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The Uniform Mean-Square Ergodic Theorem for Wide Sense Stationary Processes

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It is shown that the uniform mean-square ergodic theorem holds for the family of wide sense stationary sequences, as soon as the random process with orthogonal increments, which corresponds to the orthogonal stochastic measure generated by means of the spectral representation theorem, is of bounded variation and uniformly continuous at zero in a mean-square sense. The converse statement is also shown to be valid, whenever the process is sufficiently rich. The method of proof relies upon the spectral representation theorem, integration by parts formula, and estimation of the asymptotic behaviour of total variation of the underlying trigonometric functions. The result extends and generalizes to provide the uniform mean-square ergodic theorem for families of wide sense stationary processes.

1. Introduction

Let $\big(\, \{\xi_n(t)\}_{n\in\mathbf{Z}} \mid t \in T \,\big)$ be a family of (wide sense) stationary sequences of complex random variables defined on the probability space $(\Omega, \mathcal{F}, P)$ and indexed by the set T. Then the mean-square ergodic theorem is known to be valid:

$$(1.1) \qquad \frac{1}{n}\sum_{k=0}^{n-1} \xi_k(t) \to L_t \quad \text{in } L^2(P)$$

as $n \to \infty$, for all $t \in T$. The present paper is motivated by the following question: When does the convergence in (1.1) hold uniformly over $t \in T$? In other words, when do we have:

$$(1.2) \qquad \sup_{t\in T} \Big| \frac{1}{n}\sum_{k=0}^{n-1} \xi_k(t) - L_t \Big| \to 0 \quad \text{in } L^2(P)$$

as $n \to \infty$? It is the purpose of the paper to exhibit a solution for this problem, as well as to motivate further research in this direction. We begin by recalling some historical facts.

This sort of problem originates in the papers of Glivenko [8] and Cantelli [3] who considered the a.s. version of (1.2) in the i.i.d. case and proved the well-known Glivenko-Cantelli theorem. Various generalizations and extensions of this result are shown to be of fundamental importance in different fields ranging from Banach space theory to statistics. Here we do not wish to review the more detailed history of this development, but will point out some of the fundamental results.

In the papers of Vapnik and Chervonenkis [18] and [19] the a.s. version of (1.2) is characterized in the i.i.d. case in terms of *random entropy numbers*. This result is recently generalized and extended into ergodic theory by obtaining uniform Birkhoff's pointwise ergodic theorem (see [14]). The extension happens to be valid for stationary ergodic (in the strict sense) sequences with an

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additional weak dependence structure involving a form of mixing. This research is moreover indicated that the mixing property is important and can not be avoided. For this reason we see that problem (1.2) goes far beyond the level where the entropy numbers could be of use.

Another approach towards the a.s. version of (1.2) in the i.i.d. case appeared in the papers of Blum [2] and DeHardt [4]. They used the concept of *metric entropy with bracketing* (see [5]). In this context they obtained by now the best known sufficient condition. Following this result, a characterization of the a.s. version of (1.2) in the i.i.d. case is obtained in the paper of Hoffmann-Jørgensen [9], which involves Blum-DeHardt's theorem as a particular case. It is shown recently that this result extends to the general stationary ergodic (in the strict sense) case (see [12]), as well as to the case of general measurable dynamical systems (see [13]).

Somewhat different characterization of the a.s. version of (1.2) in the i.i.d. case is obtained in the paper of Talagrand [17]. A close look into the proof indicates that this approach also requires a form of mixing, and thus will be not taken into consideration here. To conclude the exposition as stated above, we find it convenient to recall the papers [7], [10], [11] and [20].

The main novelty of the approach towards uniform ergodic theorem (1.2) taken in the present paper relies upon the spectral representation theorem which is valid for (wide sense) stationary sequences under consideration. It makes possible to investigate the uniform ergodic theorem (1.2) in terms of the orthogonal stochastic measure which is associated with the underlying sequence by means of the theorem, or equivalently, in terms of the random process with orthogonal increments which corresponds to the measure. We think that both the problem and approach appear worthy of consideration, and moreover to the best of our knowledge it has not been studied previously.

The main result of the paper states that the uniform mean-square ergodic theorem (1.2) holds as soon as the random process with orthogonal increments which is associated with the underlying sequence by means of the spectral representation theorem is of bounded variation and uniformly continuous at zero in a mean-square sense. The converse statement is also shown to be valid whenever the process is sufficiently rich. It should be mentioned that the approach of the present paper makes no attempt to treat the case where the orthogonal stochastic measure (process with orthogonal increments) is of unbounded variation. We postpone this question for further research and leave it in general as open.

In the second part of the paper we investigate the same problem in the continuous parameter case. Let $\big(\, \{X_s(t)\}_{s \in \mathbf{R}} \mid t \in T \,\big)$ be a family of (wide sense) stationary processes of complex random variables defined on the probability space $(\Omega, \mathcal{F}, P)$ and indexed by the set T . Then the mean-square ergodic theorem is known to be valid:

$$\frac{1}{\tau} \int_0^\tau X_s(t)\, ds \rightarrow L_t \quad \text{in } L^2(P) \tag{1.3}$$

as $\tau \rightarrow \infty$, for all $t \in T$. The question under investigation is as above: When does the convergence in (1.3) hold uniformly over $t \in T$? In other words, when do we have:

$$\sup_{t \in T} \Big| \frac{1}{\tau} \int_0^\tau X_s(t)\, ds \,-\, L_t \Big| \rightarrow 0 \quad \text{in } L^2(P) \tag{1.4}$$

as $\tau \rightarrow \infty$? The main result in this context is shown to be of the same nature as the main result for sequences stated above. The same holds for the remarks following it. We will not pursue either of this more precisely here, but instead pass to the results in a straightforward way.

2. The uniform mean-square ergodic theorem in the discrete parameter case

The aim of this section is to present the uniform mean-square ergodic theorem in the discrete parameter case. Throughout we consider a family of (*wide sense*) *stationary* sequences of complex random variables $\big(\{\xi_n(t)\}_{n\in\mathbf{Z}} \mid t \in T\big)$ defined on the probability space $(\Omega, \mathcal{F}, P)$ and indexed by the set T . Thus, we have:

$$E\big|\xi_n(t)\big|^2 < \infty \tag{2.1}$$

$$E\big(\xi_n(t)\big) = E\big(\xi_0(t)\big) \tag{2.2}$$

$$\mathrm{Cov}\big(\xi_{m+n}(t), \xi_m(t)\big) = \mathrm{Cov}\big(\xi_n(t), \xi_0(t)\big) \tag{2.3}$$

for all $n, m \in \mathbf{Z}$, and all $t \in T$. For all of the well-known results which will be soon stated below, as well as for more information about the (wide sense) stationary sequences, we shall refer the reader to the classical references on the subject [1], [6], [15] and [16].

As a matter of convenience, we will henceforth suppose:

$$E\big(\xi_n(t)\big) = 0 \tag{2.4}$$

for all $n \in \mathbf{Z}$, and all $t \in T$. Thus the *covariance function* of $\{\xi_n(t)\}_{n\in\mathbf{Z}}$ is given by:

$$R_t(n) = E\big(\xi_n(t)\overline{\xi_0(t)}\big) \tag{2.5}$$

whenever $n \in \mathbf{Z}$ and $t \in T$.

By the *Herglotz theorem* there exists a finite measure $\mu_t = \mu_t(\Delta)$ on $\mathcal{B}(<\!-\pi,\pi\,])$ such that:

$$R_t(n) = \int_{-\pi}^{\pi} e^{in\lambda}\, \mu_t(d\lambda) \tag{2.6}$$

for $n \in \mathbf{Z}$ and $t \in T$. The measure μ_t is called the *spectral measure* of $\{\xi_n(t)\}_{n\in\mathbf{Z}}$ for $t \in T$.

The *spectral representation theorem* states that there exists an *orthogonal stochastic measure* $Z_t = Z_t(\omega, \Delta)$ on $\Omega \times \mathcal{B}(<\!-\pi,\pi\,])$ such that:

$$\xi_n(t) = \int_{-\pi}^{\pi} e^{in\lambda}\, Z_t(d\lambda) \tag{2.7}$$

for $n \in \mathbf{Z}$ and $t \in T$. The fundamental identity in this context is as follows:

$$E\Big|\int_{-\pi}^{\pi} \varphi(\lambda)\; Z_t(d\lambda)\Big|^2 = \int_{-\pi}^{\pi} \big|\varphi(\lambda)\big|^2\, \mu_t(d\lambda) \tag{2.8}$$

whenever the function $\varphi : <\!-\pi,\pi\,] \rightarrow \mathbf{C}$ belongs to $L^2(\mu_t)$ for $t \in T$. We also have:

$$Z_t(-\Delta) = \overline{Z_t(\Delta)} \tag{2.9}$$

for all $\Delta \in \mathcal{B}(<\!-\pi,\pi\!>)$, and all $t \in T$.

The random process defined by:

$$Z_t(\lambda) = Z_t(<\!-\pi,\lambda]) \tag{2.10}$$

for $\lambda \in <\!-\pi,\pi\,]$ is with *orthogonal increments* for every $t \in T$. Thus, we have:

$$(2.11) \qquad E\big|Z_t(\lambda)\big|^2 < \infty \text{ , for all } \lambda \in \,<\!-\pi, \pi\,]$$

$$(2.12) \qquad E\big|Z_t(\lambda_n) - Z_t(\lambda)\big|^2 \to 0 \text{ , whenever } \lambda_n \downarrow \lambda \text{ for } \lambda \in \,<\!-\pi, \pi\,]$$

$$(2.13) \qquad E\Big(\big(Z_t(\lambda_4) - Z_t(\lambda_3)\big)\overline{\big(Z_t(\lambda_2) - Z_t(\lambda_1)\big)}\Big) = 0$$

whenever $-\pi < \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 \leq \pi$, for all $t \in T$. We will henceforth put $Z_t(-\pi) = 0$ for all $t \in T$. Moreover, we will assume below that the process $\{\, Z_t(\lambda)\,\}_{-\pi \leq \lambda \leq \pi}$ is of *bounded variation* and *right continuous* (outside of a P-nullset) for all $t \in T$. In this case the integral:

$$(2.14) \qquad \int_{-\pi}^{\pi} \varphi(\lambda) \;Z_t(d\lambda)$$

may be well defined pointwise on Ω as the usual Riemann-Stieltjes integral for all $t \in T$. If $\psi : <\!\lambda_1, \lambda_2\,] \to \mathbf{C}$ is of bounded variation and right continuous for some $-\pi \leq \lambda_1 < \lambda_2 \leq \pi$, then *integration by parts formula* states:

$$(2.15) \qquad \int_{\lambda_1}^{\lambda_2} \psi(\lambda-) \;Z_t(d\lambda) + \int_{\lambda_1}^{\lambda_2} Z_t(\lambda) \;\psi(d\lambda) = \psi(\lambda_2)Z_t(\lambda_2) - \psi(\lambda_1)Z_t(\lambda_1)$$

for all $t \in T$. Moreover, if we denote by $\mathrm{V}(\Phi, <\!\lambda_1, \lambda_2\,])$ the *total variation* of the function $\Phi : <\!\lambda_1, \lambda_2] \to \mathbf{C}$, then we have:

$$(2.16) \qquad \Big|\int_{\lambda_1}^{\lambda_2} \psi(\lambda) \;Z_t(d\lambda)\Big| \leq \sup_{\lambda_1 < \lambda \leq \lambda_2} \big|\psi(\lambda)\big| \cdot \mathrm{V}(Z_t, <\!\lambda_1, \lambda_2])$$

$$(2.17) \qquad \Big|\int_{\lambda_1}^{\lambda_2} Z_t(\lambda) \;\psi(d\lambda)\Big| \leq \sup_{\lambda_1 < \lambda \leq \lambda_2} \big|Z_t(\lambda)\big| \cdot \mathrm{V}(\psi, <\!\lambda_1, \lambda_2])$$

for all $t \in T$.

The *mean-square ergodic theorem* for $\{\xi_n(t)\}_{n \in \mathbf{Z}}$ states:

$$(2.18) \qquad \frac{1}{n}\sum_{k=0}^{n-1} \xi_k(t) \to Z_t(\{0\}) \text{ in } L^2(P)$$

as $n \to \infty$, for all $t \in T$. If moreover the process $\{\, Z_t(\lambda)\,\}_{-\pi \leq \lambda \leq \pi}$ is of bounded variation and right continuous for all $t \in T$, then the convergence in (2.18) is P-a.s. as well. We also have:

$$(2.19) \qquad \frac{1}{n}\sum_{k=0}^{n-1} R_t(k) \to \mu_t(\{0\})$$

as $n \to \infty$, for all $t \in T$. Finally, it is easily seen that:

$$(2.20) \qquad Z_t(\{0\}) = 0 \;\Leftrightarrow\; \mu_t(\{0\}) = 0$$

whenever $t \in T$.

It is the main purpose of the present section to investigate when the *uniform mean-square ergodic theorem* is valid:

$$(2.21) \qquad \sup_{t \in T} \Big|\, \frac{1}{n}\sum_{k=0}^{n-1} \xi_k(t) - Z_t(\{0\}) \,\Big| \to 0 \text{ in } L^2(P)$$

as $n \to \infty$. We think that this problem appears worthy of consideration, and moreover to the best of our knowledge it has not been studied previously.

The main novelty of the approach towards uniform ergodic theorem taken here relies upon the spectral representation (2.7) which makes possible to investigate (2.21) in terms of the orthogonal stochastic measure $Z_t(\omega, \Delta)$ defined on $\Omega \times \mathcal{B}(<-\pi, \pi])$, or equivalently, in terms of the random process $\{ Z_t(\lambda) \}_{-\pi \leq \lambda \leq \pi}$ with orthogonal increments which corresponds to the measure by means of (2.10), where t ranges over T . In the sequel we find it convenient to restrict ourselves to the case where the process $\{ Z_t(\lambda) \}_{-\pi \leq \lambda \leq \pi}$ is of bounded variation and right continuous for $t \in T$. It is an open interesting question do the results which are obtained below under these hypotheses extend in some form to the general case. We leave this as worthy of consideration.

One may observe that certain measurability problems related to (2.21) could appear (when the supremum is taken over an uncountable set). It is due to our general hypothesis on the set T . Despite this drawback we will implicitly assume measurability wherever needed. We emphasize that this simplification is not essential, and might be supported in quite a general setting by using the theory of analytic spaces. Roughly speaking, if T is an analytic space and the underlying random function $\Phi(\omega, t)$ is jointly measurable, then the map $\omega \mapsto \sup_{t \in T} \Phi(\omega, t)$ is P-measurable (see [13]). Another approach could be based on a separability assumption which would reduce the set over which supremum is taken to a countable set. Finally, even the most general case of the arbitrary set T could be well treated by using the theory of non-measurable calculus involving the upper integral. All of these methods are well-known and already seen many times. We will not pursue either of them, but instead concentrate to (2.21) in a straightforward way. The following definition is shown to be useful in the main theorem below.

Definition 2.1

Let $\{\xi_n(t)\}_{n \in \mathbf{Z}}$ be a (wide sense) stationary sequence of complex random variables for which the spectral representation (2.7) is valid with the process $\{ Z_t(\lambda) \}_{-\pi \leq \lambda \leq \pi}$ being of bounded variation and right continuous for $t \in T$. Then the family $\big(\{\xi_n(t)\}_{n \in \mathbf{Z}} \mid t \in T \big)$ is said to be *variationally rich*, if for any given $-\pi \leq \lambda_1 < \lambda_2 < \lambda_3 \leq \pi$ and $t', t'' \in T$ one can find $t^* \in T$ satisfying:

$$\mathbf{V}(Z_{t'}, <\lambda_1, \lambda_2]) + \mathbf{V}(Z_{t''}, <\lambda_2, \lambda_3]) \leq \mathbf{V}(Z_{t^*}, <\lambda_1, \lambda_3]) \ . \tag{2.22}$$

It should be noted that every one point family is variationally rich. A typical non-trivial example of variationally rich family is presented in Example 2.6 below. Finally, variationally rich families satisfy the following important property.

Lemma 2.2

Let $\big(\{\xi_n(t)\}_{n \in \mathbf{Z}} \mid t \in T \big)$ *be variationally rich, and suppose that the condition is satisfied:*

$$E\Big(\sup_{t \in T} \mathbf{V}^2\big(Z_t, <-\pi, \pi]\big) \Big) < \infty \ . \tag{2.23}$$

If $I_n = <\alpha_n, \beta_n]$ *are disjoint intervals in* $<-\pi, \pi]$ *with* $\alpha_n = \beta_{n+1}$ *for* $n \geq 1$ *, then we have:*

$$\sup_{t \in T} \mathbf{V}(Z_t, I_n) \to 0 \quad \text{in} \ \ L^2(P) \tag{2.24}$$

as $n \to \infty$.

Proof. Given $\varepsilon > 0$, choose $t_n \in T$ such that:

$$\sup_{t\in T} \mathbf{V}(Z_t, I_n) - \frac{\varepsilon}{2^n} \le \mathbf{V}(Z_{t_n}, I_n)$$

for $n \ge 1$. Given t_n , $t_{n+1} \in T$, by (2.22) one can select $t^* \in T$ such that:

$$\mathbf{V}(Z_{t_n}, I_n) + \mathbf{V}(Z_{t_{n+1}}, I_{n+1}) \le \mathbf{V}(Z_{t^*}, I_n \cup I_{n+1}) \le \sup_{t\in T} \mathbf{V}(Z_t, \langle -\pi, \pi]) \ .$$

Applying the same argument to t^* and t_{n+2} , and then continuing by induction, we obtain:

$$\sum_{n=1}^{\infty} \sup_{t\in T} \mathbf{V}(Z_t, I_n) - \varepsilon \ \le \ \sup_{t\in T} \ \mathbf{V}(Z_t, \langle -\pi, \pi]) \ .$$

Letting $\varepsilon \downarrow 0$, we get:

$$\sum_{n=1}^{\infty} \sup_{t\in T} \mathbf{V}^2(Z_t, I_n) \le \Big(\sum_{n=1}^{\infty} \sup_{t\in T} \mathbf{V}(Z_t, I_n) \Big)^2 \le \sup_{t\in T} \ \mathbf{V}^2(Z_t, \langle -\pi, \pi]) \ .$$

Taking expectation and using condition (2.23), we obtain (2.24). This completes the proof. □

We may now state the main result of this section.

Theorem 2.3

Let $\{\xi_n(t)\}_{n\in\mathbf{Z}}$ *be a (wide sense) stationary sequence of complex random variables for which the spectral representation (2.7) is valid with the process* $\{ Z_t(\lambda) \}_{-\pi\le\lambda\le\pi}$ *being of bounded variation and right continuous for* $t \in T$ *. Suppose that the condition is satisfied:*

(2.25) $$E\Big(\sup_{t\in T} \mathbf{V}^2\big(Z_t, \langle -\pi, \pi]\big)\Big) \ < \ \infty \ .$$

Then the uniform mean-square ergodic theorem is valid:

(2.26) $$\sup_{t\in T} \Big| \frac{1}{n} \sum_{k=0}^{n-1} \xi_k(t) - Z_t(\{0\}) \Big| \to 0 \ \ \textit{in } L^2(P)$$

as $n \to \infty$ *, as soon as either of the following two conditions is fulfilled:*

(2.27) *There exists* $0 < \alpha < 1$ *such that:*

$$\sup_{-\frac{1}{n^\alpha} < \lambda \le \frac{1}{n^\alpha}} E\Big(\sup_{t\in T} \big|Z_t(\lambda) - Z_t(0)\big|^2\Big) = \mathrm{o}\big(n^{\alpha-1}\big)$$

as $n \to \infty$.

(2.28) *There exist* $0 < \alpha < 1 < \beta$ *such that:*

(i) $\sup_{t\in T} \big|Z_t(\lambda) - Z_t(0)\big| \to 0$ *in* P*-probability*

(ii) $\sup_{t\in T} \mathbf{V}\Big(Z_t, \big\langle n^{-\beta}, n^{-\alpha}\big]\Big) \to 0$ *in* P*-probability*

as $\lambda \to 0$ *and* $n \to \infty$.

Moreover, if $\big(\{\xi_n(t)\}_{n\in\mathbf{Z}} \mid t\in T\big)$ *is variationally rich, then the uniform mean-square ergodic theorem (2.26) holds if and only if we have:*

$$\sup_{t\in T}\big|Z_t(\lambda)-Z_t(0)+Z_t(0-)-Z_t(-\lambda)\big| \to 0 \;\; \textit{in} \;\; P\textit{-probability} \tag{2.29}$$

as $\lambda\to 0$ *. In particular, if* $\big(\{\xi_n(t)\}_{n\in\mathbf{Z}} \mid t\in T\big)$ *is variationally rich, then the uniform mean-square ergodic theorem (2.26) holds whenever the process* $\{Z_t(\lambda)\}_{-\pi\le\lambda\le\pi}$ *is uniformly continuous at zero:*

$$\sup_{t\in T}\big|Z_t(\lambda)-Z_t(0)\big| \to 0 \;\; \textit{in} \;\; P\textit{-probability} \tag{2.30}$$

as $\lambda\to 0$ *.*

Proof. Let $t\in T$ and $n\ge 1$ be given and fixed. Then by (2.7) we have:

$$\frac{1}{n}\sum_{k=0}^{n-1}\xi_k(t) = \frac{1}{n}\sum_{k=0}^{n-1}\int_{-\pi}^{\pi} e^{ik\lambda}\; Z_t(d\lambda) = \int_{-\pi}^{\pi}\varphi_n(\lambda)\; Z_t(d\lambda)$$

where $\varphi_n(\lambda)=(1/n)(e^{in\lambda}-1)/(e^{i\lambda}-1)$ for $\lambda\ne 0$ and $\varphi_n(0)=1$. Hence we get:

$$\frac{1}{n}\sum_{k=0}^{n-1}\xi_k(t) - Z_t(\{0\}) = \int_{-\pi}^{\pi}\Big(\varphi_n(\lambda)-1_{\{0\}}(\lambda)\Big)\, Z_t(d\lambda) = \int_{-\pi}^{\pi}\psi_n(\lambda)\; Z_t(d\lambda) \tag{2.31}$$

$$= \int_{-\pi}^{-\delta_n}\psi_n(\lambda)\; Z_t(d\lambda) + \int_{-\delta_n}^{\delta_n}\psi_n(\lambda)\; Z_t(d\lambda) + \int_{\delta_n}^{\pi}\psi_n(\lambda)\; Z_t(d\lambda)$$

for any $0<\delta_n<\pi$, where $\psi_n(\lambda)=\varphi(\lambda)$ for $\lambda\ne 0$ and $\psi_n(0)=0$.

We begin by showing that (2.27) is sufficient for (2.26). The proof of this fact is carried out into two steps as follows. (The first step will be of use later on as well.)

Step 1. We choose $\delta_n\downarrow 0$ in (2.31) such that:

$$\sup_{t\in T}\Big|\int_{-\pi}^{-\delta_n}\psi_n(\lambda)\; Z_t(d\lambda)\Big| \to 0 \;\text{ in }\; L^2(P) \tag{2.32}$$

$$\sup_{t\in T}\Big|\int_{\delta_n}^{\pi}\psi_n(\lambda)\; Z_t(d\lambda)\Big| \to 0 \;\text{ in }\; L^2(P) \tag{2.33}$$

as $n\to\infty$.

First consider (2.32), and note that by (2.16) we get:

$$\Big|\int_{-\pi}^{-\delta_n}\psi_n(\lambda)\; Z_t(d\lambda)\Big| \le \sup_{-\pi<\lambda\le-\delta_n}\big|\psi_n(\lambda)\big| \cdot \mathrm{V}\big(Z_t,<\!-\pi,-\delta_n]\big) \tag{2.34}$$

$$\le \frac{2}{n}\,\frac{1}{|\,e^{-i\delta_n}-1\,|}\cdot \mathrm{V}\big(Z_t,<\!-\pi,\pi]\big)\;.$$

Put $\delta_n=n^{-\alpha}$ for some $\alpha>0$, and denote $A_n=(1/n)(1/|e^{-i\delta_n}-1|)$. Then we have:

$$A_n^2 = \frac{1}{n^2}\,\frac{1}{|\,e^{-i\delta_n}-1\,|^2} = \frac{1}{n^2}\,\frac{1}{2\big(1-\cos(n^{-\alpha})\big)} \to 0 \tag{2.35}$$

as $n\to\infty$, if and only if $\alpha<1$. Hence by (2.34) and (2.25) we see that (2.32) holds with

$\delta_n = n^{-\alpha}$ for any $0 < \alpha < 1$.

Next consider (2.33), and note that by (2.16) we get:

$$\Big| \int_{\delta_n}^{\pi} \psi_n(\lambda)\ Z_t(d\lambda) \Big| \le 2A_n \cdot \mathrm{V}\big(Z_t, <-\pi,\pi]\big) \tag{2.36}$$

where A_n is clearly as above. Thus by the same argument we see that (2.33) holds with $\delta_n = n^{-\alpha}$ for any $0 < \alpha < 1$. (In the sequel δ_n is always understood in this sense.)

Step 2. Here we consider the remaining term in (2.31). First notice that from integration by parts (2.15) we obtain the estimate:

$$\Big| \int_{-\delta_n}^{\delta_n} \psi_n(\lambda)\ Z_t(d\lambda) \Big| \le \big|\psi_n(\delta_n)\big| \cdot \big|Z_t(\delta_n)\big| + \big|\psi_n(-\delta_n)\big| \cdot \big|Z_t(-\delta_n)\big|$$
$$+ \Big| \int_{-\delta_n}^{\delta_n} \big(Z_t(\lambda) - Z_t(0)\big)\psi_n(d\lambda) \Big| + \big|Z_t(0)\big| \cdot \big|\psi_n(\delta_n) - \psi_n(-\delta_n)\big| \ .$$

Hence by Jensen's inequality we get:

$$\sup_{t\in T} \Big| \int_{-\delta_n}^{\delta_n} \psi_n(\lambda)\ Z_t(d\lambda) \Big|^2 \le 4\Big(\big|\psi_n(\delta_n)\big|^2 \cdot \sup_{t\in T} \big|Z_t(\delta_n)\big|^2 + \big|\psi_n(-\delta_n)\big|^2 \cdot \sup_{t\in T} \big|Z_t(-\delta_n)\big|^2$$
$$+ \mathrm{V}\big(\psi_n, <-\delta_n, \delta_n]\big) \cdot \int_{-\delta_n}^{\delta_n} \sup_{t\in T} \big|Z_t(\lambda) - Z_t(0)\big|^2\, \mathrm{V}(\psi_n, d\lambda)$$
$$+ \sup_{t\in T} \big|Z_t(0)\big|^2 \cdot \big|\psi_n(\delta_n) - \psi_n(-\delta_n)\big|^2 \Big) \ .$$

Taking expectation and using Fubini's theorem we obtain:

$$E\Big(\sup_{t\in T} \Big| \int_{-\delta_n}^{\delta_n} \psi_n(\lambda)\ Z_t(d\lambda) \Big|^2 \Big) \le 4\Big(\big|\psi_n(\delta_n)\big|^2 \cdot E\Big(\sup_{t\in T} \big|Z_t(\delta_n)\big|^2 \Big) + \big|\psi_n(-\delta_n)\big|^2 \cdot E\Big(\sup_{t\in T} \big|Z_t(-\delta_n)\big|^2 \Big) \tag{2.37}$$
$$+ \sup_{-\delta_n < \lambda \le \delta_n} E\Big(\sup_{t\in T} \big|Z_t(\lambda) - Z_t(0)\big|^2 \Big) \cdot \mathrm{V}^2\big(\psi_n, <-\delta_n, \delta_n]\big)$$
$$+ E\Big(\sup_{t\in T} \big|Z_t(0)\big|^2 \Big) \cdot \big|\psi_n(\delta_n) - \psi_n(-\delta_n)\big|^2 \Big) \ .$$

Now note that for any $-\pi < \lambda \le \pi$ we have:

$$|Z_t(\lambda)| \le |Z_t(\lambda) - Z_t(\lambda_t)| + |Z_t(\lambda_t)| \le \mathrm{V}(Z_t, <-\pi, \pi]) + \varepsilon$$

where λ_t is chosen to be close enough to $-\pi$ to satisfy $|Z_t(\lambda_t)| \le \varepsilon$ by right continuity. Passing to the supremum and using (2.25) hence we get:

$$E\Big(\sup_{t\in T} \sup_{-\pi < \lambda \le \pi} \big|Z_t(\lambda)\big|^2 \Big) < \infty \ . \tag{2.38}$$

By (2.35) we have $\psi_n(\delta_n) \to 0$ and $\psi_n(-\delta_n) \to 0$, and thus by (2.38) we see that:

$$\big|\psi_n(\delta_n)\big|^2 \cdot E\Big(\sup_{t\in T} \big|Z_t(\delta_n)\big|^2 \Big) + \big|\psi_n(-\delta_n)\big|^2 \cdot E\Big(\sup_{t\in T} \big|Z_t(-\delta_n)\big|^2 \Big) \tag{2.39}$$
$$+ E\Big(\sup_{t\in T} \big|Z_t(0)\big|^2 \Big) \cdot \big|\psi_n(\delta_n) - \psi_n(-\delta_n)\big|^2 \to 0$$

as $n \to \infty$. From (2.37) we see that it remains to estimate the total variation of ψ_n on $<-\delta_n, \delta_n]$.

For this put $F_k(\lambda) = e^{ik\lambda}$ for $0 \leq k \leq n-1$, and notice that we have:

$$\text{(2.40)} \qquad \mathrm{V}\big(\psi_n, <\!-\delta_n, \delta_n]\big) \leq 1 + \mathrm{V}\big(\varphi_n, <\!-\delta_n, \delta_n]\big) \leq 1 + \int_{-\delta_n}^{\delta_n} |\varphi_n'(\lambda)| \; d\lambda$$

By the Cauchy-Schwarz inequality and orthogonality of F_k's on $<\!-\pi, \pi]$, we obtain from (2.40) the following estimate:

$$\text{(2.41)} \qquad \mathrm{V}\big(\psi_n, <\!-\delta_n, \delta_n]\big) \leq 1 + \sqrt{2\delta_n}\, \frac{1}{n} \Big(\int_{-\pi}^{\pi} \Big| \sum_{k=0}^{n-1} ikF_k(\lambda) \Big|^2 d\lambda \Big)^{1/2} =$$

$$= 1 + \sqrt{2\delta_n}\, \frac{1}{n} \Big(\sum_{k=1}^{n-1} k^2 \Big)^{1/2} \leq 1 + \sqrt{2\delta_n}\, \frac{1}{n}\, n^{3/2} \leq C n^{(1-\alpha)/2}$$

with some constant $C > 0$. Combining (2.31), (2.32), (2.33), (2.37), (2.39) and (2.41) we complete the proof of sufficiency of (2.27) for (2.26). This fact finishes Step 2.

We proceed by showing that (2.28) is sufficient for (2.26). For this Step 1 can stayed unchanged, and Step 2 is modified as follows.

Step 3. First split up the integral:

$$\text{(2.42)} \qquad \int_{-\delta_n}^{\delta_n} \psi_n(\lambda)\; Z_t(d\lambda) = \int_{-\delta_n}^{-\eta_n} \psi_n(\lambda)\; Z_t(d\lambda) + \int_{-\eta_n}^{\eta_n} \psi_n(\lambda)\; Z_t(d\lambda) + \int_{\eta_n}^{\delta_n} \psi_n(\lambda)\; Z_t(d\lambda)$$

for any $0 < \eta_n < \delta_n$. Put $\eta_n = n^{-\beta}$ for some $\beta > 1$. (In the sequel η_n is always understood in this sense.) Then from (2.9) and (2.16) we get by right continuity:

$$\text{(2.43)} \qquad \Big| \int_{-\delta_n}^{-\eta_n} \psi_n(\lambda)\; Z_t(d\lambda) \Big| \leq \mathrm{V}\big(Z_t, <\!-\delta_n, -\eta_n]\big) = \mathrm{V}\big(Z_t, [-\delta_n, -\eta_n]\big)$$

$$= \mathrm{V}\big(Z_t, [\eta_n, \delta_n]\big) = \mathrm{V}\big(Z_t, <\!\eta_n, \delta_n]\big) \; .$$

Similarly, by (2.16) we get:

$$\text{(2.44)} \qquad \Big| \int_{\eta_n}^{\delta_n} \psi_n(\lambda)\; Z_t(d\lambda) \Big| \leq \mathrm{V}\big(Z_t, <\!\eta_n, \delta_n]\big) \; .$$

From (2.25), (2.43), (2.44) and (ii) in (2.28) it follows by uniform integrability that:

$$\text{(2.45)} \qquad \sup_{t \in T} \Big| \int_{-\delta_n}^{-\eta_n} \psi_n(\lambda)\; Z_t(d\lambda) \Big| \to 0 \;\text{ in }\; L^2(P)$$

$$\text{(2.46)} \qquad \sup_{t \in T} \Big| \int_{\eta_n}^{\delta_n} \psi_n(\lambda)\; Z_t(d\lambda) \Big| \to 0 \;\text{ in }\; L^2(P)$$

as $n \to \infty$. Now by (2.31), (2.32), (2.33), (2.45) and (2.46) we see that it is enough to show:

$$\text{(2.47)} \qquad \sup_{t \in T} \Big| \int_{-\eta_n}^{\eta_n} \psi_n(\lambda)\; Z_t(d\lambda) \Big| \to 0 \;\text{ in }\; L^2(P)$$

as $n \to \infty$. For this recall $\psi_n(0) = 0$. Thus from integration by parts (2.15) with (2.17) we get:

$$\text{(2.48)} \qquad \Big| \int_{-\eta_n}^{\eta_n} \psi_n(\lambda)\; Z_t(d\lambda) \Big| \leq \Big| \int_{-\eta_n}^{0-} \psi_n(\lambda)\; Z_t(d\lambda) \Big| + \Big| \int_{0}^{\eta_n} \psi_n(\lambda)\; Z_t(d\lambda) \Big|$$

$$\leq \big|\psi_n(0-)Z_t(0-) - \psi_n(-\eta_n)Z_t(-\eta_n)\big| + \Big|\int_{-\eta_n}^{0-} Z_t(\lambda)\ \psi_n(d\lambda)\Big|$$

$$+ \big|\psi_n(\eta_n)Z_t(\eta_n) - \psi_n(0+)Z_t(0)\big| + \Big|\int_0^{\eta_n} Z_t(\lambda)\ \psi_n(d\lambda)\Big| \leq \big|Z_t(0-) - \psi_n(-\eta_n)Z_t(-\eta_n)\big|$$

$$+ \sup_{-\eta_n<\lambda<0} \big|Z_t(\lambda)\big| \cdot \mathrm{V}\big(\psi_n, <-\eta_n, 0>\big) + \big|\psi_n(\eta_n)Z_t(\eta_n) - Z_t(0)\big| + \sup_{0<\lambda\leq\eta_n} \big|Z_t(\lambda)\big| \cdot \mathrm{V}\big(\psi_n, <0, \eta_n]\big)\ .$$

Next we verify that:

$$\psi_n(-\eta_n) \to 1 \quad \& \quad \psi_n(\eta_n) \to 1 \tag{2.49}$$

as $n \to \infty$. For this note that we have:

$$\big|\psi_n(\pm\eta_n) - 1\big| = \Big|\frac{1}{n}\sum_{k=0}^{n-1} e^{\pm ik\eta_n} - 1\Big| \leq \max_{0\leq k\leq n-1} \big|e^{\pm ik\eta_n} - 1\big|$$

$$= \big|e^{\pm i(n-1)\eta_n} - 1\big| \leq \big|e^{\pm i(1/n^{\beta-1})} - 1\big| \to 0$$

as $n \to \infty$. This proves (2.49). In addition we show that:

$$\mathrm{V}\big(\psi_n, <-\eta_n, 0>\big) \to 0 \quad \& \quad \mathrm{V}\big(\psi_n, <0, \eta_n]\big) \to 0 \tag{2.50}$$

as $n \to \infty$. For this note that we have:

$$\mathrm{V}\big(\psi_n, <0, \eta_n]\big) \leq \frac{1}{n}\sum_{k=0}^{n-1} \mathrm{V}\big(F_k, <0, \eta_n]\big) \leq \max_{0\leq k\leq n-1} \mathrm{V}\big(F_k, <0, \eta_n]\big)$$

$$= \mathrm{V}\big(F_{n-1}, <0, \eta_n]\big) \leq \big|1 - \cos\eta_n\big| + \big|\sin\eta_n\big| \to 0$$

as $n \to \infty$. In the same way we obtain:

$$\mathrm{V}\big(\psi_n, <-\eta_n, 0>\ \big) \leq \big|1 - \cos\eta_n\big| + \big|\sin\eta_n\big| \to 0$$

as $n \to \infty$. Thus (2.50) is established. Finally, we obviously have:

$$\big|Z_t(0-) - \psi_n(-\eta_n)Z_t(-\eta_n)\big| \leq \big|Z_t(0-) - Z_t(-\eta_n)\big| + \big|1 - \psi_n(-\eta_n)\big| \cdot \big|Z_t(-\eta_n)\big| \tag{2.51}$$

$$\big|\psi_n(\eta_n)Z_t(\eta_n) - Z_t(0)\big| \leq \big|\psi_n(\eta_n) - 1\big| \cdot \big|Z_t(\eta_n)\big| + \big|Z_t(\eta_n) - Z_t(0)\big|\ . \tag{2.52}$$

Combining (2.28), (2.38), (2.48), (2.49), (2.50), (2.51) and (2.52) we obtain (2.47). This fact completes the proof of sufficiency of (2.28) for (2.26). Step 3 is complete.

A slight modification of Step 3 will show that (2.29) is sufficient for (2.26) whenever the family is variationally rich. This is done in the next step.

Step 4. First consider the left side in (2.48). By (2.15) and (2.17) we have:

$$\Big|\int_{-\eta_n}^{\eta_n} \psi_n(\lambda)\ Z_t(d\lambda)\Big| \leq \Big|\int_{-\eta_n}^{0-} \psi_n(\lambda)\ Z_t(d\lambda) + \int_0^{\eta_n} \psi_n(\lambda)\ Z_t(d\lambda)\Big|$$

$$\leq \big|\psi_n(0-)Z_t(0-) - \psi_n(-\eta_n)Z_t(-\eta_n) + \psi_n(\eta_n)Z_t(\eta_n) - \psi_n(0+)Z_t(0)\big|$$

$$+ \Big|\int_{-\eta_n}^{0-} Z_t(\lambda)\ \psi_n(d\lambda)\Big| + \Big|\int_0^{\eta_n} Z_t(\lambda)\ \psi_n(d\lambda)\Big| \leq \big|\psi_n(\eta_n) - 1\big| \cdot \big|Z_t(\eta_n)\big|$$

$$+ \big|Z_t(\eta_n) - Z_t(0) + Z_t(0-) - Z_t(-\eta_n)\big| + \big|1 - \psi_n(-\eta_n)\big| \cdot \big|Z_t(-\eta_n)\big|$$

$$+ \sup_{-\eta_n<\lambda<0} \big|Z_t(\lambda)\big| \cdot \mathrm{V}\big(\psi_n, <-\eta_n, 0>\big) + \sup_{0<\lambda\leq\eta_n} \big|Z_t(\lambda)\big| \cdot \mathrm{V}\big(\psi_n, <0, \eta_n]\big) \ .$$

Hence by the same arguments as in Step 3 we obtain (2.47).

Next consider the remaining terms in (2.42). By (2.43) and (2.44) we see that it suffices to show:

(2.53) $$\mathrm{V}\big(Z_t, <\eta_n, \delta_n]\big) \to 0 \ \text{ in } \ L^2(P)$$

as $n \to \infty$. We show that (2.53) holds with $\eta_n = n^{-3/2}$ and $\delta_n = n^{-1/2}$. The general case follows by the same pattern (with possibly a three intervals argument in (2.55) below).

For this put $I_n = <\eta_n, \delta_n]$, and let $p_n = (p_{n-1})^3$ for $n \geq 2$ with $p_1 = 2$. Then the intervals I_{p_n} satisfy the hypotheses of Lemma 2.2 for $n \geq 1$, and therefore we get:

(2.54) $$\sup_{t\in T} \mathrm{V}\big(Z_t, I_{p_n}\big) \to 0 \ \text{ in } \ L^2(P)$$

as $n \to \infty$. Moreover, for $p_n < q \leq p_{n+1}$ we have:

(2.55) $$\mathrm{V}\big(Z_t, I_q\big) \leq \mathrm{V}\big(Z_t, I_{p_n}\big) + \mathrm{V}\big(Z_t, I_{p_{n+1}}\big) \ .$$

Thus from (2.54) we obtain (2.53), and the proof of sufficiency of (2.29) for (2.26) follows as in Step 3. This fact completes Step 4.

In the last step we prove necessity of (2.29) for (2.26) under the assumption that the family is variationally rich.

Step 5. From integration by parts (2.15) we have:

$$\frac{1}{n}\sum_{k=0}^{n-1} \xi_k(t) - Z_t\big(\{0\}\big) = \int_{-\pi}^{\pi} \psi_n(\lambda)\ Z_t(d\lambda) = \int_{-\pi}^{-\delta_n} \psi_n(\lambda)\ Z_t(d\lambda) + \int_{-\delta_n}^{-\eta_n} \psi_n(\lambda)\ Z_t(d\lambda)$$
$$+ \psi_n(0-)Z_t(0-) - \psi_n(-\eta_n)Z_t(-\eta_n) + \psi_n(\eta_n)Z_t(\eta_n) - \psi_n(0+)Z_t(0)$$
$$- \int_{-\eta_n}^{0-} Z_t(\lambda)\ \psi_n(d\lambda) - \int_{0}^{\eta_n} Z_t(\lambda)\ \psi_n(d\lambda)$$
$$+ \int_{\eta_n}^{\delta_n} \psi_n(\lambda)\ Z_t(d\lambda) + \int_{\delta_n}^{\pi} \psi_n(\lambda)\ Z_t(d\lambda) \ .$$

Hence we easily get:

(2.56) $$\big|Z_t(\eta_n) - Z_t(0) + Z_t(0-) - Z_t(-\eta_n)\big| \leq \big|\psi_n(\eta_n) - 1\big| \cdot \big|Z_t(\eta_n)\big| + \big|1 - \psi_n(-\eta_n)\big| \cdot \big|Z_t(-\eta_n)\big|$$
$$+ \Big|\int_{-\pi}^{\pi} \psi_n(\lambda)\ Z_t(d\lambda)\Big| + \Big|\int_{-\pi}^{-\delta_n} \psi_n(\lambda)\ Z_t(d\lambda)\Big| + \Big|\int_{-\delta_n}^{-\eta_n} \psi_n(\lambda)\ Z_t(d\lambda)\Big| + \Big|\int_{-\eta_n}^{0-} Z_t(\lambda)\ \psi_n(d\lambda)\Big|$$
$$+ \Big|\int_{0}^{\eta_n} Z_t(\lambda)\ \psi_n(d\lambda)\Big| + \Big|\int_{\eta_n}^{\delta_n} \psi_n(\lambda)\ Z_t(d\lambda)\Big| + \Big|\int_{\delta_n}^{\pi} \psi_n(\lambda)\ Z_t(d\lambda)\Big| \ .$$

Finally, from (2.16) and (2.17) we obtain the estimates as in Step 3 and Step 4:

(2.57) $$\Big|\int_{-\eta_n}^{0-} Z_t(\lambda)\ \psi_n(d\lambda)\Big| \leq \sup_{-\eta_n<\lambda<0} \big|Z_t(\lambda)\big| \cdot \mathrm{V}\big(\psi_n, <-\eta_n, 0>\big)$$

(2.58) $$\Big|\int_{0}^{\eta_n} Z_t(\lambda)\ \psi_n(d\lambda)\Big| \leq \sup_{0<\lambda\leq\eta_n} \big|Z_t(\lambda)\big| \cdot \mathrm{V}\big(\psi_n, <0, \eta_n]\big)$$

(2.59) $$\Big| \int_{-\delta_n}^{-\eta_n} \psi_n(\lambda)\ Z_t(d\lambda) \Big| \leq \mathrm{V}\big(Z_t, \langle \eta_n, \delta_n]\big)$$

(2.60) $$\Big| \int_{\eta_n}^{\delta_n} \psi_n(\lambda)\ Z_t(d\lambda) \Big| \leq \mathrm{V}\big(Z_t, \langle \eta_n, \delta_n]\big)\ .$$

Combining (2.32), (2.33), (2.38), (2.49), (2.50), (2.53), (2.56) and (2.57)-(2.60) we complete the proof of necessity of (2.29) for (2.26). This fact finishes Step 5. The last statement of the theorem is obvious, and the proof is complete. □

Remarks 2.4

(1) Note that Theorem 2.3 reads as follows: If the convergence in (2.27) is not uniformly fast enough (but we still have it), then examine convergence of the total variation as stated in (ii) of (2.28). Characterization (2.29) with (2.30) shows that this approach is in some sense optimal.

(2) A close look into the proof shows that we have convergence P-a.s. in (2.26), as soon as we have convergence P-a.s. either in (2.27) (without the expectation and square sign, but with $(\alpha–1)/2$), or in (i) and (ii) of (2.28). Moreover, if $\big(\{\xi_n(t)\}_{n\in\mathbf{Z}} \mid t \in T\big)$ is *pointwise* variationally rich, then the same fact holds as for characterization (2.29), as well as for sufficient condition (2.30). In all of these cases the condition (2.25) could be relaxed by removing the expectation and square sign. In this way we cover a *pointwise uniform ergodic theorem for (wide sense) stationary sequences*.

(3) Under condition (2.25) convergence in P-probability in either (i) or (ii) of (2.28) is equivalent to the convergence in $L^2(P)$. The same fact holds for convergence in P-probability in either (2.29) or (2.30). It follows by uniform integrability.

(4) For condition (ii) of (2.28) note that for every fixed $t \in T$ and any $0 < \alpha < 1 < \beta$:

$$\mathbf{V}\Big(Z_t, \langle n^{-\beta}, n^{-\alpha}]\Big) \to 0 \quad P\text{-a.s.}$$

whenever $\{\, Z_t(\lambda)\,\}_{-\pi\leq\lambda\leq\pi}$ is of bounded variation and right continuous (at zero), as $n \to \infty$. Note also that under condition (2.25) the convergence is in $L^2(P)$ as well.

(5) It is easily verified by examining the proof above that characterization (2.29) remains valid under (2.25), whenever the property of being variationally rich is replaced with any other property implying condition (ii) of (2.28).

(6) It remains an open interesting question does the result of Theorem 2.3 extend in some form to the case where the associated process $\{\, Z_t(\lambda)\,\}_{-\pi\leq\lambda\leq\pi}$ is not of bounded variation for $t \in T$.

Example 2.5

Consider *an almost periodic sequence* of random variables:

(2.61) $$\xi_n(t) = \sum_{k\in\mathbf{Z}} z_k(t)\, e^{i\lambda_k n}$$

for $n \in \mathbf{Z}$ and $t \in T$. In other words, for every fixed $t \in T$ we have:

(2.62) Random variables $z_i(t)$ and $z_j(t)$ are mutually orthogonal for all $i \neq j$:

$$E\big(z_i(t)\overline{z_j(t)}\big) = 0$$

(2.63) Numbers λ_k belong to $\langle -\pi, \pi]$ for $k \in \mathbf{Z}$, and satisfy $\lambda_i \neq \lambda_j$ whenever $i \neq j$

(2.64) The condition is satisfied:

$$\sum_{k\in\mathbf{Z}} E|z_k(t)|^2 \, < \, \infty \ .$$

Note that under (2.64) the series in (2.61) converges in the mean-square sense.

From (2.61) we see that the orthogonal stochastic measure is defined as follows:

$$Z_t(\Delta) \, = \sum_{k\in\mathbf{Z},\lambda_k\in\Delta} z_k(t)$$

for $\Delta \in \mathcal{B}(<\!-\pi,\pi\,])$ and $t \in T$. The covariance function is given by:

$$R_t(n) \, = \sum_{k\in\mathbf{Z}} e^{i\lambda_k n} E|z_k(t)|^2$$

for $n \in \mathbf{Z}$ and $t \in T$.

In order to apply Theorem 2.3 we will henceforth assume:

(2.65)
$$E\biggl(\sup_{t\in T}\Bigl(\sum_{k\in\mathbf{Z}} |z_k(t)|\Bigr)^2\biggr) \, < \, \infty \ .$$

Note that this condition implies:

(2.66)
$$E\biggl(\sup_{t\in T}\Bigl(\sum_{k\in\mathbf{Z}} |z_k(t)|^2\Bigr)\biggr) \, < \, \infty \ .$$

Let $k_0 \in \mathbf{Z}$ be chosen to satisfy $\lambda_{k_0} = 0$, and otherwise conceive $z_{k_0}(t) \equiv 0$ for all $t \in T$. According to Theorem 2.3, the uniform mean-square ergodic theorem is valid:

(2.67)
$$\sup_{t\in T} \Bigl| \frac{1}{n}\sum_{k=0}^{n-1} \xi_k(t) - z_{k_0}(t) \Bigr| \rightarrow 0 \ \text{ in } L^2(P)$$

as $n \rightarrow \infty$, as soon as either of the following two conditions is fulfilled:

(2.68) There exists $0 < \alpha < 1$ such that:

$$\sup_{0<\lambda\leq\frac{1}{n^\alpha}} E\biggl(\sup_{t\in T}\Bigl|\sum_{0<\lambda_k\leq\lambda} z_k(t)\Bigr|^2\biggr) + \sup_{-\frac{1}{n^\alpha}<\lambda<0} E\biggl(\sup_{t\in T}\Bigl|\sum_{\lambda<\lambda_k\leq 0} z_k(t)\Bigr|^2\biggr) \, = \, o\bigl(n^{\alpha-1}\bigr)$$

as $n \rightarrow \infty$.

(2.69) There exist $0 < \alpha < 1 < \beta$ such that:

(i) $$\sup_{t\in T}\Bigl|\sum_{0<\lambda_k\leq\lambda} z_k(t)\Bigr| + \sup_{t\in T}\Bigl|\sum_{-\lambda<\lambda_k\leq 0} z_k(t)\Bigr| \longrightarrow 0 \quad \text{in } P\text{-probability}$$

(ii) $$\sup_{t\in T} \sum_{\frac{1}{n^\beta}<\lambda_j\leq\frac{1}{n^\alpha}} |z_j(t)| \rightarrow 0 \ \text{ in } P\text{-probability}$$

as $\lambda \downarrow 0$ and $n \rightarrow \infty$.

In particular, hence we see if zero does not belong to the closure of the sequence $\{\lambda_k\}_{k\in\mathbf{Z}}$, then (2.67) is valid. Moreover, if the condition is fulfilled:

(2.70)
$$E\Bigl(\sum_{k\in\mathbf{Z}} \sup_{t\in T} |z_k(t)|\Bigr)^2 < \, \infty$$

with $z_{k_0}(t) \equiv 0$ for $t \in T$, then clearly (i) and (ii) of (2.69) are satisfied, even though the condition (2.68) on the speed of convergence could possibly fail. Thus, under (2.70) we have again (2.67).

Example 2.6 (Variationally rich family)
Consider the Gaussian case in the preceding example. Thus, suppose that the almost periodic sequence (2.61) is given:

(2.71) $$\xi_n(t) = \sum_{k \in \mathbf{Z}} z_k(t) e^{i\lambda_k n}$$

for $n \in \mathbf{Z}$ and $t \in T$, where for every fixed $t \in T$ the random variables $z_k(t) = \sigma_k(t) \cdot g_k \sim N(0, \sigma_k^2(t))$ are independent and Gaussian with zero mean and variance $\sigma_k^2(t)$ for $k \in \mathbf{Z}$. Then (2.62) is fulfilled. We assume that (2.63) and (2.64) hold. Thus, the family $\Sigma = \big(\{\sigma_k^2(t)\}_{k \in \mathbf{Z}} \mid t \in T \big)$ satisfies the following condition:

(2.72) $$\sum_{k \in \mathbf{Z}} \sigma_k^2(t) < \infty$$

for all $t \in T$. We want to see when the family $\xi = \big(\{\xi_n(t)\}_{n \in \mathbf{Z}} \mid t \in T \big)$ is variationally rich, and this should be expressed in terms of the family Σ .

For this, take arbitrary $-\pi \leq \lambda_1 < \lambda_2 < \lambda_3 \leq \pi$ and t' , $t'' \in T$, and compute the left-hand side in (2.22). From the form of the orthogonal stochastic measure $Z_t = Z_t(\omega, \Delta)$ which is established in the preceding example, we see that:

(2.73) $$V(Z_t, \Delta) = \sum_{k \in \mathbf{Z}, \lambda_k \in \Delta} |z_k(t)|$$

for $\Delta \in \mathcal{B}(<\!-\pi, \pi\,])$ and $t \in T$. Hence we find:

(2.74) $$\mathbf{V}(Z_{t'}, <\!\lambda_1, \lambda_2]) + \mathbf{V}(Z_{t''}, <\!\lambda_2, \lambda_3]) = \sum_{\lambda_1 < \lambda_k \leq \lambda_2} \big|\sigma_k(t')\big| |g_k| + \sum_{\lambda_2 < \lambda_k \leq \lambda_3} \big|\sigma_k(t'')\big| |g_k| \; .$$

Thus, in order that ξ is variationally rich, the expression in (2.74) must be dominated by:

(2.75) $$\sum_{\lambda_1 < \lambda_k \leq \lambda_3} \big|\sigma_k(t^*)\big| |g_k|$$

for some $t^* \in T$. For instance, this will be true if the family Σ satisfies the following property:

(2.76) $$\big\{\sigma_k^2(t') \vee \sigma_k^2(t'')\big\}_{k \in \mathbf{Z}} \in \Sigma$$

for all t' , $t'' \in T$. For example, if $\sigma_k(t) = t/2^{|k|}$ for $k \in \mathbf{Z}$ and t belongs to a subset T of $<0, \infty>$, the last property (2.76) is satisfied, and the family ξ is variationally rich.

3. The uniform mean-square ergodic theorem in the continuous parameter case

The aim of this section is to present the uniform mean-square ergodic theorem in the continuous parameter case. Throughout we consider a family of (*wide sense*) *stationary* processes of complex random variables $\big(\{X_s(t)\}_{s \in \mathbf{R}} \mid t \in T \big)$ defined on the probability space $(\Omega, \mathcal{F}, P)$ and indexed by the set T. Thus, we have:

$$E\big|X_s(t)\big|^2 < \infty \tag{3.1}$$

$$E\big(X_s(t)\big) = E\big(X_0(t)\big) \tag{3.2}$$

$$\mathrm{Cov}\big(X_{r+s}(t), X_r(t)\big) = \mathrm{Cov}\big(X_s(t), X_0(t)\big) \tag{3.3}$$

for all $s, r \in \mathbf{R}$, and all $t \in T$. For the same reasons as in Section 2 we shall refer the reader to the classical references on the subject [1], [6] and [15].

As a matter of convenience, we will henceforth suppose:

$$E\big(X_s(t)\big) = 0 \tag{3.4}$$

for all $s \in \mathbf{R}$, and all $t \in T$. Thus the *covariance function* of $\{X_s(t)\}_{s\in\mathbf{R}}$ is given by:

$$R_t(s) = E\big(X_s(t)\overline{X_0(t)}\big) \tag{3.5}$$

whenever $s \in \mathbf{R}$ and $t \in T$.

By the *Bochner theorem* there exists a finite measure $\mu_t = \mu_t(\Delta)$ on $\mathcal{B}(\mathbf{R})$ such that:

$$R_t(s) = \int_{-\infty}^{\infty} e^{is\lambda}\, \mu_t(d\lambda) \tag{3.6}$$

for $s \in \mathbf{R}$ and $t \in T$. The measure μ_t is called the *spectral measure* of $\{X_s(t)\}_{s\in\mathbf{R}}$ for $t \in T$.

The *spectral representation theorem* states if R_t is continuous, then there exists an *orthogonal stochastic measure* $Z_t = Z_t(\omega, \Delta)$ on $\Omega \times \mathcal{B}(\mathbf{R})$ such that:

$$X_s(t) = \int_{-\infty}^{\infty} e^{is\lambda}\, Z_t(d\lambda) \tag{3.7}$$

for $s \in \mathbf{R}$ and $t \in T$. The fundamental identity in this context is as follows:

$$E\Big|\int_{-\infty}^{\infty} \varphi(\lambda)\; Z_t(d\lambda)\Big|^2 = \int_{-\infty}^{\infty} \big|\varphi(\lambda)\big|^2\, \mu_t(d\lambda) \tag{3.8}$$

whenever the function $\varphi : \mathbf{R} \to \mathbf{C}$ belongs to $L^2(\mu_t)$ for $t \in T$. We also have (2.9) which is valid for all $\Delta \in \mathcal{B}(\mathbf{R})$, and all $t \in T$.

The random process defined by:

$$Z_t(\lambda) = Z_t(<-\infty, \lambda]) \tag{3.9}$$

for $\lambda \in \mathbf{R}$ is with *orthogonal increments* for every $t \in T$. Thus, we have (2.11), (2.12) and (2.13) whenever $\lambda \in \mathbf{R}$ and $-\infty < \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < \infty$ for all $t \in T$. We will henceforth put $Z_t(-\infty) = 0$ for all $t \in T$. Moreover, we will assume below again that the process $\{\, Z_t(\lambda)\,\}_{\lambda\in\mathbf{R}}$ is of *bounded variation* and *right continuous* (outside of a P-nullset) for all $t \in T$. In this case the integral:

$$\int_{-\infty}^{\infty} \varphi(\lambda)\; Z_t(d\lambda) \tag{3.10}$$

may be well defined pointwise on Ω as the usual Riemann-Stieltjes integral for all $t \in T$. If

$\psi : <\lambda_1, \lambda_2\,] \to \mathbf{C}$ is of bounded variation and right continuous for some $-\infty \leq \lambda_1 < \lambda_2 \leq \infty$, then *integration by parts formula* (2.15) holds for all $t \in T$. Moreover, for the total variation $\mathrm{V}(\Phi, <\lambda_1, \lambda_2\,])$ of the function $\Phi : <\lambda_1, \lambda_2] \to \mathbf{C}$ we have (2.16) and (2.17) for all $t \in T$.

The *mean-square ergodic theorem* for $\{X_s(t)\}_{s \in \mathbf{R}}$ states:

$$\frac{1}{\tau} \int_0^\tau X_s(t)\ ds \ \to\ Z_t(\{0\}) \ \text{ in } L^2(P) \tag{3.11}$$

as $\tau \to \infty$, for all $t \in T$. If moreover the process $\{\, Z_t(\lambda)\,\}_{\lambda \in \mathbf{R}}$ is of bounded variation and right continuous for all $t \in T$, then the convergence in (3.11) is P-a.s. as well. We also have:

$$\frac{1}{\tau} \int_0^\tau R_t(s)\ ds \ \to\ \mu_t(\{0\}) \tag{3.12}$$

as $\tau \to \infty$, for all $t \in T$. Finally, it is easily seen that (2.20) is valid in the present case whenever $t \in T$.

It is the main purpose of the present section to investigate when the *uniform mean-square ergodic theorem* is valid:

$$\sup_{t \in T} \Big| \frac{1}{\tau} \int_0^\tau X_s(t)\ ds - Z_t(\{0\}) \Big| \to 0 \ \text{ in } L^2(P) \tag{3.13}$$

as $\tau \to \infty$. As before, we think that this problem appears worthy of consideration, and moreover to the best of our knowledge it has not been studied previously. It turns out that the methods developed in the last section carry over to the present case without any difficulties.

The main novelty of the approach could be explained in the same way as in Section 2. The same remark might be also directed to the measurability problems. We will not state either of this more precisely here, but instead recall that we implicitly assume measurability wherever needed.

The definition stated in Section 2 extends verbatim to the present case. Again, it is shown to be useful in the main theorem below.

Definition 3.1

Let $\{X_s(t)\}_{s \in \mathbf{R}}$ be a (wide sense) stationary process of complex random variables for which the spectral representation (3.7) is valid with the process $\{\, Z_t(\lambda)\,\}_{\lambda \in \mathbf{R}}$ being of bounded variation and right continuous for $t \in T$. Then the family $\big(\{X_s(t)\}_{s \in \mathbf{R}} \mid t \in T\big)$ is said to be *variationally rich*, if for any given $-\infty < \lambda_1 < \lambda_2 < \lambda_3 < \infty$ and $t', t'' \in T$ one can find $t^* \in T$ satisfying:

$$\mathbf{V}(Z_{t'}, <\lambda_1, \lambda_2]) + \mathbf{V}(Z_{t''}, <\lambda_2, \lambda_3]) \leq \mathbf{V}(Z_{t^*}, <\lambda_1, \lambda_3])\ . \tag{3.14}$$

We remark again that every one point family is variationally rich. A typical non-trivial example of variationally rich family in the present case may be constructed similarly as in Example 2.6 above. Finally, variationally rich families satisfy the following important property.

Lemma 3.2

Let $\big(\{X_s(t)\}_{s \in \mathbf{R}} \mid t \in T\big)$ be variationally rich, and suppose that the condition is satisfied:

$$E\Big(\sup_{t \in T} \mathbf{V}^2\big(Z_t, \mathbf{R}\big)\Big) \ <\ \infty\ . \tag{3.15}$$

If $I_n = <\alpha_n, \beta_n]$ are disjoint intervals in $\mathbf{R}$ with $\alpha_n = \beta_{n+1}$ for $n \geq 1$, then we have:

(3.16) $$\sup_{t\in T} \mathbf{V}(Z_t, I_n) \to 0 \quad in \quad L^2(P)$$

as $n \to \infty$.

Proof. The proof is exactly the same as the proof of Lemma 2.2. The only difference is that the interval $<-\pi, \pi]$ should be replaced with $\mathbf{R}$. □

We may now state the main result of this section.

Theorem 3.3

Let $\{X_s(t)\}_{s\in\mathbf{R}}$ *be a (wide sense) stationary process of complex random variables for which the spectral representation (3.7) is valid with the process* $\{Z_t(\lambda)\}_{\lambda\in\mathbf{R}}$ *being of bounded variation and right continuous for* $t \in T$ *. Suppose that the condition is satisfied:*

(3.17) $$E\Big(\sup_{t\in T} \mathbf{V}^2\big(Z_t, \mathbf{R}\big)\Big) < \infty \, .$$

Then the uniform mean-square ergodic theorem is valid:

(3.18) $$\sup_{t\in T} \Big| \frac{1}{\tau} \int_0^\tau X_s(t)\; ds - Z_t(\{0\}) \Big| \to 0 \;\; in \; L^2(P)$$

as $\tau \to \infty$ *, as soon as either of the following two conditions is fulfilled:*

(3.19) *There exists* $0 < \alpha < 1$ *such that:*

$$\sup_{-\tau^{-\alpha} < \lambda \le \tau^{-\alpha}} E\Big(\sup_{t\in T} \big|Z_t(\lambda) - Z_t(0)\big|^2\Big) = \mathrm{o}\big(\tau^{\alpha-1}\big)$$

as $\tau \to \infty$.

(3.20) *There exist* $0 < \alpha < 1 < \beta$ *such that:*

(i) $$\sup_{t\in T} \big|Z_t(\lambda) - Z_t(0)\big| \to 0 \quad in \quad P\text{-}probability$$

(ii) $$\sup_{t\in T} \mathbf{V}\Big(Z_t, \big\langle \tau^{-\beta}, \tau^{-\alpha}\big]\Big) \to 0 \quad in \quad P\text{-}probability$$

as $\lambda \to 0$ *and* $\tau \to \infty$.

Moreover, if the family $\big(\{Z_s(t)\}_{s\in\mathbf{R}} \mid t \in T\big)$ *is variationally rich, then the uniform mean-square ergodic theorem (3.18) holds if and only if we have:*

(3.21) $$\sup_{t\in T} \big|Z_t(\lambda) - Z_t(0) + Z_t(0-) - Z_t(-\lambda)\big| \to 0 \;\; in \;\; P\text{-}probability$$

as $\lambda \to 0$ *. In particular, if the family* $\big(\{Z_s(t)\}_{s\in\mathbf{R}} \mid t \in T\big)$ *is variationally rich, then the uniform mean-square ergodic theorem (3.18) holds whenever the process* $\{Z_t(\lambda)\}_{\lambda\in\mathbf{R}}$ *is uniformly continuous at zero:*

(3.22) $$\sup_{t\in T} \big|Z_t(\lambda) - Z_t(0)\big| \to 0 \quad in \quad P\text{-}probability$$

as $\lambda \to 0$.

Proof. The proof may be carried out by using precisely the same method which is presented in the proof of Theorem 2.3. Thus we find it unnecessary here to provide all of the details, but instead will turn out only the essential points which make the procedure working.

Let $\tau > 0$ be fixed. We begin by noticing that a simple Riemann approximation yields:

$$\sum_{k=1}^{n} \frac{\tau}{n} \exp\Big(ik\frac{\tau}{n}\lambda\Big) \rightarrow \int_0^{\tau} e^{is\lambda}\ ds$$

as $n \rightarrow \infty$, for all $\lambda \in \mathbf{R}$. Since the left side above is bounded by constant function τ which belongs to $L^2(\mu_t)$, then by (3.8) we get:

$$\int_{-\infty}^{\infty} \Big(\sum_{k=1}^{n} \frac{\tau}{n} \exp\Big(ik\frac{\tau}{n}\lambda\Big)\Big)\ Z_t(d\lambda) = \sum_{k=1}^{n} \frac{\tau}{n} X_{k\frac{\tau}{n}}(t) \rightarrow \int_{-\infty}^{\infty} \Big(\int_0^{\tau} e^{is\lambda}\ ds\Big)\ Z_t(d\lambda) \ \text{ in } \ L^2(P)$$

as $n \rightarrow \infty$, for all $t \in T$. Hence we get:

$$\frac{1}{\tau}\int_0^{\tau} X_s(t)\ ds = \int_{-\infty}^{\infty} f_\tau(\lambda)\ Z_t(d\lambda)$$

for all $t \in T$, where $f_\tau(\lambda) = (1/\tau)(e^{i\tau\lambda}-1)/(i\lambda)$ for $\lambda \neq 0$ and $f_\tau(0) = 1$. Thus we have:

$$\frac{1}{\tau}\int_0^{\tau} X_s(t)\ ds - Z_t(\{0\}) = \int_{-\infty}^{\infty} \Big(f_\tau(\lambda) - 1_{\{0\}}(\lambda)\Big)\ Z_t(d\lambda) = \int_{-\infty}^{\infty} g_\tau(\lambda)\ Z_t(d\lambda)$$

for all $t \in T$, where $g_\tau(\lambda) = f_\tau(\lambda)$ for $\lambda \neq 0$ and $g_\tau(0) = 0$.

Let us first reconsider Step 1. For this note that by (2.16) we get:

(3.23) $$\Big|\int_{-\pi}^{-\delta_\tau} g_\tau(\lambda)\ Z_t(d\lambda)\Big| \leq \sup_{-\infty<\lambda\leq-\delta_\tau} \big|g_\tau(\lambda)\big| \cdot \mathrm{V}\big(Z_t, <-\infty, -\delta_\tau]\big) \leq \frac{2}{\tau}\frac{1}{|\,i\delta_\tau\,|} \cdot \mathrm{V}\big(Z_t, \mathbf{R}\big)$$

for all $t \in T$. Thus putting $\delta_\tau = \tau^{-\alpha}$ for some $\alpha > 0$, we see that:

(3.24) $$\sup_{t\in T} \Big|\int_{-\infty}^{-\delta_\tau} g_\tau(\lambda)\ Z_t(d\lambda)\Big| \rightarrow 0 \ \text{ in } \ L^2(P)$$

as $\tau \rightarrow \infty$, as soon as we have $0 < \alpha < 1$. In exactly the same way as for (3.24) we find:

(3.25) $$\sup_{t\in T} \Big|\int_{\delta_\tau}^{\infty} g_\tau(\lambda)\ Z_t(d\lambda)\Big| \rightarrow 0 \ \text{ in } \ L^2(P)$$

as $\tau \rightarrow \infty$. Facts (3.24) and (3.25) complete Step 1.

Next reconsider Step 2. First, it is clear that:

(3.26) $$g_\tau(-\delta_\tau) \rightarrow 0 \quad \& \quad g_\tau(\delta_\tau) \rightarrow 0$$

as $\tau \rightarrow \infty$. Next, by the same arguments as in the proof of Theorem 2.3 we obtain:

(3.27) $$E\Big(\sup_{t\in T}\sup_{\lambda\in\mathbf{R}} |Z_t(\lambda)|^2\Big) < \infty\ .$$

Thus the only what remains is to estimate the total variation of g_τ on $<-\delta_\tau, \delta_\tau]$.

For this put $G_s(\lambda) = e^{is\lambda}$ for $0 \leq s \leq \tau$, and notice that we have:

(3.28) $$\mathrm{V}\big(g_\tau, <-\delta_\tau, \delta_\tau]\big) \leq 1 + \mathrm{V}\big(f_\tau, <-\delta_\tau, \delta_\tau]\big) \leq 1 + \int_{-\delta_\tau}^{\delta_\tau} |f_\tau'(\lambda)|\ d\lambda\ .$$

Next recall that $f_\tau(\lambda) = (1/\lambda)\int_0^\tau e^{is\lambda}\ ds$ for all $\lambda \in \mathbf{R}$. Again, by the Cauchy-Schwarz inequality (with Fubini's theorem) and orthogonality of G_s's on $<-\pi,\pi]$, we obtain from (3.28):

$$(3.29) \qquad \mathrm{V}\big(g_\tau, <-\delta_\tau, \delta_\tau]\big) \leq 1 + \sqrt{2\delta_\tau}\,\frac{1}{\tau}\left(\int_{-\pi}^{\pi}\Big|\int_0^\tau is\cdot G_s(\lambda)\ ds\Big|^2 d\lambda\right)^{1/2} =$$

$$= 1 + \sqrt{2\delta_\tau}\,\frac{1}{\tau}\Big(\int_0^\tau s^2\ ds\Big)^{1/2} = 1 + \sqrt{2\delta_\tau}\,\frac{1}{\tau}\,\frac{\tau^{3/2}}{\sqrt{3}} \leq C\tau^{(1-\alpha)/2}$$

with some constant $C > 0$. Finally, by using (3.26), (3.27) and (3.29) we can complete Step 2 as in the proof of Theorem 2.3.

To complete Step 3 as in the proof of Theorem 2.3, it is just enough to verify that:

$$(3.30) \qquad g_\tau(-\eta_\tau) \to 1 \ \ \&\ \ g_\tau(\eta_\tau) \to 1$$

$$(3.31) \qquad \mathrm{V}\big(g_\tau, <-\eta_\tau, 0>\ \big) \to 0 \ \ \&\ \ \mathrm{V}\big(g_\tau, <0, \eta_\tau]\big) \to 0$$

as $\tau \to \infty$, where $\eta_\tau = n^{-\beta}$ for $\beta > 1$.

First consider (3.30), and note that we have:

$$\big|g_\tau(\pm\eta_\tau) - 1\big| = \Big|\frac{1}{\tau}\int_0^\tau e^{\pm is\eta_\tau}\ ds - 1\Big| \leq \sup_{0\leq s\leq\tau}\big|e^{\pm is\eta_\tau} - 1\big| = \big|e^{\pm i\tau\eta_\tau} - 1\big| = \big|e^{\pm i(1/\tau^{\beta-1})} - 1\big| \to 0$$

as $\tau \to \infty$. This proves (3.30).

Next consider (3.31), and note that we have:

$$\mathrm{V}\big(g_\tau, <0,\eta_\tau]\big) \leq \frac{1}{\tau}\int_0^\tau \mathrm{V}\big(G_s, <0,\eta_\tau]\big)\ ds \leq \sup_{0\leq s\leq\tau}\mathrm{V}\big(G_s, <0,\eta_\tau]\big) = \mathrm{V}\big(G_\tau, <0,\eta_\tau]\big)$$

$$\leq \big|1 - \cos\eta_\tau\big| + \big|\sin\eta_\tau\big| \to 0$$

as $\tau \to \infty$. This proves the second part of (3.31). The first part follows by the same argument.

The rest of the proof can be carried out as it was done in the proof of Theorem 2.3. □

We conclude by pointing out that Remarks 2.4 carry over in exactly the same form to cover the present case. The same remark might be directed to Example 2.5 and Example 2.6.

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