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An Optimal Stopping Problem for Reflecting Brownian Motions

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We study the optimal stopping problem

$$\sup_\tau \mathsf{E}\big(|B^1_\tau| \vee |B^2_\tau| - c\tau\big)$$

where (B^1, B^2) is a standard two-dimensional Brownian motion and $c > 0$ is a given and fixed constant. We first show that this problem is equivalent to the one where $|B^1_\tau| \vee |B^2_\tau|$ is replaced by $|B^1_\tau| + |B^2_\tau|$. Solving the latter problem we find a closed formula for the value function expressed in terms of the optimal stopping boundary which in turn is shown to be a unique solution to a nonlinear Fredholm integral equation. A key argument in the existence proof is played by a pointwise maximisation of the expression obtained by Wald-type identities. This provides tight bounds on the optimal stopping boundary describing its asymptotic behaviour for large coordinate values of (B^1, B^2). The solution found is applied to find the best constants in the inequalities which bound $\mathsf{E}\big(|B^1_\tau| \vee |B^2_\tau|\big)$ or $\mathsf{E}\big(|B^1_\tau| + |B^2_\tau|\big)$ from above by a constant multiple of $\sqrt{\mathsf{E}(\tau)}$ for any stopping time τ of (B^1, B^2).

1. Introduction

In this paper we study the optimal stopping problem

$$\sup_\tau \mathsf{E}\big(|B^1_\tau| \vee |B^2_\tau| - c\tau\big) \tag{1.1}$$

where (B^1, B^2) is a standard two-dimensional Brownian motion and $c > 0$ is a given and fixed constant. The supremum in (1.1) is taken over all stopping times τ of (B^1, B^2) for which $\mathsf{E}(\tau) < \infty$. Under this constraint we know by the optional sampling theorem applied to the martingales $((B^1_t)^2 - t)_{t \ge 0}$ and $((B^2_t)^2 - t)_{t \ge 0}$ that $\mathsf{E}(\tau) = \lambda\, \mathsf{E}(B^1_\tau)^2 + (1-\lambda)\, \mathsf{E}(B^2_\tau)^2$ for any $\lambda \in [0,1]$. These Wald-type identities show that the Bolza formulated problem (1.1) can be Mayer reformulated (see [7, Section 7] for the terminology) by replacing $-c\tau$ in (1.1) by $-c\lambda(B^1_\tau)^2 - c(1-\lambda)(B^2_\tau)^2$ for any $\lambda \in [0,1]$ given and fixed. A central question of the study is to examine whether the Mayer reformulation of the optimal stopping problem (1.1) is helpful in finding its solution in a closed form.

The question is motivated by the fact that this is true in dimension one as shown in [3]. Renaming B^1 to B and rewriting $\mathsf{E}(|B_\tau| - c\tau)$ as $\mathsf{E}(|B_\tau| - c(B_\tau)^2)$ using the optional sampling

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theorem, one sees that a pointwise maximisation of the resulting gain function $G(x) = |x| - cx^2$ over $x \in I\!R$ yields $\pm 1/2c$ as the optimal stopping points in the problem. This shows that the first entry time of B into $(-\infty, -1/2c] \cup [1/2c, \infty)$ maximises $\mathsf{E}(|B_\tau| - c\tau)$ over all stopping times τ of B for which $\mathsf{E}(\tau) < \infty$ (see [3] for further details).

One of our main findings in the paper is that although the method of pointwise maximisation breaks down in dimension two, it is still a useful tool to provide tight bounds on the optimal stopping boundary that are helpful in establishing its existence and describing its asymptotic properties for large coordinate values of (B^1, B^2). For this, we first show that the optimal stopping problem (1.1) is equivalent to the optimal stopping problem

$$\sup_\tau \mathsf{E}\big(|B^1_\tau| + |B^2_\tau| - c\tau\big) \tag{1.2}$$

where the challenging maximum of $|B^1_\tau|$ and $|B^2_\tau|$ in (1.1) is replaced by their sum in (1.2). Solving the optimal stopping problem (1.2) using the powerful technique of its reduction to a (two-dimensional elliptic) free-boundary problem, and exploiting a number of auxiliary results developed in recent years to utilise this correspondence, we find a closed formula for the value function expressed in terms of the optimal stopping boundary, which in turn is shown to be a unique solution to a nonlinear Fredholm integral equation. A key argument in the existence proof is played by a pointwise maximisation of the expression obtained by Wald-type identities. This provides tight bounds on the optimal stopping boundary describing its asymptotic behaviour for large coordinate values of (B^1, B^2). The solution found is applied to find the best constants in the inequalities which bound $\mathsf{E}\big(|B^1_\tau| \vee |B^2_\tau|\big)$ or $\mathsf{E}\big(|B^1_\tau| + |B^2_\tau|\big)$ from above by a constant multiple of $\sqrt{\mathsf{E}(\tau)}$ for any stopping time τ of (B^1, B^2).

2. Problem formulation

In this section we formulate the optimal stopping problem to be studied. The problem will be reformulated to a more convenient form in the next section.

1. Throughout the paper we study the optimal stopping problem

$$U(x,y) = \sup_\tau \mathsf{E}_{x,y}\big(X_\tau \vee Y_\tau - c\tau\big) \tag{2.1}$$

where X and Y are independent standard Brownian motions in $[0,\infty)$ both instantaneously reflecting at 0 with $(X_0, Y_0) = (x,y)$ under $\mathsf{P}_{x,y}$ and the supremum in (2.1) is taken over all stopping times τ of (X,Y) satisfying $\mathsf{E}_{x,y}(\tau) < \infty$ for $(x,y) \in [0,\infty)^2$ given and fixed. The process (X,Y) is strong Markov/Feller and the optimal stopping problem (2.1) is Bolza formulated. Its Mayer reformulation will be addressed/discussed in the next section.

2. The process (X,Y) admits two realisations of Markovian flows on the probability space $(\Omega, \mathcal{F}, \mathsf{P})$ on which a standard two-dimensional Brownian motion (B^1, B^2) starting at $(0,0)$ is defined. The first realisation is given by

$$X^x_t = |x + B^1_t| \;\;\&\;\; Y^y_t = |y + B^2_t| \tag{2.2}$$

and the second realisation is given by

$$X^x_t = x \vee S^1_t - B^1_t \;\;\&\;\; Y^y_t = y \vee S^2_t - B^2_t \tag{2.3}$$

where $S^1_t := \sup_{0\le s\le t} B^1_s$ and $S^2_t := \sup_{0\le s\le t} B^2_s$ for $t\ge 0$ and $(x,y)\in[0,\infty)^2$. We will make use of both of these realisations in the sequel depending on the context. We will see in Section 4 below that the supremum in (2.1) is attained at the first entry time of (X,Y) into a (closed) set so that making use of different realisations of (X,Y) is justified (recall the general fact invoked at the end of the proof of Proposition 1 below).

3. The optimal stopping problem (2.1) is two-dimensional. If $X_\tau \vee Y_\tau$ in (2.1) is replaced by either X_τ or Y_τ, then the resulting problem is one-dimensional. This version of the problem is simpler and its solution is presented in [3] as discussed in Section 1 above. We will make use of this solution in what follows. A challenging feature in the two-dimensional problem (2.1) is the maximum of X_τ and Y_τ which can be replaced by the sum of X_τ and Y_τ as we show in the next section.

4. The optimal stopping problem (2.1) may be viewed as the Lagrangian associated with the constrained problem in which $\mathsf{E}_{x,y}(X_\tau \vee Y_\tau)$ is maximised over all stopping times τ of (X,Y) satisfying $\mathsf{E}_{x,y}(\tau)\le C$ for $(x,y)\in[0,\infty)^2$ where $C>0$ is a given and fixed constant. We will return to the Lagrangian interpretation of (2.1) in Section 7 below.

3. Reducing maximum to sum

In this section we reformulate the optimal stopping problem (2.1) to a more convenient form where the maximum of two stopped processes is replaced by their sum.

1. Consider the optimal stopping problem

$$V(x,y) = \sup_\tau \mathsf{E}_{x,y}\big(X_\tau + Y_\tau - c\tau\big) \tag{3.1}$$

where X and Y are independent standard Brownian motions in $[0,\infty)$ both instantaneously reflecting at 0 with $(X_0,Y_0)=(x,y)$ under $\mathsf{P}_{x,y}$ and the supremum in (3.1) is taken over all stopping times τ of (X,Y) satisfying $\mathsf{E}_{x,y}(\tau)<\infty$ for $(x,y)\in[0,\infty)^2$ given and fixed. The process (X,Y) is strong Markov/Feller and the optimal stopping problem (3.1) is Bolza formulated. Its Mayer reformulation will be addressed/discussed shortly below.

2. We show that the optimal stopping problems (2.1) and (3.1) are equivalent in the sense described in the following proposition and its proof.

Proposition 1. *Denoting the value functions in* (2.1) *and* (3.1) *by* $U(x,y;c)$ *and* $V(x,y;c)$ *respectively, the following identity holds*

$$U(x,y;c) = \frac{1}{\sqrt{2}} V\big(|x-y|/\sqrt{2}, (x+y)/\sqrt{2}; c\sqrt{2}\big) \tag{3.2}$$

for $(x,y)\in[0,\infty)^2$ *and* $c>0$. *Moreover, if the first entry time of* (X,Y) *into a (closed) set is optimal in* (2.1) *then the first entry time of* (X,Y) *into the same (closed) set is optimal in* (3.1) *and vice versa (note that the realisations of* (X,Y) *is* (2.1) *and* (3.1) *may be different).*

Proof. It is easily verified that

$$|z_1| \vee |z_2| = \frac{1}{2}\big(|z_1 - z_2| + |z_1 + z_2|\big) \tag{3.3}$$

for all $z_1, z_2 \in I\!R$. Applying (3.3) to the realisation (2.2) we see that

$$|X_t^x| \vee |Y_t^y| = \frac{1}{2}\big(|x+B_t^1-(y+B_t^2)| + |x+B_t^1+y+B_t^2|\big) \tag{3.4}$$
$$= \frac{1}{\sqrt{2}}\Big(\Big|\frac{x-y}{\sqrt{2}}+\frac{B_t^1-B_t^2}{\sqrt{2}}\Big| + \Big|\frac{x+y}{\sqrt{2}}+\frac{B_t^1+B_t^2}{\sqrt{2}}\Big|\Big)$$
$$= \frac{1}{\sqrt{2}}\Big(\Big|\frac{x-y}{\sqrt{2}}+\tilde{B}_t^1\Big| + \Big|\frac{x+y}{\sqrt{2}}+\tilde{B}_t^2\Big|\Big)$$

for $(x,y) \in [0,\infty)^2$ where $\tilde{B}_t^1 := (B_t^1-B_t^2)/\sqrt{2}$ and $\tilde{B}_t^2 := (B_t^1+B_t^2)/\sqrt{2}$ are independent standard Brownian motions for $t \ge 0$ (as is well known and easily verified). Inserting (3.4) into (2.1) and comparing the resulting expression with (3.1) we see using $-\tilde{B}^1 \sim \tilde{B}^1$ that (3.2) holds as claimed. The final (equivalence) claim follows from the general and easily verified fact (using discretisation and characteristic functions for instance) that if the (right-)continuous processes (X,Y) and $(\tilde{X},\tilde{Y})$ are equally distributed then the random vectors (X_τ, Y_τ) and $(\tilde{X}_\tau, \tilde{Y}_\tau)$ are equally distributed whenever τ is the first entry time of either (X,Y) or $(\tilde{X},\tilde{Y})$ into a (closed) set. This completes the proof. □

3. Motivated by a pointwise maximisation argument as described in Section 1 above, consider the optimal stopping problem

$$V_\lambda(x,y) = \sup_\tau \mathsf{E}_{x,y}\big(X_\tau+Y_\tau-c\lambda X_\tau^2-c(1-\lambda)Y_\tau^2\big) \tag{3.5}$$

for $\lambda \in [0,1]$ where X and Y are independent standard Brownian motions in $[0,\infty)$ both instantaneously reflecting at 0 with $(X_0,Y_0)=(x,y)$ under $\mathsf{P}_{x,y}$ and the supremum in (3.5) is taken over all stopping times τ of (X,Y) satisfying $\mathsf{E}_{x,y}(\tau) < \infty$ for $(x,y) \in [0,\infty)^2$ given and fixed. The process (X,Y) is strong Markov/Feller and the optimal stopping problem (3.5) is Mayer formulated. Moreover, the optimal stopping problem (3.5) is a Mayer reformulation of the optimal stopping problem (3.1) as the following proposition and its proof show.

Proposition 2. *The following identity holds*

$$V(x,y) = V_\lambda(x,y) + c\lambda x^2 + c(1-\lambda)y^2 \tag{3.6}$$

for all $(x,y) \in [0,\infty)^2$ *and* $c > 0$ *with* $\lambda \in [0,1]$. *Moreover, if the first entry time of* (X,Y) *into a (closed) set is optimal in* (3.1) *then the first entry time of* (X,Y) *into the same (closed) set is optimal in* (3.5) *and vice versa (note that the realisations of* (X,Y) *in* (3.1) *and* (3.5) *may be different).*

Proof. Without loss of generality we may and do assume that X and Y in (3.1) are realised as in (2.2) above under P . Let $(x,y) \in [0,\infty)^2$ and a stopping time τ of (X,Y) with $\mathsf{E}(\tau) < \infty$ be given and fixed. By the optional sampling theorem applied to the martingales $((B_t^1)^2-t)_{t\ge 0}$ and $((B_t^2)^2-t)_{t\ge 0}$ we find that $\mathsf{E}(x+B_\tau^1)^2 = x^2+\mathsf{E}(\tau)$ and $\mathsf{E}(y+B_\tau^2)^2 = y^2+\mathsf{E}(\tau)$. This shows that $\mathsf{E}(\tau) = \mathsf{E}(X_\tau^x)^2-x^2 = \mathsf{E}_{x,y}(X_\tau^2)-x^2$ and $\mathsf{E}(\tau) = \mathsf{E}(Y_\tau^y)^2-y^2 = \mathsf{E}_{x,y}(Y_\tau^2)-y^2$. It follows therefore that $\mathsf{E}(\tau) = \lambda\mathsf{E}(\tau)+(1-\lambda)\mathsf{E}(\tau) = \lambda\mathsf{E}_{x,y}(X_\tau^2)-\lambda x^2+(1-\lambda)\,\mathsf{E}_{x,y}(Y_\tau^2)-(1-\lambda)y^2$ for any $\lambda \in [0,1]$ given and fixed. Inserting this expression into (3.1) we obtain (3.6) as claimed. The final (equivalence) claim follows using the same argument as in the proof of Proposition 1 above. This completes the proof. □

Note that each $\lambda \in [0,1]$ corresponds to a different optimal stopping problem (3.5) so that in effect we have infinitely many Mayer reformulations of (3.1).

Remark 3. If we drop the instantaneous reflection at 0 for both X and Y in (2.1), then the resulting problem reduces to an optimal stopping problem in dimension one solved in [3]. Indeed, this can be seen by noting that

$$z_1 \vee z_2 = \frac{1}{2}\big(|z_1 - z_2| + z_1 + z_2\big) \tag{3.7}$$

for all $z_1, z_2 \in I\!R$ and applying (3.7) to the realisation of X and Y as

$$X_t^x = x + B_t^1 \;\;\&\;\; Y_t^y = y + B_t^2 \tag{3.8}$$

which gives

$$\begin{aligned} X_t^x \vee Y_t^y &= \frac{1}{2}\big(|x + B_t^1 - (y + B_t^2)| + x + B_t^1 + y + B_t^2\big) \\ &= \frac{1}{\sqrt{2}}\Big(\Big|\frac{x-y}{\sqrt{2}} + \frac{B_t^1 - B_t^2}{\sqrt{2}}\Big| + \frac{x+y}{\sqrt{2}} + \frac{B_t^1 + B_t^2}{\sqrt{2}}\Big) \\ &= \frac{1}{\sqrt{2}}\Big(\Big|\frac{x-y}{\sqrt{2}} + \tilde{B}_t^1\Big| + \frac{x+y}{\sqrt{2}} + \tilde{B}_t^2\Big) \end{aligned} \tag{3.9}$$

for $(x,y) \in [0,\infty)^2$ where $\tilde{B}_t^1 := (B_t^1 - B_t^2)/\sqrt{2}$ and $\tilde{B}_t^2 := (B_t^1 + B_t^2)/\sqrt{2}$ are (independent) standard Brownian motions for $t \ge 0$ (as in (3.4) above). Inserting the identity (3.9) into (2.1) with no instantaneous reflection at 0 imposed, and denoting the resulting value function by $\tilde{U}(x,y)$, we see that

$$\tilde{U}(x,y) = \frac{1}{\sqrt{2}} \sup_\tau \mathsf{E}_{x,y}\Big(\Big|\frac{x-y}{\sqrt{2}} + \tilde{B}_\tau^1\Big| - c\tau\Big) + \frac{x+y}{2} \tag{3.10}$$

for $(x,y) \in I\!R^2$ upon using that $\mathsf{E}(\tilde{B}_\tau^2) = 0$ whenever $\mathsf{E}(\tau) < \infty$. From (3.10) we see that the initial statement is satisfied as claimed.

4. Optimal stopping boundary

In this section we establish the existence of an optimal stopping time in the problem (2.1) and derive basic properties of the optimal stopping boundary. Recalling that the problem (2.1) is equivalent to the problem (3.1) which in turn is equivalent to the problem (3.5), we will exploit these equivalences by deciding which of the three equivalent problem formulations is most convenient for a particular property/claim to be derived. This will be further combined with choosing a more convenient process realisation among (2.2) and (2.3) above.

1. We begin our analysis by establishing basic regularity results about the value functions (2.1) and (3.1) as well as (3.5).

Proposition 4. *The value functions U and V from* (2.1) *and* (3.1) *are convex and continuous on $[0,\infty)^2$. The value function V_λ from* (3.5) *is continuous on $[0,\infty)^2$ for every $\lambda \in [0,1]$ given and fixed.*

Proof. Using the realisation (2.2) of (X,Y) in (2.1) and (3.1) it is a matter of routine to verify that U and V are convex functions on $[0,\infty)^2$ respectively. Hence we can conclude that U and V are continuous on the open set $(0,\infty)^2$. To see that U and V are continuous at the boundary points of $[0,\infty)^2$ we may recall the well-known (and easily verified) fact that the convex functions U and V are upper semicontinuous on the closed and convex set $[0,\infty)^2$. Moreover, recalling that (2.2) defines a Markovian functional of the initial point (x,y) of the process (X,Y) , we see that the expectations in (2.1) and (3.1) define continuous functions of the initial point (x,y) of the process (X,Y) for every stopping time τ of (X,Y) with $\mathsf{E}(\tau)<\infty$ given and fixed. Taking the supremum over all such τ in (2.1) and (3.1) we can thus conclude that the value functions U and V are lower semicontinuous on $[0,\infty)^2$. Being also upper semicontinuous it follows that U and V are continuous on $[0,\infty)^2$ as claimed. The continuity of V_λ on $[0,\infty)^2$ for $\lambda\in[0,1]$ given and fixed then follows from (3.6) above and the proof is complete. □

2. Looking at (3.1) we may conclude that the (candidate) continuation and stopping sets in this problem need to be defined as follows

$$C=\{\,(x,y)\in[0,\infty)^2\mid V(x,y)>G(x,y)\,\} \tag{4.1}$$

$$D=\{\,(x,y)\in[0,\infty)^2\mid V(x,y)=G(x,y)\,\} \tag{4.2}$$

respectively where $G(x,y)=x+y$ for $(x,y)\in[0,\infty)^2$ denotes the gain function in (3.1). Moreover, the definitions of C and D remain unchanged if we replace V and G in (4.1) and (4.2) respectively by V_λ from (3.5) and G_λ defined by $G_\lambda(x,y)=x+y-c\lambda x^2-c(1-\lambda)y^2$ for $(x,y)\in[0,\infty)^2$ with $\lambda\in(0,1)$ given and fixed. Since the functions V_λ and G_λ are continuous, it follows by [7, Corollary 2.9] that the first entry time τ_D of the process (X,Y) into the closed set D defined by

$$\tau_D=\inf\{\,t\ge 0\mid (X_t,Y_t)\in D\,\} \tag{4.3}$$

is optimal in (3.1) whenever $\mathsf{E}_{x,y}(\tau_D)<\infty$ for all $(x,y)\in[0,\infty)^2$. In the sequel we will establish this and other properties of τ_D by analysing the boundary of D .

3. To derive the shape of D we first establish a basic monotonicity result. We focus on the function V from (3.1) but the same result and proof also hold for the function U from (2.1).

Proposition 5. *We have*

(4.4) *the mapping* $x\mapsto V(x,y)-G(x,y)$ *is decreasing on* $[0,\infty)$

(4.5) *the mapping* $y\mapsto V(x,y)-G(x,y)$ *is decreasing on* $[0,\infty)$

for every y *and* x *in* $[0,\infty)$ *given and fixed respectively.*

Proof. Using the realisation (2.3) in (3.1) with any stopping time τ of (X,Y) satisfying $\mathsf{E}(\tau)<\infty$ given and fixed, we have

$$\mathsf{E}_{x,y}\big(X_\tau+Y_\tau-c\tau\big)-G(x,y)=\mathsf{E}\big(x\vee S^1_\tau-x+y\vee S^2_\tau-y-c\tau\big) \tag{4.6}$$

where we use that $\mathsf{E}(B^1_\tau)=0$ and $\mathsf{E}(B^2_\tau)=0$ by the optional sampling theorem. Noting that $x\mapsto x\vee S^1_\tau-x$ and $y\mapsto y\vee S^2_\tau-y$ are decreasing on $[0,\infty)$ and taking the supremum in (4.6) over all stopping times τ of (X,Y) satisfying $\mathsf{E}(\tau)<\infty$ we see that (4.4) and (4.5) are satisfied as claimed. $\square$

A direct consequence of the monotonicity facts (4.4) and (4.5) determining the shape of D may now be stated as follows.

Corollary 6. *If* $(x,y)\in D$ *then* $(\bar{x},\bar{y})\in D$ *for all* $\bar{x}\ge x$ *and* $\bar{y}\ge y$.

Proof. If $(x,y)\in D$ and we take $(\bar{x},\bar{y})\in[0,\infty)^2$ such that $\bar{x}\ge x$ and $\bar{y}\ge y$ then

$$0\le V(\bar{x},\bar{y})-G(\bar{x},\bar{y})\le V(x,\bar{y})-G(x,\bar{y})\le V(x,y)-G(x,y)=0 \tag{4.7}$$

where in the second inequality we use (4.4) above and in the third inequality we use (4.5) above. From (4.7) we see that $V(\bar{x},\bar{y})=G(\bar{x},\bar{y})$ so that $(\bar{x},\bar{y})\in D$ as claimed. $\square$

The topological boundary between the sets C and D is called the *optimal stopping boundary.* From the result of Corollary 6 we see that b can be parameterised as a function (of either x or y and we choose x) as follows

$$b(x):=\inf\{\,y\in[0,\infty)\mid(x,y)\in D\,\} \tag{4.8}$$

for $x\in[0,\infty)$ where we formally set $\inf\emptyset=\infty$. A reduction to one dimension establishes the following fact.

Proposition 7. *We have*

$$[0,1/2c)\times[0,\infty)\cup[0,\infty)\times[0,1/2c)\subseteq C \tag{4.9}$$

for any $c>0$ *given and fixed.*

Proof. From the one-dimensional optimal stopping problem solved in [3] we know that

$$\sup_\tau \mathsf{E}\big(|x+B^1_\tau|-c\tau\big)>x \tag{4.10}$$

for all $x\in[0,1/2c)$ where the supremum is taken over all stopping times τ satisfying $\mathsf{E}(\tau)<\infty$. On the other hand, the submartingale property of $|y+B^2|$ implies that

$$\mathsf{E}|y+B^2_\tau|\ge y \tag{4.11}$$

for all $y\in[0,\infty)$ and all stopping times τ satisfying $\mathsf{E}(\tau)<\infty$. Combining (4.10) and (4.11) we find that

$$\sup_\tau \mathsf{E}\big(|x+B^1_\tau|+|y+B^2_\tau|-c\tau\big)\ge\sup_\tau \mathsf{E}\big(|x+B^1_\tau|-c\tau\big)+y>x+y \tag{4.12}$$

from where we see that $(x,y)\in C$ so that $[0,1/2c)\times[0,\infty)\subseteq C$ as claimed. The fact that $[0,\infty)\times[0,1/2c)\subseteq C$ follows by symmetry. This completes the proof. $\square$

The previous proof can be refined to establish the following more precise fact about the location of the optimal stopping boundary.

Proposition 8. *We have*

$$\{1/2c\}\times[0,\infty)\cup[0,\infty)\times\{1/2c\}\subseteq C \tag{4.13}$$

for any $c>0$ *given and fixed.*

Proof. It is enough to prove that $\{1/2c\}\times[0,\infty)\subseteq C$ since then $[0,\infty)\times\{1/2c\}\subseteq C$ follows by symmetry. To derive the former inclusion, set $x=1/2c$ and let $y\in[0,\infty)$ be given and fixed. We need to show that (x,y) belongs to C. For this, note that Proposition 7 ensures that we may and do assume that $y\geq 1/2c$. The main idea of the proof is to make use of the one-dimensional optimal stopping problem (4.10) simultaneously for both B^1 and B^2. For this, we first choose $\lambda\in(0,1)$ close enough to 1 so that $y<1/2c(1-\lambda)$. This makes y a continuation point in the problem (4.10) for B^2 with $c(1-\lambda)$ in place of c. Moreover, since $x=1/2c<1/2c\lambda$ we see simultaneously that x is a continuation point in the problem (4.10) for B^1 with $c\lambda$ in place of c. Solution to the problem (4.10) found in [3] then shows that

$$\tau_1=\inf\left\{\,t\geq 0\mid |x+B_t^1|=1/2c\lambda\,\right\}\ \ \&\ \ \tau_2=\inf\{\,t\geq 0\mid |y+B_t^2|=1/2c(1-\lambda)\,\} \tag{4.14}$$

are optimal stopping times for B^1 and B^2 respectively. In view of the initial aim to make use of both τ_1 and τ_2 simultaneously, we set

$$\sigma:=\tau_1\wedge\tau_2\,. \tag{4.15}$$

Then σ is a stopping time of (X,Y) with $\mathsf{E}(\sigma)<\infty$ and using (2.2) we have

$$\begin{aligned}
V(x,y)&=\sup_\tau \mathsf{E}\big(|x+B_\tau^1|+|y+B_\tau^2|-c\tau\big)\geq \mathsf{E}\big(|x+B_\sigma^1|+|y+B_\sigma^2|-c\sigma\big)\\
&=\mathsf{E}\big(|x+B_\sigma^1|+|y+B_\sigma^2|-c\lambda(x+B_\sigma^1)^2-c(1-\lambda)(y+B_\sigma^2)^2\big)+c\lambda x^2+c(1-\lambda)y^2\\
&=\mathsf{E}\big[\big(|x+B_{\tau_1}^1|-c\lambda(x+B_{\tau_1}^1)^2+|y+B_{\tau_1}^2|-c(1-\lambda)(y+B_{\tau_1}^2)^2\big)I(\sigma=\tau_1)\big]\\
&\quad+\mathsf{E}\big[\big(|y+B_{\tau_2}^2|-c(1-\lambda)(y+B_{\tau_2}^2)^2+|x+B_{\tau_2}^1|-c\lambda(x+B_{\tau_2}^1)^2\big)I(\sigma=\tau_2)\big]\\
&\quad+c\lambda x^2+c(1-\lambda)y^2\\
&>\frac{1}{4c\lambda}\,\mathsf{P}(\sigma=\tau_1)+\frac{1}{2}\mathsf{E}\big[|y+B_{\tau_1}^2|\,I(\sigma=\tau_1)\big]\\
&\quad+\frac{1}{4c(1-\lambda)}\,\mathsf{P}(\sigma=\tau_2)+\frac{1}{2}\mathsf{E}\big[|x+B_{\tau_2}^1|\,I(\sigma=\tau_2)\big]+c\lambda x^2+c(1-\lambda)y^2\\
&\geq\frac{1}{4c\lambda}\,\mathsf{P}(\sigma=\tau_1)+\frac{1}{2}y\,\mathsf{P}(\sigma=\tau_1)+\frac{1}{4c(1-\lambda)}\,\mathsf{P}(\sigma=\tau_2)+\frac{1}{2}x\,\mathsf{P}(\sigma=\tau_2)\\
&\quad+c\lambda x^2+c(1-\lambda)y^2\\
&>\frac{1}{2}x\,\mathsf{P}(\sigma=\tau_1)+\frac{1}{2}y\,\mathsf{P}(\sigma=\tau_1)+\frac{1}{2}y\,\mathsf{P}(\sigma=\tau_2)+\frac{1}{2}x\,\mathsf{P}(\sigma=\tau_2)=x+y
\end{aligned} \tag{4.16}$$

where we use the fact that $|x+B_{\tau_1}^1|$ equals $1/2c\lambda$ which is a unique maximum point of $G_1(z):=z-c\lambda z^2$ for $z\in[0,\infty)$ with $G_1(1/2c\lambda)=1/4c\lambda$ and the fact that $|y+B_{\tau_2}^2|$ equals $1/2c(1-\lambda)$ which is a unique maximum point of $G_2(z):=z-c(1-\lambda)z^2$ for $z\in[0,\infty)$ with

$G_2(1/2c(1-\lambda)) = 1/4c(1-\lambda)$, while $|y+B^2_{\tau_1}|-c(1-\lambda)(y+B^2_{\tau_1})^2 \geq \frac{1}{2}|y+B^2_{\tau_1}|$ on $\{\tau_1 \leq \tau_2\}$ because $G_2(z) \geq \frac{1}{2}z$ for $z \in [0, 1/2c(1-\lambda)]$ and $|x+B^1_{\tau_2}|-c\lambda(x+B^1_{\tau_2})^2 \geq \frac{1}{2}|x+B^1_{\tau_2}|$ on $\{\tau_2 \leq \tau_1\}$ because $G_1(z) \geq \frac{1}{2}z$ for $z \in [0, 1/2c\lambda]$, upon recalling in the final inequality that $1/2c\lambda > x$ and $1/2c(1-\lambda) > y$. From (4.16) we see that $V(x,y) > x+y$ so that $(x,y) \in C$ as needed and the proof is complete. □

Combining the results of Propositions 7 and 8 we see that $b(x) = \infty$ for all $x \in [0, 1/2c]$ and, at least in principle, the stopping set D could still be empty. We now show that this is not the case. For this, we focus on the optimal stopping problem (3.5) where we recall that the gain function is given by

$$G_\lambda(x,y) = x+y-c\lambda x^2 - c(1-\lambda)y^2 \tag{4.17}$$

for $(x,y) \in [0,\infty)^2$ and $c > 0$ with $\lambda \in (0,1)$. From (4.17) it is easily verified that $(1/2c\lambda, 1/2c(1-\lambda))$ is a unique point in $[0,\infty)^2$ at which the (global) maximum of G_λ on $[0,\infty)^2$ is attained. Setting $x = 1/2c\lambda$ and $y = 1/2c(1-\lambda)$ we find that $y = \bar{b}(x)$ where the function $\bar{b} : (1/2c, \infty) \to (1/2c, \infty)$ is defined as follows

$$\bar{b}(x) = \frac{1}{2c} \frac{x}{\left(x - \frac{1}{2c}\right)} \tag{4.18}$$

for $x \in (1/2c, \infty)$. The set of all unique (global) maximum points of G_λ on $[0,\infty)^2$ when λ runs through $(0,1)$ can be parameterised by means of the function $\bar{b}$ as follows

$$M = \{\, (1/2c\lambda, 1/2c(1-\lambda)) \mid \lambda \in (0,1) \,\} = \{\, (x, \bar{b}(x)) \mid x \in (1/2c, \infty) \,\} \tag{4.19}$$

Note that $\bar{b}$ is a strictly decreasing continuous function on $(1/2c, \infty)$ with $\bar{b}(x) \uparrow \infty$ as $x \downarrow 1/2c$ and $\bar{b}(x) \downarrow 1/2c$ as $x \uparrow \infty$.

Proposition 9. *We have*

$$M \subseteq D\,. \tag{4.20}$$

Consequently, the following inequalities are satisfied

$$\frac{1}{2c} < b(x) \leq \bar{b}(x) \tag{4.21}$$

for all $x \in (1/2c, \infty)$ *with*

$$\lim_{x \downarrow 1/2c} b(x) = \infty \;\;\&\;\; \lim_{x \uparrow \infty} b(x) = \frac{1}{2c}\,. \tag{4.22}$$

Proof. Take any point $(x,y) \in M$. Then $y = \bar{b}(x)$ and by (4.19) there exists a unique $\lambda \in (0,1)$ such that $x = 1/2c\lambda$ and $y = 1/2c(1-\lambda)$. As stated following (4.17) above we know that (x,y) is a unique point at which the (global) maximum of G_λ on $[0,\infty)^2$ is attained. This implies that (x,y) is an optimal stopping point in the problem (3.5). Since the problems (3.1) and (3.5) are equivalent by the result of Proposition 2, it follows that (x,y) is an optimal

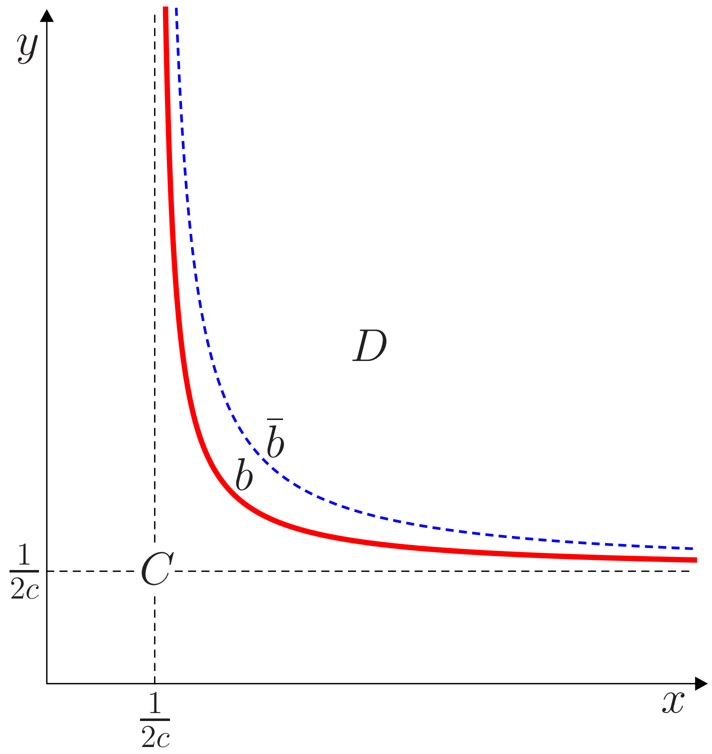

Figure 1. The optimal stopping boundary b in the problem (3.1) when $c = 1$. The upper bound $\bar{b}$ is given explicitly in (4.18) above.

stopping point in the problem (3.1) as well, i.e. $(x, y) \in D$ as claimed. This establishes (4.20). Recalling the result of Corollary 6 combined with (4.8) and the results of Propositions 7 and 8 we obtain (4.21) while (4.22) follows from (4.21) because $\bar{b}$ satisfies the same properties as noted above. This completes the proof. $\square$

Using the results derived so far we can describe the continuation and stopping sets from (4.1) and (4.2) as follows

$$C = [0, 1/2c] \times [0, \infty) \cup \{\, (x, y) \in [0, \infty)^2 \mid x > 1/2c \;\;\&\;\; y < b(x) \,\} \tag{4.23}$$

$$D = \{\, (x, y) \in [0, \infty)^2 \mid x > 1/2c \;\;\&\;\; y \geq b(x) \,\} \tag{4.24}$$

(see Figure 1). We conclude this section by showing that the optimal stopping boundary has no vertical/horisontal jump.

Proposition 10. *The mapping* $x \mapsto b(x)$ *is convex (and thus continuous) and strictly decreasing on* $(1/2c, \infty)$.

Proof. Recalling from Proposition 4 that V is convex while G is concave (linear), it is easily verified that D is convex and hence $x \mapsto b(x)$ is convex (and thus continuous) on $(1/2c, \infty)$ as claimed. Applying exactly the same arguments to $y \mapsto b^{-1}(y)$ on $(1/2c, \infty)$ (defined as $b^{-1}(y) := \inf\{\, x \in (1/2c, \infty) \mid b(x) = y \,\}$ for $y \in (1/2c, \infty)$ and understood as another parametrisation of ∂C) we find that $y \mapsto b^{-1}(y)$ is continuous on $(1/2c, \infty)$ which means that $x \mapsto b(x)$ is strictly decreasing on $(1/2c, \infty)$ as claimed. $\square$

Remark 11. Regarding the continuity of $x \mapsto b(x)$ on $(1/2c, \infty)$ one may notice in the notation of Theorem 10 from [6] that $\hat{H} = I\!\!L_Z G + H = (\frac{1}{2}\partial_{xx} + \frac{1}{2}\partial_{yy})(x+y) - c = -c$ so that $\partial_y \hat{H} = 0$. This shows that Theorem 10 from [6] is not directly applicable but a modification of its proof pointed out in Remark 13 from [6] is applicable because $\partial_y \hat{V} = \partial_y V - \partial_y G = V_y - 1 < 0$ in C due to convexity of $y \mapsto V(x,y)$ and the fact that $y \mapsto G(x,y) = x+y$ is a linear function for every $x \in [0,\infty)$ given and fixed. It follows therefore by the extended Corollary 20 from [6] as pointed out in Remark 13 from [6] that $x \mapsto b(x)$ is continuous on $(1/2c, \infty)$. Note that this fact was established above using different means.

5. Free-boundary problem

In this section we derive a free-boundary problem that stands in one-to-one correspondence with the optimal stopping problem (3.1). Using the results derived in the previous sections we show that the value function V from (3.1) and the optimal stopping boundary b from (4.8) solve the free-boundary problem. This establishes the existence of a solution to the free-boundary problem. Its uniqueness in a natural class of functions will follow from a more general uniqueness result that will be established in the next section. This will also yield an explicit integral representation of the value function V expressed in terms of the optimal stopping boundary b.

1. Consider the optimal stopping problem (3.1) where X and Y are independent standard Brownian motions in $[0,\infty)$ both instantaneously reflecting at 0 with $(X_0, Y_0) = (x,y)$ under $\mathsf{P}_{x,y}$ for $(x,y) \in [0,\infty)^2$ given and fixed. The process (X,Y) is strong Markov/Feller and its motion is determined by the *infinitesimal generator* $I\!\!L_{X,Y}$ of (X,Y) given by

$$I\!\!L_{X,Y} = \tfrac{1}{2}\partial_{xx} + \tfrac{1}{2}\partial_{yy} \tag{5.1}$$

on $(0,\infty)^2$ and the *instantaneous reflection* (Neumann) conditions

$$\partial_x = 0 \text{ at } x = 0 \;\;\&\;\; \partial_y = 0 \text{ at } y = 0 \tag{5.2}$$

within the state space $[0,\infty)^2$ of (X,Y). Looking at (3.1) and relying on other properties of V and b derived above, we are naturally led to formulate the following free-boundary problem for finding V and b:

$$I\!\!L_{X,Y} V = c \text{ in } C \tag{5.3}$$

$$V = G \text{ on } D \text{ (instantaneous stopping)} \tag{5.4}$$

$$V_x = G_x \;\&\; V_y = G_y \text{ at } \partial C \text{ (smooth fit)} \tag{5.5}$$

$$V_x(0,y) = 0 \;\&\; V_y(x,0) = 0 \text{ (instantaneous reflection)} \tag{5.6}$$

for $x,y \in [0,\infty)$ where $G(x,y) = x+y$ for $(x,y) \in [0,\infty)^2$ is the gain function in (3.1), C is the (continuation) set from (4.1) above, D is the (stopping) set from (4.2) above, and $\partial C = \{\, (x,y) \in [0,\infty)^2 \mid x > 1/2c \;\&\; y = b(x) \,\}$ is the (optimal stopping) boundary between the sets C and D.

2. To formulate the existence and uniqueness result for the free-boundary problem (5.3)-(5.6), we let $\mathcal{C}$ denote the class of functions (U, a) such that

$$U \text{ belongs to } C^2(C_a) \cap C^1(\bar{C}_a) \text{ and is continuous on } [0,\infty)^2 \tag{5.7}$$

$$a \text{ is continuous \& decreasing from } (1/2c, \infty) \text{ to } (1/2c, \infty) \tag{5.8}$$

where we set $C_a = [0, 1/2c] \times [0, \infty) \cup \{ (x, y) \in [0,\infty)^2 \mid x > 1/2c \ \& \ y < a(x) \}$ and $\bar{C}_a = [0, 1/2c] \times [0, \infty) \cup \{ (x, y) \in [0,\infty)^2 \mid x > 1/2c \ \& \ y \le a(x) \}$.

Theorem 12. *The free-boundary problem* (5.3)-(5.6) *has a unique solution* (V, b) *in the class* $\mathcal{C}$ *where* V *is given in* (3.1) *and* b *is given in* (4.8) *above.*

Proof. We first show that the pair (V, b) belongs to the class $\mathcal{C}$ and solves the free-boundary problem (5.3)-(5.6). For this, note that the optimal stopping problem (3.1) is Bolza formulated so that standard arguments (see e.g. Subsections 7.1 and 7.2 in [7]) imply that V belongs to $C^2(C)$ and satisfies (5.3). From Proposition 4 above we know that V is continuous on $[0,\infty)^2$. Moreover, recall that the process (X, Y) is strong Feller while it is evident that each point of ∂C is probabilistically regular for the set D since b is decreasing and the coordinate processes X and Y are independent standard Brownian motions (when off $x = 0$ and $y = 0$). Finally, from (2.2) and/or (2.3) we see that the process (X, Y) can be realised as a continuously differentiable stochastic flow (with probability one) so that the (first two) integrability conditions of Theorem 8 from [1] are satisfied. Recalling that V satisfies (5.4), and applying the result of that theorem, we can conclude that

$$V \text{ is continuously differentiable on } [0,\infty)^2. \tag{5.9}$$

In particular, this shows that (5.5) holds as well so that V belongs to $C^1(\bar{C})$ as required in (5.7) above. Moreover, the fact that the supremum in (3.1) is attained at the first entry time of (X, Y) into D implies that (5.6) holds. Finally, the fact that b satisfies (5.8) was established in Proposition 10 above. This shows that (V, b) belongs to $\mathcal{C}$ and solves (5.3)-(5.6) as claimed. To derive uniqueness of the solution we will first see in the next section that any solution (U, a) to (5.3)-(5.6) from the class $\mathcal{C}$ admits an explicit integral representation for U expressed in terms of a, which in turn solves a nonlinear Fredholm integral equation, and we will see that this equation cannot have other solutions satisfying the required properties. From these facts we can conclude that the free-boundary problem (5.3)-(5.6) cannot have other solutions in the class $\mathcal{C}$ as claimed. This completes the proof. □

Remark 13. Results similar to the result of Theorem 12 also hold for the equivalent optimal stopping problems (2.1) and (3.5). This can be also derived directly from Theorem 12 using (3.2) and (3.6) respectively and we omit further details.

6. Nonlinear integral equation

In this section we show that the optimal stopping boundary b from (4.23)+(4.24) can be characterised as the unique solution to a nonlinear Fredholm integral equation. This also yields an explicit integral representation of the value function V from (3.1) expressed in terms of the optimal stopping boundary b. As a consequence of the existence and uniqueness result for the nonlinear Fredholm integral equation we also obtain uniqueness of the solution to the free-boundary problem (5.3)-(5.6) as explained in the proof of Theorem 12 above. Recalling that the problems (2.1) and (3.5) are equivalent to the problem (3.1) we see that the results derived in this section solve the problems (2.1) and (3.5) as well. Applications of the derived results to sharp inequalities will be presented in the next section.

1. Consider the optimal stopping problem (3.1) where X and Y are independent standard Brownian motions in $[0,\infty)$ both instantaneously reflecting at 0 with $(X_0, Y_0)=(x,y)$ under $\mathsf{P}_{x,y}$ for $(x,y) \in [0,\infty)^2$ given and fixed. Set

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2} \quad \& \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy \tag{6.1}$$

to denote the standard normal density and distribution functions for $x \in I\!R$. Building on the results derived in the previous sections we can now disclose the solution to the problem (3.1).

Theorem 14. *The optimal stopping boundary b in (3.1) can be characterised as the unique solution to the nonlinear Fredholm integral equation*

$$\begin{aligned} \int_0^\infty \Bigg[\frac{1}{\sqrt{t}} \Big(\varphi\Big(\frac{x}{\sqrt{t}}\Big) + \varphi\Big(\frac{b(x)}{\sqrt{t}}\Big)\Big) - c\Big(\Phi\Big(\frac{x+\frac{1}{2c}}{\sqrt{t}}\Big) - \Phi\Big(\frac{x-\frac{1}{2c}}{\sqrt{t}}\Big)\Big) \\ - \frac{c}{\sqrt{t}} \int_{\frac{1}{2c}}^\infty \Big(\Phi\Big(\frac{b(x)+b(z)}{\sqrt{t}}\Big) - \Phi\Big(\frac{b(x)-b(z)}{\sqrt{t}}\Big)\Big)\Big(\varphi\Big(\frac{x+z}{\sqrt{t}}\Big) + \varphi\Big(\frac{x-z}{\sqrt{t}}\Big)\Big) dz \Bigg] dt = 0 \end{aligned} \tag{6.2}$$

for $x \in (1/2c, \infty)$ in the class of continuous & decreasing (convex) functions b on $(1/2c, \infty)$ taking values in $(1/2c, \infty)$. The value function V in (3.1) admits the representation

$$\begin{aligned} V(x,y) = x+y \\ + \int_0^\infty \Bigg[\frac{1}{\sqrt{t}} \Big(\varphi\Big(\frac{x}{\sqrt{t}}\Big) + \varphi\Big(\frac{y}{\sqrt{t}}\Big)\Big) - c\Big(\Phi\Big(\frac{x+\frac{1}{2c}}{\sqrt{t}}\Big) - \Phi\Big(\frac{x-\frac{1}{2c}}{\sqrt{t}}\Big)\Big) \\ - \frac{c}{\sqrt{t}} \int_{\frac{1}{2c}}^\infty \Big(\Phi\Big(\frac{y+b(z)}{\sqrt{t}}\Big) - \Phi\Big(\frac{y-b(z)}{\sqrt{t}}\Big)\Big)\Big(\varphi\Big(\frac{x+z}{\sqrt{t}}\Big) + \varphi\Big(\frac{x-z}{\sqrt{t}}\Big)\Big) dz \Bigg] dt \end{aligned} \tag{6.3}$$

for $(x,y) \in [0,\infty)^2$. The optimal stopping time in (3.1) is given by

$$\tau_b = \inf\{\, t \ge 0 \mid X_t > 1/2c \;\; \& \;\; Y_t \ge b(X_t)\,\} \tag{6.4}$$

under $\mathsf{P}_{x,y}$ with $(x,y) \in [0,\infty)^2$ given and fixed.

Proof. (i) *Existence.* We first show that the optimal stopping boundary b in (3.1) solves (6.2). Recalling from Proposition 10 that b satisfies the properties stated following (6.2), this will establish the existence of a solution to (6.2) in the specified class of functions.

For this, set $F := V - G$ on $[0,\infty)^2$ and note that $n \wedge b(X)$ is a continuous semimartingale because b is convex on $[b^{-1}(n), \infty)$ for every $n \ge 1$ given and fixed. Using the properties of V derived in Proposition 4 and Theorem 12, this shows that the change-of-variable formula with local time on surfaces [5, Theorem 2.1] is applicable to F composed with (X,Y) stopped at $\tau_n = \inf\{\, t \ge 0 \mid Y_t \ge n \,\}$, and due to the smooth fit condition (5.5) this formula reduces to the classic Itô formula. Letting $n \to \infty$ in the resulting identity we find that the classic Itô formula is applicable to F composed with (X, Y) . This gives

$$F(X_T^x, Y_T^y) = F(x,y) + \int_0^T F_x(X_t^x, Y_t^y)\, dX_t^x + \int_0^T F_y(X_t^x, Y_t^y)\, dY_t^y + \frac{1}{2}\int_0^T (F_{xx}+F_{yy})(X_t^x, Y_t^y)\, dt \tag{6.5}$$

for $T > 0$ and $(x, y) \in [0,\infty)^2$ given and fixed. Using (2.2) we can realise X_t^x as $|x+B_t^1|$ and Y_t^y as $|y+B_t^2|$ where B^1 and B^2 are independent standard Brownian motions. By the Itô-Tanaka formula we know that

$$|z+B_t^i| = z + \beta_t^i + \ell_t^0(z+B^i) \tag{6.6}$$

where $\beta_t^i = \int_0^t \mathrm{sign}(z+B_s^i)\, dB_s^i$ is a standard Brownian motion and $\ell_t^0(z+B^i) = \lim_{\varepsilon\to 0}(1/\varepsilon) \int_0^t I(0 \le z+B_s^i \le \varepsilon)\, ds$ is the (right) local time of $z+B^i$ at 0 upon noting that $\ell_t^0(z+B^i) = \ell_t^{-z}(B^i)$ for $t \ge 0$ where i equals 1 or 2 with z being equal to x or y respectively. Making use of (6.6) in (6.5) and recalling by (5.6) that $F_x(0,y) = V_x(0,y) - G_x(0,y) = -1$ and $F_y(x,0) = V_y(x,0) - G_y(x,0) = -1$ we find that

$$F(X_T^x, Y_T^y) = F(x,y) + \int_0^T (V_x(X_t^x, Y_t^y) - 1)\, d\beta_t^1 + \int_0^T (V_y(X_t^x, Y_t^y) - 1)\, d\beta_t^2 - \ell_T^{-x}(B^1) - \ell_T^{-y}(B^2) + \int_0^T c\, I((X_t, Y_t) \in C)\, dt \tag{6.7}$$

where in the final integral we also use (5.3) and (5.4) above. Since $x' \mapsto V(x', y)$ is convex and increasing on $[0,\infty)$ with $V(x', y) > x'+y$ for $x' \in [0, b^{-1}(y))$, we see that $0 \le V_x(x', y) \le 1$ for all $x' \in [0,\infty)$. Similarly (by symmetry) we see that $0 \le V_y(x, y') \le 1$ for all $y' \in [0,\infty)$. This shows that the stochastic integrals with respect to β^1 and β^2 in (6.7) are (continuous) martingales. Taking E on both sides in (6.7) we therefore find that

$$\mathsf{E}\big[F(X_T^x, Y_T^y)\big] = F(x,y) - \mathsf{E}\big[\ell_T^{-x}(B^1)\big] - \mathsf{E}\big[\ell_T^{-y}(B^2)\big] + \mathsf{E}\Big[\int_0^T c\, I((X_t, Y_t) \in C)\, dt\Big] \tag{6.8}$$

for $T > 0$ and $(x, y) \in [0,\infty)^2$ given and fixed.

We proceed by calculating each expected value in (6.8) individually. Firstly, note by the scaling property of B^1 and B^2 that

$$\mathsf{E}\big[F(X_T^x, Y_T^y)\big] = \mathsf{E}\big[F(|x+B_T^1|, |y+B_T^2|)\big] = \mathsf{E}\big[F(|x+\sqrt{T}B_1^1|, |y+\sqrt{T}B_1^2|)\big] \to 0 \tag{6.9}$$

as $T \to \infty$, because $(|x+\sqrt{T}B_1^1(\omega)|, |y+\sqrt{T}B_1^2(\omega)|)$ belongs to D for all $T \ge T'(\omega)$ with some $T'(\omega)$ large enough whenever $B_1^1(\omega) \ne 0$ and $B_1^2(\omega) \ne 0$ for $\omega \in \Omega$ so that $F = V - G$ takes value 0 at each such $(|x+\sqrt{T}B_1^1(\omega)|, |y+\sqrt{T}B_1^2(\omega)|)$, while by (4.4)+(4.5) above we have $0 \le F = V - G \le V(0,0) < \infty$ on $[0,\infty)^2$ so that the dominated convergence theorem is applicable. Secondly, using the scaling property of B^i we find that

$$\mathsf{E}\big[\ell_T^{-z}(B^i)\big] = \mathsf{E}\Big[\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^T I(-z \le B_t^i \le -z+\varepsilon)\, dt\Big] \tag{6.10}$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^T \mathsf{P}(-z \le \sqrt{t} B_1^i \le -z+\varepsilon)\, dt$$

$$= \int_0^T \frac{1}{\sqrt{t}} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon/\sqrt{t}} \Big(\varPhi\Big(-\frac{z}{\sqrt{t}} + \frac{\varepsilon}{\sqrt{t}}\Big) - \varPhi\Big(-\frac{z}{\sqrt{t}}\Big)\Big)\, dt$$

$$= \int_0^T \frac{1}{\sqrt{t}} \varphi\Big(-\frac{z}{\sqrt{t}}\Big)\, dt = \int_0^T \frac{1}{\sqrt{t}} \varphi\Big(\frac{z}{\sqrt{t}}\Big)\, dt$$

where the second equality above is justified by the occupation times formula (see e.g. [8, p. 224]) combined with Fubini's theorem and the third equality above follows by the dominated convergence theorem (using the mean value theorem) for $i = 1, 2$ with $z = x, y$ respectively. Thirdly, by Fubini's theorem we have

$$\text{(6.11)} \quad \mathsf{E}\Big[\int_0^T c I((X_t, Y_t) \in C)\, dt \Big] = c \int_0^T \mathsf{P}((X_t, Y_t) \in C)\, dt$$

$$= c \int_0^T \Big[\mathsf{P}(X_t^x \le 1/2c) + \mathsf{P}\big(X_t^x > 1/2c\,,\, Y_t^y < b(X_t^x)\big) \Big]\, dt$$

where we use (4.23) above. To calculate the two probabilities above note firstly that

$$\text{(6.12)} \quad \mathsf{P}(X_t^x \le 1/2c) = \mathsf{P}\big(-\tfrac{1}{2c} \le x + B_t^1 \le \tfrac{1}{2c}\big) = \mathsf{P}\Big(\frac{-x-\frac{1}{2c}}{\sqrt{t}} \le B_1^1 \le \frac{-x+\frac{1}{2c}}{\sqrt{t}} \Big)$$

$$= \varPhi\Big(\frac{-x+\frac{1}{2c}}{\sqrt{t}}\Big) - \varPhi\Big(\frac{-x-\frac{1}{2c}}{\sqrt{t}}\Big) = \varPhi\Big(\frac{x+\frac{1}{2c}}{\sqrt{t}}\Big) - \varPhi\Big(\frac{x-\frac{1}{2c}}{\sqrt{t}}\Big)$$

where in the final equality we use that $\varPhi(z) + \varPhi(-z) = 1$ for all $z \in I\!R$. Next note that

$$\text{(6.13)} \quad \mathsf{P}\big(X_t^x > 1/2c\,,\, Y_t^y < b(X_t^x)\big) = \int_{\frac{1}{2c}}^{\infty} \int_0^{b(x')} f_{X_t^x}(x') f_{Y_t^y}(y')\, dy'\, dx'$$

$$= \int_{\frac{1}{2c}}^{\infty} F_{Y_t^y}(b(x'))\, f_{X_t^x}(x')\, dx'$$

where we use that X_t^x and Y_t^y are independent. Moreover, as in (6.12) we find that

$$\text{(6.14)} \quad F_{Y_t^y}(y') = \mathsf{P}(Y_t^y \le y') = \mathsf{P}\big(-y' \le y + B_t^2 \le y'\big) = \varPhi\Big(\frac{y+y'}{\sqrt{t}}\Big) - \varPhi\Big(\frac{y-y'}{\sqrt{t}}\Big)$$

and by symmetry

$$\text{(6.15)} \quad F_{X_t^x}(x') = \mathsf{P}(X_t^x \le x') = \mathsf{P}\big(-x' \le x + B_t^1 \le x'\big) = \varPhi\Big(\frac{x+x'}{\sqrt{t}}\Big) - \varPhi\Big(\frac{x-x'}{\sqrt{t}}\Big)$$

so that by differentiating we find that

$$\text{(6.16)} \quad f_{X_t^x}(x') = \partial_{x'} F_{X_t^x}(x') = \frac{1}{\sqrt{t}} \Big(\varphi\Big(\frac{x+x'}{\sqrt{t}}\Big) + \varphi\Big(\frac{x-x'}{\sqrt{t}}\Big) \Big)$$

for $x', y' \in [0, \infty)$. Inserting (6.14) and (6.16) into (6.13) we obtain an integral expression for the probability on the left-hand side of (6.13) above. Inserting (6.12) and (6.13) back into

(6.11) we obtain a double-integral expression for the expectation on the left-hand side of (6.11) above. Using (6.10) and (6.11) and combining the three expectations on the right-hand side of (6.8) into an integral from 0 to T with respect to t, letting $T \to \infty$ and using (6.9), we obtain the representation (6.3) upon recalling that $F = V - G$ and $G(x,y) = x+y$ for $(x,y) \in [0,\infty)^2$. Moreover, the fact that τ_b is optimal in (3.1) follows from (4.24) above. Note that the right-hand side bound in (4.21) with $\bar{b}$ from (4.18) (see Figure 1) guarantees that $\mathsf{E}_{x,y}(\tau_b) < \infty$ for all $(x,y) \in [0,\infty)^2$. Finally, inserting $y = b(x)$ in (6.3) and using that $V(x,b(x)) = G(x,b(x)) = x+b(x)$ for $x > 1/2c$, we see that b solves (6.2) as claimed.

(ii) *Uniqueness.* To show that b is a unique solution to the equation (6.2) in the specified class of functions, one can adopt the four-step procedure from the proof of uniqueness given in [2, Theorem 4.1] extending and further refining the original uniqueness arguments from [4, Theorem 3.1]. Given that the present setting creates no additional difficulties, we omit further details of this verification and this completes the proof. □

Remark 15. Note from the proof above using the recurrence property of (X,Y) in $[0,\infty)^2$ that none of the three terms appearing under the integral sign on the right-hand side of (6.3) is integrable on its own over $[0,\infty)$ with respect to the time variable t. Nonetheless, when the three terms are combined into a single term, as the proof above shows, then the resulting function is integrable.

7. Sharp inequalities

In this section we present applications of the results derived in the previous sections to sharp inequalities for the maximum and sum of two independent reflecting Brownian motions.

1. To indicate dependence of the value function V from (3.1) on c we will write V_c for $c>0$ given and fixed. Recalling (2.2) and using the scaling property of B^1 and B^2 combined with the fact that the supremum in (3.1) is attained at the first entry time of (X,Y) into D, it is easily verified that

$$V_c(x,y) = \frac{1}{c} V_1(cx,cy) \tag{7.1}$$

for all $(x,y) \in [0,\infty)^2$ and $c>0$. From (3.1) and (7.1) we see that

$$\mathsf{E}\big(|B^1_\tau|+|B^2_\tau|\big) \le V_c(0,0)+c\,\mathsf{E}(\tau) = \frac{1}{c} V_1(0,0)+c\,\mathsf{E}(\tau) \tag{7.2}$$

for every $c>0$ and every stopping time τ of (B^1,B^2) with $\mathsf{E}(\tau)<\infty$ given and fixed. Minimising the right-hand side in (7.2) over $c>0$ we find that the (unique) minimum is attained at $c = \sqrt{V_1(0,0)/\mathsf{E}(\tau)}$. Evaluating the right-hand side in (7.2) at this c gives

$$\mathsf{E}\big(|B^1_\tau|+|B^2_\tau|\big) \le 2\sqrt{V_1(0,0)}\sqrt{\mathsf{E}(\tau)} \tag{7.3}$$

for all stopping times τ of (B^1,B^2) with $\mathsf{E}(\tau)<\infty$. Moreover, writing U_c to indicate dependence of the value function U from (2.1) on $c>0$ given and fixed, in exactly the same way as in (7.1) above we find that

$$V_c(x,y) = \frac{1}{c} V_1(cx,cy) \tag{7.4}$$

for all $(x,y) \in [0,\infty)^2$ and $c>0$. The same arguments as in (7.2) and (7.3) show that

$$\mathsf{E}\big(|B^1_\tau|\vee|B^2_\tau|\big) \le 2\sqrt{U_1(0,0)}\sqrt{\mathsf{E}(\tau)} \tag{7.5}$$

for all stopping times τ of (B^1,B^2) with $\mathsf{E}(\tau)<\infty$. Finally, by (3.2) and (7.1) we see that

$$U_c(0,0) = \frac{1}{\sqrt{2}}V_{c\sqrt{2}}(0,0) = \frac{1}{2c}V_1(0,0)\,. \tag{7.6}$$

Using this identity in (7.5) gives

$$\mathsf{E}\big(|B^1_\tau|\vee|B^2_\tau|\big) \le \sqrt{2}\sqrt{V_1(0,0)}\sqrt{\mathsf{E}(\tau)} \tag{7.7}$$

for all stopping times τ of (B^1,B^2) with $\mathsf{E}(\tau)<\infty$.

Corollary 16. *The constants* $2\sqrt{V_1(0,0)}$ *and* $\sqrt{2}\sqrt{V_1(0,0)}$ *are best possible in the inequalities* (7.3) *and* (7.7) *respectively.*

Proof. Let τ_1 denote the optimal stopping time in (3.1) when $c=1$. We claim that equality in (7.3) is attained at τ_1 . To see this note that

$$\begin{aligned}\mathsf{E}\big(|B^1_{\tau_1}|+|B^2_{\tau_1}|\big) &= \mathsf{E}\big(|B^1_{\tau_1}|+|B^2_{\tau_1}|-\tau_1\big)+\mathsf{E}(\tau_1) = V_1(0,0)+\mathsf{E}(\tau_1)\\ &= \Big(\frac{V_1(0,0)}{\sqrt{\mathsf{E}(\tau_1)}}+\sqrt{\mathsf{E}(\tau_1)}\Big)\sqrt{\mathsf{E}(\tau_1)} \le 2\sqrt{V_1(0,0)}\sqrt{\mathsf{E}(\tau_1)}\end{aligned} \tag{7.8}$$

where the inequality follows from (7.3) above. From (7.8) we see that

$$\frac{V_1(0,0)}{\sqrt{\mathsf{E}(\tau_1)}}+\sqrt{\mathsf{E}(\tau_1)} \le 2\sqrt{V_1(0,0)} \tag{7.9}$$

or equivalently $(\sqrt{V_1(0,0)}-\sqrt{\mathsf{E}(\tau_1)}\,)^2 \le 0$ implying that $V_1(0,0)=\mathsf{E}(\tau_1)$. Making use of this identity in (7.8) above we obtain

$$\mathsf{E}\big(|B^1_{\tau_1}|+|B^2_{\tau_1}|\big) = V_1(0,0)+\mathsf{E}(\tau_1) = 2V_1(0,0) = 2\sqrt{V_1(0,0)}\sqrt{\mathsf{E}\tau_1} \tag{7.10}$$

showing that equality in (7.3) is attained at τ_1 as claimed. Similarly, letting σ_1 denote the optimal stopping time in (7.5) and using the same arguments as above with $|B^1_{\sigma_1}|\vee|B^2_{\sigma_1}|$ in place of $|B^1_{\tau_1}|+|B^2_{\tau_1}|$, we find that equality in (7.5) and therefore in (7.7) as well is attained at σ_1 . This completes the proof. □

2. We conclude this section with a few comments on the constant $V_1(0,0)$ which plays a prominent role in the inequalities (7.3) and (7.7) above.

Remark 17. Note from (6.3) that the constant $V_1(0,0)$ appearing in both (7.3) and (7.7) above is given by

$$\begin{aligned}V_1(0,0) = \int_0^\infty \Big[\sqrt{\frac{2}{\pi t}} &- \Big(\Phi\Big(\frac{1}{2\sqrt{t}}\Big)-\Phi\Big(-\frac{1}{2\sqrt{t}}\Big)\Big)\\ &- \frac{2}{\sqrt{t}}\int_{1/2}^\infty \Big(\Phi\Big(\frac{b(z)}{\sqrt{t}}\Big)-\Phi\Big(-\frac{b(z)}{\sqrt{t}}\Big)\Big)\varphi\Big(\frac{z}{\sqrt{t}}\Big)\,dz\Big]\,dt\end{aligned} \tag{7.11}$$

where b is the unique solution to (6.2) as described above. Let a stopping time τ of (B^1, B^2) with $\mathsf{E}(\tau) < \infty$ be given and fixed. Using that $\mathsf{E}(|B^1_\tau|+|B^2_\tau|) \leq \mathsf{E}(|B^1_\tau|)+\mathsf{E}(|B^2_\tau|) \leq \sqrt{\mathsf{E}(B^1_\tau)^2} + \sqrt{\mathsf{E}(B^2_\tau)^2} \leq 2\sqrt{\mathsf{E}(\tau)}$ by Jensen's inequality and the optional sampling theorem, from the result of Corollary 16 we see that $2\sqrt{V_1(0,0)} \leq 2$ which is the same as $V_1(0,0) \leq 1$. Similarly, by (7.7) we see that $\mathsf{E}\,(|B^i_\tau|) \leq \mathsf{E}\,(|B^1_\tau| \vee |B^2_\tau|) \leq \sqrt{2}\sqrt{V_1(0,0)}\sqrt{\mathsf{E}(\tau)}$ while it is known that $\mathsf{E}(|B^i_\tau|) \leq \sqrt{\mathsf{E}(\tau)}$ for $i = 1, 2$ and the constant 1 is best possible in the previous inequality (cf. [3]) so that $1 \leq \sqrt{2}\sqrt{V_1(0,0)}$ which is the same as $1/2 \leq V_1(0,0)$. Combining the two inequalities derived we see that

$$\frac{1}{2} \leq V_1(0,0) \leq 1 \tag{7.12}$$

and clearly both inequalities are strict. More accurate numerical calculations of $V_1(0,0)$ appear to be challenging given the intricacy of the expression (7.11) above.

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