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A Change-of-Variable Formula with Local Time on Surfaces

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Let $X = (X^1, \ldots, X^n)$ be a continuous semimartingale and let $b : I\!\!R^{n-1} \rightarrow I\!\!R$ be a continuous function such that the process $b^X = b(X^1, \ldots, X^{n-1})$ is a semimartingale. Setting $C = \{ (x_1, \ldots, x_n) \in I\!\!R^n \mid x_n < b(x_1, \ldots, x_{n-1}) \}$ and $D = \{ (x_1, \ldots, x_n) \in I\!\!R^n \mid x_n > b(x_1, \ldots, x_{n-1}) \}$ suppose that a continuous function $F : I\!\!R^n \rightarrow I\!\!R$ is given such that F is $C^{i_1,\ldots,i_n}$ on $\bar{C}$ and F is $C^{i_1,\ldots,i_n}$ on $\bar{D}$ where each i_k equals 1 or 2 depending on whether X^k is of bounded variation or not. Then the following change-of-variable formula holds:

$$\begin{aligned} F(X_t) = F(X_0) &+ \sum_{i=1}^n \int_0^t \frac{1}{2}\Big(\frac{\partial F}{\partial x_i}(X_s^1, \ldots, X_s^n+) + \frac{\partial F}{\partial x_i}(X_s^1, \ldots, X_s^n-) \Big)\, dX_s^i \\ &+ \frac{1}{2} \sum_{i,j=1}^n \int_0^t \frac{1}{2}\Big(\frac{\partial^2 F}{\partial x_i \partial x_j}(X_s^1, \ldots, X_s^n+) + \frac{\partial^2 F}{\partial x_i \partial x_j}(X_s^1, \ldots, X_s^n-) \Big)\, d\langle X^i, X^j \rangle_s \\ &+ \frac{1}{2} \int_0^t \Big(\frac{\partial F}{\partial x_n}(X_s^1, \ldots, X_s^n+) - \frac{\partial F}{\partial x_n}(X_s^1, \ldots, X_s^n-) \Big) I(X_s^n = b_s^X)\ d\ell_s^b(X) \end{aligned}$$

where $\ell_s^b(X)$ is the local time of X on the surface b given by:

$$\ell_s^b(X) = I\!\!P\!-\!\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^s I(-\varepsilon < X_r^n - b_r^X < \varepsilon)\ d\langle X^n - b^X, X^n - b^X \rangle_r$$

and $d\ell_s^b(X)$ refers to integration with respect to $s \mapsto \ell_s^b(X)$. The analogous formula extends to general semimartingales X and b^X as well. A version of the same formula under weaker conditions on F is derived for the semimartingale $((t, X_t, S_t))_{t \geq 0}$ where $(X_t)_{t \geq 0}$ is an Itô diffusion and $(S_t)_{t \geq 0}$ is its running maximum.

1. Introduction

Let $(X_t)_{t \geq 0}$ be a continuous semimartingale (see e.g. [13]) and let $b : I\!\!R_+ \rightarrow I\!\!R$ be a continuous function of bounded variation. Setting $C = \{ (t, x) \in I\!\!R_+ \times I\!\!R \mid x < b(t) \}$ and $D = \{ (t, x) \in I\!\!R_+ \times I\!\!R \mid x > b(t) \}$ suppose that a continuous function $F : I\!\!R_+ \times I\!\!R \rightarrow I\!\!R$ is given such that F is $C^{1,2}$ on $\bar{C}$ and F is $C^{1,2}$ on $\bar{D}$.

Then the following change-of-variable formula is known to be valid (cf. [11]):

$$\begin{aligned} F(t, X_t) = F(0, X_0) &+ \int_0^t \frac{1}{2}\Big(F_t(s, X_s+) + F_t(s, X_s-) \Big)\, ds \\ &+ \int_0^t \frac{1}{2}\Big(F_x(s, X_s+) + F_x(s, X_s-) \Big)\, dX_s \\ &+ \frac{1}{2} \int_0^t F_{xx}(s, X_s)\ I(X_s \neq b(s))\ d\langle X, X \rangle_s \end{aligned} \tag{1.1}$$

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$$+\frac{1}{2}\int_0^t \Big(F_x(s,X_s+)-F_x(s,X_s-)\Big)\, I(X_s=b(s))\; d\ell_s^b(X)$$

where $\ell_s^b(X)$ is the local time of X on the curve b given by:

(1.2) $$\ell_s^b(X) = I\!P-\lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^s I(b(r)-\varepsilon<X_r<b(r)+\varepsilon)\, d\langle X,X\rangle_r$$

and $d\ell_s^b(X)$ refers to integration with respect to the continuous increasing function $s\mapsto \ell_s^b(X)$. A version of the same formula for an Itô diffusion X derived under weaker conditions on F has found applications in free-boundary problems of optimal stopping (cf. [11]).

The main aim of the present paper is to extend the change-of-variable formula (1.1) to a multi-dimensional setting of continuous functions F which are smooth above and below surfaces. Continuous semimartingales are considered in Section 2, and semimartingales with jumps are considered in Section 3. A version of the same formula under weaker conditions on F is derived in Section 4 for the continuous semimartingale $((t,X_t,S_t))_{t\ge 0}$ where $(X_t)_{t\ge 0}$ is an Itô diffusion and $(S_t)_{t\ge 0}$ is its running maximum. This version is useful in the study of free-boundary problems for optimal stopping of the maximum process when the horizon is finite (for the infinite horizon case see [10] with references).

The study of Section 4 serves as an example of what generally needs to be done in order to relax the smoothness conditions on F from $\bar{C}$ and $\bar{D}$ to $C\cup D$. These relaxed versions of the formula are important for applications. It is thus hoped that the programme started in Section 3 of [11] and in Section 4 of the present paper will be continued.

For related results on the local time-space calculus see [1], [5], [3], [2], [8]. Older references on the topic include [7], [14], [9], [15], [4].

2. Continuous semimartingales

Let $X=(X^1,\ldots,X^n)$ be a continuous semimartingale and let $b: I\!R^{n-1}\rightarrow I\!R$ be a continuous function such that the process $b^X=b(X^1,\ldots,X^{n-1})$ is a semimartingale. [Note that the sufficient condition $b\in C^2$ is by no means necessary.] Setting:

(2.1) $$C=\{\,(x_1,\ldots,x_n)\in I\!R^n \mid x_n<b(x_1,\ldots,x_{n-1})\,\}$$

(2.2) $$D=\{\,(x_1,\ldots,x_n)\in I\!R^n \mid x_n>b(x_1,\ldots,x_{n-1})\,\}$$

suppose that a continuous function $F: I\!R^n\rightarrow I\!R$ is given such that:

(2.3) $$F \text{ is } C^{i_1,\ldots,i_n} \text{ on } \bar{C}$$

(2.4) $$F \text{ is } C^{i_1,\ldots,i_n} \text{ on } \bar{D}$$

where each i_j equals 1 or 2 depending on whether X^j is of bounded variation or not. More explicitly, it means that F restricted to C coincides with a function F_1 which is $C^{i_1,\ldots,i_n}$ on $I\!R^n$, and F restricted to D coincides with a function F_2 which is $C^{i_1,\ldots,i_n}$ on $I\!R^n$. [We recall that a continuous function $F_k: I\!R^n\rightarrow I\!R$ is $C^{i_1,\ldots,i_n}$ on $I\!R^n$ if the partial derivatives $\partial F_k/\partial x_j$ when $i_j=1$ as well as $\partial^2 F_k/\partial x_j\partial x_{j'}$ when $i_j, i_{j'}=2$ exist and are continuous as functions from $I\!R^n$ to $I\!R$ for all $1 \le j, j' \le n$ where k equals 1 or 2 .]

Then the natural desire arising in free-boundary problems of optimal stopping (and other problems where the hitting time of D by the process X plays a role) is to apply a change-of-variable formula to $F(X_t)$ so to account for possible jumps of $(\partial F/\partial x_n)(x_1, \ldots, x_n)$ at $x_n = b(x_1, \ldots, x_{n-1})$ being measured by:

$$\ell^b_s(X) = I\!P\!-\!\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^s I(-\varepsilon < X^n_r - b^X_r < \varepsilon)\; d\langle X^n\!-b^X, X^n\!-b^X \rangle_r \tag{2.5}$$

which represents the local time of X on the surface b for $s \in [0,t]$. Note that the limit in (2.5) exists (as a limit in probability) since $X^n - b^X$ is a continuous semimartingale.

In the special case when the semimartingale equals (t, X_t) it is evident that the previous setting reduces to the setting leading to the change-of-variable formula (1.1) above. Further particular cases of the formula (1.1) are reviewed in [11]. The following theorem provides a general formula of this kind for continuous semimartingales (see also Section 3 below).

Theorem 2.1

Let $X = (X^1, \ldots, X^n)$ be a continuous semimartingale, let $b : I\!R^{n-1} \to I\!R$ be a continuous function such that the process $b^X = b(X^1, \ldots, X^{n-1})$ is a semimartingale, and let $F : I\!R^n \to I\!R$ be a continuous function satisfying (2.3) and (2.4) above.

Then the following change-of-variable formula holds:

$$\begin{aligned} F(X_t) = F(X_0) &+ \sum_{i=1}^n \int_0^t \frac{1}{2}\Big(\frac{\partial F}{\partial x_i}(X^1_s, \ldots, X^n_s+) + \frac{\partial F}{\partial x_i}(X^1_s, \ldots, X^n_s-) \Big)\, dX^i_s \\ &+ \frac{1}{2} \sum_{i,j=1}^n \int_0^t \frac{1}{2}\Big(\frac{\partial^2 F}{\partial x_i \partial x_j}(X^1_s, \ldots, X^n_s+) + \frac{\partial^2 F}{\partial x_i \partial x_j}(X^1_s, \ldots, X^n_s-) \Big)\, d\langle X^i, X^j \rangle_s \\ &+ \frac{1}{2} \int_0^t \Big(\frac{\partial F}{\partial x_n}(X^1_s, \ldots, X^n_s+) - \frac{\partial F}{\partial x_n}(X^1_s, \ldots, X^n_s-) \Big) I(X^n_s = b^X_s)\, d\ell^b_s(X) \end{aligned} \tag{2.6}$$

where $\ell^b_s(X)$ is the local time of X on the surface b given in (2.5) above, and $d\ell^b_s(X)$ refers to integration with respect to the continuous increasing function $s \mapsto \ell^b_s(X)$.

Proof. 1. Set $Z^1_t = X^n_t \wedge b^X_t$ and $Z^2_t = X^n_t \vee b^X_t$ for $t > 0$ given and fixed. Denoting $\hat{X}_t = (X^1_t, \ldots, X^{n-1}_t, Z^1_t)$, $\check{X}_t = (X^1_t, \ldots, X^{n-1}_t, Z^2_t)$ and $\tilde{X}_t = (X^1_t, \ldots, X^{n-1}_t, b^X_t)$, we see that the following identity holds:

$$F(X_t) = F_1(\hat{X}_t) + F_2(\check{X}_t) - F(\tilde{X}_t) \tag{2.7}$$

where we use that $F(x) = F_1(x) = F_2(x)$ for $x = (x_1, \ldots, x_{n-1}, b(x_1, \ldots, x_{n-1}))$. The processes $(Z^1_t)_{t \ge 0}$ and $(Z^2_t)_{t \ge 0}$ are continuous semimartingales admitting the following representations:

$$Z^1_t = \frac{1}{2}\Big(X^n_t + b^X_t - |X^n_t - b^X_t| \Big) \tag{2.8}$$

$$Z^2_t = \frac{1}{2}\Big(X^n_t + b^X_t + |X^n_t - b^X_t| \Big)\;. \tag{2.9}$$

Recalling the Tanaka formula:

$$|X_t^n - b_t^X| = |X_0^n - b_0^X| + \int_0^t \operatorname{sign}(X_s^n - b_s^X)\ d(X_s^n - b_s^X) + \ell_t^b(X) \tag{2.10}$$

where $\operatorname{sign}(0) = 0$, we find that:

$$\begin{aligned} dZ_t^1 &= \frac{1}{2}\Big(d(X_t^n + b_t^X) - \operatorname{sign}(X_t^n - b_t^X)\, d(X_t^n - b_t^X) - d\ell_t^b(X) \Big) \\ &= \frac{1}{2}\Big(\big(1 - \operatorname{sign}(X_t^n - b_t^X)\big)\, dX_t^n + \big(1 + \operatorname{sign}(X_t^n - b_t^X)\big)\, db_t^X - d\ell_t^b(X) \Big) \end{aligned} \tag{2.11}$$

$$\begin{aligned} dZ_t^2 &= \frac{1}{2}\Big(d(X_t^n + b_t^X) + \operatorname{sign}(X_t^n - b_t^X)\, d(X_t^n - b_t^X) + d\ell_t^b(X) \Big) \\ &= \frac{1}{2}\Big(\big(1 + \operatorname{sign}(X_t^n - b_t^X)\big)\, dX_t^n + \big(1 - \operatorname{sign}(X_t^n - b_t^X)\big)\, db_t^X + d\ell_t^b(X) \Big)\ . \end{aligned} \tag{2.12}$$

In the sequel we set $D_i = \partial/\partial x_i$ and $D_{ij} = \partial^2/\partial x_i \partial x_j$ as well as $D_i^2 = \partial^2/\partial x_i^2$.

2. Applying the Itô formula to $F_1(\hat{X}_t)$ and using (2.11) we get:

$$\begin{aligned} F_1(\hat{X}_t) &= F_1(\hat{X}_0) + \sum_{i=1}^n \int_0^t D_i F_1(\hat{X}_s)\, d\hat{X}_s^i + \frac{1}{2} \sum_{i,j=1}^n \int_0^t D_{ij} F_1(\hat{X}_s)\, d\langle \hat{X}^i, \hat{X}^j \rangle_s \\ &= F_1(\hat{X}_0) + \sum_{i=1}^{n-1} \int_0^t D_i F_1(\hat{X}_s)\, dX_s^i + \frac{1}{2} \int_0^t \Big(1 - \operatorname{sign}(X_s^n - b_s^X) \Big) D_n F_1(\hat{X}_s)\, dX_s^n \\ &\quad + \frac{1}{2} \int_0^t \Big(1 + \operatorname{sign}(X_s^n - b_s^X) \Big) D_n F_1(\hat{X}_s)\, db_s^X - \frac{1}{2} \int_0^t D_n F_1(\hat{X}_s)\, d\ell_s^b(X) \\ &\quad + \frac{1}{2} \sum_{i,j=1}^n \int_0^t D_{ij} F_1(X_s)\, I(X_s^n < b_s^X)\, d\langle X^i, X^j \rangle_s \\ &\quad + \frac{1}{4} \sum_{i,j=1}^n \int_0^t D_{ij} F_1(X_s)\, I(X_s^n = b_s^X)\, d\langle X^i, X^j \rangle_s \\ &\quad + \frac{1}{4} \sum_{i,j=1}^n \int_0^t D_{ij} F_1(\tilde{X}_s)\, I(X_s^n = b_s^X)\, d\langle \tilde{X}^i, \tilde{X}^j \rangle_s \\ &\quad + \frac{1}{2} \sum_{i,j=1}^n \int_0^t D_{ij} F_1(\tilde{X}_s)\, I(X_s^n > b_s^X)\, d\langle \tilde{X}^i, \tilde{X}^j \rangle_s \end{aligned} \tag{2.13}$$

where in the last four integrals we make use of the general fact:

$$I(Y_s^1 = Y_s^2)\ d\langle Y^1, Y^3 \rangle_s = I(Y_s^1 = Y_s^2)\ d\langle Y^2, Y^3 \rangle_s \tag{2.14}$$

whenever Y^1, Y^2 and Y^3 are continuous (one-dimensional) semimartingales. The identity (2.14) can easily be verified using the Kunita-Watanabe inequality and the occupation times formula (for more details see the proof following (3.11) below).

The right-hand side of (2.13) can further be expressed in terms of $\tilde{X}$ using (2.14) as follows:

$$
\begin{aligned}
(2.15)\qquad F_1(\hat{X}_t) \;=\; & F_1(\hat{X}_0) + \sum_{i=1}^{n-1} \int_0^t D_i F_1(X_s)\, I(X_s^n < b_s^X)\, dX_s^i \\
& + \frac{1}{2} \sum_{i=1}^{n-1} \int_0^t D_i F_1(X_s)\, I(X_s^n = b_s^X)\, dX_s^i + \frac{1}{2} \sum_{i=1}^{n-1} \int_0^t D_i F_1(\tilde{X}_s)\, I(X_s^n = b_s^X)\, d\tilde{X}_s^i \\
& + \sum_{i=1}^{n-1} \int_0^t D_i F_1(\tilde{X}_s)\, I(X_s^n > b_s^X)\, d\tilde{X}_s^i + \int_0^t D_n F_1(X_s)\, I(X_s^n < b_s^X)\, dX_s^n \\
& + \frac{1}{2} \int_0^t D_n F_1(X_s)\, I(X_s^n = b_s^X)\, dX_s^n + \int_0^t D_n F_1(\tilde{X}_s)\, I(X_s^n > b_s^X)\, d\tilde{X}_s^n \\
& + \frac{1}{2} \int_0^t D_n F_1(\tilde{X}_s)\, I(X_s^n = b_s^X)\, d\tilde{X}_s^n - \frac{1}{2} \int_0^t D_n F_1(X_s)\, I(X_s^n = b_s^X)\, d\ell_s^b(X) \\
& \qquad + \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_1(X_s)\, I(X_s^n < b_s^X)\, d\langle X^i, X^j \rangle_s \\
& \qquad + \frac{1}{4} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_1(X_s)\, I(X_s^n = b_s^X)\, d\langle X^i, X^j \rangle_s \\
& \qquad + \frac{1}{4} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_1(\tilde{X}_s)\, I(X_s^n = b_s^X)\, d\langle \tilde{X}^i, \tilde{X}^j \rangle_s \\
& \qquad + \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_1(\tilde{X}_s)\, I(X_s^n > b_s^X)\, d\langle \tilde{X}^i, \tilde{X}^j \rangle_s \;.
\end{aligned}
$$

By grouping the corresponding terms in (2.15) we obtain:

$$
\begin{aligned}
(2.16)\qquad F_1(\hat{X}_t) \;=\; & F_1(\hat{X}_0) \\
& + \sum_{i=1}^{n} \int_0^t D_i F_1(X_s)\, I(X_s^n < b_s^X)\, dX_s^i + \frac{1}{2} \sum_{i=1}^{n} \int_0^t D_i F_1(X_s)\, I(X_s^n = b_s^X)\, dX_s^i \\
& \qquad + \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_1(X_s)\, I(X_s^n < b_s^X)\, d\langle X^i, X^j \rangle_s \\
& \qquad + \frac{1}{4} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_1(X_s)\, I(X_s^n = b_s^X)\, d\langle X^i, X^j \rangle_s \\
& \qquad - \frac{1}{2} \int_0^t D_n F_1(X_s)\, I(X_s^n = b_s^X)\, d\ell_s^b(X) \\
& + \sum_{i=1}^{n} \int_0^t D_i F_1(\tilde{X}_s)\, I(X_s^n > b_s^X)\, d\tilde{X}_s^i + \frac{1}{2} \sum_{i=1}^{n} \int_0^t D_i F_1(\tilde{X}_s)\, I(X_s^n = b_s^X)\, d\tilde{X}_s^i \\
& \qquad + \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_1(\tilde{X}_s)\, I(X_s^n > b_s^X)\, d\langle \tilde{X}^i, \tilde{X}^j \rangle_s \\
& \qquad + \frac{1}{4} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_1(\tilde{X}_s)\, I(X_s^n = b_s^X)\, d\langle \tilde{X}^i, \tilde{X}^j \rangle_s \;.
\end{aligned}
$$

3. Applying the Itô formula to $F_2(\hat{X}_t)$ and using (2.12) we get:

$$
\begin{aligned}
(2.17)\qquad F_2(\check{X}_t) &= F_2(\check{X}_0) + \sum_{i=1}^{n} \int_0^t D_i F_2(\check{X}_s)\, d\check{X}_s^i + \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_2(\check{X}_s)\, d\langle \check{X}^i, \check{X}^j \rangle_s \\
&= F_2(\check{X}_0) + \sum_{i=1}^{n-1} \int_0^t D_i F_2(\check{X}_s)\, dX_s^i + \frac{1}{2} \int_0^t \Big(1 + \text{sign}(X_s^n - b_s^X)\Big) D_n F_2(\check{X}_s)\, dX_s^n \\
&\quad + \frac{1}{2} \int_0^t \Big(1 - \text{sign}(X_s^n - b_s^X)\Big) D_n F_2(\check{X}_s)\, db_s^X + \frac{1}{2} \int_0^t D_n F_2(\check{X}_s)\, d\ell_s^b(X) \\
&\quad + \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_2(X_s)\, I(X_s^n > b_s^X)\, d\langle X^i, X^j \rangle_s \\
&\quad + \frac{1}{4} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_2(X_s)\, I(X_s^n = b_s^X)\, d\langle X^i, X^j \rangle_s \\
&\quad + \frac{1}{4} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_2(\tilde{X}_s)\, I(X_s^n = b_s^X)\, d\langle \tilde{X}^i, \tilde{X}^j \rangle_s \\
&\quad + \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_2(\tilde{X}_s)\, I(X_s^n < b_s^X)\, d\langle \tilde{X}^i, \tilde{X}^j \rangle_s
\end{aligned}
$$

where in the last four integrals we make use of the general fact (2.14).

The right-hand side of (2.17) can further be expressed in terms of $\tilde{X}$ using (2.14) as follows:

$$
\begin{aligned}
(2.18)\qquad F_2(\check{X}_t) &= F_2(\check{X}_0) + \sum_{i=1}^{n-1} \int_0^t D_i F_2(X_s)\, I(X_s^n > b_s^X)\, dX_s^i \\
&\quad + \frac{1}{2} \sum_{i=1}^{n-1} \int_0^t D_i F_2(X_s)\, I(X_s^n = b_s^X)\, dX_s^i + \frac{1}{2} \sum_{i=1}^{n-1} \int_0^t D_i F_2(\tilde{X}_s)\, I(X_s^n = b_s^X)\, d\tilde{X}_s^i \\
&\quad + \sum_{i=1}^{n-1} \int_0^t D_i F_2(\tilde{X}_s)\, I(X_s^n < b_s^X)\, d\tilde{X}_s^i + \int_0^t D_n F_2(X_s)\, I(X_s^n > b_s^X)\, dX_s^n \\
&\quad + \frac{1}{2} \int_0^t D_n F_2(X_s)\, I(X_s^n = b_s^X)\, dX_s^n + \int_0^t D_n F_2(\tilde{X}_s)\, I(X_s^n < b_s^X)\, d\tilde{X}_s^n \\
&\quad + \frac{1}{2} \int_0^t D_n F_2(\tilde{X}_s)\, I(X_s^n = b_s^X)\, d\tilde{X}_s^n + \frac{1}{2} \int_0^t D_n F_2(X_s)\, I(X_s^n = b_s^X)\, d\ell_s^b(X) \\
&\quad + \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_2(X_s)\, I(X_s^n > b_s^X)\, d\langle X^i, X^j \rangle_s \\
&\quad + \frac{1}{4} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_2(X_s)\, I(X_s^n = b_s^X)\, d\langle X^i, X^j \rangle_s \\
&\quad + \frac{1}{4} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_2(\tilde{X}_s)\, I(X_s^n = b_s^X)\, d\langle \tilde{X}^i, \tilde{X}^j \rangle_s \\
&\quad + \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_2(\tilde{X}_s)\, I(X_s^n < b_s^X)\, d\langle \tilde{X}^i, \tilde{X}^j \rangle_s \ .
\end{aligned}
$$

By grouping the corresponding terms in (2.18) we obtain:

$$(2.19)\qquad F_2(\check{X}_t) \;=\; F_2(\check{X}_0)$$
$$+\sum_{i=1}^n \int_0^t D_iF_2(X_s)\, I(X_s^n\!<\!b_s^X)\, dX_s^i + \frac{1}{2}\sum_{i=1}^n \int_0^t D_iF_2(X_s)\, I(X_s^n\!=\!b_s^X)\, dX_s^i$$
$$+\frac{1}{2}\sum_{i,j=1}^n \int_0^t D_{ij}F_2(X_s)\, I(X_s^n < b_s^X)\, d\langle X^i,X^j\rangle_s$$
$$+\frac{1}{4}\sum_{i,j=1}^n \int_0^t D_{ij}F_2(X_s)\, I(X_s^n = b_s^X)\, d\langle X^i,X^j\rangle_s$$
$$+\frac{1}{2}\int_0^t D_nF_2(X_s)\, I(X_s^n = b_s^X)\, d\ell_s^b(X)$$
$$+\sum_{i=1}^n \int_0^t D_iF_2(\tilde{X}_s)\, I(X_s^n\!>\!b_s^X)\, d\tilde{X}_s^i + \frac{1}{2}\sum_{i=1}^n \int_0^t D_iF_2(\tilde{X}_s)\, I(X_s^n\!=\!b_s^X)\, d\tilde{X}_s^i$$
$$+\frac{1}{2}\sum_{i,j=1}^n \int_0^t D_{ij}F_2(\tilde{X}_s)\, I(X_s^n > b_s^X)\, d\langle \tilde{X}^i,\tilde{X}^j\rangle_s$$
$$+\frac{1}{4}\sum_{i,j=1}^n \int_0^t D_{ij}F_2(\tilde{X}_s)\, I(X_s^n = b_s^X)\, d\langle \tilde{X}^i,\tilde{X}^j\rangle_s\ .$$

4. Combining the right-hand sides of (2.16) and (2.19) we conclude:

$$(2.20)\qquad F(X_t) \;=\; F_1(\hat{X}_t) + F_2(\check{X}_t) - F(\tilde{X}_t) \;=\; F(X_0)$$
$$+\sum_{i=1}^n \int_0^t \frac{1}{2}\Big(D_iF(X_s^1,\ldots,X_s^n+) + D_iF(X_s^1,\ldots,X_s^n-)\Big)\, dX_s^i$$
$$+\frac{1}{2}\sum_{i,j=1}^n \int_0^t \frac{1}{2}\Big(D_{ij}F(X_s^1,\ldots,X_s^n+) + D_{ij}F(X_s^1,\ldots,X_s^n-)\Big)\, d\langle X^i,X^j\rangle_s$$
$$+\frac{1}{2}\int_0^t \Big(D_nF(X_s^1,\ldots,X_s^n+) - D_nF(X_s^1,\ldots,X_s^n-)\Big) I(X_s^n\!=\!b_s^X)\, d\ell_s^b(X) + R_t$$

where the final term is given by:

$$(2.21)\qquad R_t \;=\; F(\tilde{X}_0) + \sum_{i=1}^n \int_0^t D_iF_1(\tilde{X}_s)\, I(X_s^n > b_s^X)\, d\tilde{X}_s^i$$
$$+\frac{1}{2}\sum_{i,j=1}^n \int_0^t D_{ij}F_1(\tilde{X}_s)\, I(X_s^n\!>\!b_s^X)\, d\langle \tilde{X}^i,\tilde{X}^j\rangle_s + \frac{1}{2}\sum_{i=1}^n \int_0^t D_iF_1(\tilde{X}_s)\, I(X_s^n\!=\!b_s^X)\, d\tilde{X}_s^i$$
$$+\frac{1}{4}\sum_{i,j=1}^n \int_0^t D_{ij}F_1(\tilde{X}_s)\, I(X_s^n\!=\!b_s^X)\, d\langle \tilde{X}^i,\tilde{X}^j\rangle_s + \sum_{i=1}^n \int_0^t D_iF_2(\tilde{X}_s)\, I(X_s^n\!<\!b_s^X)\, d\tilde{X}_s^i$$

$$+\frac{1}{2}\sum_{i,j=1}^{n}\int_0^t D_{ij}F_2(\tilde{X}_s)\, I(X_s^n < b_s^X)\, d\langle \tilde{X}^i, \tilde{X}^j\rangle_s + \frac{1}{2}\sum_{i=1}^{n}\int_0^t D_iF_2(\tilde{X}_s)\, I(X_s^n = b_s^X)\, d\tilde{X}_s^i$$

$$+\frac{1}{4}\sum_{i,j=1}^{n}\int_0^t D_{ij}F_2(\tilde{X}_s)\, I(X_s^n = b_s^X)\, d\langle \tilde{X}^i, \tilde{X}^j\rangle_s - F(\tilde{X}_t)\ .$$

Hence we see that (2.6) will be proved if we show that $R_t = 0$. Note that if $F_1 = F_2$ then the identity $R_t = 0$ reduces to the Itô formula applied to $F(\tilde{X}_t)$. In the general case we may proceed as follows.

5. Since $F_1(x) = F_2(x)$ for $x = (x_1, \ldots, x_{n-1}, b(x_1, \ldots, x_{n-1}))$, we see that the two semimartingales $F_1(\tilde{X})$ and $F_2(\tilde{X})$ coincide, so that:

$$\int_0^t I(X_s^n > b_s^X)\, d(F_1(\tilde{X}_s)) = \int_0^t I(X_s^n > b_s^X)\, d(F_2(\tilde{X}_s)) \tag{2.22}$$

$$\int_0^t I(X_s^n = b_s^X)\, d(F_1(\tilde{X}_s)) = \int_0^t I(X_s^n = b_s^X)\, d(F_2(\tilde{X}_s))\ . \tag{2.23}$$

Applying the Itô formula to $F_1(\tilde{X}_s)$ and $F_2(\tilde{X}_s)$ we see that (2.22) and (2.23) become:

$$\sum_{i=1}^{n}\int_0^t D_iF_1(\tilde{X}_s)\, I(X_s^n > b_s^X)\, d\tilde{X}_s^i + \frac{1}{2}\sum_{i,j=1}^{n}\int_0^t D_{ij}F_1(\tilde{X}_s)\, I(X_s^n > b_s^X)\, \langle \tilde{X}^i, \tilde{X}^j\rangle_s \tag{2.24}$$

$$= \sum_{i=1}^{n}\int_0^t D_iF_2(\tilde{X}_s)\, I(X_s^n > b_s^X)\, d\tilde{X}_s^i + \frac{1}{2}\sum_{i,j=1}^{n}\int_0^t D_{ij}F_2(\tilde{X}_s)\, I(X_s^n > b_s^X)\, \langle \tilde{X}^i, \tilde{X}^j\rangle_s$$

$$\sum_{i=1}^{n}\int_0^t D_iF_1(\tilde{X}_s)\, I(X_s^n = b_s^X)\, d\tilde{X}_s^i + \frac{1}{2}\sum_{i,j=1}^{n}\int_0^t D_{ij}F_1(\tilde{X}_s)\, I(X_s^n = b_s^X)\, \langle \tilde{X}^i, \tilde{X}^j\rangle_s \tag{2.25}$$

$$= \sum_{i=1}^{n}\int_0^t D_iF_2(\tilde{X}_s)\, I(X_s^n = b_s^X)\, d\tilde{X}_s^i + \frac{1}{2}\sum_{i,j=1}^{n}\int_0^t D_{ij}F_2(\tilde{X}_s)\, I(X_s^n = b_s^X)\, \langle \tilde{X}^i, \tilde{X}^j\rangle_s\ .$$

Making use of (2.24) and (2.25) we see that F_1 in the first four integrals on the right-hand side of (2.21) can be replaced by F_2 . This combined with the remaining terms shows that the identity $R_t = 0$ reduces to the Itô formula applied to $F_2(\tilde{X}_t)$. This completes the proof of the theorem.

□

Remark 2.2

The change-of-variable formula (2.6) can obviously be extended to the case when instead of one function b we are given finitely many functions $b_1, b_2, \ldots, b_m$ which do not intersect.

More precisely, let $X = (X^1, \ldots, X^n)$ be a continuous semimartingale and let us assume that the following conditions are satisfied:

(2.26) $b_k : I\!R^{n-1} \rightarrow I\!R$ is continuous such that $b^{k,X} = b_k(X^1, \ldots, X^{n-1})$ is a semimartingale for $1 \le k \le m$

(2.27) $F_k : I\!R^n \rightarrow I\!R$ is $C^{i_1, \ldots, i_n}$ for $1 \le k \le m+1$ where each i_j equals 1 or 2 depending on whether X^j is of bounded variation or not

$$
\begin{aligned}
(2.28) \qquad F(x) &= F_1(x) \quad \text{if} \quad x_n < b_1(x_1,\dots,x_{n-1}) \\
&= F_k(x) \quad \text{if} \quad b_k(x_1,\dots,x_{n-1}) < x_n < b_{k+1}(x_1,\dots,x_{n-1}) \ \text{ for } \ 2 \le k \le m \\
&= F_{m+1}(x) \quad \text{if} \quad x_n > b_{m+1}(x_1,\dots,x_{n-1})
\end{aligned}
$$

where $F : I\!\!R^n \to I\!\!R$ is continuous and $x = (x_1,\dots,x_n)$ belongs to $I\!\!R^n$.

Then the change-of-variable formula (2.6) extends as follows:

$$
\begin{aligned}
(2.29) \quad F(X_t) = F(X_0) &+ \sum_{i=1}^n \int_0^t \frac{1}{2}\Big(\frac{\partial F}{\partial x_i}(X_s^1,\dots,X_s^n+) + \frac{\partial F}{\partial x_i}(X_s^1,\dots,X_s^n-)\Big)\, dX_s^i \\
&+ \frac{1}{2}\sum_{i,j=1}^n \int_0^t \frac{1}{2}\Big(\frac{\partial^2 F}{\partial x_i \partial x_j}(X_s^1,\dots,X_s^n+) + \frac{\partial^2 F}{\partial x_i \partial x_j}(X_s^1,\dots,X_s^n-)\Big)\, d\langle X^i, X^j\rangle_s \\
&+ \frac{1}{2}\sum_{k=1}^m \int_0^t \Big(\frac{\partial F}{\partial x_n}(X_s^1,\dots,X_s^n+) - \frac{\partial F}{\partial x_n}(X_s^1,\dots,X_s^n-)\Big) I(X_s^n = b_s^{k,X})\, d\ell_s^{b_k}(X)
\end{aligned}
$$

where $\ell_s^{b_k}(X)$ is the local time of X on the surface b_k given in (2.5) above, and $d\ell_s^{b_k}(X)$ refers to integration with respect to $s \mapsto \ell_s^{b_k}(X)$.

Note in particular that an open set C in $I\!\!R^n$ (such as a ball) can often be described in terms of functions $b_1, b_2, \dots, b_m$ so that (2.29) becomes applicable. Perhaps the most interesting example of a function F is obtained by looking at $\tau_D = \inf\{\, t > 0 \mid X_t \in D \,\}$ and setting $F(x) = E_x(G(X_{\tau_D}))$ where G is an admissible function and $X_0 = x$ under P_x for $x \in I\!\!R^n$. One such example will be studied in Section 4 below.

Remark 2.3

The change-of-variable formula (2.6) is expressed in terms of the *symmetric* local time (2.5). It is evident from the proof above that one could also use the *one-sided* local times defined by:

$$
(2.30) \qquad \ell_s^{b+}(X) = I\!\!P - \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\int_0^s I(0 \le X_r^n - b_r^X < \varepsilon)\, d\langle X^n - b^X, X^n - b^X\rangle_r
$$

$$
(2.31) \qquad \ell_s^{b-}(X) = I\!\!P - \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\int_0^s I(-\varepsilon < X_r^n - b_r^X \le 0)\, d\langle X^n - b^X, X^n - b^X\rangle_r \ .
$$

Then under the same conditions as in Theorem 2.1 we find that the following two equivalent formulations of (2.6) are valid:

$$
\begin{aligned}
(2.32) \qquad F(X_t) = F(X_0) &+ \sum_{i=1}^n \int_0^t \frac{\partial F}{\partial x_i}(X_s^1,\dots,X_s^n\mp)\, dX_s^i \\
&+ \frac{1}{2}\sum_{i,j=1}^n \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(X_s^1,\dots,X_s^n\mp)\, d\langle X^i, X^j\rangle_s \\
&+ \frac{1}{2}\int_0^t \Big(\frac{\partial F}{\partial x_n}(X_s^1,\dots,X_s^n+) - \frac{\partial F}{\partial x_n}(X_s^1,\dots,X_s^n-)\Big) I(X_s^n = b_s^X)\, d\ell_s^{b\pm}(X) \ .
\end{aligned}
$$

Clearly (2.29) above can also be expressed in terms of one-sided local times. Note finally that if $X^n - b^X$ is a continuous local martingale, then the three definitions (2.5), (2.30) and (2.31) coincide.

3. Semimartingales with jumps

In this section we will extend the change-of-variable formula (2.6) first to semimartingales with jumps of bounded variation (Theorem 3.1) and then to general semimartingales (Theorem 3.2).

1. Let $X = (X^1, \ldots, X^n)$ be a semimartingale (see e.g. [12]). Recall that each sample path $t \mapsto X_t^i$ is right continuous and has left limits for $1 \le i \le n$. In Theorem 3.1 below we will assume that each semimartingale X^i has jumps of bounded variation in the sense that:

$$\sum_{0<s\le t} |\Delta X_s^i| < \infty \tag{3.1}$$

where $\Delta X_s^i = X_s^i - X_{s-}^i$ for $1 \le i \le n$. In this case each X^i can be uniquely decomposed into:

$$X_t^i = X_0^i + X_t^{i,c} + X_t^{i,d} \tag{3.2}$$

where $X^{i,c} = M^{i,c} + A^{i,c}$ is a continuous semimartingale and $X^{i,d}$ is a discrete semimartingale (of bounded variation) given by:

$$X_t^{i,d} = \sum_{0<s\le t} \Delta X_s^i \;. \tag{3.3}$$

Moreover, if $F : I\!R^n \to I\!R$ is C^2 then Itô's formula takes any of the two equivalent forms:

$$\begin{aligned} F(X_t) &= F(X_0) + \sum_{i=1}^n \int_0^t \frac{\partial F}{\partial x_i}(X_{s-})\, dX_s^i + \frac{1}{2} \sum_{i,j=1}^n \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(X_{s-})\, d[X^{i,c}, X^{j,c}]_s \\ &\quad + \sum_{0<s\le t} \Big(F(X_s) - F(X_{s-}) - \sum_{i=1}^n \frac{\partial F}{\partial x_i}(X_{s-})\, \Delta X_s^i \Big) \\ &= F(X_0) + \sum_{i-1}^n \int_0^t \frac{\partial F}{\partial x_i}(X_{s-})\, dX_s^{i,c} + \frac{1}{2} \sum_{i,j=1}^n \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(X_{s-})\, d[X^{i,c}, X^{j,c}]_s \\ &\quad + \sum_{0<s\le t} \Big(F(X_s) - F(X_{s-}) \Big) \;. \end{aligned} \tag{3.4}$$

Both of these forms will be used freely below without further mentioning.

Let $b : I\!R^{n-1} \to I\!R$ be a continuous function such that the process $b^X = b(X^1, \ldots, X^{n-1})$ is a semimartingale with jumps of bounded variation. Then $X^n - b^X$ is a semimartingale with jumps of bounded variation and the local time of X on the surface b is well-defined as follows:

$$\ell_s^b(X) = I\!P\text{-}\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^s I(-\varepsilon < X_r^n - b_r^X < \varepsilon)\; d[X^n - b^X, X^n - b^X]_r^c \tag{3.5}$$

where $[X^n - b^X, X^n - b^X]^c$ is the continuous (path by path) component of $[X^n - b^X, X^n - b^X]$. Recalling that $X^{n,c}$ and $b^{X,c}$ are continuous semimartingales associated with X^n and b^X as in (3.2) above, we know that $[X^n - b^X, X^n - b^X]^c = [X^{n,c} - b^{X,c}, X^{n,c} - b^{X,c}]$.

The following theorem extends the change-of-variable formula (2.6) to semimartingales with jumps of bounded variation.

Theorem 3.1

Let $X = (X^1, \ldots, X^n)$ *be a semimartingale where each* X^i *has jumps of bounded variation, let* $b : I\!R^{n-1} \rightarrow I\!R$ *be a continuous function such that the process* $b^X = b(X^1, \ldots, X^{n-1})$ *is a semimartingale with jumps of bounded variation, and let* $F : I\!R^n \rightarrow I\!R$ *be a continuous function satisfying (2.3) and (2.4) above.*

Then the following change-of-variable formula holds:

$$
\begin{aligned}
(3.6) \quad F(X_t) = F(X_0) &+ \sum_{i=1}^n \int_0^t \frac{1}{2}\Big(\frac{\partial F}{\partial x_i}(X^1_{s-}, \ldots, X^n_{s-}+) + \frac{\partial F}{\partial x_i}(X^1_{s-}, \ldots, X^n_{s-}-)\Big)\, dX^{i,c}_s \\
&+ \frac{1}{2} \sum_{i,j=1}^n \int_0^t \frac{1}{2}\Big(\frac{\partial^2 F}{\partial x_i \partial x_j}(X^1_{s-}, \ldots, X^n_{s-}+) + \frac{\partial^2 F}{\partial x_i \partial x_j}(X^1_{s-}, \ldots, X^n_{s-}-)\Big)\, d[X^{i,c}, X^{j,c}]_s \\
&+ \sum_{0<s\le t} \Big(F(X_s) - F(X_{s-})\Big) \\
&+ \frac{1}{2} \int_0^t \Big(\frac{\partial F}{\partial x_n}(X^1_{s-}, \ldots, X^n_{s-}+) - \frac{\partial F}{\partial x_n}(X^1_{s-}, \ldots, X^n_{s-}-)\Big)\, I(X^n_{s-} = b^X_{s-},\ X^n_s = b^X_s)\, d\ell^b_s(X)
\end{aligned}
$$

where $\ell^b_s(X)$ *is the local time of* X *on the surface* b *given in (3.5) above, and* $d\ell^b_s(X)$ *refers to integration with respect to the continuous increasing function* $s \mapsto \ell^b_s(X)$ *.*

Proof. The proof can be carried out similarly to the proof of Theorem 2.1 and we will only highlight a few novel points appearing due to the existence of jumps. The remaining details are the same as in the proof of Theorem 2.1.

1. We begin as in the proof of Theorem 2.1 by introducing the processes $Z^1, Z^2, \hat{X}, \check{X}, \tilde{X}$ and observing that (2.7)-(2.9) carries over unchanged. Since X^n and b^X both have jumps of bounded variation, it is easily seen that so do Z^1 and Z^2 as well. Thus the analogue of (2.10) which is obtained by applying the Tanaka formula reads:

$$
\begin{aligned}
(3.7) \quad |X^n_t - b^X_t| = |X^n_0 - b^X_0| &+ \int_0^t \text{sign}(X^n_{s-} - b^X_{s-})\, d(X^{n,c}_s - b^{X,c}_s) + \ell^b_t(X) \\
&+ \sum_{0<s\le t} \Big(|X^n_s - b^X_s| - |X^n_{s-} - b^X_{s-}|\Big)
\end{aligned}
$$

where $\text{sign}(0) = 0$. Similarly to (2.11) and (2.12) we find that:

$$
(3.8) \quad dZ^{1,c}_t = \frac{1}{2}\Big(\big(1 - \text{sign}(X^n_{t-} - b^X_{t-})\big)\, dX^{n,c}_t + \big(1 + \text{sign}(X^n_{t-} - b^X_{t-})\big)\, db^{X,c}_t - d\ell^b_t(X)\Big)
$$

$$
(3.9) \quad dZ^{2,c}_t = \frac{1}{2}\Big(\big(1 + \text{sign}(X^n_{t-} - b^X_{t-})\big)\, dX^{n,c}_t + \big(1 - \text{sign}(X^n_{t-} - b^X_{t-})\big)\, db^{X,c}_t + d\ell^b_t(X)\Big)\ .
$$

2. Applying the Itô formula to $F_1(\hat{X}_t)$ we get:

$$
\begin{aligned}
(3.10) \quad F_1(\hat{X}_t) = F_1(\hat{X}_0) &+ \sum_{i=1}^n \int_0^t D_i F_1(\hat{X}_{s-})\, d\hat{X}^{i,c}_s + \frac{1}{2} \sum_{i,j=1}^n \int_0^t D_{ij} F_1(\hat{X}_{s-})\, d[\hat{X}^{i,c}, \hat{X}^{j,c}]_s \\
&+ \sum_{0<s\le t} F_1(\hat{X}_s) - F_1(\hat{X}_{s-})\ .
\end{aligned}
$$

Hence using (3.8) and proceeding in the same way as in (2.13) and (2.15) we obtain the analogue of the identity (2.16) where all X^i and $\tilde{X}^i$ in the integrators (including those with the angle brackets) are replaced by $X^{i,c}$ and $\tilde{X}^{i,c}$ (now written as the square brackets).

It may be noted (as in the proof of Theorem 2.1) that in the preceding derivation (and in the derivation following (3.12) below) we need to make use of the general fact:

$$I(Y^1_{s-}=Y^2_{s-})\, d[Y^{1,c},Y^{3,c}]_s \;=\; I(Y^1_{s-}=Y^2_{s-})\, d[Y^{2,c},Y^{3,c}]_s \tag{3.11}$$

whenever Y^1, Y^2 and Y^3 are (one-dimensional) semimartingales. To verify (3.11) note that the claim is equivalent to the fact that for two (one-dimensional) semimartingales Y^1 and Y^2 we have $I(Y^1_{s-}=0)\, d[Y^{1,c},Y^{2,c}]=0$. To derive the latter we may invoke the Kunita-Watanabe inequality (cf. [12; p.61]) according to which it is enough to show that $I(Y^1_{s-}=0)\, d[Y^{1,c},Y^{1,c}]=0$. This identity however follows by the occupation times formula (cf. [12; p. 168]) since $g=1_{\{0\}}$ equals zero almost everywhere with respect to Lebesgue measure on $I\!R$. This proves (3.11) in the general case (recall also (2.14) above).

3. Applying the Itô formula to $F_2(\check{X}_t)$ we get:

$$\begin{aligned} F_2(\check{X}_t) = F_2(\check{X}_0) + \sum_{i=1}^n \int_0^t D_i F_2(\check{X}_{s-})\, d\check{X}^{i,c}_s + \frac{1}{2}\sum_{i,j=1}^n \int_0^t D_{ij}F_2(\check{X}_{s-})\, d[\check{X}^{i,c},\check{X}^{j,c}]_s \\ + \sum_{0<s\le t} F_2(\check{X}_s) - F_2(\check{X}_{s-}) \;. \end{aligned} \tag{3.12}$$

Hence using (3.9) and proceeding in the same way as in (2.17) and (2.18) we obtain the analogue of the identity (2.19) where all X^i and $\tilde{X}^i$ in the integrators (including those with the angle brackets) are replaced by $X^{i,c}$ and $\tilde{X}^{i,c}$ (now written as the square brackets).

4. Combining the right-hand sides of the resulting identities we find the analogue of (2.20) to be:

$$\begin{aligned} F(X_t) &= F_1(\hat{X}_t) + F_2(\check{X}_t) - F(\tilde{X}_t) = F(X_0) \\ &+ \sum_{i=1}^n \int_0^t \frac{1}{2}\Big(D_iF(X^1_{s-},\ldots,X^n_{s-}+) + D_iF(X^1_{s-},\ldots,X^n_{s-}-)\Big)\, dX^{i,c}_s \\ &+ \frac{1}{2}\sum_{i,j=1}^n \int_0^t \frac{1}{2}\Big(D_{ij}F(X^1_{s-},\ldots,X^n_{s-}+) + D_{ij}F(X^1_{s-},\ldots,X^n_{s-}-)\Big)\, d[X^{i,c},X^{j,c}]_s \\ &+ \frac{1}{2}\int_0^t \Big(D_nF(X^1_{s-},\ldots,X^n_{s-}+) - D_nF(X^1_{s-},\ldots,X^n_{s-}-)\Big)\, d\ell^b_s(X) \\ &+ \sum_{0<s\le t} F(X_s) - F(X_{s-}) + R_t \end{aligned} \tag{3.13}$$

where we use that:

$$\begin{aligned} &\sum_{0<s\le t} F_1(\hat{X}_s) - F_1(\hat{X}_{s-}) + \sum_{0<s\le t} F_2(\check{X}_s) - F_2(\check{X}_{s-}) - \sum_{0<s\le t} F(\tilde{X}_s) - F(\tilde{X}_{s-}) \\ &= \sum_{0<s\le t} F(X_s) - F(X_{s-}) \end{aligned} \tag{3.14}$$

and the final term in (3.13) is given by:

$$
\begin{aligned}
(3.15)\qquad R_t \;=\; & F(\tilde X_0) + \sum_{i=1}^n \int_0^t D_iF_1(\tilde X_{s-})\, I(X^n_{s-} > b^X_{s-})\, d\tilde X^{i,c}_s \\
& + \frac{1}{2}\sum_{i,j=1}^n \int_0^t D_{ij}F_1(\tilde X_{s-})\, I(X^n_{s-} > b^X_{s-})\, d[\tilde X^{i,c}, \tilde X^{j,c}]_s \\
& + \frac{1}{2}\sum_{i=1}^n \int_0^t D_iF_1(\tilde X_{s-})\, I(X^n_{s-} = b^X_{s-})\, d\tilde X^{i,c}_s \\
& + \frac{1}{4}\sum_{i,j=1}^n \int_0^t D_{ij}F_1(\tilde X_{s-})\, I(X^n_{s-} = b^X_{s-})\, d[\tilde X^{i,c}, \tilde X^{j,c}]_s \\
& + \sum_{i=1}^n \int_0^t D_iF_2(\tilde X_{s-})\, I(X^n_{s-} < b^X_{s-})\, d\tilde X^{i,c}_s \\
& + \frac{1}{2}\sum_{i,j=1}^n \int_0^t D_{ij}F_2(\tilde X_{s-})\, I(X^n_{s-} < b^X_{s-})\, d[\tilde X^{i,c}, \tilde X^{j,c}]_s \\
& + \frac{1}{2}\sum_{i=1}^n \int_0^t D_iF_2(\tilde X_{s-})\, I(X^n_{s-} = b^X_{s-})\, d\tilde X^{i,c}_s \\
& + \frac{1}{4}\sum_{i,j=1}^n \int_0^t D_{ij}F_2(\tilde X_{s-})\, I(X^n_{s-} = b^X_{s-})\, d[\tilde X^{i,c}, \tilde X^{j,c}]_s - F(\tilde X_t)^c
\end{aligned}
$$

where $F(\tilde X)^c$ is the continuous semimartingale part of $F(\tilde X)$. From (3.13) and (3.15) we see that (3.6) will be proved if we show that $R_t = 0$.

5. The same arguments as those given in (2.22)-(2.25) show again that F_1 in the first four integrals on the right-hand side of (3.15) can be replaced by F_2. This combined with the remaining terms shows that the identity $R_t = 0$ reduces to applying the Itô formula to $F_2(\tilde X_t)$ and identifying the continuous part of the resulting semimartingale. This completes the proof of the theorem.

□

2. The condition (3.1) applied to the semimartingale $X^n - b^X$ is the best known sufficient condition for the local time of X on the surface b to be given by means of the explicit expression (3.5) above. In the case of general semimartingales X and b^X, however, the local time of X on the surface b (i.e. the local time of the semimartingale $X^n - b^X$ at zero) can still be defined by means of the Tanaka formula (3.17) retaining its role as the occupation density relative to the random clock $[X^n - b^X, X^n - b^X]^c$ (see [12; p. 168]) but we do not have the explicit representation (3.5) anymore and the use of the local time is somewhat less transparent.

If $X = (X^1, \ldots, X^n)$ is a general semimartingale (not necessarily satisfying (3.1) above) then each X^i can still be decomposed into (3.2) with $X^{i,c} = M^{i,c} + A^{i,c}$ and $X^{i,d} = M^{i,d} + A^{i,d}$ where $M^{i,c}$ is a continuous local martingale, $A^{i,c}$ is a continuous process of bounded variation, $M^{i,d}$ is a purely discontinuous local martingale, and $A^{i,d}$ is a pure jump process of bounded variation. Since the condition (3.1) may fail (due to the existence of many small jumps) we know that Itô's formula takes only the first form in (3.4) above. It is well-known (and easily verified by localization using Taylor's theorem) that the first series over $0 < s \le t$ in (3.4) is absolutely convergent (even if (3.1) fails to hold).

The following theorem extends the change-of-variable formula (2.6) to general semimartingales. Note that (1.1), (2.6) and (3.6) above are special cases of the general formula (3.16) below.

Theorem 3.2

Let $X = (X^1, \ldots, X^n)$ be a semimartingale, let $b : I\!R^{n-1} \to I\!R$ be a continuous function such that the process $b^X = b(X^1, \ldots, X^{n-1})$ is a semimartingale, and let $F : I\!R^n \to I\!R$ be a continuous function satisfying (2.3) and (2.4) above.

Then the following change-of-variable formula holds:

$$
\begin{aligned}
(3.16)\qquad F(X_t) = F(X_0) &+ \sum_{i=1}^{n} \int_0^t \frac{1}{2}\Big(\frac{\partial F}{\partial x_i}(X^1_{s-}, \ldots, X^n_{s-}+) + \frac{\partial F}{\partial x_i}(X^1_{s-}, \ldots, X^n_{s-}-) \Big)\, dX^i_s \\
&+ \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t \frac{1}{2}\Big(\frac{\partial^2 F}{\partial x_i \partial x_j}(X^1_{s-}, \ldots, X^n_{s-}+) + \frac{\partial^2 F}{\partial x_i \partial x_j}(X^1_{s-}, \ldots, X^n_{s-}-) \Big)\, d[X^{i,c}, X^{j,c}]_s \\
&+ \sum_{0<s\le t} \bigg(F(X_s) - F(X_{s-}) - \sum_{i=1}^{n} \frac{1}{2}\Big(\frac{\partial F}{\partial x_i}(X^1_{s-}, \ldots, X^n_{s-}+) + \frac{\partial F}{\partial x_i}(X^1_{s-}, \ldots, X^n_{s-}-) \Big)\, \Delta X^i_s \bigg) \\
&+ \frac{1}{2} \int_0^t \Big(\frac{\partial F}{\partial x_n}(X^1_{s-}, \ldots, X^n_{s-}+) - \frac{\partial F}{\partial x_n}(X^1_{s-}, \ldots, X^n_{s-}-) \Big)\, I(X^n_{s-} = b^X_{s-},\ X^n_s = b^X_s)\, d\ell^b_s(X)
\end{aligned}
$$

where $\ell^b_s(X)$ is the local time of X on the surface b given by means of (3.17) below and $d\ell^b_s(X)$ refers to integration with respect to the continuous increasing function $s \mapsto \ell^b_s(X)$.

Proof. The proof can be carried out similarly to the proof of Theorem 2.1 and Theorem 3.1 and we will only highlight a few novel points appearing due to the absence of the condition (3.1). The remaining details are the same as in the proof of Theorem 2.1 and Theorem 3.1.

1. We begin as in the proof of Theorem 2.1 by introducing the processes Z^1, Z^2, $\hat{X}$, $\check{X}$, $\tilde{X}$ and observing that (2.7)-(2.9) carries over unchanged. The analogue of (2.10) which is obtained by applying the Tanaka formula now reads:

$$
\begin{aligned}
(3.17)\qquad |X^n_t - b^X_t| = |X^n_0 - b^X_0| &+ \int_0^t \mathrm{sign}(X^n_{s-} - b^X_{s-})\, d(X^n_s - b^X_s) + \ell^b_t(X) \\
&+ \sum_{0<s\le t} \Big(|X^n_s - b^X_s| - |X^n_{s-} - b^X_{s-}| - \mathrm{sign}(X^n_{s-} - b^X_{s-})\, \Delta(X^n - b^X)_s \Big)
\end{aligned}
$$

where $\mathrm{sign}(0) = 0$. Similarly to (2.11) and (2.12) we now find that:

$$
(3.18)\qquad dZ^1_t = \frac{1}{2} \Big(\big(1 - \mathrm{sign}(X^n_{t-} - b^X_{t-})\big)\, dX^n_t + \big(1 + \mathrm{sign}(X^n_{t-} - b^X_{t-})\big)\, db^X_t - d\ell^b_t(X) - dJ_t(X) \Big)
$$

$$
(3.19)\qquad dZ^2_t = \frac{1}{2} \Big(\big(1 + \mathrm{sign}(X^n_{t-} - b^X_{t-})\big)\, dX^n_t + \big(1 - \mathrm{sign}(X^n_{t-} - b^X_{t-})\big)\, db^X_t + d\ell^b_t(X) + dJ_t(X) \Big)
$$

where we denote:

$$
(3.20)\qquad J_t(X) = \sum_{0<s\le t} \Big(|X^n_s - b^X_s| - |X^n_{s-} - b^X_{s-}| - \mathrm{sign}(X^n_{s-} - b^X_{s-})\, \Delta(X^n - b^X)_s \Big)\ .
$$

2. Applying the Itô formula to $F_1(\hat{X}_t)$ we get:

$$(3.21)\qquad F_1(\hat{X}_t) = F_1(\hat{X}_0) + \sum_{i=1}^n \int_0^t D_i F_1(\hat{X}_{s-})\, d\hat{X}_s^i + \frac{1}{2}\sum_{i,j=1}^n \int_0^t D_{ij} F_1(\hat{X}_{s-})\, d[\hat{X}^{i,c}, \hat{X}^{j,c}]_s$$
$$+ \sum_{0<s\le t} \Big(F_1(\hat{X}_s) - F_1(\hat{X}_{s-}) - \sum_{i=1}^n D_i F_1(\hat{X}_{s-})\, \Delta X_s^i \Big)\ .$$

Hence using (3.18) and proceeding in the same way as in (2.13) and (2.15), making use of the general fact (3.11), we obtain the analogue of the identity (2.16) where all X^i and $\tilde{X}^i$ in the integrators with the angle brackets are replaced by $X^{i,c}$ and $\tilde{X}^{i,c}$ now written as the square brackets, and the right-hand side of the identity contains a new term given by:

$$(3.22)\qquad -\frac{1}{2}\int_0^t D_n F_1(\hat{X}_{s-})\, dJ_s(X)$$

due to the appearance of $-dJ_t(X)$ in (3.18).

3. Applying the Itô formula to $F_2(\check{X}_t)$ we get:

$$(3.23)\qquad F_2(\check{X}_t) = F_2(\check{X}_0) + \sum_{i=1}^n \int_0^t D_i F_2(\check{X}_{s-})\, d\check{X}_s^i + \frac{1}{2}\sum_{i,j=1}^n \int_0^t D_{ij} F_2(\check{X}_{s-})\, d[\check{X}^{i,c}, \check{X}^{j,c}]_s$$
$$+ \sum_{0<s\le t} \Big(F_2(\check{X}_s) - F_2(\check{X}_{s-}) - \sum_{i=1}^n D_i F_2(\check{X}_{s-})\, \Delta X_s^i \Big).$$

Hence using (3.19) and proceeding in the same way as in (2.17) and (2.18), making use of the general fact (3.11), we obtain the analogue of the identity (2.19) where all X^i and $\tilde{X}^i$ in the integrators with the angle brackets are replaced by $X^{i,c}$ and $\tilde{X}^{i,c}$ now written as the square brackets, and the right-hand side of the identity contains a new term given by:

$$(3.24)\qquad \frac{1}{2}\int_0^t D_n F_2(\check{X}_{s-})\, dJ_s(X)$$

due to the appearance of $dJ_t(X)$ in (3.19).

4. Combining the right-hand sides of the resulting identities we find the analogue of (2.20) to be:

$$(3.25)\qquad F(X_t) \;=\; F_1(\hat{X}_t) + F_2(\check{X}_t) - F(\tilde{X}_t) \;=\; F(X_0)$$
$$+ \sum_{i=1}^n \int_0^t \frac{1}{2}\Big(D_i F(X_{s-}^1, \ldots, X_{s-}^n +) + D_i F(X_{s-}^1, \ldots, X_{s-}^n -) \Big)\, dX_s^i$$
$$+ \frac{1}{2}\sum_{i,j=1}^n \int_0^t \frac{1}{2}\Big(D_{ij} F(X_{s-}^1, \ldots, X_{s-}^n +) + D_{ij} F(X_{s-}^1, \ldots, X_{s-}^n -) \Big)\, d[X^{i,c}, X^{j,c}]_s$$
$$+ \frac{1}{2}\int_0^t \Big(D_n F(X_{s-}^1, \ldots, X_{s-}^n +) - D_n F(X_{s-}^1, \ldots, X_{s-}^n -) \Big)\, d\ell_s^b(X) + R_t$$

where the final term is given by:

$$
\begin{aligned}
(3.26)\qquad R_t \;=\;& F(\tilde{X}_0) + \sum_{i=1}^{n}\int_0^t D_iF_1(\tilde{X}_{s-})\, I(X^n_{s-} > b^X_{s-})\, d\tilde{X}^i_s \\
&+ \frac{1}{2}\sum_{i,j=1}^{n}\int_0^t D_{ij}F_1(\tilde{X}_{s-})\, I(X^n_{s-} > b^X_{s-})\, d[\tilde{X}^{i,c},\tilde{X}^{j,c}]_s \\
&+ \frac{1}{2}\sum_{i=1}^{n}\int_0^t D_iF_1(\tilde{X}_{s-})\, I(X^n_{s-} = b^X_{s-})\, d\tilde{X}^i_s \\
&+ \frac{1}{4}\sum_{i,j=1}^{n}\int_0^t D_{ij}F_1(\tilde{X}_{s-})\, I(X^n_{s-} = b^X_{s-})\, d[\tilde{X}^{i,c},\tilde{X}^{j,c}]_s \\
&+ \sum_{i=1}^{n}\int_0^t D_iF_2(\tilde{X}_{s-})\, I(X^n_{s-} < b^X_{s-})\, d\tilde{X}^i_s \\
&+ \frac{1}{2}\sum_{i,j=1}^{n}\int_0^t D_{ij}F_2(\tilde{X}_{s-})\, I(X^n_{s-} < b^X_{s-})\, d[\tilde{X}^{i,c},\tilde{X}^{j,c}]_s \\
&+ \frac{1}{2}\sum_{i=1}^{n}\int_0^t D_iF_2(\tilde{X}_{s-})\, I(X^n_{s-} = b^X_{s-})\, d\tilde{X}^i_s \\
&+ \frac{1}{4}\sum_{i,j=1}^{n}\int_0^t D_{ij}F_2(\tilde{X}_{s-})\, I(X^n_{s-} = b^X_{s-})\, d[\tilde{X}^{i,c},\tilde{X}^{j,c}]_s - F(\tilde{X}_t) \\
&+ \sum_{0<s\le t}\Big(F_1(\hat{X}_s) - F_1(\hat{X}_{s-}) - \sum_{i=1}^{n} D_iF_1(\hat{X}_{s-})\,\Delta\hat{X}^i_s\Big) \\
&+ \sum_{0<s\le t}\Big(F_2(\check{X}_s) - F_2(\check{X}_{s-}) - \sum_{i=1}^{n} D_iF_2(\check{X}_{s-})\,\Delta\check{X}^i_s\Big) \\
&+ \frac{1}{2}\int_0^t D_nF_2(\check{X}_{s-})\, dJ_s(X) - \frac{1}{2}\int_0^t D_nF_1(\hat{X}_{s-})\, dJ_s(X)\ .
\end{aligned}
$$

5. The same arguments as those given in (2.22) and (2.23) now lead to the following analogues of (2.24) and (2.25) respectively:

$$
\begin{aligned}
(3.27)\qquad & \sum_{i=1}^{n}\int_0^t D_iF_1(\tilde{X}_{s-})\, I(X^n_{s-} > b^X_{s-})\, d\tilde{X}^i_s + \frac{1}{2}\sum_{i,j=1}^{n}\int_0^t D_{ij}F_1(\tilde{X}_{s-})\, I(X^n_{s-} > b^X_{s-})\, [\tilde{X}^{i,c},\tilde{X}^{j,c}]_s \\
&+ \sum_{0<s\le t} I(X^n_{s-} > b^X_{s-})\Big(F_1(\tilde{X}_s) - F_1(\tilde{X}_{s-}) - \sum_{i=1}^{n} D_iF_1(\tilde{X}_{s-})\,\Delta\tilde{X}^i_s\Big) \\
=& \sum_{i=1}^{n}\int_0^t D_iF_2(\tilde{X}_{s-})\, I(X^n_{s-} > b^X_{s-})\, d\tilde{X}^i_s + \frac{1}{2}\sum_{i,j=1}^{n}\int_0^t D_{ij}F_2(\tilde{X}_{s-})\, I(X^n_{s-} > b^X_{s-})\, [\tilde{X}^{i,c},\tilde{X}^{j,c}]_s \\
&+ \sum_{0<s\le t} I(X^n_{s-} > b^X_{s-})\Big(F_2(\tilde{X}_s) - F_2(\tilde{X}_{s-}) - \sum_{i=1}^{n} D_iF_2(\tilde{X}_{s-})\,\Delta\tilde{X}^i_s\Big)
\end{aligned}
$$

$$
(3.28)\qquad \sum_{i=1}^{n}\int_0^t D_iF_1(\tilde{X}_{s-})\, I(X^n_{s-} = b^X_{s-})\, d\tilde{X}^i_s + \frac{1}{2}\sum_{i,j=1}^{n}\int_0^t D_{ij}F_1(\tilde{X}_{s-})\, I(X^n_{s-} = b^X_{s-})\, [\tilde{X}^{i,c},\tilde{X}^{j,c}]_s
$$

$$
\begin{aligned}
&+\sum_{0<s\le t} I(X^n_{s-}=b^X_{s-})\Big(F_1(\tilde X_s)-F_1(\tilde X_{s-})-\sum_{i=1}^n D_iF_1(\tilde X_{s-})\,\Delta\tilde X^i_s\Big)\\
&=\sum_{i=1}^n\int_0^t D_iF_2(\tilde X_{s-})\,I(X^n_{s-}=b^X_{s-})\,d\tilde X^i_s+\frac12\sum_{i,j=1}^n\int_0^t D_{ij}F_2(\tilde X_{s-})\,I(X^n_{s-}=b^X_{s-})\,[\tilde X^{i,c},\tilde X^{j,c}]_s\\
&+\sum_{0<s\le t} I(X^n_{s-}=b^X_{s-})\Big(F_2(\tilde X_s)-F_2(\tilde X_{s-})-\sum_{i=1}^n D_iF_2(\tilde X_{s-})\,\Delta\tilde X^i_s\Big)\ .
\end{aligned}
$$

Making use of (3.27) and (3.28) we see that F_1 in the first four integrals in (3.26) can be replaced by F_2 upon taking into account the four series over $0<s\le t$ appearing in (3.27) and (3.28). Adding and subtracting the same series over $0<s\le t$ we see that the first nine terms on the right-hand side of (3.26), together with the series added, assemble exactly the expression obtained by applying the Itô formula to $F_2(\tilde X_t)$. Since $F(\tilde X_t)=F_2(\tilde X_t)$ hence we see that the first ten terms obtained on the right-hand side of (3.26) equals the eleventh term which is the series subtracted. Recalling also the four series from (3.27) and (3.28) this shows that:

$$
\begin{aligned}
R_t=&\sum_{0<s\le t} I(X^n_{s-}>b^X_{s-})\Big(F_2(\tilde X_s)-F_2(\tilde X_{s-})-\sum_{i=1}^n D_iF_2(\tilde X_{s-})\,\Delta\tilde X^i_s\Big)\\
&-\sum_{0<s\le t} I(X^n_{s-}>b^X_{s-})\Big(F_1(\tilde X_s)-F_1(\tilde X_{s-})-\sum_{i=1}^n D_iF_1(\tilde X_{s-})\,\Delta\tilde X^i_s\Big)\\
&+\frac12\sum_{0<s\le t} I(X^n_{s-}=b^X_{s-})\Big(F_2(\tilde X_s)-F_2(\tilde X_{s-})-\sum_{i=1}^n D_iF_2(\tilde X_{s-})\,\Delta\tilde X^i_s\Big)\\
&-\frac12\sum_{0<s\le t} I(X^n_{s-}=b^X_{s-})\Big(F_1(\tilde X_s)-F_1(\tilde X_{s-})-\sum_{i=1}^n D_iF_1(\tilde X_{s-})\,\Delta\tilde X^i_s\Big)\\
&-\sum_{0<s\le t}\Big(F_2(\tilde X_s)-F_2(\tilde X_{s-})-\sum_{i=1}^n D_iF_2(\tilde X_{s-})\,\Delta\tilde X^i_s\Big)\\
&+\sum_{0<s\le t}\Big(F_1(\hat X_s)-F_1(\hat X_{s-})-\sum_{i=1}^n D_iF_1(\hat X_{s-})\,\Delta\hat X^i_s\Big)\\
&+\sum_{0<s\le t}\Big(F_1(\check X_s)-F_1(\check X_{s-})-\sum_{i=1}^n D_iF_1(\check X_{s-})\,\Delta\check X^i_s\Big)\\
&+\frac12\int_0^t D_nF_2(\check X_{s-})\,dJ_s(X)-\frac12\int_0^t D_nF_1(\hat X_{s-})\,dJ_s(X)\ .
\end{aligned}
\tag{3.29}
$$

From (3.25) we thus see that the proof of (3.16) reduces to verify the following identity:

$$
\begin{aligned}
R_t=\sum_{0<s\le t}\bigg(&F(X_s)-F(X_{s-})-\sum_{i=1}^n\Big(I(X^n_{s-}<b^X_{s-})\,D_iF_1(X_{s-})\,\Delta X^i_s\\
&+I(X^n_{s-}=b^X_{s-})\,\frac12\Big(D_iF_1(\tilde X_{s-})+D_iF_2(\tilde X_{s-})\Big)\,\Delta X^i_s\\
&+I(X^n_{s-}>b^X_{s-})\,D_iF_2(X_{s-})\,\Delta X^i_s\Big)\bigg)\ .
\end{aligned}
\tag{3.30}
$$

To this end it is helpful to note that:

$$
\begin{aligned}
(3.31)\quad & \frac{1}{2}\int_0^t D_nF_2(\check{X}_{s-})\,dJ_s(X) - \frac{1}{2}\int_0^t D_nF_1(\hat{X}_{s-})\,dJ_s(X) \\
&= \frac{1}{2}\sum_{0<s\le t} I(X_s^n > b_s^X,\, X_{s-}^n = b_{s-}^X)\Big(D_nF_2(\tilde{X}_{s-}) - D_nF_1(\tilde{X}_{s-})\Big)(X_s^n - b_s^X) \\
&\quad + \sum_{0<s\le t} I(X_s^n > b_s^X,\, X_{s-}^n < b_{s-}^X)\Big(D_nF_2(\tilde{X}_{s-}) - D_nF_1(X_{s-})\Big)(X_s^n - b_s^X) \\
&\quad - \sum_{0<s\le t} I(X_s^n < b_s^X,\, X_{s-}^n > b_{s-}^X)\Big(D_nF_2(X_{s-}) - D_nF_1(\tilde{X}_{s-})\Big)(X_s^n - b_s^X) \\
&\quad - \frac{1}{2}\sum_{0<s\le t} I(X_s^n < b_s^X,\, X_{s-}^n = b_{s-}^X)\Big(D_nF_2(\tilde{X}_{s-}) - D_nF_1(\tilde{X}_{s-})\Big)(X_s^n - b_s^X)\ .
\end{aligned}
$$

A lengthy but straightforward verification shows that the two sides in (3.30) coincide i.e. that the right-hand side of (3.29) equals the right-hand side of (3.30). This can be done by recalling that each series over $0 < s \le t$ in (3.29) and (3.31) is absolutely convergent so that all eleven of them appearing on the right-hand side of (3.29) can be combined into a single series of the finite sum of the eleven individual terms. Multiplying the sum by each of the indicators $I(X_s^n > b_s^X,\, X_{s-}^n = b_{s-}^X)$, $I(X_s^n = b_s^X,\, X_{s-}^n = b_{s-}^X)$, $I(X_s^n < b_s^X,\, X_{s-}^n = b_{s-}^X)$, $I(X_s^n \ge b_s^X,\, X_{s-}^n > b_{s-}^X)$, $I(X_s^n < b_s^X,\, X_{s-}^n > b_{s-}^X)$, $I(X_s^n > b_s^X,\, X_{s-}^n < b_{s-}^X)$, $I(X_s^n = b_s^X,\, X_{s-}^n < b_{s-}^X)$, $I(X_s^n < b_s^X,\, X_{s-}^n < b_{s-}^X)$ and comparing the result with the corresponding expression on the right-hand side of (3.30) it is seen that all eight of them coincide. This establishes the identity (3.30) and completes the proof of the theorem. □

Remark 3.3

It is evident that the contents of Remark 2.2 and Remark 2.3 carry over to the setting of Theorem 3.2 (or Theorem 3.1) without major change. By adding the corresponding jump terms to (2.29) and (2.32) one obtains their extension to general semimartingales (or semimartingales with jumps of bounded variation). We will omit the explicit expressions of these formulas.

4. The time-space maximum process

In this section we first apply the change-of variable formula (2.6) to a three-dimensional continuous semimartingale and then derive a version of the same formula under weaker conditions on the function. This version is useful in the study of free-boundary problems.

1. Let X be a diffusion process solving:

$$
dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dB_t \tag{4.1}
$$

in Itô's sense. The latter more precisely means that X satisfies:

$$
X_t = X_0 + \int_0^t \mu(r, X_r)\,dr + \int_0^t \sigma(r, X_r)\,dB_r \tag{4.2}
$$

for all $t \ge 0$ where μ and σ are locally bounded (continuous) functions for which the integrals in (4.2) are well-defined (the second being Itô's) so that X itself is a continuous semimartingale (the process B is a standard Brownian motion). To ensure that X is non-degenerate we will assume that $\sigma > 0$.

Associated with X we consider the maximum process S defined by:

$$S_t = \Big(\max_{0 \le r \le t} X_r \Big) \vee S_0 \ . \tag{4.3}$$

Then $((t, X_t, S_t))_{t \ge 0}$ is a continuous semimartingale taking values in $I\!R_+ \times E$ where we set $E = \{ (x,s) \in I\!R^2 \mid x \le s \}$.

2. Let $b : I\!R_+ \times I\!R \rightarrow I\!R$ be a continuous function such that the process b^X defined by $b_t^X = b(t, S_t)$ is a semimartingale. Setting:

$$C = \{ (t,x,s) \in I\!R_+ \times E \mid x > b(t,s) \} \tag{4.4}$$

$$D = \{ (t,x,s) \in I\!R_+ \times E \mid x < b(t,s) \} \tag{4.5}$$

suppose that a continuous function $F : I\!R_+ \times E \rightarrow I\!R$ is given such that:

$$F \ \text{ is } \ C^{1,2,1} \ \text{ on } \ \bar{C} \tag{4.6}$$

$$F \ \text{ is } \ C^{1,2,1} \ \text{ on } \ \bar{D} \tag{4.7}$$

in the sense explained following (2.3) and (2.4) above. [A slight notational change in the definition of the process $((t, X_t, S_t))_{t \ge 0}$ and the sets C and D in comparison with those given in Section 2 above is made to meet the notation used in [10] and related papers.]

Moreover, since $\sigma > 0$ it follows that:

$$P\big(X_r = b_r^X \big) = 0 \ \text{ for } \ r \in \langle 0, t] \tag{4.8}$$

so that under (4.6) and (4.7) the change-of-variable formula (2.6) takes the simpler form:

$$\begin{aligned} F(t, X_t, S_t) = F(0, X_0, S_0) &+ \int_0^t F_t(r, X_r, S_r)\, I(X_r \ne b_r^X)\, dr \\ &+ \int_0^t F_x(r, X_r, S_r)\, I(X_r \ne b_r^X)\, dX_r + \int_0^t F_s(r, X_r, S_r)\, I(X_r \ne b_r^X)\, dS_r \\ &+ \int_0^t F_{xx}(r, X_r, S_r)\, I(X_r \ne b_r^X)\, d\langle X, X \rangle_r \\ &+ \frac{1}{2} \int_0^t \Big(F_x(r, X_r+, S_r) - F_x(r, X_r-, S_r) \Big)\, d\ell_r^b(X) \end{aligned} \tag{4.9}$$

where $\ell_r^b(X)$ is the local time of X on the surface b given by:

$$\ell_r^b(X) = I\!P\!-\!\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^r I(-\varepsilon < X_u - b_u^X < \varepsilon)\ d\langle X - b^X, X - b^X \rangle_u \tag{4.10}$$

and $d\ell_r^b(X)$ in (4.9) refers to integration with respect to the continuous increasing function $r \mapsto \ell_r^b(X)$. [The appearance of X in $d\ell_r^b(X)$ is motivated by the fact that S_t is a functional of X .] Note also that using (4.1) the formula (4.9) can be rewritten as (4.22) below.

3. It turns out, however, that similarly to the case studied in Section 3 of [11] the conditions (4.6) and (4.7) are not always readily verified. The main example we have in mind (arising from the free-boundary problems mentioned above) is:

$$F(t,x,s) = E_{t,x,s}\big(G(t+\tau_D, X_{t+\tau_D}, S_{t+\tau_D})\big) \tag{4.11}$$

where $(X_t,S_t)=(x,s)$ under $P_{t,x,s}$, an admissible function G is given and fixed, and:

$$\tau_D = \inf\{\, r>0 \mid (t+r, X_{t+r}, S_{t+r}) \in D \,\} \ . \tag{4.12}$$

Then one directly obtains the 'interior condition' (4.13) by standard means while the 'closure condition' (4.6) is harder to verify at b since (unless we know a priori that $r \mapsto b(r,s)$ is Lipschitz continuous or even differentiable) both F_t and F_{xx} may in principle diverge when b is approached from the interior of C .

Motivated by applications in free-boundary problems we will now present a version of the formula (4.9) where (4.6) and (4.7) are replaced by the conditions:

$$F \ \text{ is } \ C^{1,2,1} \ \text{ on } \ C \tag{4.13}$$

$$F \ \text{ is } \ C^{1,2,1} \ \text{ on } \ D \ . \tag{4.14}$$

The rationale behind this version is the same as in [11]. Given that one has some basic control over F_x at b (in free-boundary problems mentioned above such a control is provided by the principle of smooth fit) even if F_t is formally to diverge when the boundary b is approached from the interior of C , this deficiency is counterbalanced by a similar behaviour of F_{xx} through the infinitesimal generator of X , and consequently the first integral in (4.22) below is still well-defined and finite.

4. Given a subset A of $I\!R_+ \times E$ and a function $f: A \rightarrow I\!R$ we say that f is *locally bounded* on A (in $I\!R_+ \times E$) if for each a in $\bar{A}$ there is an open set U in $I\!R_+ \times E$ containing a such that f restricted to $A \cap U$ is bounded. Note that f is locally bounded on A if and only if for each compact set K in $I\!R_+ \times E$ the restriction of f to $A \cap K \neq \emptyset$ is bounded. Given a function $g:[0,t] \rightarrow I\!R$ of bounded variation we let $V(g)(t)$ denote the total variation of g on $[0,t]$.

To grasp the meaning of the condition (4.19) below in the case of F from (4.11) above, letting $I\!L_X = \partial/\partial t + \mu\, \partial/\partial x + (\sigma^2/2)\, \partial^2/\partial x^2$ denote the infinitesimal generator of X , recall that the infinitesimal generator $I\!L$ of $((t,X_t,S_t))_{t \ge 0}$ can formally be described as follows (cf. [10]):

$$I\!L = I\!L_X \ \text{ in } \ x<s \tag{4.15}$$

$$\frac{\partial}{\partial s} = 0 \ \text{ at } \ x=s \ . \tag{4.16}$$

Denoting $C_s = \{\, (t,x) \mid (t,x,s) \in C \,\}$ and $D_s = \{\, (t,x) \mid (t,x,s) \in D \,\}$ hence we see that:

$$I\!L F = 0 \ \text{ in } \ C_s \tag{4.17}$$

$$I\!L F = I\!L G \ \text{ in } \ D_s \ . \tag{4.18}$$

This shows that $I\!L F$ is locally bounded on $C_s \cup D_s$ as soon as $I\!L G$ is so on D_s . The latter condition (in free-boundary problems) is easily verified since G is given explicitly.

The main result of the present section may now be stated as follows (see also Remark 4.2 below for further sufficient conditions).

Theorem 4.1

Let X be a diffusion process solving (4.1) in Itô's sense, let $b : I\!R_+ \times I\!R \to I\!R$ be a continuous function such that the process b^X defined by $b_t^X = b(t, S_t)$ is a semimartingale, and let $F : I\!R_+ \times E \to I\!R$ be a continuous function satisfying (4.13) and (4.14) above.

If the following conditions are satisfied:

$$\text{(4.19)} \qquad \big(F_t + \mu F_x + (\sigma^2/2) F_{xx}\big)(\,\cdot\,,\,\cdot\,,s) \;\; \textit{is locally bounded on} \;\; C_s \cup D_s$$

$$\text{(4.20)} \qquad F_x(\,\cdot\,, b(\,\cdot\,,s) \pm \varepsilon, s) \to F_x(\,\cdot\,, b(\,\cdot\,,s) \pm, s) \;\; \textit{uniformly on} \;\; [0,t] \;\; \textit{as} \;\; \varepsilon \downarrow 0$$

$$\text{(4.21)} \qquad \sup_{0<\varepsilon<\delta} V(F(\,\cdot\,, b(\,\cdot\,,s) \pm \varepsilon, s))(t) < \infty \;\; \textit{for some} \;\; \delta > 0$$

for each s given and fixed, then the following change-of-variable formula holds:

$$\begin{aligned}
\text{(4.22)} \qquad F(t, X_t, S_t) = F(0, X_0, S_0) &+ \int_0^t \big(F_t + \mu F_x + (\sigma^2/2) F_{xx}\big)(r, X_r, S_r)\; I(X_r \neq b_r^X)\; dr \\
&+ \int_0^t (\sigma F_x)(r, X_r, S_r)\; I(X_r \neq b_r^X)\; dB_r \\
&+ \int_0^t F_s(r, X_r, S_r)\; I(X_r \neq b_r^X\,,\; X_r = S_r)\; dS_r \\
&+ \frac{1}{2} \int_0^t \Big(F_x(r, X_r+, S_r) - F_x(r, X_r-, S_r)\Big)\; I(X_r = b_r^X)\; d\ell_r^b(X)
\end{aligned}$$

where $\ell_r^b(X)$ is the local time of X at the surface b given by (4.10) above, and $d\ell_r^b(X)$ refers to integration with respect to the continuous increasing function $r \mapsto \ell_r^b(X)$.

Proof. The key observation is that off the diagonal $x = s$ in E the process (t, X_t, S_t) can be identified with a process (t, X_t) and the surface process $b(t, S_t)$ can be identified with a curve $b(t)$. By slightly extending the 'two-map argument' given in Remark 4.2 of [6] the previous observation can be embedded rigorously in a well-defined mathematical setting. In this setting the problem becomes equivalent to the problem treated in Theorem 3.1 of [11]. Applying the same method of proof, upon making use of (2.16) and (2.19) above, and relying upon the properties of the local time and Helly's selection theorem, it is seen that the conditions (3.26)-(3.28) in Theorem 3.1 of [11] become the conditions (4.19)-(4.21) above. As this verification is lengthy, but in principle the same, further details will be omitted (for more details see [11]). □

Remark 4.2

It is evident that all of the number of *sufficient conditions* discussed in [11], which are either to imply (4.19)-(4.21) or could be used instead, can easily be translated into the present setting. We will state explicitly only one set of these conditions. *Assume that F satisfies (4.13) and (4.14)* *above. If (4.19) is satisfied and for each* s *given and fixed we have*:

(4.23) $x \mapsto F(r,x,s)$ *is convex or concave on* $[b(r,s)-\delta, b(r,s)]$ *and convex or concave on* $[b(r,s), b(r,s)+\delta]$ *for each* $r \in [0,t]$ *with some* $\delta > 0$

(4.24) $r \mapsto F_x(r, b(r,s)\pm, s)$ *is continuous on* $[0,t]$ *with values in* $\mathbb{R}$

then both (4.20) and (4.21) hold. This shows that *(4.23) and (4.24) imply (4.22) when (4.19) holds.* The condition (4.23) can further be relaxed to the form where:

$$F_{xx}(\,\cdot\,,\,\cdot\,,s) = G_1(\,\cdot\,,\,\cdot\,,s) + G_2(\,\cdot\,,\,\cdot\,,s) \quad \text{on} \quad C_s \cup D_s \tag{4.25}$$

where $G_1(\,\cdot\,,\,\cdot\,,s)$ is non-negative (non-positive) and $G_2(\,\cdot\,,\,\cdot\,,s)$ is continuous on $\bar{C}_s$ and $\bar{D}_s$ for each s given and fixed. Thus, if (4.24) and (4.25) hold, then both (4.20) and (4.21) hold implying also (4.22) when (4.19) holds.

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