SELF-ASSESSMENT TEST FOR THE M.SC. IN PURE MATHEMATICS AND MATHEMATICAL LOGIC

These questions are based on past questions from first and second year undergraduate courses at Manchester; the subject headings correspond to the respective undergraduate lecture courses. You should be able to solve the majority of these questions, possibly by referring to textbooks or to your lecture notes. Equally important: Certain words below are in **boldface**. They refer to mathematical objects and concepts that will be assumed as prerequisites in the courses of the programme. You should be able to define and understand most of them in a detailed manner without using references. There may be some small gaps in your knowledge, or some notation may be used in a way that is different to what you are used to.

The questions are merely indicative of the level of background knowledge that would be assumed at the beginning of the M.Sc in Pure Mathematics and Mathematical Logic. In particular, they do <u>not</u> cover all the prerequisite material. There are other topics in mathematics which you will be expected to be familiar with.

If you are particularly interested in the Logic component of the programme you should also have some knowledge in propositional logic (see headline 7).

1. Sets, numbers and functions

1.1. Prove that $4^n + 6n - 1$ is divisible by 9 for all $n \geq 1$.

1.2. (i) Let X be a set. What is meant by an **equivalence relation** on X?
(ii) Explain how the set $\mathbb{Z}_n$ of integers modulo n is defined by means of an equivalence relation on the set of integers $\mathbb{Z}$.
(iii) Give the addition and multiplication tables for $\mathbb{Z}_6$. Which elements are invertible in $\mathbb{Z}_6$?
(iv) Solve the equation $x^2 + x = 0$ in $\mathbb{Z}_6$, giving all solutions.

1.3. (i) Let X, Y be sets and let $f : X \to Y$ be a function. What does it mean to say that f is (a) **injective**, (b) **surjective**, (c) **bijective**? If $A \subseteq X$, what is $f(A)$? If $B \subseteq Y$, what is $f^{-1}(B)$? In the case $X = Y = \mathbb{R}$ and $f(x) = x^2$, what is $f^{-1}([-1, 2])$?
(ii) Suppose that X has 3 elements and Y has 2 elements. How many functions $f : X \to Y$ are there? How many of these are injective? How many are surjective? How many are bijective?

1.4. What is a **countable** set? Is $\mathbb{R}$ countable?

1.5. What is the **powerset** of a set? How many elements does the powerset of $\{1, 2, 3, 4, 5, 7, 8, 9\}$ have?

1.6. Compute a square root of the **imaginary unit** $i = \sqrt{-1}$.

2. Linear algebra

2.1. Define what is a **complex vector space**. Give a precise definition of a **basis** of V and of the **dimension** of V.

2.2. For a real vector space V, what does it mean to say that a map $T : V \to V$ is **linear**? If T is linear, what is the **kernel** of T?

2.3. Let V be a finite dimensional real vector space and let $T : V \longrightarrow V$ be linear.
(i) Show that T is injective if and only if T is surjective.
(ii) What is meant by an **eigenvalue** of T and what is meant by an **eigenvector** of T?
(iii) Find the eigenvalues and the corresponding eigenvectors of the linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ that maps the three unit vectors of $\mathbb{R}^3$ to $(-3, 2, 1), (-2, 6, -2)$ and $(1, -2, 3)$.

2.4. What is a **quadratic matrix** and what is the **determinant of a quadratic matrix**?

2.5. Fix $k \in \mathbb{N}$. Let V_k denote the space of all real polynomials of degree less than or equal to k. Explain why V_k is $(k + 1)$-dimensional, and give a basis.

In terms of this basis, calculate the determinant of the matrix of the linear map T, which maps a polynomial to its derivative. What is the kernel of T?

3. Real analysis

3.1. (i) <u>In terms of εs and ns</u>, what does it mean for a sequence of real numbers a_n to **converge**?
(ii) Prove from this definition that if $a_n \to a$ and $b_n \to b$, then $a_n \cdot b_n \to a \cdot b$.

3.2. What does it mean to say that $\sum_{n=1}^{\infty} a_n$ **converges**? If $\sum_{n=1}^{\infty} a_n < \infty$, prove that $a_n \to 0$. Give an example to show that the converse is false.

3.3. (i) For which real values of s does the **integral**

$$\int_1^{\infty} \frac{1}{x^s}\, dx$$

converge?
(ii) Hence explain for which real values of s the series

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

is finite.

3.4. (i) Give a precise definition <u>in terms of εs and δs</u> what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be **continuous**.
(ii) Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Is f continuous? Justify your answer.

3.5. State the **Intermediate Value Theorem**. Prove that there is a value of x in $(0, \frac{\pi}{2})$ for which

$$\frac{x}{\sin x} + \frac{1}{\cos x} = \pi.$$

Prove that this value of x is unique within $(0, \frac{\pi}{2})$.

3.6. Compute the first and the second **derivative** of $|x|^3$. Explain why this function does <u>not</u> have a third derivative.

3.7. Give an example, or sketch the graph of a differentiable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ whose derivative at 0 is 1 such that f is **not** increasing in any open interval containing 0.

4. Groups, rings and fields

4.1. What does it mean to say that a set G together with an operation $G \times G \to G$ is a **group**?

4.2. (i) For each $a \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{R}$, define the map $f_{a,b} : \mathbb{R} \to \mathbb{R}$ by

$$f_{a,b}(x) = ax + b.$$

Prove that $f_{a,b} \circ f_{c,d} = f_{ac,ad+b}$. (Here, $\circ$ denotes composition of maps.) Hence prove that the set

$$G = \{f_{a,b} \mid a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R}\}$$

forms a group under composition of maps.

(ii) Prove that the sets

$$G_1 = \{f_{a,0} \mid a \in \mathbb{R} \setminus \{0\}\}, G_2 = \{f_{1,b} \mid b \in \mathbb{R}\}$$

are both **subgroups** of G, but that only one is a **normal** subgroup.

4.3. Define what is a **commutative ring** and what is a **principal ideal domain**. Show that for every **field** K, the polynomial ring $K[X]$ is a principal ideal domain.

4.4. Give an example of a finite non-commutative ring.

4.5. For a commutative **domain** R with unit, what is the **fraction field** (also called 'quotient field') of R?

4.6. Give an example of a field K and a polynomial $P(X)$ of $K[X]$ with $P \neq 0$ such that $P(a) = 0$ for all $a \in K$.

5. Metric spaces

5.1. What does it mean to say that d is a **metric** on a set X? Give an example of a metric space X together with a nonempty subset $S \neq X$ such that S is at the same time **open** and **closed**.

5.2. Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Define

$$|f| = \sup_{x \in [0,1]} |f(x)|.$$

Explain how to use this to construct a metric on $C([0,1], \mathbb{R})$, the **space of continuous real-valued functions** defined on $[0,1]$.

5.3. What does it mean to say that a metric space is **compact**? What does it mean to say that a metric space is **connected**? Is $[0,1]$ connected? Is $[0,1]$ compact? Is $C([0,1], \mathbb{R})$ compact with the metric defined in 5.2.?

6. Complex analysis

This topic is relevant for many of the pure components of the programme.

6.1. State **De Moivre's theorem** and deduce that for all $\vartheta \in \mathbb{R}$ we have

$$\cot(5\vartheta) = \frac{1 - 10\tan^2\vartheta + 5\tan^4\vartheta}{1 - 10\tan^3\vartheta + 5\tan\vartheta}$$

Here cot is the cotangent function.

6.2. What is a **holomorphic function** and what are the **Cauchy-Riemann equations**?

6.3. By using **Cauchy's Residue theorem**, calculate

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+5)(x^2+2)}\, dx.$$

7. Propositional logic

This topic is relevant for the Logic components of the programme.

7.1. What is the **language of propositional logic**? Define what is a **tautology** and define what is a **formal proof** in propositional logic (you may choose your favorite **proof system**).

7.2. Formulate and prove the **soundness theorem** of propositional logic.

7.3. What does the **completeness theorem** of propositional logic say?