Singularities (J.P. Brasselet ed.),
L.M.S. Lecture Note Series **201**,
C.U.P. 1994. (pp. 263–284)

Deformations of maps on complete intersections, Damon's $\mathcal{K}_V$-equivalence and bifurcations

David Mond and James Montaldi

Abstract

A recent result of J. Damon's [4] relates the $\mathcal{A}_e$-versal unfoldings of a map-germ f with the $\mathcal{K}_{D(G)}$-versal unfoldings of an associated map germ which induces f from a stable map G. We extend this result to the case where the source is a complete intersection with an isolated singularity. In a similar vein, we also relate the bifurcation theoretic versal deformation of a bifurcation problem (map-germ) g to the $\mathcal{K}_\Delta$-versal deformation of an associated map germ which induces g from a versal deformation of the organizing centre g_0 of g, where Δ is the bifurcation set of this versal deformation.

The extension of Damon's theorem is used to provide an extension (again to cases where the source is an ICIS) of a result of Damon and Mond relating the discriminant Milnor number of a map to its $\mathcal{A}_e$-codimension.

Introduction

Recently Damon has introduced a generalized version of contact equivalence of map-germs, which he calls $\mathcal{K}_V$-equivalence ([3]; see the background section below for definitions). In [4] he shows how the deformation theory for left-right equivalence of a map-germ $f : \mathbf{C}^n, 0 \to \mathbf{C}^p, 0$ can be understood in terms of the $\mathcal{K}_V$-equivalence and deformation theory of an auxilliary map-germ γ which induces f from a stable unfolding G by a base change, taking as V the discriminant of G (our G and γ replace Damon's F and g):

$$\begin{array}{ccc} \mathbf{C}^{n+q}, 0 & \xrightarrow{G} & \mathbf{C}^{p+q}, 0 \supset V \\ \uparrow i & & \uparrow \gamma \\ \mathbf{C}^n, 0 & \xrightarrow{f} & \mathbf{C}^p, 0 \end{array}$$

where i is the inclusion. This identification of the two deformation theories led to the discovery, by Damon and Mond [5], of a "$\mu \geq \tau$" type theorem, relating the $\mathcal{A}_e$-codimension of a map-germ $\mathbf{C}^n, 0 \to \mathbf{C}^p, 0$ ($n \geq p$) to the vanishing topology of its discriminant.

In this paper we begin by generalizing Damon's result to include the case where the domain of f is an isolated complete intersection singularity (ICIS) X, giving in the process what we feel to be a clearer proof. This leads immediately to the following generalization of the main theorem of [5] to the case of a map $f : X, 0 \to \mathbf{C}^p$, where X is an n-dimensional ICIS with $n \geq p$ and (n,p) nice-dimensions in the sense of Mather. Let $f_t : X_t \to \mathbf{C}^p$ be a stable perturbation of the pair (X, f), i.e. X_t is a smoothing of X and f_t is a stable map. Then the discriminant of f_t has the homotopy type of a wedge of $(p-1)$-spheres the number μ_Δ of which satisfies

$$\mu_\Delta \geq \mathcal{A}_e\text{-codim}(X, f),$$

with equality if (X, f) is quasihomogeneous. Here $\mathcal{A}_e$-codim(X, f) denotes the number of parameters necessary for a versal deformation of (X, f) in the sense defined in Section 1 below. This theorem is proved in Section 2. (A more algebraic proof of Damon's theorem has recently been given by du Plessis, Gaffney and Wilson [15].)

In Section 3, we apply the idea of $\mathcal{K}_V$-equivalence to bifurcation theory. The map-germ $g(x, \lambda)$ (and associated "bifurcation problem" $g(x, \lambda) = 0$) can be obtained from a versal deformation $G(x, u)$ of $g_0(x) = g(x, 0)$, by means of a map γ from λ-space to u-space. We show that the bifurcation-theoretic deformation theory of g is isomorphic to the $\mathcal{K}_V$-deformation theory of γ, where V is the bifurcation set of the deformation G (Theorem 3.1). This provides a theoretical framework for the "path formulation" of bifurcation theory [6, §12(b)]. Combining this with the results of Section 1, we obtain the following extension of a classical theorem due to Martinet (Theorem 3.3): let $V_g = g^{-1}(0)$, and $\pi_g : V_g \to \Lambda$ be the projection (where Λ is λ-space, the parameter space of the deformation). Then the $\mathcal{A}_e$ codimension of the pair (V_g, π_g) (as defined in Section 1) is equal to the bifurcation theoretic codimension of g.

For simplicity, the results in this paper are all proved for the complex analytic category. However, except for those in Section 2, they are all valid in the real analytic category as well.

Acknowledgements We would like to thank Jacques Furter for some useful discussions, especially concerning Section 3. We would also like to take this opportunity to thank J.-P. Brasselet in particular, and also the other organizers, for such an excellent and well-organized conference.

Background

Here we describe briefly three equivalence relations on spaces of map germs and the corresponding deformation theories. In what follows, X and Y denote germs of analytic spaces. If $X \subset E$ (E smooth), Derlog(X) denotes the $\mathcal{O}_E$-submodule of Θ_E consisting of vector fields tangent to X, while Θ_X denotes the $\mathcal{O}_X$-module of vector fields on X. Thus, $\Theta_X \cong \text{Derlog}(X)/I(X)\Theta_E$. For a Cartesian product $X \times Y$ we use $\Theta_{X\times Y/Y}$ to denote the $\mathcal{O}_{X\times Y}$-module of vector fields on $X \times Y$ tangent to the fibres of the Cartesian projection $X \times Y \to Y$.

$\mathcal{A}$- (left-right) equivalence: Two map germs $f, \overline{f} : X \to Y$ are $\mathcal{A}$-equivalent if there are diffeomorphisms ϕ of X and ψ of Y with $\overline{f} = \psi \circ f \circ \phi$. The $\mathcal{O}_Y$-module

$$N\mathcal{A}_e \cdot f := \frac{f^*\Theta_Y}{tf(\Theta_X) + f^{-1}\Theta_Y}$$

is the $\mathcal{A}_e$ normal space for f. When X and Y are smooth, then by the versality theorem ([12], [17]), $\dim_{\mathbf{C}} N\mathcal{A}_e f$ is equal to the number $\mathcal{A}_e$-codim(f) of parameters necessary for an $\mathcal{A}_e$-versal unfolding of f. The same statement with X an ICIS (and Y smooth) is given in Corollary 1.6 below.

$\mathcal{K}_{\mathrm{un}}$-equivalence: Let $g_0 : X \to Y$ be a map germ and let $g, \overline{g} : X \times \Lambda \to Y$ be deformations of g_0. They are $\mathcal{K}_{\mathrm{un}}$-equivalent if there are diffeomorphisms H of $X \times \Lambda$ and h of Λ with $H(x, \lambda) = (H'(x, \lambda), h(\lambda))$ such that

$$g(x, \lambda) = S(x, \lambda).\overline{g}\,(H(x, \lambda)),$$

with $S(x, \lambda) \in GL_p$. (In bifurcation theory, this is simply called, *equivalence of the bifurcation problems* $g = 0$ and $\overline{g} = 0$. The subscript 'un' is due to Damon [2] and refers to unfoldings.)

The $\mathcal{K}_{\mathrm{un}}$ normal space of g is the $\mathcal{O}_\Lambda$-module

$$N\mathcal{K}_{\mathrm{un}} \cdot g := \frac{g^*\Theta_Y}{tg(\Theta_{X\times\Lambda/\Lambda} + \pi^{-1}\Theta_\Lambda) + g^*\boldsymbol{m}_Y\Theta_Y}$$

where $\pi : X \times \Lambda \to \Lambda$ is the Cartesian projection. (In bifurcation theory, $\dim N\mathcal{K}_{\mathrm{un}} g$ is called the codimension of the bifurcation problem $g = 0$.)

$\mathcal{K}_V$-equivalence: Let V be a subspace of Y. Two map germs $\gamma, \overline{\gamma} : X \to Y$ are said to be $\mathcal{K}_V$-equivalent if there are diffeomorphisms H of $X \times Y$ and h of X with H preserving $X \times V$, and $H(x, y) = (h(x), H'(x, y))$ satisfying

$$H(x, \gamma(x)) = (h(x), \overline{\gamma} \circ h(x)).$$

This group of diffeomorphisms was introduced by J. Damon in [3]. Damon shows that the $\mathcal{K}_V$ normal space for γ is given by

$$N\mathcal{K}_{V,e} \cdot \gamma = \frac{\gamma^*\Theta_Y}{t\gamma(\Theta_X) + \gamma^* \operatorname{Derlog}(V)}.$$

The dimension of $N\mathcal{K}_V\gamma$ is thus the number of parameters necessary for a $\mathcal{K}_{V,e}$-versal deformation of γ.

In the case that $V = \{0\} \subset Y$, then $\mathcal{K}_V$-equivalence reduces to $\mathcal{K}$-equivalence. It is perhaps worth pointing out that in contrast to the case $V = \{0\}$, isomorphism of $\gamma^{-1}(V)$ and $\overline{\gamma}^{-1}(V)$ does not in general imply the $\mathcal{K}_V$-equivalence of γ and $\overline{\gamma}$.

1 Deformations of maps on complete intersections

Let X be a germ of an isolated complete intersection singularity (ICIS), with base point 0. Let $f : X \to P$ be a map-germ, where P is a germ of a smooth space ($P = \mathbf{C}^p, 0$). We say $x \in X$ is a singular point of (X, f) either if x is a singular point of X, or, in the case that $x \in X$ is regular, if f is not a submersion at x. We denote the set of singular points by $\Sigma_{(X,f)}$. We say that f, or (X, f), has *finite singularity type* if the restriction of f to $\Sigma_{(X,f)}$ is finite-to-one.

Let $G : N \to Q$ and $\gamma : P \to Q$ be map germs (where N, P, and Q are germs of smooth spaces, of dimensions n, p and q respectively). Pulling back G by γ gives a space X as the fibre product of G and γ, and a map $f = f_\gamma : X \to P$ which is just the projection to P, giving the following commutative diagram.

$$\begin{tikzcd} N \arrow[r, "G"] & Q \\ X \arrow[u, "i"] \arrow[r, "f"] & P \arrow[u, "\gamma"'] \end{tikzcd}$$

We will say that (X, f) is induced from G by γ (by fibre product).

Recall that the fibre product of G and γ can be defined by

$$X = \{(x, y) \in N \times P \mid G(x) = \gamma(y)\},$$

or equivalently as $(G \times \gamma)^{-1}(\Delta_{Q\times Q})$, where $\Delta_{Q\times Q}$ is the diagonal in $Q \times Q$. It is clear that X is smooth of codimension q in $N \times P$ if and only if γ is transverse to G, and (provided $n + p \geq q$) is an ICIS of codimension q if and only if γ is transverse to G off $(0, 0) \in X$.

The first result of this section asserts that provided X is an ICIS and G is stable then up to isomorphism, any unfolding of $f : X \to P$ can be obtained by deforming γ. Prior to proving this, we show that in fact a large class of pairs (X, f) can be obtained as fibre products.

Lemma 1.1 *Suppose that X is an* ICIS *and $f : X \to P$ is a map germ of finite singularity type. Then there exist a stable map germ $G : N \to Q$, and an immersion germ $\gamma : P \to Q$, such that f is induced from G by γ (by fibre product).*

Remark The property that (X, f) be of finite singularity type holds in general (e.g. [17], Theorem 5.1). Moreover, a small modification of the argument given in [17] shows that for any given ICIS X the property that $f : X \to P$ is of finite singularity type also holds in general.

PROOF Suppose that $h : E \to \mathbf{C}^k, 0$ is a map-germ with $h^{-1}(0) = X$, and let $\hat{f} : E \to P$ extend f. The hypothesis on (X, f) guarantees that the map-germ $(h, \hat{f}) : E \to (\mathbf{C}^k, 0) \times P$ is of finite singularity type. It then follows by a theorem of Mather that there exists a stable map-germ

$$G : E \times (\mathbf{C}^d, 0) \to (\mathbf{C}^k, 0) \times P \times (\mathbf{C}^d, 0)$$

which is a level preserving unfolding of $(h, \hat{f})$. We can construct G by means of the following algorithmic procedure: choose $\alpha_1, \ldots, \alpha_n \in \Theta(h)$ and $\beta_1, \ldots, \beta_b \in \Theta(\hat{f})$ such that, viewing $\Theta(h)$ and $\Theta(\hat{f})$ as subspaces of $\Theta(h, \hat{f})$, the α_i and β_j together form a $\mathbf{C}$-basis for the quotient

$$\frac{\boldsymbol{m}_E \, \Theta(h, \hat{f})}{T\mathcal{K}_e(h, \hat{f}) \cap \boldsymbol{m}_E \, \Theta(h, \hat{f})}.$$

Let

$$G(x, u) = (h(x) + \sum_{i=1}^{a} u_i \alpha_i(x),\ \hat{f}(x) + \sum_{i=1}^{b} u_{a+j} \beta_{b_j}(x),\ u),$$

where $a + b = d$. Now define $\gamma : P \to (\mathbf{C}^k, 0) \times P \times (\mathbf{C}^d, 0)$ by $\gamma(y) = (0, y, 0)$. Then G is infinitesimally stable (see e.g. [13]) and it is easy to see that γ has the required properties. $\square$

We need to make precise the notions of unfolding, and versal unfolding, of the map $f : X \to P$, and of the pair (X, f).

Definition 1) An *unfolding* of (X, f) *over* a space germ S is a map $F : \mathcal{X} \to P \times S$ together with a flat projection $\pi : \mathcal{X} \to S$, such that if $\pi_2 : P \times S \to S$ is the cartesian projection, then
i) $\pi_2 \circ F = \pi$,
ii) There is an isomorphism $j : X \to \pi^{-1}(0)$, and $F \circ j : X \to P \times \{0\} = P$ is equal to f.
2) The unfoldings $(\mathcal{X}, \pi, F, j)$ and $(\mathcal{X}', \pi', F', j')$ over S are *isomorphic* if there are isomorphisms $\Phi : \mathcal{X} \to \mathcal{X}'$ and $\Psi : P \times S \to P \times S$ over S, such that the diagram below commutes.

$$\begin{tikzcd}
 & X \arrow[dl, "j"'] \arrow[dr, "j'"] & \\
\mathcal{X} \arrow[rr, "\Phi"] \arrow[dr, "\pi"] \arrow[dd, "F"'] & & \mathcal{X}' \arrow[dl, "\pi'"'] \arrow[dd, "F'"] \\
 & S & \\
P \times S \arrow[ur, "\pi_2"] \arrow[rr, "\Psi"] & & P \times S \arrow[ul, "\pi_2"']
\end{tikzcd}$$

3) If $(\mathcal{X}, \pi, F, j)$ is an unfolding of (X, f) over S, a germ $\rho : T \to S$ induces an unfolding $(X_\rho, \pi_\rho, F_\rho, j_\rho)$ of (X, f) by base change, in the obvious way. The unfolding $(\mathcal{X}, \pi, F, j)$ is *versal* if every unfolding of (X, f) is isomorphic to an unfolding induced from $(\mathcal{X}, \pi, F, j)$ by base change.

4) An *unfolding of* f is an unfolding of (X, f) with $\mathcal{X} = X \times S$, and with $\pi : \mathcal{X} \to S$ being the cartesian projection.

Canonical Construction Suppose that $f : X \to P$ is induced from the map $G : N \to Q$ by $\gamma : P \to Q$ (by fibre product). Associated to any deformation $\Gamma : P \times S \to Q$ of γ there is a fibre product $\mathcal{X} = N \times_Q (P \times S)$, a map $F : \mathcal{X} \to P \times S$ and a natural map $i : X \to \mathcal{X}$ induced by the inclusion $P \times \{0\} \hookrightarrow P \times S$, such that $\pi_2 \circ F \circ i = f$. We make $F : \mathcal{X} \to P \times S$ into an unfolding of f by defining $\pi : \mathcal{X} \to S$ by $\pi = \pi_2 \circ F$.

$$\begin{tikzcd} N \arrow[r, "G"] & Q \\ \mathcal{X} \arrow[u] \arrow[r, "F"] & P \times S \arrow[u, "\Gamma"'] \end{tikzcd}$$

Fibre product is of course unique only up to (unique) isomorphism; in this case such an isomorphism is automatically an isomorphism over S, and thus gives rise to an isomorphism of unfoldings.

Proposition 1.2 *Let X be an* ICIS *and let $f : X \to P$ be a map of finite singularity type. If (X, f) is induced from the stable map $G : N \to Q$ by $\gamma : P \to Q$, and if $(\mathcal{X}, \pi, F, j)$ is an unfolding of (X, f), then there exists a deformation Γ of γ such that $(\mathcal{X}, \pi, F, j)$ is isomorphic to the unfolding obtained from Γ by the canonical construction above.*

PROOF Since X is a complete intersection, a deformation over the base S extends to a deformation over a smooth ambient space of S, so we may assume from the outset that S is smooth. Moreover, any deformation of X can be realised as an embedded deformation, so we may suppose that there is a map $H : E \times S \to \mathbf{C}^k$, with $H(x, s) = h(x) + r(x, s)$ deforming h, such that $\mathcal{X} = H^{-1}(0)$; by extending $F : \mathcal{X} \to P \times S$ to a level-preserving map $\hat{F} : E \times S \to P \times S$, with $\hat{F}(x, s) = (\hat{f}(x) + t(x, s), s)$ (for some extension $\hat{f}$ of f) we obtain an unfolding $(H, \hat{F}) : E \times S \to \mathbf{C}^k \times P \times S$ of the map (h, f), and now we define an unfolding $L : E \times \mathbf{C}^d \times S \to \mathbf{C}^k \times P \times \mathbf{C}^d \times S$ which is the "direct sum" of $(H, \hat{F})$ and G, by

$$L(x, u, s) = (h(x) + \sum_{i=1}^{a} u_i \alpha_i(x) + r(x, s), \hat{f}(x) + \sum_{j=1}^{b} u_{a+j} \beta_j(x) + t(x, v), u, s).$$

Note that $(\mathcal{X}, \pi, F, j)$ is induced from L by the map-germ $\Gamma_0 : P \times S \to \mathbf{C}^k \times P \times \mathbf{C}^d \times S$ defined by $\Gamma_0(y, s) = (0, y, 0, s)$:

$$\begin{tikzcd} E \times \mathbf{C}^d \times S \arrow[r, "L"] & \mathbf{C}^k \times P \times \mathbf{C}^d \times S \\ \mathcal{X} \arrow[u, "I_0"] \arrow[r, "F"] & P \times S \arrow[u, "\Gamma_0"'] \end{tikzcd}$$

where $I_0(x,s) = (x,0,s)$.

Now L is an unfolding of G, with base S, and as G is stable, L is trivial. Thus, there exist S-level-preserving diffeomorphisms Φ of $E \times \mathbf{C}^d \times S$ and Ψ of $\mathbf{C}^k \times P \times \mathbf{C}^d \times S$, such that the diagram

$$\begin{tikzcd} E \times \mathbf{C}^d \times S \arrow[r, "G \times \mathrm{id}_S"] & \mathbf{C}^k \times P \times \mathbf{C}^d \times S \\ E \times \mathbf{C}^d \times S \arrow[u, "\Phi"] \arrow[r, "L"] & \mathbf{C}^k \times P \times \mathbf{C}^d \times S \arrow[u, "\Psi"'] \end{tikzcd}$$

commutes. Juxtaposing these two diagrams gives the following fibre square.

$$\begin{tikzcd} E \times \mathbf{C}^d \times S \arrow[r, "G \times \mathrm{id}_S"] & \mathbf{C}^k \times P \times \mathbf{C}^d \times S \\ \mathcal{X} \arrow[u, "\Phi \circ I_0"] \arrow[r, "F"] & P \times S \arrow[u, "\Psi \circ \Gamma_0"'] \end{tikzcd}$$

Since I_0, Γ_0, Φ and Ψ are all S-level preserving, if we jettison the trivial S component on the top row, we still have a fibre square

$$\begin{tikzcd} E \times \mathbf{C}^d \arrow[r, "G"] & \mathbf{C}^k \times P \times \mathbf{C}^d \\ \mathcal{X} \arrow[u, "\phi \circ I_0"] \arrow[r, "F"] & P \times S \arrow[u, "\psi \circ \Gamma_0"'] \end{tikzcd}$$

where ϕ and ψ are Φ and Ψ composed with the cartesian projections forgetting S. Now $\Gamma = \psi \circ \Gamma_0$ is a deformation of γ, and the proposition is proved. $\square$

For a map germ G, we denote the discriminant of G by $D(G)$. The main result of this section, generalizing Damon's theorem [4], is

Theorem 1.3 *Let X be an* ICIS *and $f : X \to P$ be of finite singularity type. Suppose that (X, f) is induced from the stable map $G : N \to Q$ by the base-change $\gamma : P \to Q$. Then a $\mathcal{K}_{D(G),e}$-versal deformation $\Gamma_0 : P \times \mathbf{C}^d, 0 \to Q$ of γ induces a versal unfolding F_0 of f. If Γ_0 is miniversal, then so is F_0.*

PROOF We begin by showing that it is enough to prove this result in the case that γ is an immersion.

Define $\tilde{G} : N \times P \to Q \times P$ by $\tilde{G}(x,y) = (G(x), y)$, and the immersion $\tilde{\gamma} : P \to Q \times P$ by $\tilde{\gamma}(y) = (\gamma(y), y)$. Then (X, f) is also induced by $\tilde{\gamma}$ from $\tilde{G}$ (by fibre product). Moreover, $D(\tilde{G}) = D(G) \times P$. It is easy to see that if $\tilde{\Gamma} = (\Gamma, \Gamma')$ is an unfolding of $\tilde{\gamma}$, then $\tilde{\Gamma}$ is $\mathcal{K}_{D(\tilde{G}),e}$-versal if and only if Γ is $\mathcal{K}_{D(G),e}$-versal (see [4, Proposition 1.5]).

It follows that it is enough to prove the theorem for γ an immersion. This we now proceed to do.

For the remainder of the proof we follow Damon closely, and give this proof mainly for the sake of completeness. Let $\Gamma_1 : P \times \mathbf{C}^c, 0 \to Q$ be any deformation of γ, and let $\Gamma : P \times \mathbf{C}^d \times \mathbf{C}^c$ be the direct sum of Γ_0 and Γ_1. Each of these deformations of γ induces an unfolding of f by the canonical construction. We represent them as F_0, F_1 and F. Since Γ_1 is obtained from Γ by restriction, F_1 is obtained from F by restriction. To show that F_0 is versal, it is enough to show that F is a trivial extension of F_0. We show that this follows from the fact that Γ is a $\mathcal{K}_{D(G),e}$-trivial extension of Γ_0, by lifting the vector fields whose integral flows trivialise Γ. As this is done one dimension at a time, we may as well suppose, in order to lighten our notation, that Γ_1 is a 1-parameter deformation of γ.

Now Γ is a $\mathcal{K}_{D(G),e}$-trivial extension of Γ_0, by the $\mathcal{K}_{D(G),e}$-versality of Γ_0. Since $\Gamma \times \mathrm{id}_{\mathbf{C}^d \times \mathbf{C}}$ is an immersion, the infinitesimal criterion for $\mathcal{K}_{D(G),e}$-triviality takes an especially simple form: there exist vector fields $\eta \in \Theta_{\mathbf{C}^d \times \mathbf{C}^c/\mathbf{C}^c}$, $\zeta \in \Theta_{P \times \mathbf{C}^d \times \mathbf{C}^c/\mathbf{C}^d \times \mathbf{C}^c}$ and $\chi \in \mathrm{Derlog}(D(F) \times \mathbf{C}^d \times \mathbf{C}^c)/\mathbf{C}^d \times \mathbf{C}^c$, such that

$$\frac{\partial \Gamma}{\partial v} = -t\Gamma(\zeta + \eta) + \chi \circ (\Gamma \times \mathrm{id}_{\mathbf{C}^d \times \mathbf{C}}) \tag{1.1}$$

(here, as in the remainder of the proof, vector fields defined initially on one factor of a cartesian product are extended trivially to the product).

As χ is tangent to $D(F \times \mathrm{id}_{\mathbf{C}^d \times \mathbf{C}})$, it can be lifted with respect to $F \times \mathrm{id}_{\mathbf{C}^d \times \mathbf{C}}$: there exists $\delta \in \Theta_{N \times \mathbf{C}^d \times \mathbf{C}^c/\mathbf{C}^d \times \mathbf{C}^c}$ such that

$$t(F \times \mathrm{id}_{\mathbf{C}^d \times \mathbf{C}^c})(\delta) = \chi \circ (F \times \mathrm{id}_{\mathbf{C}^d \times \mathbf{C}^c}). \tag{1.2}$$

By the standard construction of fibre products, the spaces $\mathcal{X}_0$ and $\mathcal{X}$, and the maps $F_0 : \mathcal{X}_0 \to P \times \mathbf{C}^d$ and $F : \mathcal{X} \to P \times \mathbf{C}^d \times \mathbf{C}$, may be described as follows: let $H_0 : N \times P \times \mathbf{C}^d \to Q \times Q \times \mathbf{C}^d$ and $H : N \times P \times \mathbf{C}^d \times \mathbf{C} \to Q \times Q \times \mathbf{C}^d \times \mathbf{C}$ be defined by $H_0(x, y, u) = (G(x), \Gamma_0(y, u), u)$, $H(x, y, u, v) = (G(x), \Gamma(y, u, v), u, v)$, and let $\Delta_0 = \mathrm{diag}(Q \times Q) \times \mathbf{C}^d$ and $\Delta = \mathrm{diag}(Q \times Q) \times \mathbf{C}^d \times \mathbf{C}$. Then $\mathcal{X}_0 = H_0^{-1}(\Delta_0)$, $\mathcal{X} = H^{-1}(\Delta)$, F_0 is the restriction of F to $\mathcal{X}_0$, and F is the restriction of F to $\mathcal{X}$.

Write $H(x, y, u, v) = (h(x, y, u, v), u, v)$. Then $\partial h/\partial v = -dH(\delta, \zeta + \eta) + (\chi_i, \chi) \circ H$ by (1.1) and (1.2). Define a vector field $\alpha \in \Theta_{Q \times Q \times \mathbf{C}^d \times \mathbf{C}^c}$ by $\alpha = (\chi, \chi) + \eta + \frac{\partial}{\partial v}$. Then α is tangent to Δ, and if $\xi = \frac{\partial}{\partial v} + \eta + (\delta, \zeta)$, we have

$$tH(\xi) = \alpha \circ H. \tag{1.3}$$

This shows that ξ is tangent to $\mathcal{X}$ at its smooth points, and thus since smooth points are dense in $\mathcal{X}$, the restriction of ξ to $\mathcal{X}$ lies in $\Theta_{\mathcal{X}}$. Now (1.3) restricted to $H^{-1}(\Delta)$, shows that $F : \mathcal{X} \to P \times \mathbf{C}^d \times \mathbf{C}$ is a trivial extension of $F_0 : \mathcal{X}_0 \to P \times \mathbf{C}^d$.

This completes the proof of versality of F_0.

Minimality is proved essentially by reversing the argument: if Γ_0 is miniversal but F_0 is not, then F_0 is a trivial extension of some sub-unfolding F_{-1}. The trivialisation may be lifted to show that Γ_0 is a $\mathcal{K}_{D(G),e}$-trivial extension of a sub-unfolding Γ_{-1}, contradicting the minimality of Γ. □

The unfoldings of $f : X \to P$ we are discussing here simultaneously deform the space X and the map f. Separating these two deformations, one arrives on the one hand at the (well understood) theory of isolated complete intersection singularities, and on the other at the rather less-studied theory of $\mathcal{A}$-equivalence of map-germs defined on singular varieties. In the former, the principal deformation-theoretic invariant is the Tjurina number $\tau(X)$; in the latter, it is the dimension of the quotient space

$$N\mathcal{A}_e f := \frac{f^*\Theta_P}{tf(\Theta_X) + \omega f(\Theta_P)},$$

which we shall refer to as $\mathcal{A}_e$-codim(f). When X is smooth, this reduces to the usual $\mathcal{A}_e$-codimension of f. In the case where X is an ICIS, we shall refer to the dimension of a miniversal unfolding of the pair (X, f) as $\mathcal{A}_e$-codim(X, f).

Theorem 1.4 *With the notation and hypotheses of Theorem 1.3, there is an exact sequence of $\mathcal{O}_P$-modules,*

$$0 \to N\mathcal{A}_e f \to N\mathcal{K}_{D(G),e}\gamma \to T^1_X \to 0.$$

Consequently, $\mathcal{A}_e$-codim$(X, f) = \mathcal{A}_e$-codim$(f) + \tau(X)$.

Proof Suppose that Γ_0 is a $\mathcal{K}_{D(G),e}$-versal deformation of γ; the present theorem is proved by identifying, at an infinitesimal level, the sub-deformation of Γ_0 in which X is deformed trivially.

We begin by noting that, as in the proof of Theorem 1.3, we can assume that γ is an immersion; for, with the notation of the proof of Theorem 1.3, it is easy to see that there is a natural isomorphism $N\mathcal{K}_{D(G),e}\gamma \to N\mathcal{K}_{D(\tilde{G}),e}\tilde{\gamma}$.

Let Δ be the diagonal in $Q \times Q$; since $X = (G, \gamma)^{-1}(\Delta)$, an easy calculation shows that

$$T^1_X \cong \frac{(G, \gamma)^*(\Theta_{Q\times Q})}{t(G, \gamma)(\Theta_{N\times P}) + (G, \gamma)^*(\mathrm{Derlog}(\Delta))}.$$

In fact, if z_i and z'_i are coordinates on the two copies of Q, and h is the map on $Q \times Q$ with component functions $z_i - z'_i$, then

$$th : (G, \gamma)^*\Theta_{Q\times Q} \to (h \circ (G, \gamma))^*\Theta_Q$$

induces an isomorphism between $N\mathcal{K}_{\Delta,e}(G, \gamma)$ and $N\mathcal{K}_e(G - \gamma)$.

We shall later make use of the fact that

$$\mathrm{Derlog}(X) = (t(G, \gamma))^{-1}((G, \gamma)^* \,\mathrm{Derlog}(\Delta)).$$

A deformation of γ induces a deformation of X, and thus, at the infinitesimal level there is a map $\phi : \gamma^*(\Theta_Q) \to T^1_X$, which is given by $\phi(\xi) = (0, \xi)$. Recall that

$$N\mathcal{K}_{D(G),e}\gamma = \frac{\gamma^*\Theta_Q}{t\gamma(\Theta_P) + \gamma^* \,\mathrm{Derlog}(D(G))}.$$

Step 1: ϕ passes down to the quotient to give a map $\overline{\phi} : N\mathcal{K}_{D(G)}\gamma \to T^1_X$. To see this, we must show that $t\gamma(\Theta_P) + \gamma^* \,\mathrm{Derlog}(D(G)) \subset \ker \phi$. First, if $\eta \in \mathrm{Derlog}(D(G))$, there exists $\delta \in \Theta_N$ such that $tG(\delta) = \eta\circ G$, and so $\phi(\eta\circ\gamma) = (0, \eta\circ\gamma) = (\eta,\eta)\circ(G,\gamma) - t(G,\gamma)(\delta, 0)$ which is of course a member of $t(G,\gamma)(\Theta_{N\times P}) + (G,\gamma)^*(\mathrm{Derlog}(\Delta))$; second, if $\zeta \in \Theta_P$, then regarding ζ as an element of $\Theta_{N\times P}$, (which is 0 in the N direction) we have $\phi(t\gamma(\zeta)) = t(G,\gamma)(\zeta)$. Thus, $\overline{\phi}$ is well-defined.

Step 2: $\overline{\phi}$ is an epimorphism. As every deformation of X is induced, up to equivalence of deformations, by an appropriate deformation of γ, by Theorem 1.3, this is immediate. In fact, it is easy to find a direct algebraic proof, avoiding Theorem 1.3.

Step 3: What is the kernel of $\overline{\phi}$? The kernel of ϕ consists of the tangent vector fields to 1-parameter deformations Γ of γ such that the induced unfolding $(\mathcal{X}, S, \pi, F)$ of (X, f) deforms X trivially. Let $f_s : X_s \to P$ be the map obtained by restricting F. If the family $h_s : X \to X_s$ of diffeomorphisms trivialises the deformation of X, then we have a 1- parameter family of maps $f_s \circ h_s : X \to P$, and thus at the infinitesimal level a tangent vector field $\frac{d(f_s\circ h_s)}{ds}|_{s=0} \in f^*(\Theta_P)$. Although the trivialising family h_s is not unique, and thus there is no natural map $\ker(\phi) \to f^*(\Theta_P)$, this construction does enable us to define a map $\psi : \ker(\overline{\phi}) \to N\mathcal{A}_e f$, by $\psi(\xi) = \frac{d(f_s\circ h_s)}{ds}|_{s=0}$. Since $f_s : X_s \to P$ is simply the restriction of the cartesian projection $N \times P \to P$, then in infinitesimal terms ψ is defined as follows: if $\xi \in \gamma^*(\Theta_Q)$, then $\overline{\phi}(\xi + T\mathcal{K}_{D(G),e}\gamma) = 0$ if and only if there exist a vector field $\rho \in \Theta_{N\times P}$ and an element $\beta \in (G,\gamma)^*(\mathrm{Derlog}(\Delta))$ such that

$$(0, \xi) = t(G,\gamma)(\rho) + \beta. \tag{1.4}$$

Write $\rho = \rho_N + \rho_P$, where $\rho_N \in \Theta_{N\times P/P}$ and $\rho_P \in \Theta_{N\times P/N}$. Then the infinitesimal deformation of f induced by $(0,\xi)$ is the class in $N\mathcal{A}_e f$ of $\rho_{P|X}$; that is,

$$\psi(\xi) = \rho_{P|X} + T\mathcal{A}_e f.$$

ψ **is well-defined**: first, the ambiguity in the choice of the solutions ρ, β to (1.4) does not affect the class of $\rho_{P|X}$ in $N\mathcal{A}_e f$; for if $(0,\xi) = t(G,\gamma)(\rho) + \beta = t(G,\gamma)(\rho') + \beta'$, then $t(G,\gamma)(\rho - \rho') \in (G,\gamma)^*(\mathrm{Derlog}(\Delta))$, and so $\rho - \rho' \in \mathrm{Derlog}(X)$ (since $X = (G,\gamma)^{-1}(\Delta)$), and $\rho_{P|X} - \rho'_{P|X} \in tf(\Theta_X)$.

Secondly, if $\xi \in T\mathcal{K}_{D(G),e}\gamma$, then $\rho_{P|X} \in tf(\Theta_X) + f^{-1}(\Theta_P)$; for if $\xi = \eta\circ\gamma$, with $\eta \in \mathrm{Derlog}(D(G))$, then there exists $\delta \in \Theta_N$ such that $tG(\delta) = \eta \circ G$. Thus, (1.4) has a solution with $\rho = -\delta$: $(0,\xi) = -t(G,\gamma)(-\delta) + (\eta,\eta)\circ(G,\gamma)$; here $\rho_P = 0$ and so $\psi(\xi) = 0$. And if $\xi = t\gamma(\alpha)$ with $\alpha \in \Theta_P$, then (1.4) has solution $(0,\xi) = t(G,\gamma)(0,\alpha)$. Here $\rho_P(x,y) = \alpha(y)$, and so $\rho_{P|X} \in f^{-1}(\Theta_P)$.

ψ **is 1-1**: If (1.4) has solution $(0,\xi) = t(G,\gamma)(\rho_N + \rho_P) + \beta$ with $\rho_{P|X} = tf(\alpha) + \eta\circ f$, then extending $\eta \in \Theta_P$ trivially to $\eta \in \Theta_{N\times P/P}$, and recalling that $tf(\alpha_N + \alpha_P)$ is just α_P, we have that for some α_N, the vector field $\rho_P - \eta + \alpha_N$ is in $\mathrm{Derlog}(X)$. As $\mathrm{Derlog}(X) = (t(G,\gamma))^{-1}((G,\gamma)^*\,\mathrm{Derlog}(\Delta))$, it follows that

$$(0,\xi) = t(G,\gamma)(\rho_N - \alpha_N + \eta) + \tilde{\beta} \tag{1.5}$$

for some $\tilde{\beta} \in (G,\gamma)^*\,\mathrm{Derlog}(\Delta)$. Write $\sigma_N = \rho_N - \alpha_N$. From (1.5) we deduce that for all $y \in P$ and $x \in G^{-1}(\gamma(y))$,

$$\xi(y) = t\gamma(\eta(y)) - tG(\sigma_N(x, G(x))); \tag{1.6}$$

for since $(x,y) \in X$, the components of $\tilde{\beta}$ in the two copies of Q are equal. Now, we may suppose that γ is an immersion, and at this point it is convenient to regard P as a smooth subvariety of Q. Thus, we regard $t\gamma(\eta) - \xi$ as an element of the restriction $\Theta_{Q|P}$. To complete the proof of the injectivity of ψ, we have to show that $t\gamma(\eta) - \xi$ is the restriction to P of some element of $\mathrm{Derlog}(D(G))$; this follows from Lemma 1.5 below.

ψ **is onto**: Given $\eta \in f^*(\Theta_P)$, choose a 1-parameter deformation $f_t : X \to P$ of f, such that $\frac{df_t}{dt}|_{t=0} = \eta$. By Proposition 1.2, there is a deformation Γ of γ such that the induced unfolding F of f is isomorphic to the unfolding $f_t \times \mathrm{id}_{\mathbf{C}} : X \times \mathbf{C}, 0 \to P \times \mathbf{C}, 0$. Since in the unfolding $f_t \times \mathrm{id}_{\mathbf{C}}$, X is deformed trivially, the same is true in the unfolding F; it follows that $\frac{d\Gamma}{dt}|_{t=0} \in \ker \phi$, and, since F is isomorphic to $f_t \times \mathrm{id}_{\mathbf{C}}$, that $\psi(\frac{d\Gamma}{dt}|_{t=0}) \equiv \eta$ modulo $df(\Theta_X) + f^{-1}(\Theta_P)$. This proves that ψ is onto, and completes the proof of the theorem. $\square$

Lemma 1.5 *Let $P \subset Q$ be a smooth subvariety, and $G : N \to Q$ be stable and such that $P \not\subset D(G)$. Then the sequence*

$$0 \to \mathrm{Derlog}(D(G)) \otimes \mathcal{O}_P \to \Theta_Q \otimes \mathcal{O}_P \to \frac{G^*\Theta_Q \otimes \mathcal{O}_P}{tG(\Theta_N) \otimes \mathcal{O}_P} \to 0$$

is exact, where $\otimes = \otimes_{\mathcal{O}_Q}$.

PROOF From the exact sequence

$$0 \to \mathrm{Derlog}(D(G)) \to \Theta_Q \to \frac{G^*\Theta_Q}{tG(\Theta_N)} \to 0$$

we obtain the exact sequence

$$0 \to \mathrm{Tor}_1\Big(\mathcal{O}_P, \frac{G^*\Theta_Q}{tG(\Theta_N)}\Big) \to \mathrm{Derlog}(D(G)) \otimes \mathcal{O}_P \to \Theta_Q \otimes \mathcal{O}_P \to \frac{G^*\Theta_Q}{tG(\Theta_N)} \otimes \mathcal{O}_P \to 0,$$

for as Θ_Q is free, $\mathrm{Tor}_1(\mathcal{O}_P, \Theta_Q) = 0$. Here all tensor products are over $\mathcal{O}_Q$, and $\mathrm{Tor}_1(-,-) = \mathrm{Tor}_1^{\mathcal{O}_Q}(-,-)$.

We claim that $\mathrm{Derlog}(D(G)) \otimes \mathcal{O}_P \to \Theta_Q \otimes \mathcal{O}_P$ is injective, for then

$$\mathrm{Tor}_1\Big(\mathcal{O}_P, \frac{G^*_Q}{tG(\Theta_N)}\Big) = 0. \tag{1.7}$$

To prove the claim, it suffices to show that $\mathrm{Derlog}(D(G)) \cap I_P\Theta_Q = I_P\,\mathrm{Derlog}(D(G))$, where I_P is the ideal of functions vanishing on P. This is proved as follows: suppose $v_1, \ldots, v_q$ are free generators of $\mathrm{Derlog}(D(G))$, with $v_i = \sum v_i^j \frac{\partial}{\partial z_j}$. If $v = \sum \alpha_i v_i$, and $v_{|P} = 0$ then by Cramer's rule $\det[v_i^j]\alpha_i \in I_P$ for all i. But $\det[v_i^j]$ is a reduced equation for the irreducible set $D(G)$, and so $\alpha_i \in I_P$, and $v \in \mathrm{Derlog}(D(G))$.

Having established (1.7), the remainder of the proof is simple: tensoring the exact sequence

$$0 \to tG(\Theta_N) \to G^*\Theta_Q \to \frac{G^*\Theta_Q}{tG(\Theta_N)} \to 0$$

over $\mathcal{O}_Q$ with $\mathcal{O}_P$, we obtain the isomorphism

$$\frac{G^*\Theta_Q}{tG(\Theta_N)} \otimes \mathcal{O}_P \cong \frac{G^*\Theta_Q \otimes \mathcal{O}_P}{tG(\Theta_N) \otimes \mathcal{O}_P},$$

whence the desired exact sequence. □

Corollary 1.6 (Infinitesimal versality ⇒ versality) *Let X be an* ICIS *and $f : X \to P$ have finite $\mathcal{A}_e$-codimension. Suppose that the unfolding $F : X \times S \to P \times S$ of f is infinitesimally versal, i.e. that the initial speeds span $N\mathcal{A}_e f$, then F is versal.*

PROOF Let H be a versal deformation of h, the defining equation for X, and extend F to $\hat{F}$ on the ambient space of X. Then (H, F) is an unfolding of (h, f) which by Theorems 1.3 and 1.4 is versal. It follows then that F is a versal unfolding of f. □

Remark In [4, Section 3], Damon proves Theorem 1.4 under the assumptions that X is smooth (so that $T^1_X = 0$) and that f is of finite $\mathcal{A}_e$-codimension. In fact the hypothesis that X be smooth does not greatly simplify the proof; however, the hypothesis of finite codimension of f allows him to avoid the most difficult part of our proof, namely the injectivity of ψ. For from his Lemmas 2.8 and 2.10 (equivalence of versality) (or our Theorem 1.3) one can deduce that $\dim_{\mathbf{C}} N\mathcal{K}_{D(G),e}\gamma = \dim_{\mathbf{C}} N\mathcal{A}_e f$, and thus it is enough to establish the surjectivity of ψ. The proof of Damon's theorem in [15, 3.14] also does not need the assumption that f be of finite $\mathcal{A}_e$-codimension.

2 $\mathcal{A}_e$-codimension and the vanishing topology of the discriminant

In this section we will show how to generalise results of [5] which relate the topology of a stabilisation of an unstable map germ f, to the $\mathcal{A}_e$-codimension of f.

Definition Suppose that $(\mathcal{X}, F, i, \pi)$ is an unfolding of the map-germ $f : X \to P$, with the property that $(\mathcal{X}, \pi)$ is a smoothing of X, and such that for each smooth fibre X_s, the corresponding map $f_s : X_s \to P$ has only stable singularities. We shall call each such map $f_s : X_s \to P$ a *stabilisation* of $f : X \to P$.

If $\mathcal{A}_e$-codimension(X, f) is finite, then all stabilisations of (X, f) are topologically equivalent, for, up to isomorphism, all can be found in a versal unfolding of (X, f), and in the base of the versal unfolding, the parameter values of any two stabilisations can be joined by a path avoiding the bifurcation set; the corresponding family of mappings is locally analytically trivial, and the local analytic trivialisations can be pieced together to give a (global) topological trivialisation.

If Z is a *reduced* analytic space and $f : Z \to P$ is a mapping, then the *discriminant* of f is the set $D(f) = f(\Sigma_{(Z,f)})$, where $\Sigma_{(Z,f)} = Z_{\text{sing}} \cup \Sigma(f_{|Z_{\text{reg}}})$. In [5], the first author and J. Damon proved the following two results.

Theorem 2.1 *Suppose that (n,p) are nice dimensions in the sense of Mather [14], with $n \geq p$, and that f_s is a stabilisation of a map-germ $f : N \to P$ of finite $\mathcal{A}_e$-codimension. Then $D(f_s) \cap B_\epsilon(0)$ has the homotopy type of a wedge of spheres of dimension $p-1$.*

The number of spheres making up the wedge is independent of the choice of stabilisation, and is called the *discriminant Milnor number* of f, and denoted $\mu_\Delta(f)$.

Theorem 2.2 *Suppose that (n,p) are nice dimensions, with $n \geq p$, and that $f : N \to P$ is a map germ of finite $\mathcal{A}_e$-codimension. Then $\mu_\Delta(f) \geq \mathcal{A}_e\text{-codim}(f)$, with equality if f is quasihomogeneous.*

The proof of Theorem 2.1 given in [5] relies on a theorem of Lê:

Theorem 2.3 (Lê) *If Y is a germ of a complete intersection of dimension m (not necessarily with isolated singularity) and $h : Y \to \mathbf{C}, 0$ has an isolated singularity at 0 (with respect to the canonical Whitney stratification of Y, in the sense explained in [9]), then the fibres $h^{-1}(t)$ for $t \neq 0$, of a good representative of h, have the homotopy type of a wedge of spheres of dimension m.*

This theorem was announced in [9], and recently a proof was given in [10]; a proof of an equivalent result can be found in [16].

The proof of Theorem 2.1 is then: let f_t, $t \in \mathbf{C}$, be a 1- parameter stabilisation of f, let F be the family $f_t \times \mathrm{id}_{\mathbf{C}}$, let $Y = D(F)$ (which is a hypersurface in $P \times \mathbf{C}$), and let $h : D(F) \to \mathbf{C}$ be the projection to the parameter space. Then (one checks) h has isolated singularity, in the sense of [9]. Since $h^{-1}(t) = D(f_t)$, Theorem 2.1 follows from Theorem 2.3.

Observe now that the same proof shows that the conclusion of Theorem 2.1 still holds if $f_t : X_t \to P$ is a stabilisation of the germ of finite $\mathcal{A}_e$-codimension $f : X \to P$, where X is now an ICIS: namely that $D(f_t) \cap B_\epsilon(0)$ has the homotopy type of a wedge of spheres of dimension $p-1$. We will denote the number of these spheres by $\mu_\Delta(X, f)$.

Our main theorem is a generalization of Theorem 2.2:

Theorem 2.4 *Let X be an ICIS of dimension n, and let $f : X \to P$ be a map germ of finite $\mathcal{A}_e$-codimension. Suppose that (n,p) are nice dimensions, with $n \geq p$. Then $\mu_\Delta(X, f) \geq \mathcal{A}_e\text{-codim}(X, f)$, with equality if (X, f) is quasihomogeneous.*

(Here, (X, f) is quasihomogeneous if there are so-called good $\mathbf{C}^*$-actions on X and P with respect to which f is equivariant.)

PROOF The proof of Theorem 2.2 given in [5] carries over almost verbatim, and here we give no more than a sketch. First, as described in Lemma 1.1, choose $\gamma : P \to Q$ which induces $f : X \to P$ from the stable germ $G : N \to Q$. Then $\mathcal{A}_e\text{-codim}(X, f) = \dim_{\mathbf{C}} N\mathcal{K}_{D(G),e}\gamma$, by Theorem 1.3. Next, introduce the auxilliary module $N\mathcal{K}_{H,e}\gamma$, which is defined as follows: if $H : Q \to \mathbf{C}, 0$ is a reduced equation for the hypersurface $D(G)$, let

$$\mathrm{Derlog}(H) = \{\chi \in \Theta_Q \mid \chi(H) = 0\};$$

then set
$$N\mathcal{K}_{H,e}\gamma = \frac{\gamma^*(\Theta_Q)}{d\gamma(\Theta_P) + \gamma^*(\mathrm{Derlog}(H))}.$$
As shown in [5] Example 3.2, we may, and do, suppose that there exists a vector field χ on Q such that $\chi(H) = H$, from which it follows that $\mathrm{Derlog}(D(G))$ splits as a direct sum, $\mathrm{Derlog}(H) \oplus \mathcal{O}_Q\{\chi\}$. If G is quasihomogeneous, then we can take the vector field χ to be the Euler vector field generating the $\mathbf{C}^*$-action on Q. If also γ is quasihomogeneous with respect to the same $\mathbf{C}^*$-action on Q, then χ is tangent to the image of γ and it follows that $N\mathcal{K}_{H,e}\gamma = N\mathcal{K}_{D(G),e}\gamma$. This is the case if (X, f) is quasihomogeneous.

Now as G is stable, $\mathrm{Derlog}(D(G))$ is a free $\mathcal{O}_Q$-module on q generators, and it follows that $\mathrm{Derlog}(H)$ is free on $q-1$ generators. Thus, the exact sequence
$$\gamma^*(\mathrm{Derlog}(H)) \oplus \Theta_P \longrightarrow \gamma^*(\Theta_Q) \longrightarrow N\mathcal{K}_{H,e}\gamma \longrightarrow 0$$
defining $N\mathcal{K}_{H,e}\gamma$ in fact gives us a presentation
$$\mathcal{O}_P^{p+q-1} \longrightarrow \mathcal{O}_P^q \to N\mathcal{K}_{H,e}\gamma \longrightarrow 0.$$
It follows from the form of this presentation that if $\dim_{\mathbf{C}} N\mathcal{K}_{H,e}\gamma < \infty$, then it is conserved in a deformation [5, Corollary 5.5]. In fact, with $(\dim X, p)$ nice dimensions, the finiteness of $\mathcal{A}_e$-codim(X, f) implies that $\dim_{\mathbf{C}} N\mathcal{K}_{H,e}\gamma < \infty$; for if not $\dim \mathrm{supp} N\mathcal{K}_{H,e}\gamma \geq 1$, and hence there exists a parametrized curve σ in $\gamma(P)$, at each point of which P fails to be transverse to the fibres of H. It follows that $H \circ \gamma$ is constant (equal to 0). Now, at all points $y \in \gamma(P) \cap (D(G) \setminus \{0\})$, the map defined by the fibre product of γ and G is stable, and therefore in appropriate coordinates is quasihomogeneous (indeed, quasihomogeneity of all stable maps characterizes the nice dinmensions). It follows that $(N\mathcal{K}_{H,e}\gamma)_y = (N\mathcal{K}_{D(G),e}\gamma)_y = 0$ (see [4, Lemma 3.4]), and thus such a curve σ cannot exist, and $\dim_{\mathbf{C}} N\mathcal{K}_{H,e}\gamma < \infty$.

If γ_t is a $\mathcal{K}_{D(G),e}$-stabilisation of γ, corresponding to a stabilisation of (X, f), then provided $(\dim X, p)$ are nice dimensions, $N\mathcal{K}_{H,e}\gamma_t$ is not supported anywhere on $D(f_t) = \gamma^{-1}(D(G))$, for the reason observed above. If on the other hand $y \notin \gamma_t^{-1}(D(G))$, then $(N\mathcal{K}_{H,e}\gamma_t)_y \cong \mathcal{O}_{P,y}/J_h$, where $h = H \circ \gamma$ is a defining equation for $D(f_t)$ [5, Lemma 5.6]; from this one now concludes, by an argument of Siersma [16], that $\mu_\Delta(X, f)$ is equal to $\dim_{\mathbf{C}} N\mathcal{K}_{H,e}\gamma$. Since it is clear that $\dim_{\mathbf{C}} N\mathcal{K}_{H,e}\gamma \geq \dim_{\mathbf{C}} N\mathcal{K}_{D(G),e}\gamma$, the inequality of the statement of the theorem is proved. Equality when (X, f) is quasihomogeneous follows from the fact that in this case $N\mathcal{K}_{H,e}\gamma = N\mathcal{K}_{D(G),e}$. □

One case of interest is where $p = 1$. In this case $\mu_\Delta(X, f)$ has another topological interpretation, by an argument of Lê and Greuel:

Proposition 2.5 *If $p = 1$, then $\mu_\Delta(X, f) + 1 = \mu(X) + \mu(X')$, where $X' = f^{-1}(0)$.*

Proof Since $p = 1$, $\mu_\Delta(X, f) + 1$ is just the number of critical values of the non-degenerate function f_t on the Milnor fibre X_t of X. If $h : E \to \mathbf{C}^k$ defines X, then $\mu_\Delta(X, f) + 1$ is thus the number of intersection points of the line $\{0\} \times \mathbf{C} \subset \mathbf{C}^k \times \mathbf{C}$ with the discriminant of $h \times f$; hence, as shown for example in [11, §(5.11)], $\mu_\Delta(X, f) + 1$ is equal to $\mu(X) + \mu(X')$. $\square$

Corollary 2.6 *If $p = 1$, then*

$$\mu(X) + \mu(X') \geq \tau(X) + \mathcal{A}_e\text{-codim}(f) + 1,$$

with equality if (X, f) is quasihomogeneous. $\square$

Combining this with the theorem of Greuel, that $\tau(X) = \mu(X)$ if X is quasihomogeneous [8], gives:

Corollary 2.7 *If $p = 1$ and (X, f) is quasihomogeneous, then*

$$\mu(X') = \mathcal{A}_e\text{-codim}(f) + 1.$$

$\square$

This fact was already obtained by Bruce and Roberts [1, Proposition 7.7]; their proof also uses Greuel's theorem, together with an explicit description of Θ_X (their Θ_X^-) for X quasihomogeneous. It is also implicit in the paper of Goryunov [7].

3 Bifurcations and $\mathcal{K}_\Delta$-equivalence

Let $g_0 : N \to P$ be a $\mathcal{K}$-finite map-germ, and let $g : N \times \Lambda \to P$ be any deformation of g_0. Then g can be induced from a versal deformation $G : N \times U \to P$ of g_0 by a map-germ $\gamma : \Lambda \to U$. Here Λ and U are germs of smooth spaces. Thus we have the following commutative diagram.

$$\begin{tikzcd} N \times \Lambda \arrow[d, "(\mathrm{id}_N, \gamma)"'] \arrow[dr, "g"] & \\ N \times U \arrow[r, "G"'] & P \end{tikzcd}$$

Let $\Delta \subset U$ denote the local bifurcation set of G, that is,

$$\Delta = \{u \in U \mid G_u^{-1}(0) \text{ is not smooth}\}.$$

It is well known that Δ is the discriminant of the projection $\pi_G : V_G \to U$, where V_G is the zero set of G, and π_G is the restriction to V_G of the projection $N \times U \to U$. Let $\mathrm{Derlog}(\Delta)$ denote the $\mathcal{O}_U$-module of vector fields tangent to Δ. Because G is $\mathcal{K}$-versal, π_G is a stable map [12], so that $v \in \mathrm{Derlog}(\Delta)$ if and only if it is liftable for π_G.

In this section we formalize the close relationship that exists between the deformation g and the map γ which induces it.

For a deformation $g : N \times \Lambda \to P$ of $g_0 : N \to P$, define the *relative* $\mathcal{K}$-tangent space by,

$$T\mathcal{K}_{\mathrm{rel}}.g = tg(\Theta_{N\times\Lambda/\Lambda}) + g^*\boldsymbol{m}_P\Theta_P.$$

It is an $\mathcal{O}_{N\times\Lambda}$-module. Thus $T\mathcal{K}_{\mathrm{un}}.g = T\mathcal{K}_{\mathrm{rel}}.g + t_2g(\Theta_\Lambda)$, where t_2g means differentiating g with respect to the second (λ) variables.

Any map $\gamma : \Lambda \to U$ induces a deformation $g = g_\gamma : N \times \Lambda \to P$. A deformation of γ induces a deformation of g. At the infinitesimal level, this defines a homomorphism of $\mathcal{O}_\Lambda$-modules

$$\begin{array}{rcl}\Psi_\gamma : \Theta(\gamma) & \longrightarrow & \Theta(g),\\ \zeta & \longmapsto & t_2G(\zeta).\end{array}$$

We often omit the subscript γ on Ψ if the context is clear. More explicitly, if $[\lambda \mapsto \zeta(\lambda)] \in \Theta(\gamma)$, then $\Psi(\zeta)$ is the vector field along g given by, $[(x,\lambda) \mapsto d_2G_{(x,\gamma(\lambda))}(\zeta(\lambda))]$. In coordinates, if $G(x,u) = g_0(x) + \sum u_i\phi_i(x)$ then

$$\Psi(\zeta_1(\lambda),\ldots,\zeta_d(\lambda)) = \sum \zeta_i(\lambda)\phi_i(x).$$

Theorem 3.1 *The homomorphism $\Psi = \Psi_\gamma$ introduced above induces an isomorphism of $\mathcal{O}_\Lambda$-modules:*

$$\psi : N\mathcal{K}_\Delta.\gamma \xrightarrow{\simeq} N\mathcal{K}_{\mathrm{un}}.g.$$

Note that we do not assume that γ be of finite $\mathcal{K}_\Delta$ codimension, but only that g_0 be $\mathcal{K}$-finite. We begin the proof with a useful lemma.

Lemma 3.2 *With $g, G, \Psi, \gamma, \Delta$ as defined above,*

$$\Psi_\gamma^{-1}(T\mathcal{K}_{\mathrm{rel}}.g) = \gamma^*\mathrm{Derlog}(\Delta).$$

Proof The proof is divided into three parts.
(i) We first show that the lemma holds for $g = G$ and $\gamma = \mathrm{id}_U$.

Suppose $\Psi(\eta) \in T\mathcal{K}_{\mathrm{rel}}.G$. Then there exist $\xi \in \Theta_{N\times\Lambda/\Lambda}$ and $\alpha \in G^*\boldsymbol{m}_P\Theta_P$ such that

$$\Psi(\eta) = -tG(\xi) + \alpha.$$

Thus, $tG(\xi,\eta) = \alpha$, whence $(\xi,\eta) \in \Theta_{N\times U}$ is tangent to V_G. Thus η is liftable and hence in Derlog(Δ). To obtain the converse, just reverse the argument.
(ii) We now show that the statement holds if γ is an immersion. Via γ, consider Λ as a subspace of U. Let $\eta \in \Theta(\gamma)$ be such that

$$\Psi(\eta) \in T\mathcal{K}_{\mathrm{rel}}.g = \gamma^{-1}T\mathcal{K}_{\mathrm{rel}}.G$$

Extend η to $\tilde\eta \in \Theta_U$. Then $\Psi_{\mathrm{id}}(\tilde\eta)\circ\gamma = \Psi_\gamma(\eta)$. Thus,

$$\Psi_{\mathrm{id}}(\tilde\eta) \in I_\Lambda.\Theta(G) + T\mathcal{K}_{\mathrm{rel}}.G,$$

where I_Λ is the ideal in $\mathcal{O}_U$ of functions vanishing on Λ. By the preparation theorem, $\Theta(G) = \Psi_{\mathrm{id}}(\Theta_U) + T\mathcal{K}_{\mathrm{rel}}.G$, whence

$$\Psi_{\mathrm{id}}(\tilde{\eta}) \in I_\Lambda.\Psi_{\mathrm{id}}(\Theta_U) + T\mathcal{K}_{\mathrm{rel}}.G.$$

Consequently, modulo $T\mathcal{K}_{\mathrm{rel}}.G$, we can write

$$\Psi_{\mathrm{id}}(\tilde{\eta}) = \sum r_i \Psi_{\mathrm{id}}(e_i)$$

with $r_i \in I_\Lambda$ and $e_i \in \Theta_U$. Let $\eta' = \tilde{\eta} - \sum r_i e_i$. Then $\Psi_{\mathrm{id}}(\eta') \in T\mathcal{K}_{\mathrm{rel}}.G$, so by Part (i), $\eta' \in \mathrm{Derlog}(\Delta)$. Moreover, η' is also an extension of η, i.e. $\eta = \eta' \circ \gamma$, so $\eta \in \gamma^*\mathrm{Derlog}(\Delta)$ as required.
(iii) This final step reduces the general case to that of γ being an immersion. To this end, define

$$\begin{array}{rcl} \tilde{G} : N \times U \times \Lambda & \longrightarrow & P \\ (x, u, u') & \longmapsto & G(x, u) \end{array}$$

and $\tilde{\gamma} : \Lambda \to U \times \Lambda$ by $\tilde{\gamma}(\lambda) = (\gamma(\lambda), \lambda)$. Now, $\tilde{G}$ is versal because G is, and $\tilde{\Delta} = \Delta \times \Lambda \subset U \times \Lambda$. Since $\tilde{\gamma}$ is an immersion we can apply Part (ii) to get:

$$\Psi_{\tilde{\gamma}}^{-1}(T\mathcal{K}_{\mathrm{rel}}.g) = \tilde{\gamma}^*\,\mathrm{Derlog}(\tilde{\Delta}),$$

since γ and $\tilde{\gamma}$ both induce g. For $\tilde{\eta} \in \Theta(\tilde{\gamma})$ write $\tilde{\eta} = (\eta, \eta')$. Then

$$\Psi_{\tilde{\gamma}}(\tilde{\eta}) = \Psi_\gamma(\eta)$$

as $\tilde{G}$ is independent of u', and so

$$\begin{array}{rcl} \Psi_\gamma(\eta) \in T\mathcal{K}_{\mathrm{rel}}.g & \Longleftrightarrow & (\eta, \eta') \in \tilde{\gamma}^*\,\mathrm{Derlog}(\tilde{\Delta}) \\ & \Longleftrightarrow & \eta \in \gamma^*\mathrm{Derlog}(\Delta). \end{array}$$

□

Proof (of Theorem 3.1) We must show that Ψ_γ induces an isomorphism ψ. To do so we must show three things:

(i) $\Psi(T\mathcal{K}_\Delta.\gamma) \subset T\mathcal{K}_{\mathrm{un}}.g$, whence Ψ induces a homomorphism, $\psi : N\mathcal{K}_\Delta.\gamma \to N\mathcal{K}_{\mathrm{un}}.g$.

(ii) ψ is injective, and

(iii) ψ is surjective.

The proof is divided into three parts accordingly.
(i) First note that for $\xi \in \Theta_\Lambda$, one has

$$\Psi(t\gamma(\xi)) = t_2 G \circ t\gamma(\xi) = t_2 g(\xi),$$

so $\Psi(t\gamma(\Theta_\Lambda)) = t_2 g(\Theta_\Lambda)$. The remainder of this part follows from the lemma above. Now define

$$\psi : N\mathcal{K}_\Delta.\gamma \to N\mathcal{K}_{\mathrm{un}}.g$$

to be the morphism induced from Ψ.

(ii) Suppose $\Psi(\xi) \in T\mathcal{K}_{\mathrm{un}}.g$. We wish to show that $\xi \in \gamma^*\mathrm{Derlog}(\Delta)$. Write $\Psi(\xi) = \zeta_1 + \zeta_2$, where $\zeta_1 \in T\mathcal{K}_{\mathrm{rel}}.g$ and $\zeta_2 \in t_2g(\Theta_\Lambda)$. Since $\Psi|_{t\gamma(\Theta_\Lambda)} : t\gamma(\Theta_\Lambda) \to t_2g(\Theta_\Lambda)$ is an isomorphism, there is a unique $\xi_2 \in t\gamma(\Theta_\Lambda)$ with $\Psi(\xi_2) = \zeta_2$. Since Ψ is linear, it follows that $\zeta_1 \in \mathrm{image}(\Psi)$, and so by Lemma 3.2, $\xi_1 := \xi - \xi_2 \in \gamma^*\mathrm{Derlog}(\Delta)$.

(iii) By definition, ψ is surjective if and only if,

$$T\mathcal{K}_{\mathrm{un}}.g + t_2G(\Theta(\gamma)) = \Theta(g).$$

Consider the quotient $\Theta(g)/T\mathcal{K}_{\mathrm{un}}.g$. Reducing modulo $\boldsymbol{m}_\Lambda$ (the maximal ideal in $\mathcal{O}_\Lambda$), this becomes

$$\frac{\Theta(g)}{T\mathcal{K}_{\mathrm{un}}.g + \boldsymbol{m}_\Lambda\Theta(g)} \subset \frac{\Theta(g_0)}{T\mathcal{K}.g_0} = N\mathcal{K}.g_0$$

Now, since G is a versal deformation of g_0 we have that

$$T\mathcal{K}.g_0 + t_2G(U) = \Theta(g_0)$$

so that $t_2G(U)$ spans $N\mathcal{K}.g_0$. Here we have identified U with $\Theta_U/\boldsymbol{m}_U\Theta_U$. It follows from the preparation theorem that $t_2G(\Theta(\gamma))$ spans $\Theta(g)/T\mathcal{K}_{\mathrm{un}}.g$. □

We conclude this article by combining the main theorems of Sections 1 and 3 to refine a classical theorem of Martinet relating stable mappings and versal deformations. First we recall this theorem.

Let $g : N \times \Lambda \to P$ be a regular deformation of g_0 ("regular" means that as a map g is a submersion). Let $V_g = g^{-1}(0)$ and $\pi_g : V_g \to \Lambda$ be the restriction to V_g of the projection $\pi : N \times \Lambda \to \Lambda$. Martinet's Theorem [12] states that g is $\mathcal{K}$-versal if and only if π_g is stable.

Now, lack of versality of g is measured by $N\mathcal{K}_{\mathrm{un}}.g$, and lack of stability of π_g is measured by the $\mathcal{A}_e$-codim(V_g, π_g) ($= \dim N\mathcal{A}_e\pi_g$ if g is regular), and it is not unreasonable to expect these two numbers to coincide. Indeed one has,

Theorem 3.3 *Let $g_0 : N \to P$ be a $\mathcal{K}$-finite map germ and let $g : N \times \Lambda \to P$ be a deformation of g_0, then*

$$\mathcal{A}_e\text{-codim}(V_g, \pi_g) = \mathcal{K}_{\mathrm{un}}-\mathrm{codim}(g).$$

If, moreover, g is regular then the following two $\mathcal{O}_\Lambda$-modules are isomorphic:

$$N\mathcal{A}_e.\pi_g \simeq N\mathcal{K}_{\mathrm{un}}.g.$$

Proof Let $G : N \times U \to P$ be a versal deformation of g_0 and $\gamma : \Lambda \to U$ a map which induces g from G. The first part of the theorem follows from combining Theorems 1.3 and 3.1. The second part follows from combining Theorems 1.4 and 3.1 (since $T^1_{V_g} = 0$ if g is regular). □

References

[1] J.W. Bruce & M. Roberts, Critical points of functions on analytic varieties. *Topology* **27** (1988), 57–90.

[2] J. Damon, *The Unfolding and Determinacy Theorems for Subgroups of $\mathcal{A}$ and $\mathcal{K}$.* Memoirs A.M.S. **50**, no. 306 (1984).

[3] J. Damon, Deformations of sections of singularities and Gorenstein surface singularities. *Am. J. Math.* **109** (1987), 695–722.

[4] J. Damon, $\mathcal{A}$-equivalence and the equivalence of sections of images and disriminants. In *Singularity Theory and its Applications, Part I*, Springer Lecture Notes **1462** (1991), 93–121.

[5] J. Damon & D. Mond, $\mathcal{A}$-codimension and the vanishing topology of discriminants. *Invent. math.* **106** (1991), 217–242.

[6] M. Golubitsky & D. Schaeffer, *Singularities and Groups in Bifurcation Theory, Vol. I.* Springer-Verlag, New York etc., 1985.

[7] V.V. Goryunov, Projection and vector fields tangent to the discriminant of a complete intersection. *Func. An. Appl.* **22** (1988), 104–113.

[8] G.-M. Greuel, Dualität in der lokalen Kohomologie isolierter Singularitäten. *Math. Ann.* **250** (1980), 157–173.

[9] Lê Dũng Tráng, Le concept de singularité isolée de fonction analytique. *Adv. Studies in Pure Math.* **8** (1986), 215–227.

[10] Lê Dũng Tráng, Complex analytic functions with isolated singularities. *J. Alg. Geom.* **1** (1992), 83–100.

[11] E.J.N. Looijenga, *Isolated Singular Points on Complete Intersections.* L.M.S. Lecture Notes, **17**, C.U.P., 1984.

[12] J. Martinet, Déploiements versels des applications différentiables et classification des applications stables. Lecture Notes in Math. **535**, Springer, Heidelberg etc., 1976.

[13] J. Mather, Stability of C^∞ mappings, IV: Classification of stable germs by $\mathbf{R}$-algebras. *Publ. Math. I.H.E.S.* **37** (1969), 223–248

[14] J. Mather, Stability of C^∞ mappings, VI: The nice dimensions. Springer Lecture Notes **192** (1971), 207–253.

[15] A. du Plessis, T. Gaffney & L. Wilson, Map-germs determined by their discriminant. Preprint, Aarhus, 1991.

[16] D. Siersma, Vanishing cycles and special fibres. In *Singularity Theory and its Applications, Part I*, Springer Lecture Notes **1462** (1991), 292–301.

[17] C.T.C. Wall, Finite determinacy of smooth map-germs. *Bull. L.M.S.* **13** (1981), 481–539.

David Mond
Mathematics Institute
University of Warwick
Coventry CV4 7AL
U.K.

James Montaldi
Institut Non-Linéaire de Nice
1361 route des Lucioles
Sophia Antipolis
06560 Valbonne
France