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★**Optimal stopping and free-boundary problems.**

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Until the appearance of the present book, two monographs had been published on the theory of optimal stopping and its applications [A. N. Shiryaev, *Statistical sequential analysis. Optimal stopping rules* (Russian), Izdat. "Nauka", Moscow, 1969; MR0293789 (45 #2865); Second edition, revised, Izdat. "Nauka", Moscow, 1976; MR0445744 (56 #4078); English translation, Springer, New York, 1978; MR0468067 (57 #7906); Y. S. Chow, H. Robbins and D. Siegmund, *Great expectations: the theory of optimal stopping*, Houghton Mifflin, Boston, Mass., 1971; MR0331675 (48 #10007)]. While the first monograph was mainly devoted to the Markov approach (i.e., the underlying processes are Markov with discrete or continuous time parameter), the second one focused on the so-called martingale approach (for general gain processes with discrete time parameter) based on the fundamental papers of J. L. Snell [Trans. Amer. Math. Soc. **73** (1952), 293–312; MR0050209 (14,295a)] and G. W. Haggstrom [Ann. Math. Statist. **37** (1966), 7–29; MR0195221 (33 #3424)].

After more than three decades, this third monograph on optimal stopping presented by the authors is a remarkable book in many aspects, covering much of the research progress of that period. While the general theory was mainly developed in the 1970s with the general continuous time martingale approach, later research focused on more special models such as optimal stopping of diffusions or jump processes and, above all, on the solution of challenging concrete problems. A very strong tool for solving concrete problems with continuous time is the free-boundary problem (Stefan problem), which is reflected in the second half of the title of this book. This method, already discovered in the late 1950s, was developed and refined in the last decades in numerous papers mainly by the authors of this book and their co-authors. Especially in this last period, interest in optimal stopping was renewed and is still enforced by the challenges of financial mathematics where optimal stopping naturally appears for the pricing of American options.

The present monograph reflects these developments and is mainly based on the results obtained by the authors and their co-authors published in earlier papers. Historical comments are given at the end of each chapter or section. The book contains an almost complete bibliography of 224 items—very helpful for a reader who would like to deepen the material. It presents the theory of optimal stopping in discrete and continuous time parameters and for both the martingale and Markov approaches, including proofs of the results. The core of the book, however, is the treatment and complete solution of a series of concrete problems of general interest from probability, mathematical statistics, and mathematical finance which can be recognized as optimal stopping problems and which are completely solved by reducing them to free-boundary problems (Stefan problems).

In Chapter I, the general theory of optimal stopping is treated. Here, a standard assumption is that the gain process is bounded by an integrable random variable, which simplifies the presentation considerably. Chapter II is a brief review (without proofs) of stochastic processes: martingales, Markov processes with discrete and continuous time parameter and, in particular, diffusions and Lévy processes, and basic transformations such as change of time and space, change of measure, and killing. Chapters III and IV introduce the basic principles of optimal stopping problems (for finite and infinite horizon) in a Markov setting and their solution by the associated boundary value problems with the special feature that the boundary is unknown and part of the solution (free-boundary problems). The gain functional is given in the so-called MLS-formulation where M stands for Mayer, L for Lagrange and S for supremum, i.e., the gain consists of a terminal part, an integral part and a supremum part. As key principles for the solution of the free-boundary problems, smooth fit for diffusions and continuous fit for processes with jumps are derived. Furthermore, the methods of time change, space change and measure change for solving optimal stopping problems are discussed. Special emphasis is put on the optimal stopping of the supremum process for certain one-dimensional diffusions. Nonlinear integral equations and the first passage problem and their role in solving optimal stopping problems are dealt with. Chapter V gives applications of concrete optimal stopping problems for deriving sharp inequalities in stochastic analysis: Wald inequalities, Bessel inequalities, Doob inequalities, Hardy–Littlewood inequalities, and Burkholder–Davis–Gundy inequalities. Chapter VI is devoted to applications of optimal stopping in mathematical statistics: sequential testing and quickest detection of a Wiener process (infinite and finite horizons) and of a Poisson process (infinite horizon). Complete solutions with full proofs are given. In Chapter VII, the theory of optimal stopping is applied to solve concrete problems in mathematical finance. For the Black–Scholes model, the authors derive the explicit form of the arbitrage-free price of the American put option (infinite horizon). In the case of a finite horizon the stopping boundary for the optimal exercise time is characterized as the unique solution of a nonlinear Volterra integral equation of the second kind. The same problems are solved for Russian options and, in the case of a finite horizon, also for Asian options. Finally, the main goal of Chapter VIII is to find a stopping time τ^* of a Brownian motion B over a finite time interval (with drift, in general) such that B_{τ^*} is as close as possible (in mean-square distance) to the maximum of B. This problem could be reduced to an optimal stopping problem, and its complete solution is given. Problems of this type are (continuous) counterparts of the (discrete) problem of the best choice (secretary problem); they have great practical and theoretical interest and could be applied, e.g., in financial engineering.

To conclude, the present book is a splendid monograph unusually rich in hard concrete problems and their masterly and rigorously derived solutions. The book presents the state of the art of today's optimal stopping theory and can be recommended to everyone who is working or only interested in this challenging subject.

Reviewed by *H.-J. Engelbert*