s is invertible.

(iii). We first observe that

(†) For all $h \in A$ with $h = h^*$ and $|h| \leq 1$ we have $h \geq 0 \iff |1 - h| \leq 1$,

as can be seen by switching to continuous functions on $\sigma_A(h)$. It is therefore enough to show that

$$\left| \mathbb{1} - \frac{x+y}{|x| + |y|} \right| \leq 1.$$

We have

$$\begin{aligned} \left| \mathbb{1} - \frac{x+y}{|x|+|y|} \right| &= \left| \frac{|x| - x + |y| - y}{|x| + |y|} \right| \leq \\ &\leq \left| \frac{||x| - x| + ||y| - y|}{|x| + |y|} \right| = \\ &= \left| \frac{|x| \cdot |\mathbb{1} - \frac{x}{|x|}| + |y| \cdot |\mathbb{1} - \frac{y}{|y|}|}{|x| + |y|} \right| = \text{ using (†) and } x, y \geq 0 \\ &\leq \left| \frac{|x| \cdot 1 + |y| \cdot 1}{|x| + |y|} \right| = 1 \end{aligned}$$

(iv) is obvious from (iii).

(v) By taking Gel'fand transforms we see that $-|b| \cdot \mathbb{1} \leq -b \leq b \leq |b| \cdot \mathbb{1}$. It follows $-|b| \cdot \mathbb{1} \leq a \leq |b| \cdot \mathbb{1}$ and by taking Gel'fand transforms again we see that $|a| \leq |b|$.

(vi) By taking Gel'fand transforms (and taking into account that $x^{-1} \in \overline{\mathbb{C}[x]}$) we see that $\hat{x}$ is real valued and attains a minimum $\lambda > 0$. Then $|\hat{x}^{-1}| = \lambda^{-1}$ and $\lambda \cdot \mathbb{1} \leq x$. It follows $\lambda \cdot \mathbb{1} \leq y$ and therefore $\lambda \leq \hat{y}$ pointwise. Thus

$$|x^{-1}| = |\hat{x}^{-1}| = \lambda^{-1} \geq |\hat{y}^{-1}| = |y^{-1}|.$$

□

In fact, in the situation of 7.3(vi) it is true that $y^{-1} \leq x^{-1}$. However we can not prove this in a single commutative subalgebra, or at least not in an equally simple manner as in 7.3(vi). Also, 7.3 does not say a lot about the compatibility of positivity with respect to multiplication. The crucial property, needed to address both issues is that all so-called hermitian squares x^*x of a C^*-algebra are positive.

7.4. **Lemma.** *Let A be a ring with 1. If $x, y \in A$, then $1 - xy$ is invertible if and only if $1 - yx$ is invertible: If $a = (1 - xy)^{-1}$, then $1 + yax$ is the inverse of $1 - yx$.*

Proof. This is a straightforward calculation:

$$(1 + yax)\cdot(1 - yx) = 1 - xy + yax - yaxyx = 1 - xy + y\underbrace{(a - axy)}_{=1}x = 1$$

$$(1 - yx)\cdot(1 + yax) = 1 - yx + yax - yxyax = 1 - yx + y\overbrace{(a - xya)}x = 1.$$

□

7.5. **Corollary.** *Let A be a unital Banach algebra over $\mathbb{K}$. Then for all $x, y \in A$ we have*

$$\sigma(xy) \cup \{0\} = \sigma(yx) \cup \{0\}$$

Proof. For each $\lambda \in \mathbb{C}$ with $\lambda \neq 0$ we need to show that

$$xy - \lambda\cdot\mathbb{1} \text{ is invertible} \iff yx - \lambda\cdot\mathbb{1} \text{ is invertible.}$$

However, this is the same as to show that

$$\mathbb{1} - \lambda xy \text{ is invertible} \iff \mathbb{1} - \lambda yx \text{ is invertible,}$$

which holds true by 7.4. □

7.6. **Theorem.** *The positive elements of a unital C^*-algebra A are precisely those of the form $x^*\cdot x$, with $x \in A$.*

Proof. Obviously every positive element is of this form. We need to show that indeed $x^*\cdot x \geq 0$. Suppose this is not the case.
Claim. There is some $y \in A$ with $0 \neq y^*y \leq 0$.

Proof of the claim. Write $h = x^*x$. Then h is self-adjoint and not positive by assumption. We work in $\overline{\mathbb{C}[h]}$ and see that $\hat{h} : \sigma_A(h) \longrightarrow \mathbb{C}$ is real valued and not positive. Thus $\hat{h}(\varphi) < 0$ for some $\varphi \in \sigma_A(h)$. Let U be the set of all points where $\hat{h}$ is strictly negative. Take a continuous function $f : U \longrightarrow \mathbb{R}$ with $f \geq 0$ and $f(\varphi) > 0$. Since the C^*-algebra $\overline{\mathbb{C}[h]}$ is isomorphic to $C(\sigma_A(h), \mathbb{C})$ (by 6.7), there is an element $b \in \overline{\mathbb{C}[h]}$ which is mapped onto f. Since $f^* = f$ we have $b^* = b$ and we choose $y = xb$. Then

$$y^*y = bx^*xb \in \overline{\mathbb{C}[h]}$$

has Gel'fand transform $f\cdot\hat{h}\cdot f$. By choice of f we have $f\cdot\hat{h}\cdot f \leq 0$ and $f\cdot\hat{h}\cdot f \neq 0$. Hence y^*y also satisfies $y^*y \leq 0$ and $y^*y \neq 0$. □

Pick y as in the claim and write $y = a + ib$ with self-adjoint elements $a, b \in A$. Then

$$y^*y = (a - ib)(a + ib) = a^2 + b^2 + iba - iab \text{ and}$$

$$yy^* = (a + ib)(a - ib) = a^2 + b^2 - iba + iab.$$

It follows $y^*y + yy^* = 2(a^2 + b^2) \geq 0$ (by 7.3(iii)) and as $-y^*y \geq 0$ then $yy^* \geq 0$ (again by 7.3(iii)). From 7.2 we get

$$\sigma_A(yy^*) \subseteq [0, \infty) \text{ and } \sigma_A(y^*y) \subseteq (-\infty, 0].$$

But now, 7.5 says $\sigma(y^*y) = \{0\}$. Since y^*y is self-adjoint, this means $y^*y = 0$ (look at the Gel'fand transform). This contradicts the choice of y. □

7.7. **Corollary.** *If x is a positive element of a unital C^*-algebra and $y \in A$, then y^*xy is again positive.*

Proof. Write $x = z^*z$, so $y^*xy = y^*z^*zy = (zy)^*(zy) \geq 0$. □

7.8. **Proposition.** *(Real algebraic properties of C^*-algebras II)*
Let A be a unital C^*-algebra.

(i) If $x, y \in A$ with $0 \leq x \leq y$ and x is invertible, then also y is invertible and $0 \leq y^{-1} \leq x^{-1}$.
(ii) If $x, y \in A$ with $0 \leq x \leq y$ then

$$x(\mathbb{1} + x)^{-1} \leq y(\mathbb{1} + y)^{-1}$$

Proof. (i) From $0 \leq x \leq y$ we get, using 7.7,

$$0 \leq z := y^{-\frac{1}{2}} x y^{-\frac{1}{2}} \leq \mathbb{1}$$

Then $z = z^*$ is invertible with $0 < \hat{z} \leq 1$ pointwise. So $\hat{z}^{-1} \geq 1$, thus

$$y^{\frac{1}{2}} x^{-1} y^{\frac{1}{2}} = z^{-1} \geq \mathbb{1}$$

and so $y^{-1} \leq x^{-1}$.
(ii) From (i) we get $(\mathbb{1} + y)^{-1} \leq (\mathbb{1} + x)^{-1}$, hence $-(\mathbb{1} + x)^{-1} \leq -(\mathbb{1} + y)^{-1}$ and so

$$x(\mathbb{1} + x)^{-1} = \mathbb{1} - (\mathbb{1} + x)^{-1} \leq \mathbb{1} - (\mathbb{1} + y)^{-1} = y(\mathbb{1} + y)^{-1}.$$

□

7.9. **Proposition.** *Let A be a unital C^*-algebra and let $x \in A$ be self-adjoint. Then there are uniquely determined positive elements $x_+, x_- \in A$, called the* **positive** *and the* **negative part** *of x, with the property $x = x_+ - x_-$ and $x_+ \cdot x_- = 0$. We have*

$$x_+ = \frac{1}{2}(\sqrt{x^2} + x)$$
$$x_- = \frac{1}{2}(\sqrt{x^2} - x)$$

Proof. First note that $x^2 \geq 0$, hence by 7.3, x^2 has a unique positive square root $\sqrt{x^2}$ in A.

Existence and the representation of x_+, x_- follow by looking at Gel'fand transforms. Note that for continuous functions, x_+ is the supremum of x and 0, whereas x_- is the negative of the infimum of x and 0.

To see uniqueness, suppose $y_+, y_- \in A$ are positive with the property $x = y_+ - y_-$ and $y_+ \cdot y_- = 0$. Then also $y_- \cdot y_+ = (y_+ \cdot y_-)^* = 0$ and so

$$x^2 = (y_+ - y_-)^2 = (y_+)^2 + (y_-)^2 = (y_+ + y_-)^2.$$

Since $y_+ + y_- \geq 0$, the uniqueness of positive square roots in A gives

$$\sqrt{x^2} = y_+ + y_-.$$

Adding this to $x = y_+ - y_-$ gives $\sqrt{x^2} + x = 2y_+$. Hence $y_+ = x_+$ and similarly $y_- = x_-$. □

Further characterisations of positive elements and properties of those can be found in [Conway, Chapter VIII, §3].

7.10. **Definition.** A **homomorphism** $\varphi : A \longrightarrow B$ between unital C^*-algebras is a $\mathbb{C}$-algebra homomorphism that respects the involution and maps $\mathbb{1}_A$ to $\mathbb{1}_B$.

For example, in 5.9 we have seen that each character of A is a C^*-algebra homomorphism $A \longrightarrow \mathbb{C}$.

7.11. **Lemma.**
Every C^*-algebra homomorphism $\varphi : A \longrightarrow B$ is **order-preserving***, i.e. $x \geq 0$ implies $\varphi(x) \geq 0$ (hence also $x \leq y \Rightarrow \varphi(x) \leq \varphi(y)$) and* **norm-decreasing***, i.e. $|\varphi(x)| \leq |x|$. In other words, φ is continuous of norm ≤ 1.*

Proof. Since φ respects multiplication and involution, all elements of the form x^*x are mapped onto elements of the form y^*y. Hence by 7.6, φ is order-preserving.

In order to show $|\varphi(x)| \leq |x|$ we first assume that x is self-adjoint. In this case $-|x|\cdot\mathbb{1}_A \leq x \leq |x|\cdot\mathbb{1}_A$, since this is true for continuous functions (and self-adjoint elements). Since φ is order preserving, also

$$-|x|\cdot\mathbb{1}_B \leq \varphi(x) \leq |x|\cdot\mathbb{1}_B.$$

Since x is self-adjoint we also know that $\varphi(x) = \varphi(x^*) = \varphi(x)^*$ is self-adjoint. Thus $|\varphi(x)| \leq |x|$ by 7.3(v), for all self-adjoint elements $x \in A$. In general we then have

$$|\varphi(x)|^2 = |\varphi(x)^*\varphi(x)| = |\varphi(x^*x)| \overset{\text{as } x^*x \text{ is self-adjoint}}{\leq} |x^*x| = |x|^2$$

as required. □

7.12. **Proposition.** *Every injective C^*-algebra homomorphism $\varphi : A \longrightarrow B$ is norm preserving and therefore $\varphi(A)$ is a C^*-subalgebra of B.*

Proof. We first show that the compositional inverse $\psi : \varphi(A) \longrightarrow A$ is order preserving: Take $x \in A$ and assume that $y := \varphi(x) \geq 0$. We must show $x \geq 0$. Firstly, x is self-adjoint, because y is self-adjoint, φ is injective and $\varphi(x^*) = \varphi(x)^* = y^* = y = \varphi(x)$.

In order to prove $x \geq 0$ we show $x_- = 0$ (see 7.9). We have $y = \varphi(x) = \varphi(x_+ - x_-) = \varphi(x_+) - \varphi(x_-)$ and since φ is order preserving, $\varphi(x_+), \varphi(x_-) \geq 0$. Moreover $\varphi(x_+)\cdot\varphi(x_-) = \varphi(x_+\cdot x_-) = 0$. Hence by the uniqueness in 7.9, we see that $\varphi(x_-) = y_-$. However, $y \geq 0$ and therefore $y_- = 0$.

Now we show that $|\varphi(x)| = |x|$ for all $x \in A$ By 7.11 we already know $|\varphi(x)| \leq |x|$. We note that it suffices to prove $|\varphi(x)| = |x|$ for self-adjoint elements, because in general

$$|\varphi(x)|^2 = |\varphi(x)^*\varphi(x)| = |\varphi(x^*x)| \overset{\text{as } x^*x \text{ is self-adjoint}}{=} |x^*x| = |x|^2.$$

So we may assume that x is self-adjoint and therefore also $y = \varphi(x) = \varphi(x^*) = \varphi(x)^* = y^*$ is self-adjoint. Then $-|y|\cdot\mathbb{1}_B \leq y \leq |y|\cdot\mathbb{1}_B$, since this is true for continuous functions (and self-adjoint elements). Since ψ is order preserving we get

$$-|y|\cdot\mathbb{1}_A \leq \psi(y) \leq |y|\cdot\mathbb{1}_A.$$

By 7.3(v) again we get $|\psi(y)| \leq |y|$, which means $|x| \leq |\varphi(x)|$. □

In connection with 7.12 it should be mentioned that also the following is true (we don't need this later on and omit a full proof):

7.13. **Fact.** *For every homomorphism* $\varphi : A \longrightarrow B$ *between* C^**-algebras, the image of* φ *is a* C^**-subalgebra of* B.

Proof. This follows from 7.12, once we know that for every closed and two-sided $*$-ideal I of A, the C^*-algebra structure of A can be pushed down to A/I. This means:

The ring A/I, together with the assignment

$$(x + I)^* = x^* + I \tag{†}$$

is a (well-defined) involution, the Banach space A/I together with the ring structure of A/I and the involution given by (†) is a C^*-algebra, and the residue map $A \longrightarrow A/I$ is a C^*-algebra homomorphism. □

8. States

Let A be a unital C^*-algebra and let $\omega : A \longrightarrow \mathbb{C}$ be a $\mathbb{C}$-linear (not necessarily continuous) map. Recall that the operation

$$\langle x, y\rangle_\omega := \omega(y^*{\cdot}x)$$

is a sesquilinear form on A (see page 8).

8.1. **Lemma.** *Let A be a unital C^*-algebra and let $\omega : A \longrightarrow \mathbb{C}$ be a $\mathbb{C}$-linear map. The following are equivalent:*

(i) ω is **positive**, *i.e. $x \geq 0 \Rightarrow \omega(x) \geq 0$.*
(ii) $\langle x, y\rangle_\omega$ is a positive hermitian form on A (see page 8)

If this is the case, then

(a) ω respects the involution,
(b) for all $x, y, a \in A$ we have $\langle ax, y\rangle_\omega = \langle x, a^*y\rangle_\omega$ and
(c) ω satisfies **Schwarz' inequality**

$$|\omega(y^*x)|^2 \leq \omega(x^*x){\cdot}\omega(y^*y).$$

Proof. (ii)⇒(i) follows from $\omega(x^*x) = \langle x, x\rangle_\omega$.

(i)⇒(ii). We have $\langle x, x\rangle = \omega(x^*x) \geq 0$ since ω is positive and x^*x is positive by 7.6. In order to show that $\langle x, y\rangle_\omega$ is hermitian we must show that $\langle y, x\rangle_\omega = \overline{\langle x, y\rangle_\omega}$, in other words $\omega(x^*{\cdot}y) = \overline{\omega(y^*{\cdot}x)}$. We show that ω preserves the involution (this will also show (a) and (b)): Take $x \in A$ and write $x = a + i{\cdot}b$ with self-adjoint $a, b \in A$. Then

$$\begin{aligned}\omega(x^*) = \omega(a - i{\cdot}b) &= \omega(a_+ - a_- - i{\cdot}(b_+ - b_i)) = \\ &= \omega(a_+) - \omega(a_-) - i{\cdot}(\omega(b_+) - \omega(b_i)).\end{aligned}$$

Since ω is positive, $\omega(a_+), \omega(a_-), \omega(b_+)$ and $\omega(b_i)$ are all positive. Hence $\omega(a_+) - \omega(a_-)$ is the real part of $\omega(x^*)$ and $-(\omega(b_+) - \omega(b_i))$ is the imaginary part of $\omega(x^*)$. A similar computation of $\omega(x)$ proves $\omega(x^*) = \overline{\omega(x)}$.

Finally, the Schwarz inequality holds, since it holds for all positive hermitian forms. □

Observe that all characters of a unital C^*-algebra are positive (by 7.6). Another example of a positive linear map on the commutative C^*-algebra $C([0, 1], \mathbb{C})$ is given by integration:

$$f \mapsto \int f \, \mathrm{d}t.$$

8.2. **Theorem.** *Let A be a unital C^*-algebra and let $\omega : A \longrightarrow \mathbb{C}$ be a linear map. The following are equivalent:*

(i) ω is positive
(ii) ω is bounded and $|\omega| = \omega(\mathbb{1})$.

If this is the case, then $\omega(x) \in \mathbb{R}$ for all self-adjoint $x \in A$.

Proof. (i)⇒(ii). As $|\mathbb{1}| = 1$, it suffices to show $|\omega(x)| \leq \omega(\mathbb{1}) \cdot |x|$ for all $x \in A$.

If $x = x^*$, then $-|x| \cdot \mathbb{1} \leq x \leq |x| \cdot \mathbb{1}$ and using the positivity of ω, gives $-|x| \cdot \omega(\mathbb{1}) \leq \omega(x) \leq |x| \cdot \omega(\mathbb{1})$, thus $|\omega(x)| \leq \omega(\mathbb{1}) \cdot |x|$ by 7.3(v)

For general $x \in A$ we then get

$$|\omega(x)|^2 = |\omega(\mathbb{1}^* x)|^2 \overset{\text{by } 8.1}{\leq} \omega(\mathbb{1}^* \cdot \mathbb{1}) \cdot \omega(x^* \cdot x) \leq \omega(\mathbb{1}) \cdot \omega(\mathbb{1}) \cdot |x^* \cdot x|,$$

since $x^* x$ is self-adjoint. Thus $|\omega(x)|^2 \leq \omega(\mathbb{1})^2 \cdot |x^* \cdot x| = \omega(\mathbb{1})^2 \cdot |x|^2$ as required.

(ii)⇒(i). We first show

Claim. If $x = x^* \in A$, then $\omega(x) \in \mathbb{R}$.

Proof of the claim: Write $\omega(x) = r + i \cdot s$ with $r, s \in \mathbb{R}$. Then

$$\omega(x + i \cdot t \cdot \mathbb{1}) = r + i \cdot (s + t \cdot \omega(\mathbb{1})) \text{ for all } t \in \mathbb{R}.$$

As $\omega(\mathbb{1}) = |\omega| \in \mathbb{R}$ it follows

$$|\omega(x + i \cdot t \cdot \mathbb{1})| \geq |s + t \cdot \omega(\mathbb{1})| \text{ for all } t \in \mathbb{R}.$$

Using the Gel'fand transform of x we also have

$$|\omega(x + i \cdot t \cdot \mathbb{1})| \leq |\omega| \cdot |x + i \cdot t \cdot \mathbb{1}| = \omega(\mathbb{1}) \cdot \sqrt{|x|^2 + t^2} \text{ for all } t \in \mathbb{R}.$$

Thus

$$\omega(\mathbb{1})^2 \cdot (|x|^2 + t^2) \geq |s + t \cdot \omega(\mathbb{1})|^2 = \omega(\mathbb{1})^2 \cdot t^2 + 2st\omega(\mathbb{1}) + s^2 \text{ for all } t \in \mathbb{R}.$$

However, this is only possible if $s = 0$. □

From the claim we obtain the positivity of ω as follows: Pick $x \geq 0$, say $|x| \leq 1$. Then also $|\mathbb{1} - x| \leq 1$ (look at the Gel'fand transform of x) and therefore

$$|\omega(\mathbb{1}) - \omega(x)| = |\omega(\mathbb{1} - x)| \leq |\omega| \cdot |\mathbb{1} - x| = \omega(\mathbb{1}) \cdot |\mathbb{1} - x| \leq \omega(\mathbb{1}).$$

Since $\omega(x) \in \mathbb{R}$ by the claim, this is only possible if $\omega(x) \geq 0$. □

8.3. **Corollary.** *Let A be a unital C^*-subalgebra of a unital C^*-algebra B. Then every positive linear map $\omega : A \longrightarrow \mathbb{C}$ can be extended to a positive linear map $\tilde{\omega} : B \longrightarrow \mathbb{C}$ and all these extensions have norm $|\omega|$.*

Proof. By 8.2(i)⇒(ii), ω is a bounded linear functional with norm $|\omega| = \omega(\mathbb{1}_A)$. By the Hahn-Banach extension theorem for bounded functionals 3.6, there is a bounded functional $\tilde{\omega} : B \longrightarrow \mathbb{C}$ with $|\tilde{\omega}| = |\omega|$. It follows that

$$|\tilde{\omega}| = |\omega| = \omega(\mathbb{1}_A) = \tilde{\omega}(\mathbb{1}_B).$$

By 8.2(ii)⇒(i), $\tilde{\omega}$ is positive.

Now if $\tilde{\omega}$ is any positive extension of ω, then

$$|\tilde{\omega}| = \tilde{\omega}(\mathbb{1}_B)| = \omega(\mathbb{1}_A) = |\omega|$$

By 8.2(i)⇒(ii). □

8.4. **Corollary.** *If ω is a positive linear functional on a unital C^*-algebra A, then for all $a, x \in A$ we have*

$$|\langle ax, x \rangle_\omega| \leq |a| \cdot \langle x, x \rangle_\omega$$

Proof. Fix $x \in A$ and define $\rho : A \longrightarrow \mathbb{C}$ by $\rho(a) = \langle ax, x \rangle = \omega(x^* ax)$. Using 7.7 and the positivity of ω, we know that ρ is again a positive linear functional of A. Hence by 8.2, ρ is bounded and $|\rho| = \rho(\mathbb{1})$. Thus

$$|\langle ax, x \rangle_\omega| = |\rho(a)| \leq \rho(\mathbb{1}) \cdot |a| = |a| \cdot \langle x, x \rangle_\omega.$$

□

8.5. **Definition.** A **state** on a unital C^*-algebra A, is a positive linear map $\omega : A \longrightarrow \mathbb{C}$ with $\omega(\mathbb{1}_A) = 1$.

For example, if $A = C(X, \mathbb{C})$ for a compact space X, then evaluation at a given point of X, is a state of A.

8.6. **Theorem.** *Let A be a unital C^*-algebra and let $x, y \in A$ with $x \neq y$. Then there is a state ω of A with $\omega(x) \neq \omega(y)$.*

Proof. It suffices to find a state that does not annihilate $z = x - y$. Write $z = a + ib$ with self-adjoint elements $a, b \in A$. Since all states of A evaluate in $\mathbb{R}$ at self-adjoint elements, it is therefore enough to show that for every nonzero, self-adjoint $x \in A$, there is a state ω of A with $\omega(x) \neq 0$. By 8.3 we may replace A by $\overline{\mathbb{C}[x]}$. Since $x \neq 0$ and A is commutative, there is a character ω of A with $\omega(x) \neq 0$. Now recall that all characters are states (by 7.6). □

9. The Gel'fand-Naimark-Segal construction

Recall that for every Hilbert space H, the set of bounded operators $\mathcal{B}(H)$ is a C^*-algebra, where multiplication is given by composition and the involution is given by the transpose L^* of $L \in \mathcal{B}(H)$: By the Riesz representation theorem (see [Alt, 4.1]) we know that the map $H \longrightarrow H'; a \mapsto (x \mapsto \langle x, a \rangle)$ is an isometric isomorphism. Hence for each $y \in H$, there is a unique element $a \in H$ such that the bounded functional

$$x \mapsto \langle L(x), y \rangle$$

is equal to the functional $x \mapsto \langle x, a \rangle$. We define $L^*(y) = a$ and obtain

$$\langle L(x), y \rangle = \langle x, L^*(y) \rangle \ (x \in H).$$

It is straightforward to see that $L^* \in \mathcal{B}(H)$ and that $L \mapsto L^*$ is an involution, satisfying $|L^*L| = |L|^2$.

For every unit vector $\xi \in H$ (i.e. $|\xi| = 1$), the map

$$\mathcal{B}(H) \longrightarrow \mathbb{C}, \ x \mapsto \langle x\xi, \xi \rangle$$

obviously defines a state on $\mathcal{B}(H)$. These states are called **vector states**.

Let A be a unital C^*-algebra and let $\omega : A \longrightarrow \mathbb{C}$ be a state. We shall attach a Hilbert space to ω as follows: By 8.1, the operation

$$\boxed{\langle x, y \rangle_\omega := \omega(y^* \cdot x)}$$

defines a positive hermitian form on the Banach space A. We divide by the Null space of this form. So let

$$\boxed{N_\omega := \{x \in A \mid \langle x, x \rangle_\omega = 0\}}$$

Claim 1. N_ω is a left ideal of A (i.e. N_ω is closed under addition and for all $a \in A$ and $x \in N_\omega$ we have $ax \in N_\omega$).

Proof of claim 1. By 8.1 we have

$$\langle ax, ax \rangle_\omega = \langle x, a^*ax \rangle_\omega.$$

and by the Schwarz'-inequality (cf. 8.1), we have

$$|\langle x, a^*ax \rangle_\omega| \leq \langle x, x \rangle_\omega \cdot \langle a^*ax, a^*ax \rangle_\omega = 0.$$

Hence $|\langle ax, ax \rangle_\omega| = 0$ and $ax \in N$.

□

We define

$$\boxed{K_\omega := A/N_\omega} \text{ as a } \mathbb{C}\text{-vector space}$$

and for $x, y \in A$,

$$\boxed{\langle\langle x + N_\omega, y + N_\omega \rangle\rangle_\omega := \langle x, y \rangle_\omega.}$$

Claim 2. $\langle\langle \ , \ \rangle\rangle_\omega$ is a well-defined positive definite, hermitian form on K_ω.

Proof of claim 2. This is straightforward, since N is a $\mathbb{C}$-vector space: Suppose $x', y' \in A$ with $x - x', y - y' \in N_\omega$, then

$$\langle x, y\rangle_\omega = \langle x - x', y\rangle_\omega + \langle x', y\rangle_\omega = \langle x', y\rangle_\omega,$$

as $\langle x - x', y\rangle_\omega = 0$ from $x - x' \in N_\omega$ and the Schwarz inequality for $\langle\ ,\ \rangle_\omega$. Similarly, $\langle x', y\rangle_\omega = \langle x', y'\rangle_\omega$ and so $\langle\langle\ ,\ \rangle\rangle_\omega$ is indeed well-defined. Linearity of $\langle\langle\ ,\ \rangle\rangle_\omega$ in x, anti-linearity of $\langle\langle\ ,\ \rangle\rangle_\omega$ in y, positivity of $\langle\langle\ ,\ \rangle\rangle_\omega$ and $\langle\langle y + N_\omega, x + N_\omega\rangle\rangle_\omega = \overline{\langle\langle x + N_\omega, y + N_\omega\rangle\rangle}$ readily transfer from $\langle\ ,\ \rangle_\omega$. If $\langle\langle x + N_\omega, x + N_\omega\rangle\rangle_\omega = 0$, then $x \in N_\omega$ by definition, which shows claim 2. □

Let $\boxed{||x + N||_\omega := \langle\langle x, x\rangle\rangle_\omega}$ $(x \in A)$ be the norm induced by $\langle\langle\ ,\ \rangle\rangle_\omega$ on K_ω and let

$$\boxed{H_\omega \text{ be the completion of the normed space } (K_\omega, ||\ .\ ||_\omega).}$$

The natural extensions of $\langle\langle\ ,\ \rangle\rangle_\omega$ and $||\ .\ ||$ to H_ω, again denoted by $\langle\langle\ ,\ \rangle\rangle_\omega$ and $||\ .\ ||$, equip H_ω with a complex Hilbert space structure and we consider H_ω as this Hilbert space from now on.

We shall now construct a homomorphism of C^*-algebras $A \longrightarrow \mathcal{B}(H_\omega)$.

For $a \in A$, let $L_a : K_\omega \longrightarrow K_\omega$ be defined by

$$\boxed{L_a(x + N_\omega) = ax + N_\omega.}$$

Claim 3. L_a is a well-defined bounded linear map $K_\omega \longrightarrow K_\omega$ with

(i) $|L_a| \leq |a|$ for all $a \in A$ and

(ii) $\langle\langle L_{a^*}(\xi), \eta\rangle\rangle_\omega = \langle\langle \xi, L_a(\eta)\rangle\rangle_\omega$ for all $a \in A,\ \xi, \eta \in K_\omega$.

Proof of claim 3. L_a is well defined, because N_ω is a left ideal of A: If $x - x' \in N_\omega$, then also $ax - ax' \in N_\omega$, and so $ax + N_\omega = ax' + N_\omega$.

L_a is linear, because for $\alpha \in \mathbb{C}$ and $x \in A$ we have

$$L_a(\alpha(x + N_\omega)) = a\cdot(\alpha\cdot x) + N_\omega = \alpha\cdot L_a(x + N_\omega).$$

(i) For $x \in A$ we have

$$\begin{aligned} ||L_a(x + N_\omega)||^2 = ||ax + N_\omega||^2 = \langle ax, ax\rangle_\omega = &\text{ by } 8.1(b) \\ = \langle a^*ax, x\rangle_\omega \leq &\text{ by } 8.4 \\ \leq |a^*a|\cdot\langle x, x\rangle_\omega = |a|^2\cdot||x + N_\omega||^2, \end{aligned}$$

which shows that L_a is bounded with $|L_a| \leq |a|$.

(ii) Take $x, y \in A$ with $x + N_\omega = \xi$ and $y + N_\omega = \eta$. Then

$$\langle\langle L_{a^*}(\xi), \eta\rangle\rangle_\omega = \langle\langle a^*x + N_\omega, y + N_\omega\rangle\rangle_\omega = \langle a^*x, y\rangle_\omega = \langle x, ay\rangle_\omega = \langle\langle \xi, L_a(\eta)\rangle\rangle_\omega.$$

□

For each $a \in A$, the bounded linear map $L_a : K_\omega \longrightarrow K_\omega$ extends uniquely to a continuous map $H_\omega \longrightarrow H_\omega$ and we define

$$\boxed{\Phi_\omega(a) = \text{ the unique extension of } L_a \text{ to } H_\omega}$$

Obviously, $\Phi_\omega(a) \in \mathcal{B}(H_\omega)$ and so we have constructed a map

$$\boxed{\Phi_\omega : A \longrightarrow \mathcal{B}(H_\omega) \text{ uniquely determined by } \Phi_\omega(a)(x + N_\omega) = ax + N_\omega.}$$

This finishes the GNS-construction

9.1. **Theorem.** *Φ_ω is a C^*-algebra homomorphism.*

Proof. It follows directly from the definition that $a \mapsto L_a$ is a $\mathbb{C}$-linear ringhomomorphism $A \longrightarrow \mathcal{B}(K_\omega)$ which maps $\mathbb{1}_A$ to $\mathbb{1}_{\mathcal{B}(K_\omega)}$ $(= \mathrm{id}_{K_\omega})$. Since H_ω is the completion of K_ω, also Φ_ω has these properties. Moreover, property (ii) of claim 3 remains true when we replace L_a by $\Phi_\omega(a)$. Now this property says that $\Phi_\omega(a^*)$ is the transpose of the operator $\Phi_\omega(a)$. In other words, Φ_ω respects the involution. $\square$

Let's work out how the state ω can be expressed through Φ_ω: We define

$$\boxed{\Omega_\omega := \mathbb{1}_A + N_\omega \in K_\omega}$$

For $a \in A$ we have

$$\begin{aligned}\omega(a) &= \langle\langle L_a(\mathbb{1}_A + N_\omega), \mathbb{1}_A + N_\omega\rangle\rangle_\omega = \langle\langle \Phi_\omega(a)(\mathbb{1}_A + N_\omega), \mathbb{1}_A + N_\omega\rangle\rangle_\omega = \\ &= \langle\langle \Phi_\omega(a)\Omega_\omega, \Omega_\omega\rangle\rangle_\omega.\end{aligned}$$

In other words, ω is the composition of $\Phi_\omega : A \longrightarrow \mathcal{B}(H_\omega)$ with the vector state $\mathcal{B}(H_\omega) \longrightarrow \mathbb{C}$ given by Ω_ω.

The triple $(H_\omega, \Phi_\omega, \Omega_\omega)$ is called the **GNS-representation of A with respect to ω**.

We observe that Ω_ω is a so-called **cyclic vector** of this representation, i.e. the set

$$\{\Phi_\omega(a)\Omega_\omega \mid a \in A\}$$

is dense in H_ω (Observe that this set is just K_ω and K_ω is dense in H_ω by definition).

9.2. **Theorem.** *Up to unitary equivalence, the GNS-representation of A w.r.t. ω is unique. This means: If $\Phi : A \longrightarrow \mathcal{B}(H)$ is a C^*-algebra homomorphism for some Hilbert space H and $\xi \in H$ is of norm 1 such that*

(a) $\omega(a) = \langle\Phi(a)\xi, \xi\rangle$ for all $a \in A$ and

(b) the set

$$\{\Phi(a)\xi \mid a \in A\}$$

is dense in H,

then there is a unitary (i.e. scalar product preserving) isomorphism $u : H \longrightarrow H_\omega$ mapping ξ to Ω_ω, such that

$$\Phi(a) = u^{-1} \circ \Phi_\omega(a) \circ u \ (a \in A).$$

Proof. We first define u on $\Phi(A)\xi$ (which is dense in H) by

$$u(\Phi(a)\xi) := \Phi_\omega(a)\Omega_\omega.$$

Note that this is well-defined because property (a) holds for Φ (and Φ_ω): If $\Phi(a)\xi = \Phi(b)\xi$, then $\Phi(a-b)\xi = 0$, so $\omega(a-b) = 0$ by (a), which implies $a - b \in N_\omega$, i.e. $\Phi_\omega(a)\Omega_\omega = \Phi_\omega(b)\Omega_\omega$.

Because

$$||u(\Phi(a)\xi)||^2_{H_\omega} = ||\Phi_\omega(a)\Omega_\omega||^2 = \omega(a^*a) = \langle\Phi(a^*a)\xi, \xi\rangle = |\Phi(a)\xi|_{H'},$$

u is a norm preserving linear map between dense subsets of H and H_ω. So u extends uniquely to a norm preserving linear map between H and H_ω. Since the scalar product on a Hilbert space is given by

$$\langle x, y\rangle = \frac{1}{4}(|x+y|^2 - |x-y|^2) \text{ (see [AlbKal, Prop. 4.2.5])},$$

u is unitary. The equation $\Phi(a) = u^{-1} \circ \Phi_\omega(a) \circ u$ is readily verified. □

Finally we can state and prove the **universal representation** of any unital C^*-algebra:

9.3**. Gel'fand Naimark Segal**
Let A be a unital C^*-algebra and let $S(A)$ be the set of all states of A. Let H be the Hilbert space

$$H = \bigoplus_{\omega \in S(A)} H_\omega.$$

Then the map

$$\Phi = \bigoplus_{\omega \in S(A)} \Phi_\omega : A \longrightarrow \mathcal{B}(H),$$

which sends a to $\bigoplus_{\omega \in S(A)} \Phi_\omega(a) : H \longrightarrow H$ is an isometric isomorphism of A onto a C^*-subalgebra of $\mathcal{B}(H)$.

Proof. By 7.12, it suffices to show that Φ is an injective C^*-algebra homomorphism. The injectivity follows from 8.6. That Φ is a C^*-algebra homomorphism can be checked coordinate wise using 9.1. □

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The University of Manchester, Department of Mathematics, Oxford Road, Manchester M13 9PL, UK
Homepage: http://personalpages.manchester.ac.uk/staff/Marcus.Tressl/
Email address: marcus.tressl@manchester.ac.uk