LYMOTS PROBLEMS

Abstract. Record of the problem session of the LYMOTS meeting on June 20 and 21, 2024 in Manchester. Three problems were posed and below is a (rough) summary of the subsequent discussion containing comments, remarks and subproblems. There is no warranty that any claim made here is correct as these are only minutes.

Contents

1. Extend the Hrushovski-Loeser construction of pro-definable sets to different logics

1.1. **Problem** Extend the Hrushovski-Loeser construction of pro-definable sets to different logics and discuss how that would connect to adic and Berkovich spaces.

Very few people attended this question. We reviewed the key points of the Hrushovski-Loeser construction but did not get far enough to be able to attack the question.

2. Ordered Abelian Groups with an Automorphism

2.1. Problem (Asked by Jan Dobrowolski)

Are all ordered abelian group with an automorphism NIP?
Bonus question: Characterize the o-minimal ones!

2.2. Some information on the problem

A) Quantifier-free formulas are NIP; this follows from the general fact that, if a theory has quantifier-free NIP formulas, and we expand it by 1-ary function symbols, in the expansion quantifier-free formulas are still NIP.

B) Let $\mathbb{Z}[\sigma]$ act in the natural way, let L be an element of it, and let $L(x)$ be the associated definable function, e.g. if $L = 7\sigma^2 - 2$ then $L(x) = 7\sigma(\sigma(x)) - 2x$. Suppose that every $L(x)$ is either constantly 0 or surjective and that, for every L, our structure satisfies $(\forall x(x > 0) \to L(x) > 0) \vee (\forall x(x > 0) \to L(x) = 0) \vee (\forall x(x > 0) \to L(x) < 0)$. Then the structure is o-minimal. This follows from Pal, "Multiplicative valued difference fields", see also Laskowski–Pal, "Model companion of ordered theories with an automorphism".

2.3. Comments, Remarks and Subproblems

1) Linear orders are monadically NIP, so a formula with IP must necessarily involve the group operation in a fundamental way, e.g. not just to define the fixed group.
2) The structures described in point B) above are in fact all the o-minimal ones. We were unable to find references to this in Pal or Laskowski–Pal, but the proof simply consists in observing that if e.g. A is the set defined by $x > 0 \wedge L(x) > 0$, then $A \cup -A \cup \{0\}$ is a subgroup, so by o-minimality it must be trivial or improper. This solves the bonus question.
3) We lack examples of IP expansions by automorphisms of NIP ordered structures. For example, we do not even know of an IP real closed field with an automorphism.
4) Without the automorphisms, the qe for oags is quite involved, so if one believes the question has a positive answer it may be reasonable to first try to address the divisible case.
5) It may even be possible that expanding an oag by arbitrary subgroups of its cartesian power (note that the graph of σ is one such) still yields a NIP structure.
6) Existential formulas are positively NIP, but not known to be NIP. Subproblem: find an interesting formula with at least two alternating quantifiers.
7) At the very end of the session, this formula was written down as an attempt at IP. We did not think about it, so chances are that the attempt is silly. In particular, note that NIP formulas are closed under conjunctions, so if this has IP then one of the conjuncts must have IP. $\exists x \in (b_{W,0}, b_{W,1}) f(x) = a_{i,0} \wedge \forall x \in (b_{W,0}, b_{W,1}) g(x) \neq a_{i,1})$

3. Endomorphisms of $\mathbb{R}\langle t\rangle$

3.1. **Problem** (Asked by Marcus Tressl) Let S be a real closed field extending $\mathbb{R}$ and of transcendence degree 1 over $\mathbb{R}$ (note that all such fields are isomorphic over $\mathbb{R}$ by o-minimality, using completeness of $\mathbb{R}$).

First formulation: Suppose K is a real closed subfield of S realizing every cut[1] of $\mathbb{Q}$. Is K isomorphic to S?

Second formulation: Let $\varphi : S \longrightarrow S$ be a ring homomorphism and let K be a real closed field with $\varphi(S) \subseteq K \subseteq S$. Is K isomorphic to S?

3.2. **Explanation** For every cardinal $\kappa \leq 2^{\aleph_0}$ there is an endomorphism φ of S such that $\operatorname{tr.deg}(S/\varphi(S)) = \kappa$.

Proof. By completeness of $\mathbb{R}$, there is some $\mu \in S$ that is positive infinitesimal (meaning $0 < \mu < \frac{1}{n}$ for all $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$). Choose countable pairwise disjoint infinite subsets $B_\alpha \subseteq \mathbb{R}$ for $\alpha < \kappa$ and some $T \subseteq \mathbb{R}$ disjoint from all the B_α such that $T \cup \bigcup_{\alpha<\kappa} B_\alpha$ is a transcendence basis of $\mathbb{R}$ (over $\mathbb{Q}$). Hence

$$C = \{\mu\} \cup T \cup \bigcup_{\alpha<\kappa} B_\alpha$$

is a transcendence basis of S. For $\alpha < \kappa$, choose an enumeration $(b_{\alpha,i})_{i\in\mathbb{N}}$ of B_α, rational numbers $q_{\alpha,i} > 0$ and define

$$\begin{aligned}
b^*_{\alpha,i} &= b_{\alpha,i} + \mu^{q_{\alpha,i}} \cdot b_{\alpha,i+1} \ (i \in \mathbb{N}) \\
B^*_\alpha &= \{b^*_{\alpha,i} \mid i \in \mathbb{N}\} \\
C^* &= \{\mu\} \cup T \cup \bigcup_{\alpha<\kappa} B^*_\alpha \\
S_0 &= \text{the algebraic closure of } C^* \text{ in } S.
\end{aligned}$$

The set $\{b_{\alpha,1} \mid \alpha < \kappa\}$ is algebraically independent over S_0 because otherwise $b_{\alpha,1}$ is algebraic over

$$\{\mu\} \cup T \cup \bigcup_{\alpha\neq\beta<\kappa} B^*_\beta \cup \{b_{\alpha,1} + \mu^{q_{\alpha,2}} \cdot b_{\alpha,2}, \ldots, b_{\alpha,n} + \mu^{q_{\alpha,n+1}} \cdot b_{\alpha,n+1}\}$$

for some $\alpha < \kappa$ and some $n \in \mathbb{N}$, contradicting the assumption that $b_{\alpha,1}, \ldots, b_{\alpha,n+1}$ are algebraically independent over $\{\mu\} \cup T \cup \bigcup_{\alpha\neq\beta<\kappa} B_\beta$.

We see that $\operatorname{tr.deg}(S/S_0) = \kappa$ and it remains to show that the map $\varphi : C \longrightarrow C^*$ that is the identity on $\{\mu\} \cup T$ and sends $b_{\alpha,i}$ to $b^*_{\alpha,i}$ has a (necessarily unique) extension to an isomorphism $S \longrightarrow S_0$. This follows from the fact that

(a) For all $c \in C$, the 1-type of c is equal to the 1-type of $\varphi(c)$ (all types here are meant over the empty set in RCF; recall from o-minimality that the non-isolated 1-types are in bijection with the non-principal cuts of the real closure $\mathbb{R}_{\text{alg}}$ of $\mathbb{Q}$).

(b) For all $n \in \mathbb{N}$ and any distinct $c_1, \ldots, c_n \in C$, the sequence of 1-types $\operatorname{tp}(c_1), \ldots, \operatorname{tp}(c_n)$ is orthogonal, meaning they imply a unique n-type.

In order to verify this, we first do the case where μ is not among the c_i; then notice that in that case, the 1-type of μ is not realized in the algebraic closure of $\{c_1, \ldots, c_n\}$. Details are left to the reader. □

[1]a **cut** of a totally ordered set X here is a partition $\xi = (\xi^L, \xi^R)$ of X with $\xi^L < \xi^R$ and ξ is **principal** just if ξ^L has a supremum in $X \cup \{\pm\infty\}$. Such a cut is **realized** by y from a totally ordered set Y extending X if $\xi^L < y < \xi^R$.

3.3. Here are some comments/remarks/subproblems that came out of the discussion (using the terminology from above)

(1) The case $\kappa = \operatorname{tr.deg}(S/S_0) = 2$ should be looked at first.
(2) For $n \in \mathbb{N}$ we write $\varphi^n = \varphi \circ \ldots \circ \varphi$ (n times). Here φ could be any map as in 3.1, or the map φ explicitly constructed in 3.2 for $\kappa = 1$. Then we have a descending chain of real closed fields $S \supsetneq S_0 = \varphi(S) \supsetneq S_1 = \varphi^2(S) \supsetneq S_2 = \varphi^3(S) \supsetneq \ldots$.
Subquestion: Is every 1-type (again: over the empty set in RCF) realized in $\bigcap_n \varphi^n(S)$? We conjectured that this is not the case. (If it were the case for some φ, it was unclear what this implies, but it would be a candidate for a counterexample.)
(3) The convex hull V of $\mathbb{Z}$ in S is the unique proper convex subring of S and the associated valuation has residue field $\mathbb{R}$ and value group $\mathbb{Q}$. If we consider $S_0 = \varphi(S)$ as a valued subfield of S, then the extension S/S_0 is immediate (because S_0 realizes all cuts of $\mathbb{Q}$ and so the residue field of S_0 realizes all cuts of $\mathbb{Q}$ that are realized in $\mathbb{R}$).
(4) Note that in the construction 3.2 above, with $q_{\alpha,i} = 1$ for all α, i, the ordered set S_0 is dense in S, because $b_{\alpha,1} + \mu b_{\alpha,2} - \mu(b_{\alpha,2} + \mu b_{\alpha,3}) = b_{\alpha,1} - \mu^2 b_{\alpha,3} \in S_0$ and continuing in this way we see that $b_{\alpha,1}$ is the limit of the sequence $b_{\alpha,1} + (-1)^{k+1}\mu^k b_{\alpha,k+1} \in S_0$.
However if $q_{\alpha,i} = \frac{1}{2^i}$ for all α, i, this argument does not work anymore.
Subquestion(s): Find an explicit example where S_0 is not dense in S. Can we use the construction from 3.2 for some suitable choice of the data? Can we show $K \cong S$ under the assumption that S_0 is dense in K?
(5) Let $\iota : S_0 \hookrightarrow S$ be the inclusion map. It might be informative to write the homomorphism $\iota \circ \varphi : S \longrightarrow S$ as an inclusion $S \hookrightarrow \mathbb{R}((t^{\mathbb{Q}}))$.
This means the following. If we identify S with the subfield S_0 of S along the embedding $\iota \circ \varphi$ we get a real closed field L containing S such that L is isomorphic to S. By (3) this embedding is immediate when all fields are equipped with the valuation whose valuation ring is the convex hull of $\mathbb{Z}$. By general valuation theory, there is an $\mathbb{R}$-embedding of L into $\mathbb{R}((t^{\mathbb{Q}}))$ (where we set $t = \mu$) and so we may assume that $L \subseteq \mathbb{R}((t^{\mathbb{Q}}))$ all along.
So the question then reads as follows:
(†): Let $\mathbb{R} \subseteq S \subseteq K \subseteq L \subseteq \mathbb{R}((t^{\mathbb{Q}}))$ be real closed fields, where S is the algebraic closure of t in $\mathbb{R}((t^{\mathbb{Q}}))$ and suppose L is isomorphic to S. Is K also isomorphic to S? (This is obviously the case if the isomorphism $S \longrightarrow L$ is over $\mathbb{R}$, because then $L = S$.)
[By (4), S might in general not be dense in L. Exercise: S is not isomorphic to $S\langle\exp(t)\rangle$.]