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THE NECESSARY AND SUFFICIENT CONDITION FOR THE EXTENSION OF A FINITE MEASURE IN THE CASE OF A TWO-ELEMENT DISJOINT PERTURBATION

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Abstract. A necessary and sufficient condition for the existence of a countably additive extension of a finite measure in the case of a two-element disjoint perturbation of its σ-algebra is given. A technique of proving which is based on a direct construction and which uses the outer and inner traces of a finite measure is developed. As a consequence, several characterizations involving some typical extensions in this case are obtained.

Introduction. Let $(X, \mathcal{A})$ be a measurable space, and let $\mathcal{C}$ be an arbitrary family of subsets of X. Then we say that the operation

$$\mathcal{A} \xrightarrow{c} \sigma(\mathcal{A} \cup \mathcal{C})$$

is *a perturbation of the σ-algebra $\mathcal{A}$ by the family $\mathcal{C}$*.

If $\mathcal{C} \subset \mathcal{A}$, then we say that c is *a trivial perturbation*. If c is not a trivial perturbation, consider $\Theta = \{\operatorname{card} \mathcal{D} \mid \mathcal{D} = (D_1, D_2, \dots) \subset \mathcal{C}$, $D_1 \notin \mathcal{A}$, $D_2 \notin \sigma(\mathcal{A} \cup \{D_1\}) \dots$ and $\sigma(\mathcal{A} \cup \mathcal{D}) = \sigma(\mathcal{A} \cup \mathcal{C})\}$. If $\Theta \cap \mathrm{N} \neq \emptyset$, put $n = \min \Theta$, and in this case we say that c is *a n-element perturbation*. If $\Theta \cap \mathrm{N} = \emptyset$, but $\aleph_0 \in \Theta$, then we say that c is *a countable perturbation*. If $\Theta \cap (\mathrm{N} \cup \{\aleph_0\}) = \emptyset$, then we say that c is a *uncountable perturbation*.

Let us note if $\mathcal{C}$ is a finite family, then c is a trivial or a n-element perturbation for some $n \geqslant 1$. In this case, without loss of generality in our considerations, we can assume that card $\mathcal{C} = n$.

We say that c is *a disjoint perturbation*, if the elements of the family $\mathcal{C}$ are disjoint.

It is easy to see that each n-element perturbation is a m-element disjoint perturbation for some $m \geqslant n \geqslant 1$.

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If μ is a finite measure on $(X, \mathcal{A})$, it is natural to ask when does an extension of the measure μ in the case of a perturbation of the σ-algebra $\mathcal{A}$ by a family $\mathcal{C}$ exist, i.e. when does a measure ν on $\sigma(\mathcal{A} \cup \mathcal{C})$ exist, such that $\nu\mid_{\mathcal{A}} = \mu$?

Our investigation is devoted to this problem. In the case of an one-element perturbation it is well-known the necessary and sufficient condition for the existence of the desired extension. We will deduce this result in terms of the outer and inner traces of the given measure which we find useful in the considerations involving multi-element perturbations.

Directly by induction this result implies that in the case of a finite perturbation, an extension always exists, and we will turn out one of its inductive formulas.

In this paper we find a necessary and sufficient condition for the existence of a countably additive extension of a finite measure in the case of a two-element disjoint perturbation of its σ-algebra, and give some of consequences. The technique of proving is based on a direct construction which uses the outer and inner traces of a finite measure and the given result on the one-element perturbation of a σ-algebra.

We will also see that a quite simple example shows that in all these cases the extension, even though exists, is not unique in general.

In [5] we can also find a necessary and sufficient condition for the extension of a finite measure in the case of a general finite perturbation of its σ-algebra. However, let us mention that this result relies upon a quite different approach which essentially uses the Hahn–Banach theorem.

1. Extension of a finite measure in the case of an one-element perturbation

Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and let C be an arbitrary subset of X. *The trace of the σ-algebra $\mathcal{A}$ on the set C* is a σ-algebra defined by

$$\operatorname{tr}(\mathcal{A}, C) = \{A \cap C \mid A \in \mathcal{A}\}.$$

A set $C^* \in \mathcal{A}$ such that $\mu(C^*) = \mu^*(C)$ is called *the μ-hull* of C, while a set $C_* \in \mathcal{A}$ such that $\mu(C_*) = \mu_*(C)$ is called *the μ-kernel of* C.

It is easy to see that each subset C of X has the μ-hull and the μ-kernel, uniquely determined up to a μ-nullset, i.e. if C_1^* and C_2^* are the μ-hulls of C, then $\mu(C_1^* \Delta C_2^*) = 0$, and similarly if C_*^1 and C_*^2 are the μ-kernels of C, then $\mu(C_*^1 \Delta C_*^2) = 0$.

If $C \in \mathcal{A}$, then *the trace of the measure* μ *on the set* C is a measure $\operatorname{tr}(\mu, C) : \mathcal{A} \to [0, \infty\rangle$ defined $\forall A \in \mathcal{A}$ by

$$\operatorname{tr}(\mu, C)(A) = \mu(A \cap C).$$

If the set C is not necessarily from the σ-algebra $\mathcal{A}$, then we distinguish the following two possibilities:

(*i*) *The outer trace of the measure* μ *on the set* C is a measure

$$\operatorname{tr}^*(\mu, C) : \sigma(\mathcal{A} \cup \{C\}) \to [0, \infty\rangle$$

defined by

$$\operatorname{tr}^*(\mu, C)([A, B]) = \mu(A \cap C^*),$$

where $[A, B] = (A \cap C) \cup (B \cap C^c)$, with $A, B \in \mathcal{A}$, is the general representation of an element from the σ-algebra $\sigma(\mathcal{A} \cup \{C\})$.

(*ii*) *The inner trace of the measure* μ *on the set* C is a measure

$$\operatorname{tr}_*(\mu, C) : \sigma(\mathcal{A} \cup \{C\}) \to [0, \infty\rangle$$

defined for any $[A, B] \in \sigma(\mathcal{A} \cup \{C\})$ by

$$\operatorname{tr}_*(\mu, C)([A, B]) = \mu(A \cap C_*).$$

It is easy to see that the outer and inner traces are well-defined, i.e. that do not depend of a particular choice of the μ-hull and the μ-kernel of the set C, as well as of the representation of the elements from $\sigma(\mathcal{A} \cup \{C\})$.

The next proposition gives us elementary properties of the traces.

PROPOSITION 1.1. *Let* $(X, \mathcal{A}, \mu)$ *be a finite measure space, and let* C *be an arbitrary subset of* X. *Then the following statements are satisfied:*

(*i*) $\operatorname{tr}^*(\mu, C)([A, B]) = \mu^*(A \cap C)$, $\forall [A, B] \in \sigma(\mathcal{A} \cup \{C\})$.
In particular, we have $\operatorname{tr}^*(\mu, C)(C) = \mu^*(C)$.

(*ii*) $\operatorname{tr}_*(\mu, C)([A, B]) = \mu_*(A \cap C)$, $\forall [A, B] \in \sigma(\mathcal{A} \cup \{C\})$.
In particular, we have $\operatorname{tr}_*(\mu, C)(C) = \mu_*(C)$.

(*iii*) $\operatorname{tr}^*(\mu, C)(C^c) = \operatorname{tr}_*(\mu, C)(C^c) = 0$.

(*iv*) *If* $C \in \mathcal{A}$, *then* $\operatorname{tr}^*(\mu, C) = \operatorname{tr}_*(\mu, C) = \operatorname{tr}(\mu, C)$.

Proof. A straightforward verification.

PROPOSITION 1.2. *Let* $(X, \mathcal{A}, \mu)$ *be a finite measure space, and let* C *be an arbitrary subset of* X. *Then by*

$$\mu_0 = \operatorname{tr}_*(\mu, C) + \operatorname{tr}^*(\mu, C^c) \ \ \textit{and}\ \mu_1 = \operatorname{tr}^*(\mu, C) + \operatorname{tr}_*(\mu, C^c)$$

finite measures on the σ*-algebra* $\sigma(\mathcal{A} \cup \{C\})$ *are defined, such that:*

(*i*) $\mu_0, \mu_1 \,|_{\mathcal{A}} = \mu$,

(*ii*) $\mu_0(C) = \mu_*(C)$ *and* $\mu_1(C) = \mu^*(C)$.

Proof. Straightforward by Proposition 1.1.

The next theorem gives us a necessary and sufficient condition for the existence of a countably additive extension of a finite measure in the case of an one-element perturbation. Let us note that an extension always exists.

THEOREM 1.3. *Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and let C be an arbitrary subset of X. Then for each $\alpha \in [\mu_*(C),\ \mu^*(C)]$ there exists an extension ν_α of μ to the σ-algebra $\sigma(\mathcal{A} \cup \{C\})$ satisfying*

$$\nu_\alpha(C) = \alpha. \tag{1}$$

Moreover, an extension ν_α may be defined as follows

$$\nu_\alpha = \begin{cases} \mathrm{tr}_*(\mu, C) + \mathrm{tr}^*(\mu, C^c), & \text{if } \alpha = \mu_*(C) \\ \mathrm{tr}^*(\mu, C) + \mathrm{tr}_*(\mu, C^c), & \text{if } \alpha = \mu^*(C) \\ \dfrac{\mu^*(C) - \alpha}{\mu^*(C) - \mu_*(C)}\{\mathrm{tr}_*(\mu, C) + \mathrm{tr}^*(\mu, C^c)\} + \\ \qquad + \dfrac{\alpha - \mu_*(C)}{\mu^*(C) - \mu_*(C)}\{\mathrm{tr}^*(\mu, C) + \mathrm{tr}_*(\mu, C^c)\} \\ \qquad\qquad \text{if } \alpha \in \langle \mu_*(C), \mu^*(C) \rangle \end{cases}$$

Finally, any countably additive extension ν_α of the measure μ to the σ-algebra $\sigma(\mathcal{A} \cup \{C\})$ satisfies (1), *for some $\alpha \in [\mu_*(C),\ \mu^*(C)]$.*

Proof. With the notation from Proposition 1.2, for a given $\alpha \in [\mu_*(C),\ \mu^*(C)]$, put $\nu_\alpha = t_\alpha \mu_0 + (1 - t_\alpha)\mu_1$ and $\nu_\alpha(C) = \alpha$, where $t_\alpha \in [0, 1]$. Then we easily find that $t_\alpha = 1$ if $\alpha = \mu_*(C)$, $t_\alpha = 0$ if $\alpha = \mu^*(C)$, and $t_\alpha = \dfrac{\mu^*(C) - \alpha}{\mu^*(C) - \mu_*(C)}$ if $\alpha \in \langle \mu_*(C),\ \mu^*(C) \rangle$. Therefore the first two statements follow easily by Proposition 1.2.

The last statement follows straightforward by the definition of the μ-hull and μ-kernel of the set C.

Example 1.4. Already in the case of an one-element perturbation the extension is not unique in general.

Indeed, if $(X, \mathcal{A}, \mu) = ([0, 1], \sigma(\{[0, \frac{1}{2}\rangle, [\frac{1}{2}, 1]\}, \lambda)$ and $C = [\frac{1}{4}, \frac{3}{4}]$, then $\mu_*(C) = 0$ and $\mu^*(C) = 1$. For a given $\alpha \in [0, 1]$ and for each choice of α_1, α_2, β and γ such that $\alpha_1 + \alpha_2 = \alpha$, $\beta + \alpha_1 = \frac{1}{2}$ and $\alpha_2 + \gamma = \frac{1}{2}$, we may define $\nu([0, \frac{1}{4}\rangle) = \beta$, $\nu([\frac{1}{4}, \frac{1}{2}\rangle) = \alpha_1$, $\nu([\frac{1}{2}, \frac{3}{4}]) = \alpha_2$, $\nu(\langle \frac{3}{4}, 1]) = \gamma$, and we see that ν is an extension of the measure μ to the σ-algebra $\sigma(\mathcal{A} \cup \{C\})$.

In this way the problem of extension of a finite measure in the case of an one-element perturbation is completely solved.

2. Extension of a finite measure in the case of a two-element disjoint perturbation

Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and let C and D be arbitrary subsets of X. Suppose that there exists an extension $\nu_{\alpha,\beta}$ of the measure μ to the σ-algebra $\sigma(\mathcal{A} \cup \{C, D\})$ such that $\nu_{\alpha,\beta}(C) = \alpha$ and $\nu_{\alpha,\beta}(D) = \beta$. Let us see what kind of information about the extension $\nu_{\alpha,\beta}$ may be provided by using the upper and lower μ-integrals.

For $a \geqslant b > 0$ we have

$$\begin{aligned}
\int^{*} (a1_C + b1_D)d\mu &= \int^{*} (a1_{C\cup D} + a1_{C\cap D} + (b-a)1_D)d\mu \\
&\leqslant \int^{*} a1_{C\cup D}d\mu + \int^{*} a1_{C\cap D}d\mu + \int^{*} (b-a)1_D d\mu \\
&= a\int^{*} 1_{C\cup D}d\mu + a\int^{*} 1_{C\cap D}d\mu + (b-a)\int_{*} 1_D d\mu \\
&= a\mu^{*}(C\cup D) + a\mu^{*}(C\cap D) + (b-a)\mu_{*}(D)
\end{aligned}$$

while

$$\int^{*} (a1_C + b1_D)d\mu \geqslant \int (a1_C + b1_D)d\nu_{\alpha,\beta} = a\nu_{\alpha,\beta}(C) + b\nu_{\alpha,\beta}(D),$$

which together imply

$$a\{\mu^{*}(C\cup D) + \mu^{*}(C\cap D) - \mu_{*}(D)\} + b\mu_{*}(D) \geqslant a\alpha + b\beta.$$

Letting $b \to 0$, we get

$$\alpha \leqslant \mu^{*}(C\cup D) + \mu^{*}(C\cap D) - \mu_{*}(D).$$

By symmetry we have

$$\beta \leqslant \mu^{*}(C\cup D) + \mu^{*}(C\cap D) - \mu_{*}(C).$$

Similarly, computing $\int_{*}(a1_C + b1_D)d\mu = \ldots$, we may easily conclude that the following conditions are satisfied:

$$(A)\quad \begin{cases}
\mu_{*}(C) \vee \{\mu_{*}(C\cup D) + \mu_{*}(C\cap D) - \mu^{*}(D)\} \\
\qquad\qquad \leqslant \alpha \leqslant \mu^{*}(C) \wedge \{\mu^{*}(C\cup D) + \mu^{*}(C\cup D) - \mu_{*}(D))\} \\
\mu_{*}(D) \vee \{\mu_{*}(C\cup D) + \mu_{*}(C\cap D) - \mu^{*}(C)\} \\
\qquad\qquad \leqslant \beta \leqslant \mu^{*}(C) \wedge \{\mu^{*}(C\cup D) + \mu^{*}(C\cap D) - \mu_{*}(C)\}
\end{cases}$$

It is clear that conditions (A) are necessary for the existence of an extension $\nu_{\alpha,\beta}$. But the following simple example shows that they are not optimal, i.e. these conditions are not sufficient for the existence of a desired extension $\nu_{\alpha,\beta}$, already in the case where C and D are disjoint.

Example 2.1. Let $(X, \mathcal{A}, \mu) = ([0,1], \sigma(\{[0,\frac{1}{3}], \langle\frac{1}{3},\frac{2}{3}\rangle, [\frac{2}{3},1]\}), \lambda)$, $C = [0,\frac{1}{2}]$ and $D = \langle\frac{1}{2},1]$. Then $\mu_*(C) = \mu_*(D) = \frac{1}{3}$, $\mu^*(C) = \mu^*(D) = \frac{2}{3}$, $\mu_*(C \cap D) = \mu^*(C \cap D) = 0$ and $\mu_*(C \cup D) = \mu^*(C \cup D) = 1$. So we find

$$(A) \qquad \frac{1}{3} \leqslant \alpha \leqslant \frac{2}{3}, \qquad \frac{1}{3} \leqslant \beta \leqslant \frac{2}{3},$$

but the extensions $\nu_{1/3,1/3}$ and $\nu_{2/3,2/3}$ evidently do not exist.

So it is natural to ask what are the necessary and sufficient conditions for the existence of a countably additive extension $\nu_{\alpha,\beta}$ of the measure μ to the σ-algebra $\sigma(\mathcal{A} \cup \{C, D\})$ satisfying:

$$\nu_{\alpha,\beta}(C) = \alpha \quad \text{and} \quad \nu_{\alpha,\beta}(D) = \beta.$$

Since each finite perturbation is, without loss of generality, a disjoint finite perturbation, we will assume that C and D are disjoint. Otherwise our problem eventually reduces to the problem of a three-element disjoint perturbation. Moreover our main aim is devoted to the development of a method which may be useful in the investigations involving the extension of a finite measure in the case of a multi-element perturbation. Let us begin by introducing some definitions.

Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and let C and D be arbitrary disjoint subsets of X.

The outer μ-identity of C and D is defined by

$$\mu^*(C, D) = \mu(C^* \cap D^*),$$

where C^* and D^* are the μ-hulls of the sets C and D.

The inner μ-identity of C and D is defined by

$$\mu_*(C, D) = \mu_*\{(C \setminus C_*) \cup (D \setminus D_*)\},$$

where C_* and D_* are the μ-kernels of the sets C and D.

Since the μ-hulls and μ-kernels are uniquely determined up to a μ-nullset, we can easily verify that the outer and inner μ-identity is well-defined.

Moreover, with the following notation

$$\partial^*(C, D) = C^* \cap D^* \text{ and}$$
$$\partial_*(C, D) = \{(C \setminus C_*) \cup (D \setminus D_*)\}_*,$$

where $\{(C \setminus C_*) \cup (D \setminus D_*)\}_*$ is the μ-kernel of $(C \setminus C_*) \cup (D \setminus D_*)$, without loss of generality we may assume:

$$C \subset C^* \ \& \ D \subset D^*, \tag{1}$$

$$C_* \subset C,\ D_* \subset D \ \&\ \partial^*(C, D) \text{ are disjoint, and} \tag{2}$$

$$\partial_*(C, D) \subset \partial^*(C, D) \cap (C \cup D) \tag{3}$$

We say that the disjoint sets C and D are *μ-separated*, if there exist sets C_1 and $D_1 \in \mathcal{A}$, such that $C \subset C_1$, $D \subset D_1$ and $\mu(C_1 \cap D_1) = 0$.

The following proposition gives us basic properties and relationships of the introduced objects.

PROPOSITION 2.2. *Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and let C and D be arbitrary disjoint subsets of X. Then the following statements are satisfied:*

$$(i) \qquad 0 \leqslant \mu_*(C, D) \leqslant \mu^*(C, D)$$
$$\leqslant (\mu^*(C) - \mu_*(C)) \wedge (\mu^*(D) - \mu_*(D)),$$

$$(ii) \qquad \mu_*(C \cup D) = \mu_*(C) + \mu_*(D) + \mu_*(C, D),$$
$$\mu^*(C \cup D) = \mu^*(C) + \mu^*(D) - \mu^*(C, D),$$

$$(iii) \qquad \mu^*(C, D) = 0 \textit{ if and only if } C \textit{ and } D \textit{ are } \mu\textit{-separated.}$$

Proof. The first two inequalities in (i) and statements (ii) and (iii) follow by definitions and relations (1), (2) and (3).

Let us prove the third inequality in (i). According to (1), (2) and (3) we have $\mu_*(C) + \mu^*(C, D) = \mu(C_* \cup \partial^*(C, D)) \leqslant \mu(C^*) = \mu^*(C)$, and similarly $\mu_*(D) + \mu^*(C, D) \leqslant \mu^*(D)$. Hence the statement follows straightforward.

The next theorem contains the answer to our problem, i.e. the necessary and sufficient condition for the existence of a countably additive extension of a finite measure in the case of a two-element disjoint perturbation of its σ-algebra.

Before the proof we turn to some of its useful applications in the next corollary.

THEOREM 2.3. *Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and let C and D be arbitrary disjoint subsets of X. Then there exists a countably additive extension $\nu_{\alpha,\beta}$ of μ to the σ-algebra $\sigma(\mathcal{A} \cup \{C, D\})$, such that*

$$\nu_{\alpha,\beta}(C) = \alpha \quad \textit{and} \quad \nu_{\alpha,\beta}(D) = \beta$$

if and only if

$$(B) \qquad \begin{cases} \mu_*(C \cup D) \leqslant \alpha + \beta \leqslant \mu^*(C \cup D), \\ \mu_*(C) \quad \leqslant \quad \alpha \quad \leqslant \mu^*(C), \\ \mu_*(D) \quad \leqslant \quad \beta \quad \leqslant \mu^*(D). \end{cases}$$

COROLLARY 2.4. *Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and let C and D be arbitrary disjoint subsets of X. Then we have:*

(i) There exists a countably additive extension ν^+ of μ to the σ-algebra $\sigma(\mathcal{A} \cup \{C, D\})$ such that

$$\nu^+(C) = \mu^*(C) \text{ and } \nu^+(D) = \mu^*(D)$$

if and only C and D are μ-separated, i.e. $\mu^*(C, D) = 0$.

(ii) There exists a countably additive extension ν_+ of μ to the σ-algebra $\sigma(\mathcal{A} \cup \{C, D\})$ such that

$$\nu_+(C) = \mu_*(C) \text{ and } \nu_+(D) = \mu_*(D)$$

if and only $\mu_*(C, D) = 0$.

(iii) For each $\alpha \in [\mu_*(C),\ \mu^*(C)]$ and for each $\beta \in [\mu_*(D),\ \mu^*(D)]$ there exists a countably additive extension $\nu_{\alpha,\beta}$ of μ to the σ-algebra $\sigma(\mathcal{A} \cup \{C, D\})$ such that

$$\nu_{\alpha,\beta}(C) = \alpha \text{ and } \nu_{\alpha,\beta}(D) = \beta$$

if and only if C and D are μ-separated.

(iv) There always exist countably additive extensions ν_0 and ν_1 of μ to the σ-algebra $\sigma(\mathcal{A} \cup \{C, D\})$ such that

$$\nu_0(C) = \mu_*(C) \text{ and } \nu_0(D) = \mu^*(D),$$
$$\nu_1(C) = \mu^*(C) \text{ and } \nu_1(D) = \mu_*(D).$$

Let us first note that the statements of Corollary 2.4 follow directly by Theorem 2.3 and relations (i), (ii) and (iii) from Proposition 2.2.

Proof. Let us first show that conditions (A) are sufficient for the existence of a desired extension $\nu_{\alpha,\beta}$ of μ to the σ-algebra $\sigma(\mathcal{A} \cup \{C, D\})$. The key point in our construction of $\nu_{\alpha,\beta}$ relies upon the following two lemmas:

LEMMA 1. *For a given $\beta^- \in [\mu_*(D),\ \mu^*(D) - \mu^*(C, D)]$ there exists a countably additive extension ν_{α^*,β^-} of μ to the σ-algebra $\sigma(\mathcal{A} \cup \{C, D\})$ such that*

$$\nu_{\alpha^*,\beta^-}(C) = \mu^*(C) \text{ and } \nu_{\alpha^*,\beta^-}(D) = \beta^-.$$

LEMMA 2. *For a given* $\beta^- \in [\mu_*(D),\ \mu^*(D) - \mu^*(C,D)]$ *there exists a countably additive extension* ν_{α^+,β^-} *of* μ *to the* σ*-algebra* $\sigma(\mathcal{A} \cup \{C, D\})$ *such that*

$$\nu_{\alpha^+,\beta^-}(C) = \mu_*(C) + \mu_*(C,D) \text{ and } \nu_{\alpha^+,\beta^-}(D) = \beta^-.$$

Proof of lemma 1. Let C^* and D^* be the μ-hulls of C and D. Assign to μ its traces $\mathrm{tr}(\mu, C^*)$ and $\mathrm{tr}(\mu, D^* \setminus \partial^*(C,D))$. According to Theorem 1.3, for given $\alpha \in [\mathrm{tr}(\mu, C^*)_*(C),\ \mathrm{tr}(\mu, C^*)^*(C)]$ and $\beta^- \in [\mathrm{tr}(\mu, D^* \setminus \partial^*(C,D))_*(D), \mathrm{tr}(\mu, D^* \setminus \partial^*(C,D))^*(D)]$ there exist the extensions $\nu_{C,\alpha}$ and ν_{D,β^-} of $\mathrm{tr}(\mu, C^*)$ and $\mathrm{tr}(\mu, D^* \setminus \partial^*(C,D))$ to $\sigma(\mathcal{A} \cup \{C\})$ and $\sigma(\mathcal{A} \cup \{D\})$ such that

(1) $\quad \nu_{C,\alpha} |_{\mathcal{A}} = \mathrm{tr}(\mu, C^*)$ and $\nu_{D,\beta^-} |_{\mathcal{A}} = \mathrm{tr}(\mu, D^* \setminus \partial^*(C,D))$,

(2) $\quad \nu_{C,\alpha}(C) = \alpha$ and $\nu_{D,\beta^-}(D) = \beta^-$.

Evidently $\mathrm{tr}(\mu, D^*)_*(C) = \mu_*(C)$ and $\mathrm{tr}(\mu, C^*)^*(C) = \mu^*(C)$. Let us prove that $\mathrm{tr}(\mu, D^* \setminus \partial^*(C,D))_*(D) = \mu_*(D)$, and $\mathrm{tr}(\mu, D^* \setminus \partial^*(C,D))^*(D) = \mu^*(D) - \mu^*(C,D)$. Indeed, if $A \in \mathcal{A}$ and $A \subset D$, then $\mathrm{tr}(\mu, D^* \setminus \partial^*(C,D))(A) = \mu(A \cap (D^* \setminus \partial^*(C,D))) = \mu(A \cap D^*) - \mu(A \cap \partial^*(C,D)) = \mu(A \cap D^*) = \mu(A)$, since $A \cap \partial^*(C,D) \in \mathcal{A}$, $A \cap \partial^*(C,D) \subset C^* \setminus C$ and $A \subset D^*$. So, if we take supremum it follows $\mathrm{tr}(\mu, D^* \setminus \partial^*(C,D))_*(D) = \sup\{\mathrm{tr}(\mu, D^* \setminus \partial^*(C,D))(A) \mid A \in \mathcal{A}, A \subset D\} = \sup\{\mu(A) \mid A \in \mathcal{A},\ A \subset D\} = \mu_*(D)$.

Further, if $A \in \mathcal{A}$ and $D \subset A$, then $\mathrm{tr}(\mu, D^* \setminus \partial^*(C,D))(A) = \mu(A \cap (D^* \setminus \partial^*(C,D))) = \mu(A \cap D^*) - \mu(A \cap \partial^*(C,D)) = \mu(A \cap D^*) - \mu(\partial^*(C,D))$, since $A^c \cap \partial^*(C,D) \in \mathcal{A}$ and $A^c \cap \partial^*(C,D) \subset D^* \setminus D$. So, if we take infimum it follows $\mathrm{tr}(\mu, D^* \setminus \partial^*(C,D))^*(D) = \inf\{\mathrm{tr}(\mu, D^* \setminus \partial^*(C,D))(A) \mid A \in \mathcal{A},\ D \subset A\} = \inf\{\mu(A \cap D^*) - \mu(\partial^*(C,D)) \mid A \in \mathcal{A},\ D \subset A\} = \inf\{\mu(A) \mid A \in \mathcal{A},\ D \subset A\} - \mu(\partial^*(C,D)) = \mu^*(D) - \partial^*(C,D)$. This proves relations (1) and (2).

Assign to μ its trace $\mathrm{tr}(\mu, X \setminus (C^* \cup D^*))$. Complete the σ-algebras $\sigma(\mathcal{A} \cup \{C\})$, $\sigma(\mathcal{A} \cup \{D\})$ and $\mathcal{A}$ with respect to the measures $\nu_{C,\alpha}$, ν_{D,β^-} and $\mathrm{tr}(\mu, X \setminus (C^* \cup D^*))$ to the σ-algebras $\sigma(\mathcal{A} \cup \{C\})[\nu_{C,\alpha}]$, $\sigma(\mathcal{A} \cup \{D\})[\nu_{D,\beta^-}]$ and $\mathcal{A}[\mathrm{tr}(\mu, X \setminus (C^* \cup D^*))]$, and the corresponding extensions denote by $\overline{\nu}_{C,\alpha}$ $\overline{\nu}_{D,\beta^-}$ and $\overline{\mathrm{tr}}(\mu, X \setminus (C^* \cup D^*))$.

If we put $\alpha = \alpha^* = \mu^*(C)$, then we get $D \in \sigma(\mathcal{A} \cup \{C\})[\nu_{C,\alpha^*}]$ and $\overline{\nu}_{C,\alpha}(D) = 0$. Indeed, then $\nu_{C,\alpha^*}(C^* \setminus C) = \nu_{C,\alpha^*}(C^*) - \nu_{C,\alpha}(C) = \mu(C^*) - \alpha^* = 0$, so $D \cap C^* = D \cap (C^* \setminus C) \in \sigma(\mathcal{A} \cup \{C\})[\nu_{C,\alpha^*}]$, while $D \setminus C^* \in \sigma(\mathcal{A} \cup \{C\})[\nu_{C,\alpha^*}]$, since $D \setminus C^* \subset X \setminus C^*$ and $\nu_{C,\alpha^*}(X \setminus C^*) = \mathrm{tr}(\mu, C^*)(X \setminus C^*) = 0$. Consequently, we may conclude $D \in \sigma(\mathcal{A} \cup \{C\})[\nu_{C,\alpha^*}]$ and $\overline{\nu}_{C,\alpha^*}(D) = \overline{\nu}_{C,\alpha^*}(D \cap C^*) + \overline{\nu}_{C,\alpha^*}(D \setminus C^*) = 0$.

Further, since $C \subset X \setminus (D^* \setminus \partial^*(C,D))$, and $\nu_{D,\beta^-}(X \setminus (D^* \setminus \partial^*(C,D))) = \mathrm{tr}(\mu, D^* \setminus \partial^*(C,D))(X \setminus (D^* \setminus \partial^*(C,D))) = 0$, we see that $C \in \sigma(\mathcal{A} \cup \{D\})[\nu_{D,\beta^-}]$ and $\overline{\nu}_{D,\beta^-}(C) = 0$. Define a measure ν_{α^*,β^-} on the σ-algebra $\sigma(\mathcal{A} \cup \{C\})$ by

$$\nu_{\alpha^*,\beta^-} = \{\overline{\nu}_{C,\alpha^*} + \overline{\nu}_{D,\beta^-} + \overline{\mathrm{tr}}(\mu, X \setminus (C^* \cup D^*))\} \mid_{\sigma(\mathcal{A}\cup\{C,D\})} .$$

Note that the definition is correct, since each of the σ-algebras $\sigma(\mathcal{A} \cup \{C\})[\nu_{C,\alpha^*}]$, $\sigma(\mathcal{A} \cup \{D\})[\nu_{D,\beta^-}]$ and $\mathcal{A}[\mathrm{tr}(\mu, X \setminus (C^* \cup D^*))]$ contains the σ-algebra $\sigma(\mathcal{A} \cup \{C,D\})$. Further, by (1), (2) and the definition of the measure $\overline{\mathrm{tr}}(\mu, X \setminus (C^* \cup D^*))$, we easily find that the following relations are satisfied:

$$(3) \quad \nu_{\alpha^*,\beta^-}(A) = \mathrm{tr}(\mu, C^*)(A) + \mathrm{tr}(\mu, D^* \setminus \partial^*(C,D))(A) + \mathrm{tr}(\mu, X \setminus (C^* \cup D^*))(A) = \mu(A),$$

for each $A \in \mathcal{A}$,

$$(4) \quad \nu_{\alpha^*,\beta^-}(C) = \overline{\nu}_{C,\alpha^*}(C) + \overline{\nu}_{D,\beta^-}(C) + \overline{\mathrm{tr}}(\mu, X\setminus(C^*\cup D^*))(C) = \alpha^* = \mu^*(C), \text{ and}$$

$$(5) \quad \nu_{\alpha^*,\beta^-}(D) = \overline{\nu}_{C,\alpha^*}(D) + \overline{\nu}_{D,\beta^-}(D) + \overline{\mathrm{tr}}(\mu, X \setminus (C^* \cup D^*))(D) = \beta^-.$$

This completes the proof of Lemma 1.

Proof of lemma 2. Let C_* and D_* be the μ-kernels of C and D. Assign to μ its trace $\mathrm{tr}(\mu, C_* \cup \partial_*(C,D))$. By Theorem 1.3, for a given $\alpha \in [\mathrm{tr}(\mu, C_* \cup \partial_*(C,D))^*(C), \mathrm{tr}(\mu, C_* \cup \partial_*(C,D))^*(C)]$ there exists an extension $\nu_{C,\alpha}$ of $\mathrm{tr}(\mu, C_* \cup \partial_*(C,D))$ to the σ-algebra $\sigma(\mathcal{A} \cup \{C\})$ such that

$$(1) \qquad \nu_{C,\alpha} \mid_{\mathcal{A}} = \mathrm{tr}(\mu, C_* \cup \partial_*(C,D)) \text{ and}$$

$$(2) \qquad \nu_{C,\alpha}(C) = \alpha.$$

It is evident that $\mathrm{tr}(\mu, C_* \cup \partial_*(C,D))_*(C) = \mu_*(C)$, and let us prove that

$$\mathrm{tr}(\mu, C_* \cup \partial_*(C,D))^*(C) = \mu(C_* \cup \partial_*(C,D)) = \mu_*(C) + \mu_*(C,D).$$

Indeed, if $A \in \mathcal{A}$ and $C \subset A$, then $\mathrm{tr}(\mu, C_* \cup \partial_*(C,D))(A) = \mu(A \cap (C_* \cup \partial_*(C,D))) \geqslant \mu(A \cap C^* \cap (C_* \cup \partial_*(C,D))) = \mu(C_* \cup \partial_*(C,D))$, since $(X \setminus (A \cap C^*)) \cap \partial_*(C,D) \in \mathcal{A}$ and $(X \setminus (A \cap C^*)) \cap \partial_*(C,D) \subset C^* \setminus C$. Taking infimum we find that $\mathrm{tr}(\mu, C_* \cup \partial_*(C,D))^*(C) = \inf\{\mathrm{tr}(\mu, C_* \cup \partial_*(C,D))(A) \mid A \in \mathcal{A},\ C \subset A\} = \mu(C_* \cup \partial_*(C,D)) = \mu_*(C) + \mu_*(C,D)$.

Let $\sigma(\mathcal{A} \cup \{C\})[\nu_{C,\alpha}]$ be the completion of $\sigma(\mathcal{A} \cup \{C\})$ with respect to $\nu_{C,\alpha}$, and let $\overline{\nu}_{C,\alpha}$ be the corresponding extension of $\nu_{C,\alpha}$

to $\sigma(\mathcal{A}\cup\{C\})[\nu_{C,\alpha}]$. If we put $\alpha = \alpha^+ = \mu_*(C)+\mu_*(C,D)$, then we have $D \in \sigma(\mathcal{A}\cup\{C\})[\nu_{C,\alpha^+}]$ and $\overline{\nu}_{C,\alpha^+}(D) = 0$.

Indeed, then $\nu_{C,\alpha^+}(C^*\setminus C) = \nu_{C,\alpha^+}(C^*)-\nu_{C,\alpha^+}(C) = \mathrm{tr}(\mu, C_*\cup \partial_*(C,D))(C^*) - \alpha^+ = \mu(C_*\cup\partial_*(C,D)) - \alpha^+ = 0$. Since $D\cap C^* \subset C^*\setminus C$, it follows $D\cap C^* \in \sigma(\mathcal{A}\cup\{C\})[\nu_{C,\alpha^+}]$ and $\overline{\nu}_{C,\alpha^+}(D\cap C^*) = 0$. On the other hand $D\setminus C^* \subset X\setminus C^*$ and $\nu_{C,\alpha^+}(X\setminus C^*) = \mathrm{tr}(\mu, C_*\cup\partial_*(C,D))(X\setminus C^*) = 0$, so $D\setminus C^* \in \sigma(\mathcal{A}\cup\{C\})[\nu_{C,\alpha^+}]$ and $\overline{\nu}_{C,\alpha^+}(D\setminus C^*) = 0$, and hence $D \in \sigma(\mathcal{A}\cup\{C\})[\nu_{C,\alpha^+}]$ and $\overline{\nu}_{C,\alpha^+}(D) = \overline{\nu}_{C,\alpha^+}(D\cap C^*) + \overline{\nu}_{C,\alpha^+}(D\setminus C^*) = 0$.

Assign to μ its trace $\mathrm{tr}(\mu, X\setminus(C_*\cup\partial_*(C,D)\cup(D^*\setminus\partial^*(C,D))))$. Since for each $A\in\mathcal{A}$, $A\subset(C\cup D)\setminus(C_*\cup\partial_*(C,D)\cup(D^*\setminus\partial^*(C,D))))$ we have $\mu(A) = 0$, taking supremum it follows $\mathrm{tr}(\mu, X\setminus(C_*\cup\partial_*(C,D)\cup(D^*\setminus\partial_*(C,D))_*(C\cup D) = \sup\{\mu(A) \mid A\in\mathcal{A},\ A\subset (C\cup D)\setminus(C_*\cup\partial_*(C,D)\cup(D^*\setminus\partial^*(C,D)))\} = 0$. By Theorem 1.3 we find that there exists an extension λ of the measure $\mathrm{tr}(\mu, X\setminus(C_*\cup\partial_*(C,D)\cup(D^*\setminus\partial^*(C,D))))$ to the σ-algebra $\sigma(\mathcal{A}\cup\{C\cup D\})$ such that the following relations are satisfied:

(3) $\quad \lambda\,|_{\mathcal{A}} = \mathrm{tr}(\mu, X\setminus(C_*\cup\partial_*(C,D)\cup(D^*\setminus\partial^*(C,D))))$ and

(4) $\quad \lambda(C\cup D) = 0.$

Let $\sigma(\mathcal{A}\cup\{C\cup D\})[\lambda]$ be the completion of the σ-algebra $\sigma(\mathcal{A}\cup\{C\cup D\})$ with respect to λ, and let $\overline{\lambda}$ be the corresponding extension of λ to the σ-algebra $\sigma(\mathcal{A}\cup\{C\cup D\})[\lambda]$.

Further, in the proof of Lemma 1 a measure $\overline{\nu}_{D,\beta^-}$ on the σ-algebra $\sigma(\mathcal{A}\cup\{D\})[\nu_{D,\beta^-}]$ is constructed in such a way that

(5) $\quad \overline{\nu}_{D,\beta^-}\,|_{\mathcal{A}} = \mathrm{tr}(\mu, D^*\setminus\partial^*(C,D)),$

(6) $\quad \overline{\nu}_{D,\beta^-}(D) = \beta^-$ and

(7) $\quad C\in\sigma(\mathcal{A}\cup\{D\})[\nu_{D,\beta^-}]$ and $\nu_{D,\beta^-}(C) = 0.$

Define a measure ν_{a^+,β^-} on the σ-algebra $\sigma(\mathcal{A}\cup\{C,D\})$ by

$$\nu_{\alpha^+,\beta^-} = (\overline{\nu}_{C,\alpha} + \overline{\nu}_{D,\beta^-} + \overline{\lambda})\,|_{\sigma(\mathcal{A}\cup\{C,D\})}\,.$$

Let us note that the definition is correct, since each of σ-algebras $\sigma(\mathcal{A}\cup\{C\})[\nu_{C,\alpha^+}]$, $\sigma(\mathcal{A}\cup\{D\})[\nu_{D,\beta^-}]$ and $\sigma(\mathcal{A}\cup\{C\cup D\})[\lambda]$ contains the σ-algebra $\sigma(\mathcal{A}\cup\{C,D\})$. Further, according to relations (1)–(6) we find:

$$\begin{aligned}(8)\qquad \nu_{\alpha^+,\beta^-} &= \mathrm{tr}(\mu, C_*\cup\partial_*(C,D))(A) + \mathrm{tr}(\mu, D^*\setminus\partial^*(C,D))(A)\\ &\quad + \mathrm{tr}(\mu, X\setminus(C_*\cup\partial_*(C,D)\cup(D^*\setminus\partial^*(C,D))))(A)\\ &= \mu(A), \text{ for each } A\in\mathcal{A}.\end{aligned}$$

$$(9)\qquad \nu_{\alpha^+,\beta^-}(C) = \overline{\nu}_{C,\alpha^+}(C) + \overline{\nu}_{D,\beta^-}(C) + \overline{\lambda}(C)$$
$$= \mu(C_* \cup \partial_*(C,D)) = \mu_*(C) + \mu_*(C,D),$$
$$(10)\qquad \nu_{\alpha^+,\beta^-}(D) = \overline{\nu}_{C,\alpha^+}(D) + \overline{\nu}_{D,\beta^-}(D) + \overline{\lambda}(D) = \beta^-.$$

This proves Lemma 2.

We proceed with the main proof. For a given $\alpha \in [\mu_*(C) + \mu_*(C,D),\ \mu^*(C)]$ and $\beta^- \in [\mu_*(D),\ \mu^*(D) - \mu^*(C,D)]$ let us choose $t_\alpha \in [0,1]$ such that $\alpha = t_\alpha(\mu_*(C) + \mu_*(C,D)) + (1-\alpha)\mu^*(C)$. Note if $\mu^*(C) \neq \mu_*(C) - \mu_*(C,D)$, then $t_\alpha = \dfrac{\mu^*(C) - \alpha}{\mu^*(C) - \mu_*(C) - \mu_*(C,D)}$.

Let ν_{α^*,β^-} and ν_{α^+,β^-} be the measures from Lemma 1 and Lemma 2. Then by the following convex combination

$$\nu_{\alpha,\beta} = t_\alpha \nu_{\alpha^+,\beta^-} + (1 - t_\alpha)\nu_{\alpha^*,\beta^-}$$

a measure on the σ-algebra $\sigma(\mathcal{A} \cup \{C, D\})$ is defined, such that:

(1) $\nu_{\alpha,\beta}\,|_{\mathcal{A}} = \mu$,

(2) $\nu_{\alpha,b}(C) = t_\alpha(\mu_*(C) + \mu_*(C,D)) + (1 - t_\alpha)\mu^*(C) = \alpha$ and

(3) $\nu_{\alpha,\beta}(D) = t_\alpha\beta^- + (1 - t_\alpha)\beta^- = \beta^-$.

Since in all of the preceding consideratons the sets C and D have played a symmetric role, we find that for given $\alpha \in [\mu_*(C),\ \mu^*(C) - \mu^*(C,D)]$ and $\beta \in [\mu_*(D) + \mu_*(C,D),\ \mu^*(D)]$ there exists an extension $\nu_{\alpha,\beta}$ of μ to the σ-algebra $\sigma(\mathcal{A} \cup \{C,D\})$ such that

$$\nu_{\alpha,\beta}(C) = \alpha \text{ and } \nu_{\alpha,\beta}(D) = \beta.$$

However, this does not complete the proof. The basic problem relies upon the fact that in general we do not have $\mu^*(C) - \mu^*(C,D) \geqslant \mu_*(C) + \mu_*(C,D)$ and $\mu^*(D) - \mu^*(C,D) \geqslant \mu_*(D) + \mu_*(C,D)$, respectively. Thus we can not conclude that for all α and β which satisfy conditions (B), there exists a desired extension $\nu_{\alpha,\beta}$.

However, the given partial result can be "translated" to the whole intervals $[\mu_*(C),\ \mu^*(C)]$ and $[\mu_*(D),\ \mu^*(D)]$ by using $\partial_*(C,D)$ us a "balance" set. That can be done in the following way.

Let us apply the preceding result to the measure $\mu_0 = \mathrm{tr}(\mu, X \setminus \partial_*(C,D))$. Note that $(\mu_0)^*(C) = \mu^*(C) - \mu_*(C,D)$, $(\mu_0)_*(C) = \mu_*(C)$, $(\mu_0)^*(D) = \mu^*(D) - \mu_*(C,D)$, $(\mu_0)_*(D) = \mu_*(D)$, $(\mu_0)^*(C,D) = \mu^*(C,D) - \mu_*(C,D)$ and $(\mu_0)_*(C,D) = 0$, and hence $\forall \alpha \in [\mu_*(C),\ \mu^*(C) - \mu_*(C,D)]$ or $[\mu_*(C),\ \mu^*(C) - \mu^*(C,D)]$ and $\forall \beta \in [\mu_*(D),\ \mu^*(D) - \mu^*(C,D)]$ or $[\mu_*(D),\ \mu^*(D) - \mu_*(C,D)]$ respectively, there exists an extension $\nu^0_{\alpha,\beta}$ of μ_0 to the σ-algebra $\sigma(\mathcal{A} \cup \{C,D\})$ such that:

(4) $\nu^0_{\alpha,\beta}\mid_{\mathcal{A}} = \mu_0$,

(5) $\nu^0_{\alpha,\beta}(C) = \alpha$ and

(6) $\nu^0_{\alpha,\beta}(D) = \beta$.

Further, let us consider the measure $\mu_1 = \mathrm{tr}(\mu, \partial_*(C, D))$ and let us try to extend it to the σ-algebra $\sigma(\mathcal{A}\cup\{C\cap\partial_*(C, D)\}) = \sigma(\mathcal{A}\cup\{C\cap \partial_*(C, D),\ D\cap\partial_*(C, D)\})$. Since $\mathrm{tr}(\mu, \partial_*(C, D))_*(C\cap\partial_*(C, D)) = 0$ and $\mathrm{tr}(\mu, \partial_*(C, D))^*(C\cap\partial_*(C, D)) = \mu(\partial_*(C, D)) = \mu_*(C, D)$, by Theorem 1.3, for each $x \in [0, \mu_*(C, D)]$ there exists an extension ν_x of μ_1 to the σ-algebra $\sigma(\mathcal{A}\cup\{C\cap\partial_*(C, D)\})$ such that:

(7) $\nu_x\mid_{\mathcal{A}} = \mu_1$,

(8) $\nu_x(C\cap\partial_*(C, D)) = x$ and

(9) $\nu_x(D\cap\partial_*(C, D)) = \mu_*(C, D) - x = y$.

Complete the σ-algebra $\sigma(\mathcal{A}\cup\{C\cap\partial_*(C, D)\})$ with respect to ν_x up to the σ-algebra $\sigma(\mathcal{A}\cup\{C\cap\partial_*(C, D)\})[\nu_x]$, and the corresponding extension ν_x to $\sigma(\mathcal{A}\cup\{C\cap\partial_*(C, D)\})[\nu_x]$ denote by $\overline{\nu}_x$. Since $\overline{\nu}_x(X\setminus \partial_*(C, D)) = \mu_1(X\setminus\partial_*(C, D)) = \mathrm{tr}(\mu, \partial_*(C, D))(X\setminus\partial_*(C, D)) = 0$, we see that $C\setminus\partial_*(C, D)$ and $D\setminus\partial_*(C, D) \in \sigma(\mathcal{A}\cup\{C\cap\partial_*(C, D)\})[\nu_x]$, and that $\overline{\nu}_x(C\setminus\partial_*(C, D)) = \overline{\nu}_x(D\setminus\partial_*(C, D)) = 0$. This means that C and $D \in \sigma(\mathcal{A}\cup\{C\cap\partial_*(C, D)\})[\nu_x]$, and $\overline{\nu}_x(C) = x$ and $\overline{\nu}_x(D) = y$.

Define a measure $\nu_{\alpha+x,\beta+y}$ on the σ-algebra $\sigma(\mathcal{A}\cup\{C, D\})$ by

$$\nu_{\alpha+x,\beta+y} = \nu^0_{\alpha,\beta} + \overline{\nu}_x\mid_{\sigma(\mathcal{A}\cup\{C,D\})}.$$

Then the following relations are satisfied:

$$(10)\qquad \nu_{\alpha+x,\beta+y}(A) = \mathrm{tr}(\mu, X\setminus\partial_*(C, D))(A) + \mathrm{tr}(\mu, \partial_*(C, D))(A) = \mu(A), \quad \forall A\in\mathcal{A},$$

$$(11)\qquad \nu_{\alpha+x,\beta+y}(C) = \nu^0_{\alpha,\beta}(C) + \overline{\nu}_x(C) = \alpha + x \text{ and}$$

$$(12)\qquad \nu_{\alpha+x,\beta+y}(D) = \nu^0_{\alpha,\beta}(D) + \overline{\nu}_x(D) = \beta + y.$$

By using convex combinations we may easily conclude that $\forall\alpha\in [\mu_*(C),\ \mu^*(C)]$ and $\forall\beta$ such that conditions (B) are satisfied, there exists an extension $\nu_{\alpha,\beta}$ of μ to the σ-algebra $\sigma(\mathcal{A}\cup\{C, D\})$ such that $\nu_{\alpha,\beta}(C) = \alpha$ and $\nu_{\alpha,\beta}(D) = \beta$.

According to the symmetric role of the sets C and D, we also find that $\forall\beta\in[\mu_*(D),\ \mu^*(D)]$ and $\forall\alpha$ such that conditions (B) are satisfied, there exists a desired extension $\nu_{\alpha,\beta}$.

This proves that conditions (B) are sufficient for the existence of a desired extension, an let us now show that they are necessary too.

So, let us suppose that there exists a given extension $\nu_{\alpha,\beta}$, then by the definition of the μ-hull and μ-kernel of the sets C and D we easily find

$$\mu_*(C) \leqslant \alpha \leqslant \mu^*(C) \text{ and } \mu_*(D) \leqslant \beta \leqslant \mu^*(D).$$

Furthermore, we have $\mu_*(C \cup D) = \mu(C_* \cup D_* \cup \partial_*(C, D)) \leqslant \nu_{\alpha,\beta}(C \cup D) = \alpha + \beta \leqslant \mu(C^* \cup D^*) = \mu^*(C \cup D)$, and the proof is complete.

In this way the problem of extension of a finite measure in the case of a two-element disjoint perturbation is completely solved.

3. Extension of a finite measure in the case of general finite perturbation

In the case of a general finite perturbation, we may deduce one of inductive formulas of an extension. The result may be stated as follows.

PROPOSITION 3.1. *Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and let $C_1, C_2, \dots C_n$ be arbitrary subsets of X. Then there exist $\alpha_i \in [\mu_*(C_i),\ \mu^*(C_i)]$, $i = 1, 2, \dots, n$ and a countably additive extension $\nu_{\alpha_1,\alpha_2,\dots,\alpha_n}$ of μ to the σ-algebra $\sigma(\mathcal{A} \cup \{C_1, C_2, \dots, C_n\})$ such that*

$$\nu_{\alpha_1,\alpha_2,\dots,\alpha_n}(C_i) = \alpha_i, \quad i = 1, 2, \dots, n. \tag{1}$$

Moreover, an extension $\nu_{\alpha_1,\alpha_2,\dots,\alpha_n}$ may be defined in the following way:

$$C_0 = \emptyset, \quad \mu_0 = \mu; \tag{2}$$

(3) *If μ_{k-1} is already defined on the σ-algebra $\sigma(\mathcal{A} \cup \{C_1, C_2, \dots, C_{k-1}\})$, then we define a measure μ_k on $\sigma(\mathcal{A} \cup \{C_1, C_2, \dots, C_k\})$ by:*

$$\mu_k = \begin{cases} \mathrm{tr}_*(\mu_{k-1}, C_k) + \mathrm{tr}^*(\mu_{k-1}, C_k^c), & \textit{if } \alpha_k = (\mu_{k-1})_*(C_k); \\ \mathrm{tr}^*(\mu_{k-1}, C_k) + \mathrm{tr}_*(\mu_{k-1}, C_k^c), & \textit{if } \alpha_k = (\mu_{k-1})^*(C_k); \\ \dfrac{(\mu_{k-1})^*(C_k) - \alpha_k}{(\mu_{k-1})^*(C_k) - (\mu_{k-1})_*(C_k)} \times & \\ \qquad \times \{\mathrm{tr}_*(\mu_{k-1}, C_k) + \mathrm{tr}^*(\mu_{k-1}, C_k^c)\} & \\ \quad + \dfrac{\alpha_k - (\mu_{k-1})_*(C_k)}{(\mu_{k-1})^*(C_k) - (\mu_{k-1})_*(C_k)} \times & \\ \qquad \times \{\mathrm{tr}^*(\mu_{k-1}, C_k) + \mathrm{tr}_*(\mu_{k-1}, C_k^c)\}, & \\ \qquad\qquad \textit{if } \alpha_k \in \langle (\mu_{k-1})_*(C_k), (\mu_{k-1})^*(C_k) \rangle & \end{cases}$$

for each $k = 1, 2, \dots, n$.

$$\nu_{\alpha_1,\alpha_2,\dots,\alpha_n} = \mu_n. \tag{4}$$

Finally, any countably additive extension ν of μ to the σ-algebra $\sigma(\mathcal{A} \cup \{C_1, C_2, \dots, C_n\})$ satisfies

$$\nu(C_i) \in [\mu_*(C_i), \mu^*(C_i)], \quad \forall i = 1, 2, \dots, n. \tag{5}$$

Proof. The proof may be carried out by induction using Theorem 1.3, since for an arbitrary subset C of X, by definition of the outer and inner measure the following inequalities are satisfied

$$\begin{aligned}\mu_*(C) \leqslant (\mu_{k-1})_*(C) \leqslant (\mu_k)_*(C)& \\ \leqslant (\mu_k)^*(C) \leqslant (\mu_{k-1})^*(C) \leqslant \mu^*(C).&\end{aligned}$$

for all $k = 1, 2, \dots, n$.

The last statement follows easily by the definition of the μ-hulls and μ-kernels of the sets $C_1, C_2, \dots, C_n$.

Consequently, we can state *the following problem*: What are the necessary and sufficient conditions for the existence of a countably additive extension $\nu_{\alpha_1,\alpha_2,\dots,\alpha_n}$ of μ to the σ-algebra $\sigma(\mathcal{A} \cup \{C_1, C_2, \dots, C_n\})$ satisfying:

$$\nu_{\alpha_1,\alpha_2,\dots,\alpha_n}(C_i) = \alpha_i, \quad i = 1, 2, \dots, n.$$

The length of the proof of Theorem 2.3 indicates that in this general case it seems better to use some of the classical existential results in order to find a desired extension.

Indeed, in [5] we can find the answer to this problem, and the main point in those considerations has been played by the well-known Hahn–Banach theorem.

4. The problem of the existence of an extension of a finite measure in the case of a countable perturbation

While we know that in the case of a finite perturbation of the σ-algebra an extension always exists, it seems that in the case of a general countable perturbation the answer to this question is unknown.

Let us show that this problem actually reduces to the problem of the extension of a cilindric measure.

Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and let $\mathcal{C} = \{C_1, C_2, \dots\}$ be a countable family of subsets of X, such that the operation

$$\mathcal{A} \xrightarrow{c} \sigma(\mathcal{A} \cup \mathcal{C})$$

forms a countable perturbation.

Put $\mu_0 = \mu$, $\mathcal{A}_0 = \mathcal{A}$ and $\mathcal{A}_n = \sigma(\mathcal{A}_{n-1} \cup \{C_n\})$, $\forall n \geqslant 1$. Then by Theorem 1.3, for each $n = 1, 2, \dots$ there exists an extension μ_n of μ_{n-1} to $\mathcal{A}_n$. In this way we get the following sequence of σ-algebras:

$$\mathcal{A}_0 \subset \mathcal{A}_1 \subset \mathcal{A}_2 \subset \dots \tag{1}$$

and a sequence of measures μ_n on $\mathcal{A}_n$, $n = 0, 1, 2, \dots$ such that

$$\mu_n \mid_{\mathcal{A}_{n-1}} = \mu_{n-1}. \tag{2}$$

for all $n \geqslant 1$.

Then $\mathcal{A}_\infty = \cup_{n=0}^\infty \mathcal{A}_n$ is an algebra and by $\mu_\infty \mid_{\mathcal{A}_n} = \mu_n$, $n \geqslant 0$, a finitely additive (cilindric) measure μ_∞ on $\mathcal{A}_\infty$ is defined, such that $\mu_\infty \mid_{\mathcal{A}} = \mu$. Since $\sigma(\mathcal{A}_\infty) = \sigma(\mathcal{A} \cup \mathcal{C})$, we see that our starting problem of the existence of an extension of the measure μ to the σ-algebra $\sigma(\mathcal{A} \cup \mathcal{C})$ reduces to the problem of the existence of an extension of a cilindric measure μ_∞ to the σ-algebra $\sigma(\mathcal{A}_\infty)$. According to Hopf's theorem for this it suffices to show that μ_∞ is σ-additive on $\mathcal{A}_\infty$, i.e. for each decreasing sequence $A_n \in \mathcal{A}_\infty$, $n \geqslant 1$ such that $\cap_{n=1}^\infty A_n = \emptyset$ it follows $\lim_{n\to\infty} \mu_\infty(A_n) = 0$.

Without loss of generality we can assume that $A_1 \in \mathcal{A}_{n_1}$, $A_2 \in \mathcal{A}_{n_2}, \dots$ with $n_1 \leqslant n_2 \leqslant \dots$. Hence we get $\mu_\infty(A_i) = \mu_{n_i}(A_i) = \mu_{n_{i+1}}(A_i) \geqslant \mu_{n_{i+1}}(A_{i+1}) = \mu_\infty(A_{i+1})$ for all $i \geqslant 1$. So $\{\mu_\infty(A_n), n \geqslant 1\}$ is a decreasing sequence of real numbers. However, we have:

Example 4.1. There exist a strictly increasing sequence $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \dots$ of σ-algebras on the set X and a sequence of finite measures μ_n on $\mathcal{A}_n$, $n \geqslant 1$ satisfying:

(i) $\mu_{n+1} \mid_{\mathcal{A}_n} = \mu_n$, $\forall n \geqslant 1$ and

(ii) The map μ_∞ which is defined on the algebra $\mathcal{A}_\infty = \cup_{n=1}^\infty \mathcal{A}_n$ by

$$\mu_\infty \mid_{\mathcal{A}_n} = \mu_n \quad , \quad \forall n \geqslant 1,$$

is only finitely additive, but not σ-additive.

Indeed, put $X = \mathrm{N}$, $\mathcal{A}_n = \{A \cup B \mid A \subset \{1, 2, \dots, n\}, B \in \{\mathrm{N}, \mathrm{N} \setminus \{1\}, \mathrm{N} \setminus \{1, 2\}, \dots, \mathrm{N} \setminus \{1, 2, \dots, n\}, \emptyset\}\}$ and for each $A \in \mathcal{A}_n$ define

$$\mu_n(A) = \begin{cases} 0, & \text{if } A \text{ is a finite set,} \\ 1, & \text{if } A^c \text{ is a finite set.} \end{cases}$$

Then $\mathcal{A}_\infty = \{A \subset \mathrm{N} \mid A \text{ is finite or } A^c \text{ is finite}\}$ and for each $A \in \mathcal{A}_\infty$ we have

$$\mu_\infty(A) = \begin{cases} 0, & \text{if } A \text{ is a finite set,} \\ 1, & \text{if } A^c \text{ is a finite set.} \end{cases}$$

Now (i) and (ii) follow straightforward.

This example shows that the main weight of the starting problem of the extension of a measure μ, or a cilindric measure μ_∞, to the σ-algebra $\sigma(\mathcal{A} \cup \mathcal{C})$ is contained in the possibility of a various inductive choice of measures $\mu_1, \mu_2, \ldots$ which form a measure μ_∞, and this fact shows a signification of the investigation of extension of finite measures in the case of finite perturbations.

It can be shown, see [5], that if $\mathcal{C}$ is a countable partition of X or there exists a countable partition $\mathcal{D}$ of X such that $\sigma(\mathcal{A} \cup \mathcal{C}) = \sigma(\mathcal{A} \cup \mathcal{D})$, then there exists an extension ν of the measure μ to the σ-algebra $\sigma(\mathcal{A} \cup \mathcal{C})$. This fact actually shows that we can choose measures $\mu_1, \mu_2 \ldots$ inductively in such a way that the resulting cilindric measure μ_∞ allows the extension from the algebra $\mathcal{A}_\infty$ to a σ-additive measure ν on the σ-algebra $\sigma(\mathcal{A}_\infty)$.

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NUŽNI I DOVOLJNI UVJETI ZA EGZISTENCIJU KONAČNE MJERE U SLUČAJU DVOELEMENTNE DISJUNKTNE PERTURBACIJE

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Sadržaj

Dani su nužni i dovoljni uvjeti za egzistenciju prebrojivo aditivnog proširenja konačne mjere u slučaju dvoelementne disjunktne perturbacije. Razvijena je tehnika dokaza koja se osniva na direktnoj konstrukciji koristeći vanjski i unutarnji trag konačne mjere. Kao posljedica, izvedene su karakterizacije za neka tipična proširenja.