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A True Buyer's Risk and Classification of Options

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Accepting the classic Black-Scholes model for a financial market consisting of a riskless bank account $(B_t)_{0 \le t \le T}$ and a risky stock $(S_t)_{0 \le t \le T}$, and considering the problem of pricing an option of American type associated with the reward process $f = (f_t)_{0 \le t \le T}$, we address and discuss the question of the option risk. Motivated by the basic facts of the option pricing theory in complete markets reviewed below, and taking the formal fair-game standpoint of a "true buyer", we are naturally led to identify *the option risk* (*of a first kind*) with the distribution law of the rational payment under the equivalent martingale measure:

$$\mathcal{R}(f) = Law\left(\frac{f_{\tau_*}}{B_{\tau_*}} \mid \widetilde{P}\right)$$

where τ_* is the optimal stopping time for the buyer to exercise the option, and $\widetilde{P}$ is the equivalent martingale measure. Two options are then said to be *equivalent* if they have the same risk. This is an equivalence relation, and the set of all options splits into equivalence classes, two options being in the same class if and only if they are equivalent. Since from the formal fair-game standpoint of a "true buyer" two options belonging to the same equivalence class may be thought of as the same, this relation offers an exact mathematical tool for comparing different options and classifying them. A more realistic description of the option risk must also account for a possible random displacement of the appreciation rate μ around the interest rate r according to a distribution function F which is subject to statistical observations of each specific stock. Given that μ takes its value independently from a Wiener process driving the stock price, *the option risk* (*of a second kind*) is obtained by replacing $\widetilde{P}$ in the definition above by the actual probability measure P_F. When $F \sim \delta_{\{r\}}$ the two definitions coincide. All these facts extend to options of European type with a reward variable f_T, if in the preceding definitions one replaces τ_* with T. A natural problem is then formulated as follows: Given any probability measure ϱ on $\mathbf{R}_+$, construct an option with the option risk equal to ϱ. This problem has a simple solution if the option is of European type. If the option is of American type this problem is referred to as *the optimal Skorokhod-embedding problem* (see [10]).

1. Fair price

There is a large number of financial options, but there is no exact concept developed on how to compare them and classify. In this note we present a simple idea of how this could be done. Our considerations are devoted to *complete* markets.

Considering the classic Black-Scholes model for a financial market, and starting from basic facts upon which the fair price of an option is defined, we introduce and describe a natural concept

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of *risk* which may be associated with the option. All options can then be compared and classified according to the risk they incur, and from a formal point of view two options incurring the same risk are the same. A natural question is then to ask: Given a risk, can one construct an option with this risk? While in the context of European options this question has a simple answer (Section 3), in the context of American options such a problem is more sophisticated (see [10]).

We elaborate our presentation by considering options of American type (Sections 1-2). These considerations are then easily extended to options of European type (Section 3).

1. Consider *the Black-Scholes model* for a financial market consisting of a *riskless* bank account with value $B = (B_t)_{t \geq 0}$ and a *risky* stock with value $S = (S_t)_{t \geq 0}$. The equations which govern B and S are respectively given by:

$$dB_t = rB_t \, dt \tag{1.1}$$

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t \tag{1.2}$$

where $r > 0$ (the *interest rate*), $\mu \in I\!R$ (the *appreciation rate*), $\sigma > 0$ (the *volatility coefficient*) and $W = (W_t)_{t \geq 0}$ is a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$.

The bank-account value is deterministic, and is given by:

$$B_t = B_0 \, e^{rt} \tag{1.3}$$

where $B_0 > 0$. The stock value is random, and is given as a *geometric* Brownian motion:

$$S_t = S_0 \exp\Big(\sigma W_t + \big(\mu - \sigma^2/2\big)t\Big) \tag{1.4}$$

where $S_0 > 0$. By convention we assume that $B_0 = 1$ in (1.3).

In order to reach the central point of our exposition as simply as possible, we will drop some regularity assumptions (on measurability, integrability, etc.) in the sequel. Such details, if not self-evident, may be found in standard references on the subject quoted below.

2. **American options.** Given a *reward process* $f = (f_t)_{0 \leq t \leq T}$, consider *an option of American type* as a contract between the seller and a buyer which entitles the buyer to exercise the option at any (stopping) time $\tau \in [0, T]$ and receive the payment f_τ from the seller (if evaluated at time 0). After selling the option at a price x , the seller has at disposal *self-financing* strategies $\pi = (\beta_t, \gamma_t)$ with (non-negative) *consumption* $C = (C_t)$ which, after starting with $X_0^{\pi,c} = x$, at time t brings him the (non-negative) value:

$$\begin{aligned} X_t^{\pi,c} &= \beta_t B_t + \gamma_t S_t \\ &= x + \int_0^t \beta_r \, dB_r + \int_0^t \gamma_r \, dS_r - C_t \quad \text{(self-financing with consumption)} \end{aligned} \tag{1.5}$$

if evaluated at time 0 , or equivalently, the discounted (real) value:

$$Y_t^{\pi,c} = \frac{X_t^{\pi,c}}{B_t} = \beta_t + \gamma_t \frac{S_t}{B_t} \tag{1.6}$$

if evaluated at time t .

The central questions about such an option contract are:

(A) What is the "fair" price x ? ("Fair" refers to both the seller and the buyer.)

(B) What is the optimal strategy $\pi^* = (\beta_t^*, \gamma_t^*)$ with consumption $C^* = (C_t^*)$, and what is the optimal exercise time τ_* ?

The general option pricing theory ([1]-[6], [8]-[9], [12]-[13]) gives the following answers to these questions.

3. **Fair price.** A self-financing strategy $\pi = (\beta_t, \gamma_t)$ with consumption $C = (C_t)$ is called a *hedge* (with respect to x and f given), if $X_0^{\pi,c} = x$ and we have:

$$X_t^{\pi,c} \geq f_t \quad P\text{-a.s.} \tag{1.7}$$

for all $t \in [0,T]$. The minimal x , denoted by $V^*(f)$, for which there exists a hedge (with respect to f) is called *the fair price* of the option.

4. In order to determine the fair price $V^*(f)$, and answer the questions (A) and (B) above, the following facts and observations are shown essential.

First, the requirement (1.7) is invariant under a measure change from P to $\widetilde{P}$, as long as $\widetilde{P}$ and P are *locally equivalent*, which means that $\widetilde{P}$ and P have the same null-sets in $\mathcal{F}_t := \sigma(W_s \mid s \leq t)$ for $t \leq T$. By *the Girsanov theorem* there exists such a measure $\widetilde{P}$ on $\mathcal{F} := \sigma(\cup_{t\geq 0}\mathcal{F}_t)$ satisfying yet another good property described below. The measure $\widetilde{P}$ is determined by its values on $\mathcal{F}_t$ through the identity:

$$d\widetilde{P} = \exp\left(-\frac{\mu - r}{\sigma} W_t - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 t \right) dP \tag{1.8}$$

whenever $t \geq 0$. Under $\widetilde{P}$ the process $\widetilde{W_t} = W_t + ((\mu - r)/\sigma)t$ is a standard Brownian motion, and the process (1.6) admits the following supermartingale representation:

$$Y_t^{\pi,c} = Y_0^{\pi,c} + \int_0^t \sigma\gamma_r \frac{S_r}{B_r}\, d\widetilde{W}_r - \int_0^t \frac{dC_r}{B_r} \tag{1.9}$$

where the first integral defines a local martingale and the second integral defines an increasing process. For this reason the measure $\widetilde{P}$ is called an *equivalent martingale measure*. (In fact, such a measure is unique. Its existence guarantees that the market is *arbitrage-free*, its uniqueness is expressed by saying that the market is *complete*.)

Second, since we want a minimal x for which there is a hedge, it is clear that we shall look for a hedge π with consumption C for which the values $X_t^{\pi,c}$ satisfying (1.7) (or the values $Y_t^{\pi,c}$ satisfying $Y_t^{\pi,c} \geq f_t/B_t$ $\widetilde{P}$-a.s. for all $t \in [0,T]$) are "minimal" in some sense.

Third, it is well-known that the smallest supermartingale (under $\widetilde{P}$) which dominates f_t/B_t $\widetilde{P}$-a.s. on $[0,T]$ is given as *the Snell envelope*:

$$Y_t^* = \operatorname*{ess\,sup}_{t\leq\tau\leq T} \widetilde{E}\left(\frac{f_\tau}{B_\tau} \;\Big|\; \mathcal{F}_t \right) \tag{1.10}$$

for $0 \leq t \leq T$, where the (essential) supremum is taken over all stopping times τ taking values in $[t, T]$. By *the Doob-Meyer decomposition* this further splits into:

$$Y_t^* = Y_0^* + M_t^* - A_t^* \tag{1.11}$$

where $M^* = (M_t^*)$ is a local martingale (under $\widetilde{P}$), and $A^* = (A_t^*)$ is an increasing process, both started at zero. Moreover, by *the Itô-Clark theorem* we have:

$$M_t^* = \int_0^t \alpha_r \, d\widetilde{W}_r \tag{1.12}$$

for some process $\alpha = (\alpha_t)$. Finally, the Snell envelope process (Y_t^*) is generally known to be a martingale until it hits (f_t / B_t) , and thus during this period $A_t^* \equiv 0$ $\widetilde{P}$-a.s.

5. **Rational performance.** From the arguments just presented it is evident that the optimal $\pi^* = (\beta_t^*, \gamma_t^*)$, $C^* = (C_t^*)$ and τ_* are obtained by identifying:

$$\int_0^t \sigma \gamma_r \frac{S_r}{B_r} \, d\widetilde{W}_r = M_t^* \tag{1.13}$$

$$\int_0^t \frac{dC_r}{B_r} = A_t^* \ . \tag{1.14}$$

This gives the following explicit answers to the questions (A) and (B) stated above:

$$V^*(f) = \sup_{0 \le \tau \le T} \widetilde{E} \left(\frac{f_\tau}{B_\tau} \right) \tag{1.15}$$

$$\beta_t^* = Y_t^* - \frac{\alpha_t}{\sigma} \tag{1.16}$$

$$\gamma_t^* = \frac{\alpha_t}{\sigma} \frac{B_t}{S_t} \tag{1.17}$$

$$C_t^* = \int_0^t B_r \, dA_r^* \tag{1.18}$$

$$\tau_* = \inf \left\{ t > 0 \;\middle|\; Y_t^* = \frac{f_t}{B_t} \right\} . \tag{1.19}$$

Moreover, we have:

(1.20) The stopping time τ_* is optimal for the problem (1.15). It is pointwise the smallest possible with this property.

From (1.19) and the martingale property of Snell's envelope noted above, we see that if the buyer acts rationally and exercises the option at τ_* , there will be no consumption for the seller, that is:

$$C_t^* \equiv 0 \quad \text{for} \quad t \in [0, \tau_*] \quad \widetilde{P}\text{-a.s.} \tag{1.21}$$

Thus the fair price $V^*(f)$ is indeed "fair" from this point of view as well.

Finally, since $Law(S(\mu) \mid \widetilde{P}) = Law(S(r) \mid P)$, it follows from (1.15) that the fair price

$V^*(f)$ does not depend on μ, neither does the optimal strategy $\pi^* = (\beta_t^*, \gamma_t^*)$ with consumption $C^* = (C_t^*)$, nor does the optimal stopping time τ_*. This property is not surprising, since the requirement (1.7) is invariant under an equivalent measure change.

6. It should be realized that the process $Y^* = (Y_t^*)$ (and therefore M^* and A^* too) is computable (at least in principle) and known a priori before the option contract has been signed (in much the same way as the fair price $V^*(f)$ itself). This is important since the optimal strategy $\pi^* = (\beta_t^*, \gamma_t^*)$ with consumption $C^* = (C_t^*)$ is expressed in terms of Y^*, and its existence in an explicit form should show the seller how to act in order to keep up with the demand of the reward process f, as well as to provide both the seller and the buyer with a guarantee (and needed comfort) that the option contract can be realized at the fair price $V^*(f)$.

The problem of an explicit computation of the Snell envelope $Y^* = (Y_t^*)$ is closely linked to the problem of computing the fair price $V^*(f)$. Solving the optimal stopping problem (1.15) in a Markovian setting (which is to be found in each concrete case of the reward process f), we get the fair price $V^*(f)$ as a function of the initial position of the underlying Markov process. Composing then this function with the Markov process itself, we obtain (Y_t^*) as the smallest supermartingale which dominates the gain process (f_t/B_t). Thus the fact (1.20) is in accordance with the general optimal stopping theory for Markov processes (see [14]).

7. These considerations show that *the problem of design of an option* is closely related to the fact that the reward process f should be chosen in such a way that the optimal stopping problem (1.15) admits an explicit solution. Some experience of work with optimal stopping problems shows that this is very difficult to achieve, the main two obstacles in (1.15) being the finite horizon T and the presence of discounting (B_t). Moreover, criteria given in the literature for the choice of a reward process f are very often based rather on a subjective view than on an exact mathematical concept. Although such an approach may be well suited to the spirit of option trading, we believe that the general theory should offer exact criteria.

Our main aim in the next section is to point out and describe a simple concept of *risk* which may be associated with the option and upon which the choice of a reward process can be based. This concept offers an exact criterion for the choice of a reward process and leads to a *classification* of all options according to the risk they incur. It should be noted, however, that this criterion does not determine the reward process uniquely, but rather gives an admissible class of such processes. Other criteria may then be used (solvability of the optimal stopping problem (1.15), path properties of f providing comfort, etc.) to determine a reward process from this class. From a formal point of view, however, any other reward process from the admissible class generates an option with the same risk, which then may be thought of as the same option.

2. The concept of the option risk

Having computed (at least in principle) the fair price $V^*(f)$, the optimal strategy $\pi^* = (\beta_t^*, \gamma_t^*)$ with consumption $C^* = (C_t^*)$, and the optimal stopping time τ_*, the option is ready for use. A new natural question which is missing in the analysis above and which we want to address now may be stated as follows:

(C) What is the risk? (“Risk” refers to a “true buyer” before concluding the option contract, and afterwards, until it is exercised. By "true buyer" we mean a buyer who has no ability or desire to sell the option. Thus every "true buyer" will exercise the option according to the rational performance.)

We note that despite its appealing simplicity and necessity, this question has not been considered in such a form before. Perhaps one of the main reasons for this is that in complete markets the seller and the buyer are usually identified, i.e. a buyer "can always protect himself" by selling the same option to another buyer. We find this argument somewhat dubious, as any buyer could "protect himself" even better by simply not buying the option in the first place.

Before making any attempts to answer this question, we first want to make it clear that the bank-account value B and the stock value S (with some μ which we do not have to know a priori) are indeed given by the equations (1.1) and (1.2) respectively, and we will not be interested in answering the question as what a risk would be if some of these assumptions fail. Such a risk certainly exists, but its description will not be the subject of our discussion. Thus, we want to answer the question on what the risk is if we know that the given B and S are authentic. In other words, the risk we want to describe is a risk within the given model, and not the risk of having a model which is not genuine, or possibly a combination of these two.

1. In order to see where the risk is hidden, assume that the seller and a buyer sign the option contract at time $t = 0$. Accordingly, the buyer pays the fair price $V^*(f)$ to the seller, and the option performance starts. The seller has at disposal self-financing strategies $\pi = (\beta_t, \gamma_t)$ with consumption $C = (C_t)$, and he rationally attempts to hedge the reward process $f = (f_t)_{0 \leq t \leq T}$ given by the option contract. In fact, in our ideal world, the seller has not other choice but to apply the optimal strategy $\pi^* = (\beta_t^*, \gamma_t^*)$ with consumption $C^* = (C_t^*)$ given by (1.16)-(1.18), while the buyer has no other choice either but to exercise the option contract at the optimal stopping time τ_* given by (1.19). From the results of general theory exposed above, it is clear that if either of them does not apply the optimal tools just described, the other can do so and gain more in the expectation. Such cases are not of interest for general theory, however, since we expect a rational behaviour from both parties as a basic hypothesis. Therefore suppose that both the seller and the buyer act rationally and follow the optimal tools they have at disposal. Thus, the seller applies π^* with C^* and the buyer exercises the option contract at τ_*. At the time of exercise τ_*, the seller pays the value $Y_{\tau_*}^{\pi^*, c^*}$ to the buyer which is obtained by applying π^* with C^* and which by (1.19) equals the discounted reward value f_{τ_*}/B_{τ_*}. The consumption to the seller up to τ_* is identically zero by (1.21). Moreover, the seller is not exposed to any risk, as the fair price $V^*(f)$ by its definition enables him to pay f_{τ_*}/B_{τ_*} to the buyer as stipulated by the option contract. Since by (1.10) we have $Y_0^{\pi^*, c^*} = V^*(f)$, we see by (1.20) that in the expectation under $\widetilde{P}$ the buyer receives as much as he paid for the option contract through the fair price $V^*(f)$. The option performance is "fair" under $\widetilde{P}$, since the buyer neither gains nor loses anything extra (see *Fig. 1* above). However, it is necessary to observe that *this statement is true only in the expectation* (under the equivalent martingale measure).

2. Here we come up to the central point of our discussion. Imagine two players A and B playing a random game where the outcome is a random variable R with zero mean. Positive values of R correspond to the wealth taken from the player B and given to the player A, while

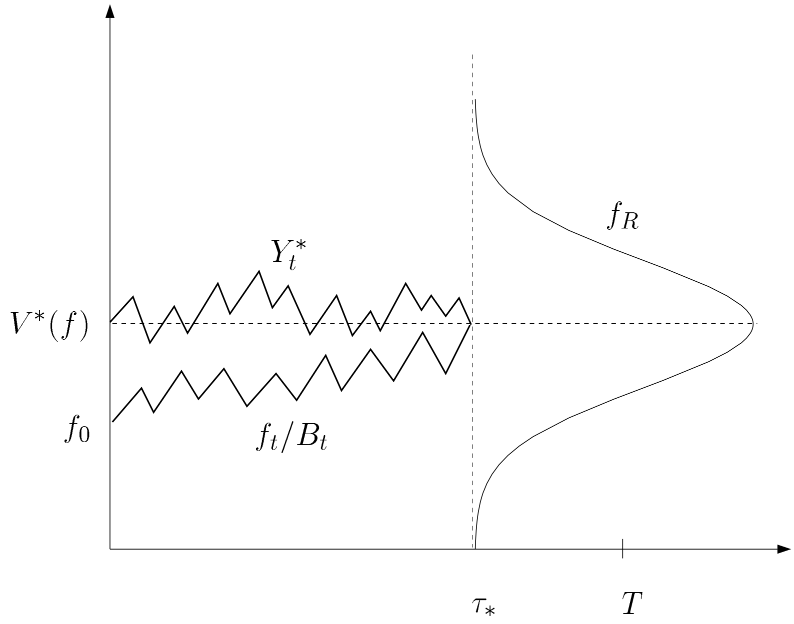

Figure 1: A schematic drawing of the rational performance at the option of American type as described in Paragraph 1 below. The rational payment $R = Y^*_{\tau_*} = f_{\tau_*}/B_{\tau_*}$ is assumed to have a density function f_R for convenience. The expectation of the rational payment R equals the fair price $V^*(f)$, and the option performance is "fair". Our understanding is that the size and the shape of the displacement of the rational payment R around its mean $V^*(f)$ determine *the option risk*.

negative values of R correspond to the wealth taken from the player A and given to the player B . First we may ask: Is the game "fair"? Second: What is the "risk"? These are exactly the questions we are trying to answer above.

Consider a special case where R takes two values -1 and 1 with probability $1/2$. Ask the same question: Is the game fair? There is no doubt that most of people would say that this game is fair. Why? Possibly because neither of the players can gain something by swapping the roles in the game if such a right is given before the game has been started. Now ask the second question: What is the "risk" which either or both players take by playing such a game? If the number 1 above represents a unit of money, then most of people would say that such a game is not "risky" (we assume that the game is played only once). However, if you replace number 1 above by the number 1000 , then most of people would say that this game is very "risky" and would not play it. It shows that in our perception of the word "risk" we do not think only of "chance" (which in this case corresponds to the probabilities $1/2$) but also of consequences (which corresponds to the values 1 or 1000) and potential (which corresponds to the total wealth of the players). As a consequence we obtain that the concept of risk cannot be solely expressed in terms of the *entropy* of R . (The concept of entropy is recalled in Paragraph 6 below.)

The special cases just considered are symmetrical, and neither of the players can gain something by swapping the roles in the game. Consider now a typical asymmetrical case where R takes two values -999 and 1 with probabilities $1/1000$ and $999/1000$ respectively. Ask the same questions: Is the game fair, and what is the risk? While most of people would say that this game is fair (although not symmetric and the players may like to swap the roles), the question of risk seems a bit more tricky. Imagine the game is played once. What could the players expect from the game? Player A wins the game 999 times out of 1000 plays on the average, that is, “almost surely” if the game is played only once. However, even so, in the win he gains only 1 unit of money. Player B wins the game only once out of 1000 plays. However, when winning the game he gains the fortune of 999 units of money. Is it clear who is in a more risky situation? We believe that most of people would agree that this is the player A. Thus, here we have a case where the risk is not the same for both players. So, what is the risk, and how to define it?

It is now easy to continue the list of examples by considering more and more general (asymmetrical) distributions (with densities, etc.), and soon one can realize that these matters get rather complicated. Instead of attempting to give a general answer to these questions, we shall confine ourselves to the problem of option pricing considered above, and note that whatever the risk is, it is contained in the distribution law of R. In other words, knowing the distribution law of R both players can read from it everything about the risk they want to know. For this reason *we shall identify the risk with the distribution law*. It would be perhaps more precise to call this risk by *the risk distribution*, or *the risk law*, and we shall do it now and then as well.

3. **Risk.** Motivated by these considerations we return back to the problem of option pricing and define *the risk variable* as the value of the payment (obtained by a rational behaviour):

$$R := \frac{f_{\tau_*}}{B_{\tau_*}} \tag{2.1}$$

which also equals the optimal value $Y_{\tau_*}^{\pi^*,c^*}$ with τ_* being the optimal stopping time. Although the expectation of R equals the fair price $V^*(f)$, and the option “game” is “fair”:

$$\widetilde{E}\big(R\big) = V^*(f) \tag{2.2}$$

it is clear that R is a random variable which generally may in its values deviate from its expectation. For instance, it is not the same if R takes values 0 and 2000 with probability $1/2$, or if R takes values 999 and 1001 with probability $1/2$. In both cases the fair price is 1000, while clearly the risk in the first case is much higher for the buyer. By simple observations presented above, we are naturally led to formulate the following definition.

Definition 2.1 (A true buyer’s risk of the American option)
The risk of the American option is the distribution law of the rational payment R under the equivalent martingale measure $\widetilde{P}$. In other words, the option risk $\mathcal{R}(f)$ is defined to be:

$$\mathcal{R}(f) = Law(R \,|\, \widetilde{P}) \tag{2.3}$$

where R is given by (2.1) with τ_* from (1.19).

Remark 2.2 Observe that the preceding definition is made from the standpoint that the rational performance defines a fair game for the buyer. The actual probability measure P may be, and usually is, different from the equivalent martingale measure $\widetilde{P}$, and in each such a case the rational performance will typically be either more or less favourable to the buyer. For example, consider $f_t = (S_t - K)^+$ for $0 \le t \le T$ where $K > 0$. Then the case $\mu > r$ is more favourable to the buyer as he receives $E(f_{\tau_*}/B_{\tau_*})$ which is strictly larger than the fair price $\widetilde{E}(f_{\tau_*}/B_{\tau_*})$ that he pays for the option contract, and similarly the case $\mu < r$ is less favourable to the buyer for exactly the same reason. *In neither of these cases the rational option performance leads to a fair game for the buyer.* However, as one would expect that there should be equal chance for μ to be above or below r, the hypothesis made in the definition that the rational payment R should be considered under $\widetilde{P}$ is a reasonable approximation of *all* risk exposures associated with the rational option performance that the buyer faces. This idealised approach is very much in the spirit of the Black-Scholes model (where μ may be unknown) as the option risk so defined does not depend on μ in any way.

For these reasons the option risk of Definition 2.1 could also be called *the American option risk of a first kind.* A less idealised and somewhat more realistic hypothesis could be made by extending the Black-Scholes model and assuming that μ takes an arbitrary real value at time $t = 0$ (or later) independently of $W = (W_t)_{t \geq 0}$ in accordance with some distribution function F (which is subject to observation and statistical estimates of each specific stock). Assuming that the random variable μ is defined on $(\Omega', \mathcal{F}', P')$ and that W is defined on $(\Omega'', \mathcal{F}'', P'')$, we could realise the stock price process $S = (S_t)_{t \geq 0}$ on $(\Omega, \mathcal{F}, P_F) := (\Omega' \times \Omega'', \mathcal{F}' \times \mathcal{F}'', P' \times P'')$. A natural hypothesis would be that μ is centered around r, i.e. that $E'(\mu) = r$ (although real data seem to indicate that $E'(\mu) > r$ in most cases), and possibly that $E_F(f_{\tau_*}/B_{\tau_*}) = \widetilde{E}(f_{\tau_*}/B_{\tau_*})$. (The latter assumption is very restrictive. In the case $f_t = (S_t - K)^+$ for $0 \leq t \leq T$ it is possible to verify using Jensen's inequality that under $E'(\mu) \geq r$ we must have $E_F(f_{\tau_*}/B_{\tau_*}) \geq \widetilde{E}(f_{\tau_*}/B_{\tau_*})$ and that this inequality is always strict unless $F \sim \delta_{\{r\}}$. This fact discerns a somewhat unexpected but rather desirable property of the option.) In this case it is reasonable to extend (2.3) by setting $\mathcal{R}_F(f) = Law(R \,|\, P_F)$ and call it *the American option risk of a second kind.* Observe that this definition of the risk depends on the distribution F of μ given a priori. Clearly, when $F = \delta_r$ then the two definitions coincide. More involved descriptions of the appreciation rates μ, being also functions of t and s, will lead to more sophisticated descriptions of the option risk. The knowledge of μ could also be continuously updated as the time passes by, which in turn will influence the risk and make it a function of time as well. We shall not go here any deeper into these more complicated considerations, but will mainly concentrate on the option risk of a first kind.

Thus, to conclude, in order to obtain a more precise description of the option risk, one should additionally to (2.3) also account for a displacement of μ around r. This can be illustrated through Figure 1 above by viewing the density function f_R as a random outcome of P corresponding to some μ, and then averaging over all such μ's according to a probability distribution which accounts for all possible displacements of μ around r that are typical for the given stock.

The option is said to be *risk-free* if the risk variable R is degenerated at a point, that is, if $R = c$ $\widetilde{P}$-a.s. for some positive c. Note, however, that any such c must then be equal to the fair price $V^*(f)$. Thus, the option is risk-free if the option risk is degenerated at the fair price.

4. *The mean-square risk of the option.* While it was clear how to define a risk-free option, it is less obvious how to define a *risky* option. This can be achieved, however, by looking at various functionals of the option risk. For instance, the *mean-square risk of the option* may be defined as the variance of the option risk:

$$(2.4) \qquad \widetilde{Var}\big(\mathcal{R}(f)\big) = \widetilde{E}\Big(R - \widetilde{E}(R)\Big)^2 = \widetilde{E}\left(\frac{f_{\tau_*}}{B_{\tau_*}} - V_*(f)\right)^2 .$$

The option may now be called "risky" (in the mean-square sense), if the mean-square risk (2.4) is "large". Note that the mean-square risk quantifies the risk of a big loss and a big gain at the same time. It treats both players equally, and no information from it can be obtained on the size of the "individual risks" indicated in the example above. We believe that it is clear how to extend this concept further by replacing the square function $(r,e) \mapsto (r-e)^2$ in (2.4) with other functions of interest (which can measure the size of the displacement of R from its mean). In this way we can obtain other (more sophisticated) risk functionals of the option risk according to which the option itself may then be called "risky". We shall omit further details in this direction.

Remark 2.3 A partial answer to the question above as how to define a *risky* option can also be given using the well-developed concept of *utility functions* (see e.g. [5] pp.1-37). We recall that an individual's preferences admit an expected utility representation if there exists a utility function $U = U(c)$ such that the random consumption C_1 is preferred to a random consumption C_2 if and only if $E(U(C_1)) \geq E(U(C_2))$. Individuals who prefer "more" wealth to "less" have *increasing* utility functions. *Risk averse* individuals are characterized by *concave* utility functions. The Arrow-Pratt coefficients of *absolute* and *relative* risk aversion are defined respectively as $-U''(c)/U'(c)$ and $-c\,U''(c)/U'(c)$. These coefficients measure the individual's attitude toward risk.

In this context the concept of *stochastic dominance* (see e.g. [5] pp.39-57) arises as a useful tool for comparing the *riskiness* of risky assets. As an illustration we shall note that the following result of Rothschild and Stieglitz (see e.g. [5] p.49) directly applies to the problem of *risky* option addressed above. Let R_1 and R_2 be two rational payments corresponding to two American options with the same fair price. Then the following statements are equivalent: (i) All risk averse individuals (buyers) having utility functions whose first derivatives are continuous except on a countable set prefer R_1 to R_2 ; (ii) $\widetilde{E}(R_1) = \widetilde{E}(R_2)$ and $\int_0^z (F_{R_1}(x) - F_{R_2}(x))\, dx \leq 0$ for all $z \geq 0$; (iii) $Law(R_2) = Law(R_1 + Z)$ where $\widetilde{E}(Z \mid R_1) = 0$. This concept, however, does not allow us to compare any two rational payments i.e. options. (For two similar results when $\widetilde{E}(R_1) \geq \widetilde{E}(R_2)$ see [5] p.45 and p.50). Observe that (iii) above implies that $\widetilde{V}ar(R_1) \leq \widetilde{V}ar(R_2)$, which is to be expected. The converse, however, is not true.

5. *The option risk per unit of the fair price.* Generally, it can also be of interest to know the option risk per unit of the fair price. To introduce such a risk concept one can modify the risk variable (2.1) in the following way:

$$(2.5) \qquad \widehat{R} = \frac{R}{V^*(f)}$$

and define *the option risk per unit of the fair price* as:

$$(2.6) \qquad \widehat{\mathcal{R}}(f) = Law(\widehat{R} \mid \widetilde{P}) .$$

Note that $\widehat{E}(R) = 1$ and from $\widehat{R}$ one can not read the fair price $V^*(f)$. Note also that the option is *risk-free* (in the sense described above) if and only if $\widehat{R}$ is degenerated at 1 , or in other words, if and only if $\widehat{\mathcal{R}}(f) = \delta_1$. In analogy with (2.4) we can now define *the mean-square* *risk of the option per unit of the fair price* as:

$$\widetilde{Var}\big(\widehat{\mathcal{R}}(f)\big) = \widetilde{E}\big(\widehat{R}-1\big)^2 = \widetilde{E}\big(\widehat{R}^2\big) - 1 = \widetilde{E}\left(\frac{f_{\tau_*}}{V^*(f)\,B_{\tau_*}}\right)^2 - 1 \ . \tag{2.7}$$

One could now interpret and extend this concept further in a manner similar to our treatment of the risk variable R following (2.4) above, and in this way obtain other (more sophisticated) risk functionals of the option risk per unit of the fair price.

6. **Entropy.** Another important concept associated with the option risk is *the option entropy*:

$$\widetilde{H}(R) = -\int_{\mathbf{R}} f(x)\log f(x)\ dx \tag{2.8}$$

if R is absolutely continuous (under $\widetilde{P}$) with a density function f, or:

$$\widetilde{H}(R) = -\sum_k p_k \log p_k \tag{2.9}$$

if R is a discrete variable taking some values x_k (being irrelevant) with probabilities p_k (under $\widetilde{P}$). The option entropy is a "measure" of uncertainty of the outcome of the rational payment R. It is a "value-free" measure, since it is expressed only through the "law of chance" of R. (For more information on the concept of entropy we refer to [7].)

In order to illustrate why this concept is of interest for design of options, we shall quote without proof the following well-known result. In the first three cases below R is assumed absolutely continuous, in the fourth final case R is assumed discrete.

(2.10) *1. If* $\widetilde{Var}(R) = \sigma^2 < \infty$ *, then the inequality holds:*

$$\widetilde{H}(R) \le \log\sqrt{2\pi e\sigma^2}$$

with equality iff $R \sim N(\mu,\sigma^2)$ *with* $\mu \in \mathbf{R}$.

2. If $R \ge 0$ *and* $\widetilde{E}(R) = \mu < \infty$ *, then the inequality holds:*

$$\widetilde{H}(R) \le \log(e\mu)$$

with equality iff $R \sim Exp(1/\mu)$.

3. If $a \le R \le b$ *for some* a *and* b *, then the inequality holds:*

$$\widetilde{H}(R) \le \log(b-a)$$

with equality iff $R \sim U(a,b)$.

4. If R *takes* n *values with probabilities* $p_k > 0$ *, then the inequality holds:*

$$\widetilde{H}(R) \le log(n)$$

with equality iff R *is uniformly distributed with* $p_k = 1/n$ *for all* k .

These facts have a beautiful interpretation in the problem of option design. To illustrate this, note that the rational payment R is non-negative in our model, so if the fair price V^* is given, then by (2.10.2) we see that the most uncertain option (in terms of the outcome of the rational payment) is the option with a reward process $f = (f_t)_{0 \le t \le T}$ for which (under $\widetilde{P}$):

$$R = \frac{f_{\tau_*}}{B_{\tau_*}} \sim Exp\left(\frac{1}{V^*}\right) \tag{2.11}$$

with $V^*(f) = V^*$. It does not mean that this option is with the highest (mean-square) risk. It rather means that if the (mean-square) risk is given and fixed, then this option is the most uncertain with respect to the outcome of the rational payment within the class of all options having the given mean-square risk. Note that the mean-square risk must then be equal to $V^*(f)^2$, since $\widetilde{Var}(R) = (\widetilde{E}R)^2$ if $R \sim Exp(1/V^*)$. Similar interpretations may now be given, subject to other restrictions on the rational payment R , if one uses remaining facts from (2.10).

7. *A triplet of the fundamental characteristics* of the option with a reward process $f = (f_t)_{0 \le t \le T}$ is defined as the three-dimensional vector $(\widetilde{E}(R), \widetilde{Var}(R), \widetilde{H}(R))$ consisting of the fair price, the mean-square risk, and the option entropy. From our considerations above it follows that these three numbers offer a good deal of information on the option character that is of interest to a buyer.

8. **Classification of options.** In view of the previous considerations, the following definition is natural. Two options $\mathcal{O}_1 = \mathcal{O}_1(f)$ and $\mathcal{O}_2 = \mathcal{O}_2(g)$ with reward processes $f = (f_t)_{0 \le t \le T'}$ and $g = (g_t)_{0 \le t \le T''}$ are said to be *equivalent* (or *risk-equivalent*) if they have the same risk:

$$Law\left(\frac{f_{\tau_*}}{B_{\tau_*}} \mid \widetilde{P}\right) = Law\left(\frac{g_{\sigma_*}}{B_{\sigma_*}} \mid \widetilde{P}\right) \tag{2.12}$$

where τ_* and σ_* are the optimal stopping times associated with $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively. In other words, we have:

$$\mathcal{O}_1 \sim \mathcal{O}_2 \quad \text{iff} \quad \mathcal{R}(f) = \mathcal{R}(g) \; . \tag{2.13}$$

Clearly, this is an equivalence relation on the set of all options within the given model, and in this way the set of all options splits into equivalence classes, two options being in the same class if and only if they are equivalent. By means of this equivalence relation we obtain a tool for comparing different options and classifying them. We note that *from a formal fair-game standpoint* two equivalent options may be thought of as the same.

9. **The optimal Skorokhod-embedding problem.** As an option may be identified with its reward process, we also see that the equivalence relation (2.13) offers an exact mathematical criterion on how to choose the reward process when designing an option. In this context the following question appears fundamental:

(2.14) *Given a distribution law ϱ on $\mathbf{R}_+$, find a reward process $f = (f_t)_{0 \le t \le T}$ such that:*

$$Law\left(\frac{f_{\tau_*}}{B_{\tau_*}} \mid \widetilde{P}\right) = \varrho$$

where τ_* is the optimal stopping time (1.19) for the problem (1.15).

In other words (and less formally) we ask: Given any risk, is there an option with this risk? We note that if the answer is positive, and we are able to indicate a reward process, then at least from a formal fair-game standpoint all options would be designed.

For obvious reasons we shall refer to the problem (2.14) as *the optimal Skorokhod-embedding problem*. (It is clear that $\mathbf{R}_+$ in (2.14) may be replaced by $\mathbf{R}$ generally.) We note that this problem involves more difficulty than the classic Skorokhod-embedding problem (see [11] p.258), since for a given ϱ we are not only supposed to find a stopping time τ_* at which to stop the underlying process in order to get ϱ , but also an optimal stopping problem (a functional of the underlying process) for which the stopping time τ_* is optimal. It is indeed a highly sophisticated machinery for design of options (or games), and although this problem is complex and difficult generally, it is shown in [10] that such a problem in principle can be solved. We note that by solving the optimal Skorokhod-embedding problem we also solve the classic Skorokhod-embedding problem.

10. It is an interesting question to determine a class of reward processes $f = (f_t)_{0 \le t \le T}$ among which one should try to find the optimal one in (2.14). (We note that a finite horizon T can also be given in advance together with the law ϱ , and then one has to find solely the functional "rule" f_t for t within $[0,T]$. Similarly, other constraints may be imposed on the admissible class of reward processes.) Since it is unlikely (if impossible) that functionals of the form $f_t = f(S_t(\mu))$ can have the right power, we restrict ourselves to path-dependent functionals of the form:

$$f_t = f\Big(\big(S_u(\mu)\big)_{0 \le u \le t} \Big) \tag{2.15}$$

where f is some map on the space of paths.

Since $Law(S(\mu) \,|\, \widetilde{P}) = Law(S(r) \,|\, P)$, we see that:

$$Law\left(\frac{f_{\tau_*}(\mu)}{B_{\tau_*(\mu)}} \,\Big|\, \widetilde{P} \right) = Law\left(\frac{f_{\tau_*}(r)}{B_{\tau_*(r)}} \,\Big|\, P \right) \tag{2.16}$$

with a clear interpretation of the notation. In other words, since the risk variable (2.1) satisfies:

$$R \sim Law\left(\frac{f_{\tau_*}(r)}{B_{\tau_*(r)}} \,\Big|\, P \right) \tag{2.17}$$

it is enough to consider the path-dependent functional:

$$f_t = f\Big(\big(S_u(r)\big)_{0 \le u \le t} \Big) \tag{2.18}$$

under P . This enables one to reformulate the problem (2.14) and consider it under the initial measure P where μ from (1.2) has to be replaced by r from (1.1).

The integral $\int_0^t g(S_u(r))\, du$, although a path dependent functional, does not seem to have the right power either (due to its Markovian nature in the expectation which follows by Itô formula and optional sampling). Therefore it seems reasonable to work with the maximum functional:

$$f_t = \max_{0 \le u \le t} S_u(r) \tag{2.19}$$

as being the next on the list of natural functionals, and which is known to produce enough randomness for the solution of the classic Skorokhod-embedding problem (see [11] p.258). This functional has also another well-known feature of interest in option trading: it provides comfort of exercising at a maximum thus reducing regret for not exercising earlier.

11. In [10] we formulate and solve the problem (2.14) in a relaxed form. We assume that the horizon T equals $+\infty$, and we drop discounting B. Such a simplified setting admits explicit computations and closed formulas throughout. A natural stock process is standard Brownian motion (W_t), and a natural reward process is the maximum process discounted linearly:

$$f_t = \left(\max_{0 \leq r \leq t} W_r \right) - \int_0^t c(W_r) \, dr \tag{2.20}$$

where $x \mapsto c(x)$ is a positive function. The option pricing problem in this setting can be formalized within the original *Bachelier model*. Even though this model is no longer accepted, it gives a good insight into (merely) technically more complicated matters of the Black-Scholes model.

3. Design of options of European type

All that was said above for options of American type extends to options of European type in an obvious manner. For the sake of completeness we shall indicate this extension below in detail. We further note that the optimal embedding problem which corresponds to (2.14) is easily solved in this case (Proposition 3.2), and although it may be of some interest for applications, this result does not require deep mathematics. (It could be that some constraints of interest, being imposed on the choice of the functional f, would make this problem more interesting mathematically as well.) Nonetheless, from the formal fair-game standpoint of a "true buyer" this simple result offers a construction of all options of European type within the given model. A European option with the most uncertain outcome of the rational payment (in the sense of entropy), within the class of options having the fixed fair price, is then easily obtained as a consequence (Corollary 3.3).

1. The pricing theory for options of European type is less sophisticated than for options of American type. We present the essential facts of this theory in complete analogy with the essential facts on American options stated in Section 1 above.

Consider *the Black-Scholes model* for a financial market consisting of a *riskless* bank account with value $B = (B_t)_{t \geq 0}$ and a *risky* stock with value $S = (S_t)_{t \geq 0}$ which are given by (1.1) and (1.2) respectively.

2. **European options.** Given a *reward variable* $f = f_T$, where $T > 0$ (the *expiration date*) is given and fixed, consider *an option of European type* as a contract between the seller and a buyer which entitles the buyer to exercise the option at time T and receive the payment f_T from the seller (if evaluated at time 0). After selling the option at a price x, the seller has at disposal *self-financing* strategies $\pi = (\beta_t, \gamma_t)$ which, after starting with $X_0^\pi = x$, at time t brings him the (non-negative) value:

$$\begin{aligned} X_t^\pi &= \beta_t B_t + \gamma_t S_t \\ &= x + \int_0^t \beta_r \, dB_r + \int_0^t \gamma_r \, dS_r \quad \text{(self-financing)} \end{aligned} \tag{3.1}$$

if evaluated at time 0 , or equivalently, the discounted (real) value:

$$Y_t^\pi = \frac{X_t^\pi}{B_t} = \beta_t + \gamma_t \frac{S_t}{B_t} \tag{3.2}$$

if evaluated at time t .

The central questions about such an option contract are:

(A) What is the “fair” price x ? (“Fair” refers to both the seller and the buyer.)

(B) What is the optimal strategy $\pi^* = (\beta_t^*, \gamma_t^*)$?

The general option pricing theory ([1]-[6], [8]-[9], [12]-[13]) gives the following answers to these questions.

3. **Fair price.** A self-financing strategy $\pi = (\beta_t, \gamma_t)$ is called a *hedge* (with respect to x and f given), if $X_0^\pi = x$ and we have:

$$X_T^\pi \geq f_T \quad P\text{-a.s.} \tag{3.3}$$

The minimal x , denoted by $V(f)$, for which there exists a hedge (with respect to f) is called *the fair price* of the option.

4. In order to determine the fair price $V(f)$, and answer the questions (A) and (B) above, we note that the observations and facts stated in Paragraph 1.4 above may be repeated here as well with no essential difference. Thus, by *the Girsanov theorem* there exists an equivalent martingale measure $\widetilde{P}$ given by (1.8), under which the process $\widetilde{W}_t = W_t + ((\mu - r)/\sigma)\, t$ is a standard Brownian motion, and the process (3.2) admits the following representation:

$$Y_t^\pi = Y_0^\pi + \int_0^t \sigma \gamma_r \frac{S_r}{B_r}\, d\widetilde{W}_r \tag{3.4}$$

where the integral defines a local martingale. Therefore we consider the smallest martingale (under $\widetilde{P}$) which dominates f_T/B_T $\widetilde{P}$-a.s. and which is given as *the Snell envelope*:

$$Y_t^* = \widetilde{E}\Big(\frac{f_T}{B_T} \;\Big|\; \mathcal{F}_t \Big) = Y_0^* + M_t^* \tag{3.5}$$

for $0 \leq t \leq T$. By *the Itô-Clark theorem* we have:

$$M_t^* = \int_0^t \alpha_r \, d\widetilde{W}_r \tag{3.6}$$

for some process $\alpha = (\alpha_t)$.

5. **Rational performance.** From the arguments indicated above it is evident that the optimal $\pi^* = (\beta_t^*, \gamma_t^*)$ is obtained by identifying:

$$\int_0^t \sigma \gamma_r \frac{S_r}{B_r}\, d\widetilde{W}_r = M_t^* \; . \tag{3.7}$$

This gives the following explicit answers to the questions (A) and (B) stated above:

$$V(f) = \widetilde{E}\Big(\frac{f_T}{B_T}\Big) \tag{3.8}$$

$$\beta_t^* = Y_t^* - \frac{\alpha_t}{\sigma} \tag{3.9}$$

$$\gamma_t^* = \frac{\alpha_t}{\sigma}\,\frac{B_t}{S_t}\ . \tag{3.10}$$

Since $Law(S(\mu)\,|\,\widetilde{P}) = Law(S(r)\,|\,P)$, it follows from (3.8) that neither the fair price $V(f)$, nor the optimal strategy $\pi^* = (\beta_t^*, \gamma_t^*)$, does depend on μ .

Finally, it should be noted that the process $Y^* = (Y_t^*)$ is computable (at least in principle) and known a priori before the option contract has been signed (as well as the fair price $V(f)$ itself). This provides both the seller and the buyer with a guarantee (and needed comfort) that the option contract can be realized at the fair price $V(f)$.

6. **Risk.** In complete analogy with American options in the beginning of Section 2, a new natural question which is missing above is stated as follows:

(C) What is the risk? ("Risk" refers to a "true buyer" before concluding the option contract, and afterwards, until it is exercised. By "true buyer" we mean a buyer who has no ability or desire to sell the option. Thus every "true buyer" will exercise the option according to the rational performance.)

For exactly the same reasons as in the context of American options in Section 2, we are naturally led to define *the risk variable* as the value of the payment:

$$R := \frac{f_T}{B_T} \tag{3.11}$$

at the expiration date T . We note again that although the expectation of R equals the fair price $V(f)$, and the option "game" is "fair":

$$\widetilde{E}\big(R\big) = V(f) \tag{3.12}$$

it is clear that R is a random variable which generally may deviate in its values from its expectation. Therefore we are again naturally led to formulate the following definition.

Definition 3.1 (A true buyer's risk of the European option)

The risk of the European option is the distribution law of the rational payment R under the equivalent martingale measure $\widetilde{P}$. In other words, the option risk $\mathcal{R}(f)$ is defined to be:

$$\mathcal{R}(f) = Law(R\,|\,\widetilde{P}) \tag{3.13}$$

where R is given by (3.8).

The considerations presented in Remark 2.2 above apply fully in the present case as well, and for exactly the same reasons the option risk of Definition 3.1 could also be called *the European* *option risk of a first kind*. This also leads to a definition of *the European option risk of a second kind* as $\mathcal{R}_F(f) = Law(R \,|\, P_F)$, where F is a distribution function which accounts for possible displacements of μ around r . We omit all remaining details and refer the reader directly to Remark 2.2 above for a complete account.

The option is said to be *risk-free* if the risk variable R is degenerated at the fair price $V(f)$. The *mean-square risk* of the option, the *option risk per unit of the fair price*, the *mean-square risk of the option per unit of the fair price*, the *option entropy*, and the *triplet of the fundamental characteristics* of the option, can now all be defined exactly as in Paragraphs 2.4-2.7 above.

7. **Classification of options.** In exactly the same way as American options, all European options are clearly classified through the equivalence relation:

$$\mathcal{O}_1(f) \sim \mathcal{O}_2(g) \quad \text{iff} \quad \mathcal{R}(f) = \mathcal{R}(g) \tag{3.14}$$

where $\mathcal{R}(f) = Law(f_{T'}/B_{T'} \,|\, \widetilde{P})$ and $\mathcal{R}(g) = Law(g_{T''}/B_{T''} \,|\, \widetilde{P})$. In this way the set of all European options splits into equivalence classes, two options being in the same class if and only if they are equivalent. By means of this equivalence relation we obtain a tool for comparing different European options and classifying them. We again note that *from a formal fair-game standpoint* two equivalent options may be thought of as the same.

8. **Option design of European type.** As European options may be identified with its reward variables, we see that the equivalence relation (3.11) offers an exact mathematical criterion on how to choose the reward variable when designing a European option. In this context the following reformulation of the optimal embedding problem (2.14) appears fundamental:

(3.15) *Given a distribution law* ϱ *on* $\mathbf{R}_+$ *, find a reward variable* $f = f_T$ *such that:*

$$Law\Big(\frac{f_T}{B_T} \,|\, \widetilde{P}\Big) = \varrho$$

where T *is the expiration time of the option.*

In other words (and less formally) we ask: Given any risk, is there an option with this risk?

In the next proposition we note that the answer to this question is positive. This simple result shows that at least from a formal fair-game standpoint all options of European type are easily designed. Below we use the standard notation:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt \tag{3.16}$$

to denote the distribution function of a standard normal random variable $X \sim N(0,1)$.

Proposition 3.2

Consider the Black-Scholes model for a financial market consisting of a riskless bank account with value $B = (B_t)_{t \geq 0}$ *and a risky stock with value* $S = (S_t)_{t \geq 0}$ *which are given by (1.1) and (1.2) respectively. Let* ϱ *be a probability measure on* $\mathbf{R}_+$ *associated with a strictly increasing and continuous distribution function* F *.*

1. Then there exists an option of European type associated with the reward variable:

$$f_T = f\big(S_T\big) \tag{3.17}$$

such that the option risk $\mathcal{R}(f)$ *from (3.10) is equal to* ϱ *.*

2. The function f *in (3.17) is explicitly given by the following formula:*

$$f(s) = e^{rT}\Big(F^{-1}\circ G\Big)\big(e^{-rT}s\big) \tag{3.18}$$

where the map G *is defined as follows:*

$$G(s) = \Phi\bigg(\frac{\sigma}{2}\sqrt{T} + \frac{1}{\sigma\sqrt{T}}\log(s)\bigg) \tag{3.19}$$

for all $s > 0$ *.*

Proof. Consider the random variable:

$$Z = \exp\bigg(\sigma W_T - \frac{\sigma^2}{2}\,T\bigg)\ . \tag{3.20}$$

Then by using $W_T \sim \sqrt{T}\,W_1$ it is easily verified that the function G given in (3.19) above is the distribution function of Z . Note that:

$$S_T(r) = \exp\bigg(\sigma W_T + \Big(r - \frac{\sigma^2}{2}\Big)T\bigg) = e^{rT}Z\ . \tag{3.21}$$

Therefore $G(e^{-rT}S_T(r)) \sim U(0,1)$, and hence $F^{-1}(G(e^{-rT}S_T(r))) \sim F$. Thus:

$$\frac{f_T(r)}{B_T} = \frac{f\big(S_T(r)\big)}{B_T} = F^{-1}(G(e^{-rT}S_T(r))) \sim F\ . \tag{3.22}$$

It remains to recall that $Law(S(\mu)\,|\,\widetilde{P}) = Law(S(r)\,|\,P)$ so that:

$$\mathcal{R}(f) = Law\bigg(\frac{f_T(\mu)}{B_T}\,\Big|\,\widetilde{P}\bigg) = Law\bigg(\frac{f_T(r)}{B_T}\,\Big|\,P\bigg) \tag{3.23}$$

and the proof is complete. □

Remarks: 1. In the theorem above it is assumed that the interest rate $r > 0$ and the expiration time $T > 0$ are given and fixed a priori (before the option has been constructed). Therefore the reward function (3.18) depends on r and T . If $r > 0$ and $T > 0$ are not given a priori, then for given $\varrho \sim F$ one may consider the reward function:

$$f(s) = c\Big(F^{-1}\circ G\Big)\big(s/c\big) \tag{3.24}$$

where $c > 0$ is a constant given and fixed. The option which correspond to the reward variable (3.17) has then the property that whenever the interest rate $r > 0$ and the termination time $T > 0$ are taken to satisfy $rT = \log(c)$, the option risk $\mathcal{R}(f)$ will be equal to ϱ .

2. These considerations can be further extended and formalized. Suppose we are given a family of measures ϱ_r with the corresponding distribution functions F_r for $r > 0$. (Such a family is typically obtained by any reward variable $f_T = f_T(r)$ which generally depends on the interest rate $r > 0$ under P , and therefore the risk $\mathcal{R}(f)$ depends also on r so that $\mathcal{R}(f) = \varrho_r$ for $r > 0$.) Then the map (3.18) could also be seen as a map from $]0,\infty[$ into the space of (continuous) functions on $]0,\infty[\times]0,\infty[$ upon identifying:

$$\text{(3.25)} \qquad \Big(f(s)\Big)(r,T) = e^{rT}\Big(F_r^{-1}\circ G\Big)\big(e^{-rT}s\big)$$

for $s > 0$. With this generalized interpretation of the reward variable (3.13) we see that whenever $r > 0$ and $T > 0$ are given before the option contract has been started, we may think of the reward function f to be given by (3.24) above with $c = e^{rT}$ (which is just a constant). Thus, from a formal fair-game standpoint we may look at (3.18) as a generalized reward function which generates all options of European type in regard to the risk equivalence (3.14).

3. We note that ϱ in the theorem above is assumed to have a strictly increasing and continuous distribution function for simplicity (the inverse is then easily written down). It is clear how this result can be extended to the case of more general ϱ by means of standard techniques. □

9. We close this section by considering the question of constructing an option with the most uncertain outcome of the rational payment R (in the sense of mathematical entropy as explained in Paragraph 2.6 above). If the fair price V is given a priori and fixed, then by (2.10.2) we see that the most uncertain option (with respect to the outcome of the rational payment R) is the option with a reward variable f_T for which (under $\widetilde{P}$):

$$\text{(3.26)} \qquad R = \frac{f_T}{B_T} \sim Exp\Big(\frac{1}{V}\Big) \ .$$

We note that the mean-square risk of such an option must then be equal to V^2 . The answer to this question is now easily obtained from the result of Proposition 3.2.

Corollary 3.3

Consider the Black-Scholes model for a financial market consisting of a riskless bank account with value $B = (B_t)_{t\geq 0}$ and a risky stock with value $S = (S_t)_{t\geq 0}$ which are given by (1.1) and (1.2) respectively.

A European option with the most uncertain outcome of the rational payment R (in the sense of mathematical entropy) within the class of all options having the fair price V (which is given and fixed) is the option with the reward process:

$$\text{(3.27)} \qquad f_T = f\big(S_T\big)$$

where f is given explicitly as follows:

$$\text{(3.28)} \qquad f(s) = V\, e^{rT} \log\left(\frac{1}{1 - G\big(e^{-rT}s\big)}\right)$$

for $s > 0$ with G from (3.19).

Proof. By (3.22) above we see that F in Proposition 3.2 should satisfy $F \sim Exp(1/V)$. Thus $F(x) = 1 - e^{-x/V}$ and hence $F^{-1}(y) = V \log(1/(1-y))$. Hence we see that (3.27)+(3.28) follows from (3.17)+(3.18). This completes the proof. □

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Summary

We examine the question of the *option risk* in complete markets from the standpoint of a true buyer. By “true buyer” we mean a buyer who has no ability or desire to sell the option; thus every true buyer will exercise the option according to the rational performance. We show that this approach offers exact mathematical means for comparing different options and classifying them. As an application of this methodology we present a simple construction of options with the most uncertain outcomes. While these considerations have some practical implications which are yet to be fully tested, the study itself has gone some way towards understanding two new avenues for research: *classification of options* and *option design*.