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Maximal Inequalities for Reflected Brownian Motion with Drift

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Let $\beta = (\beta_t)_{t \ge 0}$ denote the unique strong solution of the equation

$$d\beta_t = -\mu \, sign(\beta_t) \, dt + dB_t$$

satisfying $\beta_0 = 0$, where $\mu > 0$ and $B = (B_t)_{t \ge 0}$ is a standard Brownian motion. Then $|\beta| = (|\beta_t|)_{t \ge 0}$ is known to be a realisation of the reflected Brownian motion with drift $-\mu$. Using this representation we show that there exist universal constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 E\Big(H_\mu(\tau)\Big) \le E\Big(\max_{0 \le t \le \tau} |\beta_t|\Big) \le c_2 E\Big(H_\mu(\tau)\Big)$$

for all stopping times τ of β, where $H_\mu(x) = F_\mu^{-1}(x)$ denotes the inverse of the map $F_\mu(x) = (e^{2\mu x} - 2\mu x - 1)/2\mu^2$. In addition, we show that

$$E\Big(\max_{0 \le t \le \tau} |\beta_t|\Big) \le G_\mu\Big(E(\tau)\Big)$$

for all stopping times τ of β, where $G_\mu(x) = \inf_{c>0}(cx + (1/2\mu)\log(1+\mu/c))$. Both inequalities have their well-known analogues for Brownian motion (obtained by letting $\mu \downarrow 0$). The method of proof relies upon Lenglart's domination principle, Itô calculus, and optimal stopping techniques.

1. Introduction

A classic definition in the theory of Markov processes states that the process $X = (X_t)_{t \ge 0}$ is a *reflected Brownian motion with drift* $\lambda \in I\!R$, if X is a non-negative diffusion Markov process associated with the infinitesimal operator $I\!L_X$ acting on a space of C^2-functions $f : [0, \infty) \to I\!R$ satisfying $f'(0+) = 0$ according to the rule:

$$(I\!L_X f)(x) = \lambda f'(x) + \frac{1}{2} f''(x) \tag{1.1}$$

(see e.g. [4]). If such a process X also satisfies $X_0 = x$ for some $x \in I\!R$ given and fixed, then it is customary to write $X \sim RBM_x(\lambda)$.

In the case $\lambda = 0$ it is well-known that such a process can be realised as

$$X_t = |x + B_t| \tag{1.2}$$

where $B = (B_t)_{t \ge 0}$ is a standard Brownian motion (see e.g. [4] or [7]).

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In the case of general $\lambda \in I\!R$ it was shown only recently (see [3]) that the corresponding analogue of (1.2) takes the following form:

$$X_t = |\beta_t| \tag{1.3}$$

where $\beta = (\beta_t)_{t \geq 0}$ is the unique strong solution of the "bang-bang" equation:

$$d\beta_t = \lambda \, sign(\beta_t) \; dt + dB_t \tag{1.4}$$

with $\beta_0 = x$. Then again $X \sim RBM_x(\lambda)$ and thus each reflected Brownian motion with drift can be realised as the modulus of the "bang-bang" process β . This representation is useful in many ways as it enables one to employ the well-known methods and techniques from the theory of stochastic differential equations and stochastic calculus (e.g. Itô-Tanaka formula).

In this note we shall demonstrate this fact by establishing two inequalities for reflected Brownian motion *with drift* as a counterpart of the well-known inequalities for Brownian motion *without drift*. The first inequality we have in mind is the *Burkholder-Gundy inequality* [1] stating that

$$c_1 E\Big(\sqrt{\tau}\Big) \leq E\bigg(\max_{0 \leq t \leq \tau} |B_t|\bigg) \leq c_2 E\Big(\sqrt{\tau}\Big) \tag{1.5}$$

for all stopping times τ of B (see Theorem 2.1). The second inequality is the well-known Doob-type inequality for Brownian motion stating that

$$E\bigg(\max_{0 \leq t \leq \tau} |B_t|\bigg) \leq \sqrt{2}\,\sqrt{E(\tau)} \tag{1.6}$$

for all stopping times τ of B (see Theorem 2.4). This inequality was established independently by several authors, and the constant $\sqrt{2}$ is known to be best possible (see [2]).

2. The results and proof

1. We first establish an analogue of the Burkholder-Gundy inequality (1.5). For this we shall define a function F_μ on $I\!R_+$ by setting

$$F_\mu(x) = \frac{e^{2\mu x} - 2\mu x - 1}{2\mu^2} \tag{2.1}$$

for $x \geq 0$. Then $x \mapsto F_\mu(x)$ is strictly increasing on $I\!R_+$, and we shall set

$$H_\mu(x) = F_\mu^{-1}(x) \tag{2.2}$$

to denote its inverse for $x \geq 0$ (see Remarks 2.3 below).

Theorem 2.1

Let $X = (X_t)_{t \geq 0}$ be a reflected Brownian motion with drift $-\mu$ such that $X_0 = 0$ where $\mu > 0$ is given and fixed. Then the following inequality is satisfied:

$$c_1 E\Big(H_\mu(\tau)\Big) \leq E\bigg(\max_{0 \leq t \leq \tau} X_t\bigg) \leq c_2 E\Big(H_\mu(\tau)\Big) \tag{2.3}$$

for all stopping times τ of X , where $c_1 > 0$ and $c_2 > 0$ are some universal constants.

Proof. We shall present a proof which is based upon the following domination principle established by Lenglart [5] (see [7] p.155-156).

Lemma 2.2
Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ *be a filtered probability space, let* $Z = (Z_t)_{t\geq 0}$ *be an* $(\mathcal{F}_t)$*-adapted non-negative continuous process, let* $A = (A_t)_{t\geq 0}$ *be an* $(\mathcal{F}_t)$*-adapted increasing continuous process satisfying* $A_0 = 0$ *, and let* $H : \mathbf{R}_+ \to \mathbf{R}_+$ *be an increasing continuous function satisfying* $H(0) = 0$ *. Suppose that it is known that*

$$E(Z_\tau) \leq E(A_\tau) \tag{2.4}$$

for all bounded $(\mathcal{F}_t)$*-stopping times* τ *such that* $\big(Z_{t\wedge\tau}\big)_{t\geq 0}$ *is bounded. Then we have:*

$$E\Big(\sup_{0\leq t\leq\tau} H(Z_t)\Big) \leq E\Big(\widetilde{H}(A_\tau)\Big) \tag{2.5}$$

for all $(\mathcal{F}_t)$*-stopping times* τ *, where*

$$\widetilde{H}(x) = x \int_x^\infty \frac{1}{s}\, dH(s) + 2H(x) \tag{2.6}$$

for all $x \geq 0$ *.*

Proof. By Fubini's theorem we find:

$$\begin{aligned} E\Big(\sup_{0\leq t\leq\tau} H(Z_t)\Big) &= E\Big(H\Big(\sup_{0\leq t\leq\tau} Z_t\Big)\Big) = E\Big(\int_0^\infty 1_{\big\{\sup_{0\leq t\leq\tau} Z_t\, \geq\, s\big\}}\, dH(s)\Big) \\ &\leq \int_0^\infty \Big(P\Big\{\sup_{0\leq t\leq\tau} Z_t \geq s\, ,\; A_\tau \leq s\Big\} + P\Big\{A_\tau > s\Big\}\Big)\, dH(s) \end{aligned} \tag{2.7}$$

since $s \mapsto H(s)$ is increasing and continuous. Consider the stopping times

$$\begin{aligned} \tau_1 &= \inf\,\{\, t\geq 0 \mid Z_t \geq s\,\} \\ \tau_2 &= \inf\,\{\, t\geq 0 \mid A_t \geq s\,\}\ . \end{aligned} \tag{2.8}$$

Then Markov's inequality and (2.4) imply:

$$\begin{aligned} P\Big\{\sup_{0\leq t\leq\tau} Z_t \geq s\, ,\; A_\tau \leq s\Big\} &\leq P\Big\{\,\tau_1 \leq \tau\, ,\; \tau_2 \geq \tau\,\Big\} \leq P\Big\{\, Z_{\tau_1\wedge\tau_2\wedge\tau} \geq s\,\Big\} \\ &\leq \frac{1}{s}\, E\big(A_{\tau_1\wedge\tau_2\wedge\tau}\big) \end{aligned} \tag{2.9}$$

whenever τ is bounded. From (2.7) and (2.9) we conclude:

$$\begin{aligned} E\Big(\sup_{0\leq t\leq\tau} H(Z_t)\Big) &\leq \int_0^\infty \Big(\frac{1}{s}\, E\Big(A_\tau\, 1_{\big\{A_\tau\leq s\big\}}\Big) + 2P\Big\{A_\tau > s\Big\}\Big)\, dH(s) \\ &= E\Big(A_\tau \int_{A_\tau}^\infty \frac{1}{s}\, dH(s)\Big) + 2E\Big(H(A_\tau)\Big) = E\Big(\widetilde{H}(A_\tau)\Big) \end{aligned} \tag{2.10}$$

for all bounded τ . Finally, observe that $x \mapsto \widetilde{H}(x)$ is increasing, and pass to the limit when $k \to \infty$ to reach any τ through bounded ones $\tau \wedge k$. This completes the proof of the lemma. $\square$

From (1.3) and (1.4) above we know that the process X can be realised as $X_t = |\beta_t|$ where $\beta = (\beta_t)_{t \geq 0}$ solves the equation:

$$d\beta_t \; = \; -\mu \; sign(\beta_t) \; dt + dB_t \tag{2.11}$$

with $\beta_0 = 0$. The infinitesimal operator $I\!\!L_X$ of X is given by (1.1) with $\lambda = -\mu$.

Extend the map F_μ from (2.1) to $I\!\!R_-$ by setting $F_\mu(x) = F_\mu(-x)$ for $x < 0$. Note that $x \mapsto F_\mu(x)$ is even and satisfies $I\!\!L_X(F_\mu) = 1$ on $I\!\!R$ with $F_\mu(0) = 0$. Moreover, since $x \mapsto F_\mu(x)$ is C^2 , Itô formula can be applied to $F_\mu(\beta_t)$ and this yields:

$$F_\mu(X_t) = F_\mu\big(|\beta_t|\big) = F_\mu(\beta_t) = \int_0^t \Big(I\!\!L_X\big(F_\mu\big)\Big)(\beta_s) \; ds + \int_0^t F_\mu'(\beta_s) \; dB_s = t + M_t \tag{2.12}$$

where $M = (M_t)_{t \geq 0}$ is a continuous local martingale given by $M_t = \int_0^t F_\mu'(\beta_s) \; dB_s$.

Let τ be a bounded stopping time such that $(F_\mu(X_{t\wedge\tau}))_{t \geq 0}$ is bounded. Passing to a localising sequence of stopping times for M if needed, we find from (2.12) that

$$E\Big(F_\mu(X_\tau)\Big) \; = \; E(\tau) \tag{2.13}$$

by means of the optional sampling theorem (see e.g. [7]).

Elementary calculations based on the L'Hospital rule show that

$$\sup_{x>0} \left(\frac{x}{H_\mu(x)} \int_x^\infty \frac{dH_\mu(s)}{s} \right) \; = \; 1 \tag{2.14}$$

for all $\mu > 0$. By definition (2.6) this fact implies that

$$\widetilde{H}_\mu(x) \; \leq \; 3 \; H_\mu(x) \tag{2.15}$$

for all $x \geq 0$. It enables us to conclude the proof in two steps as follows.

Setting $Z_t = F_\mu(X_t)$ and $A_t = t$ we easily see by (2.13) that all hypotheses in Lemma 2.2 are satisfied, and thus by (2.5) and (2.15) we find:

$$E\bigg(\max_{0 \leq t \leq \tau} X_t\bigg) = E\bigg(\max_{0 \leq t \leq \tau} H_\mu(Z_t)\bigg) \leq E\Big(\widetilde{H}_\mu(A_\tau)\Big) \leq 3\, E\Big(H_\mu(A_\tau)\Big) = 3\, E\Big(H_\mu(\tau)\Big) \tag{2.16}$$

for all stopping times τ of X . This establishes the right-hand side inequality in (2.3).

On the other hand, setting $Z_t = t$ and $A_t = \max_{0 \leq s \leq t} F_\mu(X_s)$ we again see easily by (2.13) that all hypotheses in Lemma 2.2 are satisfied, and thus by (2.5) and (2.15) we find:

$$E\Big(H_\mu(\tau)\Big) = E\bigg(\max_{0 \leq t \leq \tau} H_\mu(Z_t)\bigg) \leq E\Big(\widetilde{H}_\mu(A_\tau)\Big) \leq 3\, E\Big(H_\mu(A_\tau)\Big) = 3\, E\bigg(\max_{0 \leq t \leq \tau} X_t\bigg) \tag{2.17}$$

for all stopping times τ of X . This establishes the left-hand side inequality in (2.3), and the proof of the theorem is complete. $\square$

Remarks 2.3

1. It is easily seen that $x \mapsto F_\mu(x)$ is convex on $I\!R_+$, and thus $x \mapsto H_\mu(x)$ is concave on $I\!R_+$. Hence applying Jensen's inequality in (2.3) we obtain:

$$E\Big(\max_{0\le t\le\tau} X_t\Big) \le c_2\, H_\mu\Big(E(\tau)\Big) \tag{2.18}$$

for all stopping times τ of X .

2. Recalling further (2.13) we see that (2.18) implies the following Doob-type bound:

$$E\Big(\max_{0\le t\le\tau} X_t\Big) \le c_2\, H_\mu\Big(E\big(F_\mu(X_\tau)\big)\Big) \tag{2.19}$$

which is valid for all stopping times τ of X such that $E(\tau) < \infty$.

3. It is easily verified that $F_\mu(x) \to x^2$ as $\mu \downarrow 0$, and thus

$$\lim_{\mu\downarrow 0} H_\mu(x) = \sqrt{x} \tag{2.20}$$

for all $x \ge 0$. Passing to the limit in (2.3) when $\mu \downarrow 0$ we thus recover (1.5). In exactly the same way we see that (2.18) implies (1.6) with the constant $c_2 = 3$ on the right-hand side.

4. Using that $(2\mu x)^k/(2\mu^2 k!) \le F_\mu(x) \le (e^{2\mu x}-1)/(2\mu^2)$ we find easily that

$$\frac{1}{2\mu}\log\big(1+2\mu^2 x\big) \le H_\mu(x) \le \frac{k/2}{(2\mu)^{1-2/k}}\; x^{1/k} \tag{2.21}$$

for all $x \ge 0$ and all $k \ge 2$. The left-hand side estimate in (2.21) is more accurate for large x , and the right-hand side estimate in (2.21) is more accurate for small x .

2. We turn to establishing an analogue of the Doob-type inequality (1.6). Note that although (2.18) above provides such a bound, it fails to capture (1.6) with the best constant $\sqrt{2}$ in the limit when $\mu \downarrow 0$. In the following theorem we present a result which repairs this deficiency.

Theorem 2.4

1. Let $X = (X_t)_{t\ge 0}$ be a reflected Brownian motion with drift $-\mu$ such that $X_0 = 0$ where $\mu > 0$ is given and fixed. Then the following inequality is satisfied:

$$E\Big(\max_{0\le t\le\tau} X_t\Big) \le \inf_{c>0}\Big(c\,E(\tau) + \frac{1}{2\mu}\log\Big(1+\frac{\mu}{c}\Big)\Big) \tag{2.22}$$

for all stopping times τ of X .

Moreover, the following inequalities are satisfied:

$$E\Big(\max_{0\le t\le\tau} X_t\Big) \le \sqrt{\frac{1}{2}E(\tau)} + \frac{1}{2\mu}\log\Big(1+\mu\sqrt{2E(\tau)}\Big) \tag{2.23}$$

$$E\Big(\max_{0\le t\le\tau} X_t\Big) \le \frac{1}{2\mu}\Big(1+\log\Big(1+2\mu^2 E(\tau)\Big)\Big) \tag{2.24}$$

for all stopping times τ of X . The inequality (2.23) is more accurate for small μ (letting $\mu \downarrow 0$ in (2.23) we obtain (1.6)), and the inequality (2.24) is more accurate for large μ (letting $\mu \downarrow 0$ in (2.24) the right-hand side tends to zero).

Proof. We shall only sketch the main points in the proof, and for remaining details and more information we shall refer to [2] (Theorem 3) and [6] (Corollary 3.2)

From (1.3) and (1.4) above we know that the process X can be realised as $X_t = |\beta_t|$ where $\beta = (\beta_t)_{t \geq 0}$ solves (2.11) with $\beta_0 = x \geq 0$. Set $S_t = (\max_{0 \leq r \leq t} X_r) \vee s$ for $s \geq x$ and consider the optimal stopping problem:

$$V_*(x,s) = \sup_\tau E_{x,s}\Big(S_\tau - c\tau\Big) \tag{2.25}$$

where $X_0 = x$ and $S_0 = 0$ under $P_{x,s}$. The supremum in (2.25) is taken over all stopping times τ of X satisfying $E_{x,s}(\tau) < \infty$, and the constant $c > 0$ is given and fixed.

This problem belongs to the general theory of optimal stopping for Markov processes that leads to the following *free-boundary* problem:

$$(\mathbb{L}_X V_*)(x,s) = c \quad (g_*(s) < x < s) \tag{2.26}$$

$$\frac{\partial V_*}{\partial s}(x,s)\Big|_{x=s-} = 0 \quad (\textit{normal reflection}) \tag{2.27}$$

$$V_*(x,s)\Big|_{x=g_*(s)+} = s \quad (\textit{instantaneous stopping}) \tag{2.28}$$

$$\frac{\partial V_*}{\partial x}(x,s)\Big|_{x=g_*(s)+} = 0 \quad (\textit{smooth fit}) \tag{2.29}$$

where $s \mapsto g_*(s)$ is an optimal stopping boundary (to be found). Since X is a non-negative diffusion and 0 is an instantaneously-reflecting (regular) boundary point for X, the following stopping time is to be optimal in (2.25):

$$\tau_* = \inf\Big\{\, t > 0 \mid S_t \geq s_*\,,\ X_t \leq g_*(S_t)\,\Big\} \tag{2.30}$$

where $s_* \geq 0$ satisfies $g_*(s_*) = 0$.

The solution of (2.26)-(2.29) is given by

$$V_*(x,s) = s + c\int_{g_*(s)}^{x} \big(L(x) - L(y)\big)\, m(dy) \tag{2.31}$$

for $0 \leq g_*(s) \leq x \leq s$, where the optimal boundary $s \mapsto g_*(s)$ is the *maximal* solution of the first-order (nonlinear) differential equation:

$$g'(s) = \frac{L'(g(s))}{2\,c\,\big(L(s) - L(g(s))\big)} \tag{2.32}$$

staying strictly below the diagonal in $\mathbb{R}^2_+$. In (2.31) and (2.32) we set $L = L(x)$ denote the *scale function* of X, and $m = m(dx)$ denotes the *speed measure* of X. From (2.11) we read that

$$L(x) = \frac{e^{2\mu x} - 1}{2\mu} \quad (x \geq 0) \tag{2.33}$$

$$m(dx) = 2\, e^{-2\mu x}\, dx\,. \tag{2.34}$$

A strong Markov property implies that $V_*(x,s) = V_*(s_*,s_*) - cE_{x,s}(\tau_{s_*})$ for $0 \le x \le s \le s_*$, where $\tau_{s_*} = \inf\{\, t > 0 \mid X_t = s_* \,\}$, and this leads to the following explicit expression:

$$V_*(x,s) \;=\; s_* + c \int_0^x \big(L(x) - L(y) \big)\, m(dy) \tag{2.35}$$

for $0 \le x \le s \le s_*$.

Inserting (2.33) into (2.32) we easily see that the linear function $g_*(s) = s - s_*$ is the maximal solution of (2.32), where $s_* > 0$ is explicitly given by

$$s_* \;=\; \frac{1}{2\mu} \log\left(1 + \frac{\mu}{c} \right) \;. \tag{2.36}$$

Thus $g_*(s) = s - s_*$ is the optimal stopping boundary i.e. the stopping time (2.30) is optimal in (2.25), and from (2.35) we see that

$$V_*(0,0) \;=\; s_* \;. \tag{2.37}$$

By means of (2.36) this fact establishes (2.22), and the first part of the proof is complete.

In order to derive the remaining statements, note first that for each stopping time τ of X such that $E(\tau) < \infty$, the infimum in (2.22) is attained at

$$c_* \;=\; c_*(\mu, E(\tau)) \;=\; -\frac{\mu}{2} + \sqrt{\frac{1}{2E(\tau)} + \mu^2} \;. \tag{2.38}$$

Inserting this expression into the right-hand side of (2.22) we obtain a sharp inequality (where equality is attained at the stopping time (2.30) for each $\mu > 0$ given and fixed). Moreover, letting $\mu \downarrow 0$ in this inequality we obtain (1.6) with $\sqrt{2}$ on the right-hand side.

Unfortunately, the right-hand side of the inequality obtained by inserting (2.37) into (2.22) defines a complicated function of $E(\tau)$. Its simplification can be achieved upon observing that

$$c_*(\mu, E(\tau)) \;\rightarrow\; \frac{1}{\sqrt{2E(\tau)}} \qquad (\, \mu \downarrow 0 \,) \tag{2.39}$$

$$\mu\, c_*(\mu, E(\tau)) \;\rightarrow\; \frac{1}{2E(\tau)} \qquad (\, \mu \uparrow \infty \,) \;. \tag{2.40}$$

These facts indicate that letting $c = 1/\sqrt{2E(\tau)}$ in (2.22) we get an inequality precise for small μ, and letting $c = 1/(2\mu E(\tau))$ in (2.22) we obtain an inequality precise for large μ. As these two inequalities are just those written in (2.23) and (2.24) respectively, the proof is complete.

□

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