Proc. Funct. Anal. V (Dubrovnik 1997), *Various Publ. Ser.* Vol. 44, 1998, (77-104)
Preprint Ser. No. 25, 1991, *Math. Inst. Aarhus*

Extending Finite Measures to Perturbed σ-Algebras

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Necessary and sufficient conditions are given for the existence of a countably additive extension of a given finite measure in the case of any finite or disjoint countable perturbation of its σ-algebra. It provides a complete description of all countably additive extensions of finite measures in such cases. In particular, simple characterizations are obtained for the existence of some extremal countably additive extensions taking the outer or inner measure on members from the perturbation. In addition, one construction of a countably additive extension is presented in the case of any disjoint countable perturbation. Some parts of the calculus of non-measurable functions and sets needed in the proofs of the main results are developed. The main emphasis of the paper is on the method of proof, and our approach in general, towards the solution of the problem under consideration.

1. Introduction

Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and let $C_1, C_2, \ldots, C_n$ be arbitrary subsets of X. Then the following problem* was formulated in [19]:

What are necessary and sufficient conditions for the existence of a countably additive extension $\nu_{\alpha_1,\alpha_2,\ldots,\alpha_n}$ *of the given finite measure* μ *to the* σ*-algebra* $\sigma(\mathcal{A} \cup \{C_1, C_2, \ldots, C_n\})$ *such that* $\nu_{\alpha_1,\alpha_2,\ldots,\alpha_n}(C_i) = \alpha_i$ *for all* $i = 1, 2, \ldots, n$ *?*

In the case $n=1$ the answer to this question is well-known (see [5] p.43)**; in the case $n=2$ with C_1 and C_2 disjoint, a necessary and sufficient condition is given in [19]. Both of these results rely upon a direct construction of adequate extensions which uses facts about the so-called outer and inner traces of a finite measure. While the first proof is quite simple and direct, the second one is rather long and technically complicated. This fact has been indicated that for answering the general problem with not necessarily disjoint $C_1, C_2, \ldots, C_n$ for $n \geq 2$, one should search for a method of proof which would possibly be based on some deeper classical result in mathematical analysis dealing with extensions of functions. Indeed, in this paper we give an answer to this general problem by using a method of extension which essentially relies upon the well-known Hahn-Banach theorem; we also make use of some facts from a non-measurable calculus that is for this purpose developed (in particular the calculus of an upper and inner integral).

MR 1991 Mathematics Subject Classifications. Primary 28A12, 28A20. Secondary 46B20.
Key words and phrases: Extension, countably (finitely) additive, finite, measure, perturbation, the outer (inner) μ-measure, the upper (lower) μ-integral, the μ-hull, the μ-kernel, the Hahn-Banach theorem, outer (inner) μ-separated. © goran@imf.au.dk

* Closely related problems were treated in [3] and [8] (see also [20]). For this reason our main emphasis in this paper is rather on the method of proof (which provides particular necessary and sufficient conditions) than on the novelty of a solution to this problem.

** More general results may be found in [16] (see also [11]).

We shall begin by embedding the problem stated above in a general framework and recalling some basic definitions from [19]. If $(X, \mathcal{A})$ is a measurable space and $\mathcal{C}$ is an arbitrary family of subsets of X , then $\sigma(\mathcal{C})$ denotes the smallest σ-algebra on X that includes $\mathcal{C}$; the mapping

$$\mathcal{A} \overset{c}{\longmapsto} \sigma(\mathcal{A} \cup \mathcal{C})$$

is called *a perturbation of the σ-algebra $\mathcal{A}$ by the family $\mathcal{C}$* . In that case we shall often say that the family $\mathcal{C}$ itself is a perturbation. If $\mathcal{C} \subset \mathcal{A}$, then c is said to be *a trivial perturbation*, and if c is not trivial, we define

$$\mathrm{K} = \Big\{ \ \mathrm{card}\ (\mathcal{D}) \ \mid \ \mathcal{D} = (D_1, D_2, \dots) \subset \mathcal{C} \ , D_1 \notin \mathcal{A} \ , \ D_2 \notin \sigma(\mathcal{A} \cup \{D_1\}) \ , \\ D_3 \notin \sigma(\mathcal{A} \cup \{D_1, D_2\}) \ \dots \ \text{ and } \ \sigma(\mathcal{A} \cup \mathcal{D}) = \sigma(\mathcal{A} \cup \mathcal{C}) \ \Big\} \ .$$

If $\mathrm{K} \cap \mathbf{N} \neq \emptyset$, we put $n = \min\ \mathrm{K}$. In that case c is said to be an *n-element perturbation*; sometimes we shall also briefly say that such c is a *finite perturbation*. If $\mathrm{K} \cap \mathbf{N} = \emptyset$ but $\aleph_0 \in \mathrm{K}$, then c is said to be a *countable perturbation.* If $\mathrm{K} \cap (\mathbf{N} \cup \{\aleph_0\}) = \emptyset$, then c is said to be an *uncountable perturbation.* If the elements of the family $\mathcal{C}$ are disjoint, then c is said to be a *disjoint perturbation*.

Suppose now that a finite measure μ on the measurable space $(X, \mathcal{A})$ is defined, and let $\mathcal{C}$ be an arbitrary family of subsets of X . Then the question stated above becomes a particular instance of the following general problem*:

When does an extension of the given finite measure μ in the case of a perturbation of the σ-algebra $\mathcal{A}$ by the family $\mathcal{C}$ exist? In other words: *When does a measure ν on the σ-algebra $\sigma(\mathcal{A} \cup \mathcal{C})$ exist, such that the restriction of ν to the σ-algebra $\mathcal{A}$ equals μ*?

In this paper, as a consequence of the Hahn-Banach theorem, we shall first deduce a necessary and sufficient condition for the existence of a finitely additive extension ν of μ to 2^X such that $\nu(C) = \alpha(C)$ for all $C \in \mathcal{C}$, where $\alpha : \mathcal{C} \to \mathbf{R}_+$ is any given and fixed function. In the case when $\mathcal{C}$ is a finite family we shall actually show that the restriction of ν to $\sigma(\mathcal{A} \cup \mathcal{C})$ is countably additive. Moreover, we shall see that the given necessary and sufficient condition becomes particularly simple in the case when the elements of the family $\mathcal{C}$ are disjoint. Since every finite perturbation is obviously a disjoint finite perturbation, this fact will solve the extension problem of a finite measure in the case of any finite perturbation of its σ-algebra.

In addition, we shall find a necessary and sufficient condition for a finitely additive measure ν defined on $\sigma(\mathcal{A} \cup \mathcal{C})$, which agrees with μ on $\mathcal{A}$, to be countably additive, provided that $\mathcal{C}$ is a disjoint countable family. Together with the preceding necessary and sufficient condition for the existence of a finitely additive extension ν of μ to 2^X , this will give us a necessary and sufficient condition for the existence of a countably additive extension of a finite measure in the case of any disjoint countable perturbation.

Furthermore, as a consequence of the results derived, we shall present one construction of a countably additive extension of a given finite measure in the case of any disjoint countable perturbation of its σ-algebra. In this way we shall be able to conclude that the necessary and sufficient condition mentioned above covers real cases indeed.

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* The problem has a negative answer if $\mathcal{C}$ is a general countable family of sets (see [21] or [3]); it has provided a nice explanation why through the whole manuscript we were only able to treat perturbations which are in essence linked to the countable disjoint ones.

Finally, by applying some of the results obtained, we shall briefly study some typical maximal and minimal countably additive extensions of a given finite measure taking the outer or inner measure on members from the given perturbation. This will be done in the cases where this perturbation is generated by a finite or countable (non-measurable) partition of X . We will actually see that in these cases the given extensions are unique.

2. Preliminary facts

2.1. Non-measurable calculus

Let $(X, \mathcal{A}, \mu)$ be a finite measure space; then μ^* and μ_* denote *the outer* and *the inner μ-measure* respectively; if $C \in 2^X$ is an arbitrary subset of X , then C^* denotes *the μ-hull* of C and C_* denotes *the μ-kernel* of C ; thus C^* , $C_* \in \mathcal{A}$, $C_* \subset C \subset C^*$ and we have

$$\mu(C^*) = \mu^*(C) \ , \ \ \mu(C_*) = \mu_*(C) \ .$$

It is easy to see that each subset of X has the μ-hull and the μ-kernel, uniquely determined up to a μ-nullset.

Let $L^1(\mu)$ be the set of all $\bar{\mathbf{R}}$-valued μ-integrable functions on X ; then by

$$\int^* f \ d\mu \ = \ \inf \{ \ \smallint g \ d\mu \ : \ g \in L^1(\mu) \ , \ f \leq g \ \}$$

$$\int_* f \ d\mu \ = \ \sup \{ \ \smallint g \ d\mu \ : \ g \in L^1(\mu) \ , \ g \leq f \ \}$$

the upper and *the lower μ-integral* of an arbitrary $\bar{\mathbf{R}}$-valued function f on X are defined. We want to point out that the upper and lower μ-integrals can be much easier handled by means of the so-called upper and lower μ-envelopes of functions under consideration.

The main point in the construction of the envelopes relies upon the well-known *Segal's localization principle* of a finite measure μ , which provides the existence of the μ-essential supremum μ - $ess \ sup \ (\mathcal{F})$ and the μ-essential infimum μ - $ess \ inf \ (\mathcal{F})$ of an arbitrary family $\mathcal{F}$ of $\bar{\mathbf{R}}$-valued $\mathcal{A}$-measurable functions on X . Consequently, let $\bar{\mathcal{M}}(\mathcal{A})$ be the set of all $\bar{\mathbf{R}}$-measurable $\mathcal{A}$-measurable functions on X , then by

$$f^* = \ \mu \text{ - } ess \ inf \ \{ \ g \in \bar{\mathcal{M}}(\mathcal{A}) \ : \ f \leq g \ \}$$

$$f_* = \ \mu \text{ - } ess \ sup \ \{ \ g \in \bar{\mathcal{M}}(\mathcal{A}) \ : \ g \leq f \ \}$$

the upper and *the lower μ-envelope* of an arbitrary $\bar{\mathbf{R}}$-valued function f on X may be well-defined. A basic connection between the upper and lower μ-integral, upper and lower μ-envelope, μ-hull and μ-kernel, and outer and inner μ-measure is established by the following statements:

(1) $$\int^* f \ d\mu \ = \ \begin{cases} \smallint f^* \ d\mu \ , & \text{if } \ f^* \in L(\mu) \\ +\infty \ , & \text{otherwise} \end{cases}$$

(2) $$\int_* f \ d\mu = \begin{cases} \int f_* \ d\mu & , \ \text{if} \ f_* \in L(\mu) \\ -\infty & , \ \text{otherwise} \end{cases}$$

(3) $$(1_C)^* = 1_{C^*} \quad \text{and} \quad (1_C)_* = 1_{C_*}$$

(4) $$\mu^*(C) = \mu(C^*) = \int^* 1_C \ d\mu \quad \text{and} \quad \mu_*(C) = \mu(C_*) = \int_* 1_C \ d\mu$$

for all $f \in \bar{\mathbf{R}}^X$ and all $C \in 2^X$, where $L(\mu)$ denotes the set of all $\bar{\mathbf{R}}$-valued functions on X for which the μ-integral exists in $\bar{\mathbf{R}}$. We shall refer the reader to [18] for basic properties of these concepts (we shall use them throughout without making any further reference) as well as for more details in this direction. Also, let us clarify that all relations involving hulls and kernels of sets and envelopes of functions should be understood in the $a.a.$-sense, i.e. like relations involving suitably chosen representants from the corresponding $a.a.$-equivalence classes.

Finally, let us mention that $(X, \mathcal{A}^\mu, \bar{\mu})$ denotes the completion of the finite measure space $(X, \mathcal{A}, \mu)$. If $\mathcal{B}$ is a sub-σ-algebra of $\mathcal{A}$, then $r(\mu, \mathcal{B})$ denotes the restriction of μ to $\mathcal{B}$. If $\mathcal{C}$ is an arbitrary family of subsets of X , then $\sigma(\mathcal{C})$ denotes the smallest σ-algebra on X that includes $\mathcal{C}$.

2.2. Measure-algebraic preliminaries

In this section we turn to some algebraic properties of measures and integrals which appear naturally in our next considerations. If X is a real linear space, then X^+ denotes the linear space of all *linear* functionals on X , and X^* denotes the Banach space of all *linear continuous* functionals on X . If Y is a subset of X , then $sp(Y)$ denotes the smallest linear subspace of X that includes Y . If X is a set, then $B(X)$ denotes the Banach space of all bounded real valued functions on X with the usual sup-norm defined by $\| b \| = \sup_{x \in X} | b(x) |$. If $\mathcal{A}$ is an algebra of subsets of X , then $B_s(X, \mathcal{A})$ denotes the set of all real valued $\mathcal{A}$-measurable simple functions on X , and we define $B(X, \mathcal{A}) = cl^\infty\{B_s(X, \mathcal{A})\}$, where cl^∞ denotes the closure operator in $B(X)$ with respect to the sup-norm $\| \cdot \|$. Then $B(X, \mathcal{A})$ becomes a Banach space which is a natural domain for definition of an integral with respect to a finitely additive measure on $(X, \mathcal{A})$. Moreover, if $ba(X, \mathcal{A})$ resp. $ba_+(X, \mathcal{A})$ denotes the set of all real valued resp. non-negative finitely additive bounded functions defined on $\mathcal{A}$, then $ba(X, \mathcal{A})$ is a Banach space with respect to the total variation norm, which is isometrically isomorphic to $B(X, \mathcal{A})^*$, see [7]. We shall refer the reader to [18] for facts on integration with respect to finitely additive measures; we shall use freely throughout without making any further reference.

Let us recall that the well-known *Hahn-Banach theorem* states: If X is a real linear space, Y a subspace of X , $f \in Y^+$ and p a positively homogeneous subadditive map on X such that $f(y) \leq p(y)$, $\forall y \in Y$, then there exists $F \in X^+$ such that $F(y) = f(y)$, $\forall y \in Y$ and $F(x) \leq p(x)$, $\forall x \in X$. It should be noted that if $(X, \mathcal{A}, \mu)$ is a finite measure space, then by

$$p^*(b) = \int^* b \ d\mu \ , \quad b \in B(X)$$

a positively homogeneous and subadditive map on $B(X)$ is defined, and the dual functional p_* of p^* (see [10]) is given by

$$p_*(b) = \int_* b \, d\mu \ , \quad b \in B(X) \quad .$$

For more details in this direction see [18]. If $(X, \mathcal{A}, \mu)$ is a finite measure space, then for given subsets $C_1, \ldots, C_n$ of X by

$$p^*{}_{c_1 \ldots c_n}(x) \ = \ p^*\big(\sum_{i=1}^n x_i 1_{C_i} \big) \ , \quad x = (x_1, \ldots, x_n) \in \mathbf{R}^n$$

a positively homogeneous and subadditive (and thus convex and continuous) map from $\mathbf{R}^n$ into $\mathbf{R}$ is defined. A direct application of the Hahn-Banach theorem yields:

If $\int^* \big(\sum_{i=1}^n \alpha_i 1_{C_i} \big) \, d\mu \ = \ \sum_{i=1}^n \int^* \alpha_i 1_{C_i} \, d\mu$ *for all* $\alpha_1, \ldots, \alpha_n \in \mathbf{R}$ *, then there exists* F *in* $B^*(X)$ *satisfying*

$$F(\sum_{i=1}^n \alpha_i 1_{C_i}) = \sum_{i=1}^n \alpha_i \mu^*(C_i)$$

for all $\alpha_1, \ldots, \alpha_n \in \mathbf{R}$ *and* $F(b) \ \leq \int^* b \, d\mu$ *for all* $b \in B(X)$ *. In particular, we have*

$$\int_* b \, d\mu \leq \ F(b) \ \leq \int^* b \, d\mu$$

for all $b \in B(X)$ *, and hence* $F(b) = \int_X b \, d\mu$ *, if* b *is* μ*-measurable.*

Note that $F(1_{C_i}) = \mu^*(C_i)$ for all $i = 1, \ldots, n$, and the first assumption is trivially satisfied when $n = 1$. But we shall see later on in this paper that in the case where $n \geq 2$ further restrictions on the sets $C_1, \ldots, C_n$ seem to be necessary to satisfy this condition.

3. Basic results

We shall begin by computing some typical upper and lower integrals which appear naturally in our next considerations.

Proposition 1.
Let $(X, \mathcal{A}, \mu)$ *be a finite measure space, let* C *and* D *be disjoint subsets of* X *, and let* $f, g : X \to \mathbf{R}_+$ *be functions such that* $\int^* f \cdot 1_C \, d\mu < \infty$ *and* $\int_* g \cdot 1_D \, d\mu < \infty$ *. Then we have:*

(1) $$\int^* (f \cdot 1_C - g \cdot 1_D) \, d\mu \ = \ \int^* f \cdot 1_C \, d\mu - \int_* g \cdot 1_D \, d\mu$$

(2) $$\int_* (f \cdot 1_C - g \cdot 1_D) \, d\mu \ = \ \int_* f \cdot 1_C \, d\mu - \int^* g \cdot 1_D \, d\mu \ .$$

Proof. Put $f_0 = f \cdot 1_C$ and $g_0 = -g \cdot 1_D$. Let ${f_0}^*$ and ${g_0}^*$ be the upper μ-envelopes of f_0 and g_0 , resp., and let C^* and D^* be the μ-hulls of C and D , resp. Then $f \cdot 1_C - g \cdot 1_D \leq {f_0}^* + {g_0}^*$

and without loss of generality we may assume that $f_0{}^* \geq 0$ with $f_0{}^*(x) = 0$ for $x \in X \setminus C^*$, $g_0{}^* \leq 0$ with $\{g_0{}^* < 0\} \subset D_*$, and $C \subset C^*$, $D_* \subset D$ with $C^* \cap D_* = \emptyset$. Furthermore, since $\int f_0{}^* \, d\mu = \int^* f_0 \, d\mu = \int^* f \cdot 1_C \, d\mu < \infty$ and $\int g_0{}^* \, d\mu = \int^* g_0 \, d\mu = -\int_* g \cdot 1_D \, d\mu > -\infty$, we may conclude that $f_0{}^* + g_0{}^* \in L^1(\mu)$.

Take now $h \in L^1(\mu)$ such that $f_0 + g_0 = f \cdot 1_C - g \cdot 1_D \leq h$. Then $\{h < 0\} \subset X \setminus C$ and $\{h < 0\} \in \mathcal{A}$, and thus $\mu(\{h < 0\} \cap C^*) = 0$, while by definition of $f_0{}^*$ we have $\mu\{f_0{}^* > h\} = 0$. These facts taken together imply

$$\begin{aligned} \int f_0{}^* \, d\mu \;&=\; \int_{\{h \geq 0\}} f_0{}^* \, d\mu \;+\; \int_{\{h<0\}} f_0{}^* \, d\mu \;= \\ &=\; \int_{\{h \geq 0\}} f_0{}^* \, d\mu \;+\; \int_{\{h<0\} \cap C^*} f_0{}^* \, d\mu \;+\; \int_{\{h<0\} \cap (X \setminus C^*)} f_0{}^* \, d\mu \;= \\ &=\; \int_{\{h \geq 0\}} f_0{}^* \, d\mu \;=\; \int_{\{h \geq 0, f_0{}^* \leq h\}} f_0{}^* \, d\mu \;\leq\; \int_{\{h \geq 0\}} h \, d\mu. \end{aligned}$$

Similarly, then we have $\{h < 0\} \subset D$ and $\{h < 0\} \in \mathcal{A}$, and thus we may suppose $\{h < 0\} \subset D_*$, while by definition of $g_0{}^*$ we have $\mu\{g_0{}^* > h\} = 0$. These facts taken together imply

$$\begin{aligned} \int g_0{}^* \, d\mu \;&=\; \int_{D_*} g_0{}^* \, d\mu \leq \int_{\{h<0\}} g_0{}^* \, d\mu \;= \\ &=\; \int_{\{h<0, g_0{}^* \leq h\}} g_0{}^* \, d\mu \;\leq\; \int_{\{h<0\}} h \, d\mu. \end{aligned}$$

Therefore we may deduce

$$\begin{aligned} \int (f_0{}^* + g_0{}^*) \, d\mu \;&=\; \int f_0{}^* \, d\mu + \int g_0{}^* \, d\mu \;\leq \\ &\leq\; \int_{\{h \geq 0\}} h \, d\mu \;+\; \int_{\{h<0\}} h \, d\mu \;=\; \int h \, d\mu \end{aligned}$$

and hence by definition of the upper μ-integral, we may conclude

$$\begin{aligned} \int^* (f \cdot 1_C - g \cdot 1_D) d\mu \;&=\; \int^* (f_0 + g_0) d\mu \;=\; \int (f_0{}^* + g_0{}^*) d\mu = \\ &=\; \int^* f_0 \, d\mu - \int_* (-g_0) \, d\mu \;=\; \int^* f \cdot 1_C d\mu - \int_* g \cdot 1_D d\mu \end{aligned}$$

These facts complete the proof of (1), while (2) is an easy consequence of (1).

□

Corollary 2.

Let $(X, \mathcal{A}, \mu)$ be a finite measure space, let C and D be disjoint subsets of X, and let $a, b \geq 0$. Then we have:

$$\int^* (a \cdot 1_C - b \cdot 1_D)\, d\mu \;=\; a \cdot \mu^*(C) - b \cdot \mu_*(D) \tag{1}$$

$$\int_* (a \cdot 1_C - b \cdot 1_D)\, d\mu \;=\; a \cdot \mu_*(C) - b \cdot \mu^*(D)\ . \tag{2}$$

Proof. The given equalities follow straightforwardly by (1) and (2) in Proposition 1 respectively, taking into account (2.1.4).

□

Proposition 3.

Let $(X, \mathcal{A}, \mu)$ be a finite measure space, let $C_1, C_2, \ldots, C_n$ be disjoint subsets of X, and let $x_1 \geq x_2 \geq \ldots \geq x_n \geq 0$ be given real numbers, for some $n \geq 1$. Let us define:

$$D_1 = {C_1}^*\ ,\ D_2 = {C_2}^* \setminus {C_1}^*,\ \ldots\ , D_n = {C_n}^* \setminus (\textstyle\bigcup_{i=1}^{n-1} {C_i}^*)\ .$$

Then the following equality is satisfied:

$$\int^* \sum_{i=1}^n x_i \cdot 1_{C_i}\, d\mu \;=\; \sum_{i=1}^n x_i \cdot \mu(D_i)$$

In particular, we have

$$\mu^*\big(\bigcup_{i=1}^n C_i\big) \;=\; \sum_{i=1}^n \mu(D_i) \;=\; \mu\big(\bigcup_{i=1}^n {C_i}^*\big)$$

and thus we may conclude

$$\big(\bigcup_{i=1}^n C_i\big)^* = \bigcup_{i=1}^n {C_i}^*\ .$$

Proof. Put $f = \sum_{i=1}^n x_i \cdot 1_{C_i}$ and $f_0 = \sum_{i=1}^n x_i \cdot 1_{D_i}$. Then $f_0 \geq f$ and $f_0 \in L^1(\mu)$. Take $g \in L^1(\mu)$ such that $g \geq f$. Since $x_1 \geq x_2 \geq \ldots \geq x_n \geq 0$, it is easy to check that $g(x) \geq x_1$ for μ-$a.s.$ $x \in D_1$, $f(x) \geq x_2$ for μ-$a.s.$ $x \in D_2$ $\ldots$ $f(x) \geq x_n$ for μ-$a.s.$ $x \in D_n$. Thus we have $g \geq f_0$ μ-$a.s.$, and therefore $\int g\, d\mu \geq \int f_0\, d\mu$. By definition of the upper μ-integral hence we directly find

$$\int^* \sum_{i=1}^n x_i \cdot 1_{C_i}\, d\mu \;=\; \int^* f\, d\mu \;=\; \int f_0\, d\mu \;=\; \sum_{i=1}^n x_i \cdot \mu(D_i)\ .$$

The rest of the proof is straightforward, and we shall leave its verification to the reader.

□

Proposition 4.

Let $(X, \mathcal{A}, \mu)$ *be a finite measure space, let* $C_1, C_2, \ldots, C_n$ *be disjoint subsets of* X *, and let* $x_1 \geq x_2 \geq \ldots \geq x_n \geq 0$ *be given real numbers, for some* $n \geq 1$ *. Put* $\partial_*(C_1) = \emptyset$ *, and let* $\partial_*(C_1 \ldots C_k)$ *be the* μ*-kernel of the set* $\big(\cup_{i=1}^k C_i \big) \setminus \big(\cup_{i=1}^k (C_i)_* \big)$ *for every* $k = 2, \ldots, n$ *. Let us define:*

$$D_1 = (C_1)_* \ , \ D_2 = (C_2)_* \cup \{\partial_*(C_1, C_2) \setminus \partial_*(C_1)\}, \ \ldots$$

$$D_n = (C_n)_* \cup \{\partial_*(C_1 ... C_n) \setminus \partial_*(C_1 ... C_{n-1})\} \ .$$

Then the following equality is satisfied:

$$\int_* \sum_{i=1}^n x_i \cdot 1_{C_i} \, d\mu \ = \ \sum_{i=1}^n x_i \cdot \mu(D_i) \ .$$

In particular, we have

$$\mu_*\big(\bigcup_{i=1}^n C_i \big) \ = \ \sum_{i=1}^n \mu(D_i) \ = \ \sum_{i=1}^n \mu_*(C_i) + \mu\{\partial_*(C_1 \ldots C_n)\}$$

and thus we may conclude

$$\big(\bigcup_{i=1}^k C_i \big)_* \ = \ \big(\bigcup_{i=1}^k (C_i)_* \big) \cup \partial_*(C_1 \ldots C_k)$$

for all $k = 1, \ldots, n$ *, where it is not a restriction to assume* $\partial_*(C_1 \ldots C_{k+1}) = \partial_*(C_1 \ldots C_k) \cup B_k$*, with* $B_k = \emptyset$ *or* $\mu(B_k) > 0$ *, and* $\cup_{i=1}^{k+1} B_k \cap (C_i)_* = \emptyset$ *for all* $k = 1, \ldots, n-1$ *.*

Proof. Without loss of generality we may assume $\partial_*(C_1 ... C_{k-1}) \subset \partial_*(C_1 ... C_k)$ for all $k = 2, \ldots, n$, and hence it is not a restriction to assume $x_1 > x_2 > \ldots > x_n \geq 0$. Put $f = \sum_{i=1}^n x_i \cdot 1_{C_i}$ and $f_0 = \sum_{i=1}^n x_i \cdot 1_{D_i}$. Then $f_0 \leq f$ and $f_0 \in L^1(\mu)$. Take $g \in L^1(\mu)$ such that $g \leq f$. Then $g(x) \leq x_n$, $\forall x \in C_n$, and $g(x) \leq 0$, $\forall x \in X \setminus (\cup_{i=1}^n C_i)$. Thus we have $g \cdot 1_{\{g \leq x_n\}} \leq x_n \cdot 1_{D_n}$ μ-$a.s.$ Similarly, since $g(x) \leq x_{n-1}$, $\forall x \in C_{n-1}$, we have $g \cdot 1_{\{x_n < g \leq x_{n-1}\}} \leq x_{n-1} \cdot 1_{D_{n-1}}$ μ-$a.s.$ We can continue in this way by induction and at the end of this procedure we get $g \cdot 1_{\{x_2 < g \leq x_1\}} \leq x_1 \cdot 1_{D_1}$ μ-$a.s.$, which together with the preceding relations implies

$$g \ = \ g \cdot 1 \ = \ g \cdot 1_{\{g \leq x_n\}} + \sum_{i=1}^{n-1} g \cdot 1_{\{x_{i+1} < g \leq x_i\}} \ \leq \ \sum_{i=1}^n x_i \cdot 1_{D_i}$$

or in other words $g \leq f_0$ μ-$a.s.$ Thus $\int g \, d\mu \leq \int f_0 \, d\mu$, and by definition of the inner μ-integral we directly find

$$\int_* \sum_{i=1}^n x_i \cdot 1_{C_i} \, d\mu \ = \ \int_* f \, d\mu \ = \ \int f_0 \, d\mu \ = \ \sum_{i=1}^n x_i \cdot \mu(D_i) \ .$$

The rest of the proof is straightforward, and we shall leave its verification to the reader.

$\square$

Corollary 5.

Let $(X, \mathcal{A}, \mu)$ *be a finite measure space, let* $C_1, C_2, \ldots, C_n$ *be disjoint subsets of* X *, and let* $x_1 \geq x_2 \geq \ldots \geq x_k \geq 0 \geq x_{k+1} \geq \ldots \geq x_n$ *be given real numbers, for some* $1 \leq k \leq n$ *. Let us put* $\partial_*(C_{k+1}) = \emptyset$ *, and let us define:*

$$D_1 = {C_1}^* \ , \ D_2 = {C_2}^* \setminus {C_1}^* \ldots D_k = {D_k}^* \setminus (\bigcup_{i=1}^{k-1} {C_i}^*)$$

$$D_{k+1} = (C_{k+1})_* \ , \ D_{k+2} = (C_{k+2})_* \cup \{\partial_*(C_{k+1}, C_{k+2}) \setminus \partial_*(C_{k+1})\}$$

$$D_n = (D_n)_* \cup \{\partial_*(C_{k+1} \ldots C_n) \setminus \partial_*(C_{k+1} \ldots C_{n-1})\}$$

where $\partial_*(C_{k+1} \ldots C_{k+i})$ *is the* μ*-kernel of the set* $\big(\cup_{j=1}^{i} C_{k+j} \big) \setminus \big(\cup_{j=1}^{i} (C_{k+j})_* \big)$ *, for all* $i = 2, \ldots, n-k$ *. Then the following inequality is satisfied:*

$$\int^* \sum_{i=1}^{n} x_i \cdot 1_{C_i} \, d\mu \ = \ \sum_{i=1}^{n} x_i \cdot \mu(D_i) \ .$$

Proof. By Proposition 1 we have

$$\int^* \sum_{i=1}^{n} x_i \cdot 1_{C_i} \, d\mu \ = \ \int^* \sum_{i=1}^{k} x_i \cdot 1_{C_i} \, d\mu \ - \ \int_* \sum_{i=k+1}^{n} (-x_i) \cdot 1_{C_i} \, d\mu$$

and hence the proof follows straightforwardly by using results in Proposition 3 and Proposition 4. □

Remark 1. Since $\int_* f \, d\mu = - \int^* (-f) \, d\mu$, we could note that Corollary 5 itself also contains a possibility of computing the corresponding μ-inner integrals by changing the numeration of given sets only.

Remark 2. Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and let $C_1, C_2, \ldots, C_n$ be arbitrary (not necessarily disjoint) subsets of X , for some $n \geq 1$. Let $I_n = \{ 1, 2, \ldots, n \}$ and let 2^{I_n} denote the family of all subsets of I_n . Let us for a given $\pi \in 2^{I_n}$ define

$$C_\pi \ = \ \big(\bigcap_{i \in \pi} C_i \big) \cap \big(\bigcap_{i \in I_n \setminus \pi} {C_i}^c \big).$$

Then $\{ C_\pi \ : \ \pi \in 2^{I_n} \}$ is a finite (disjoint) partition of X and for every real valued function of the form $f = \sum_{i=1}^{n} x_i \cdot 1_{C_i}$, there exist unique $y_1, y_2, \ldots, y_{2^n}$ such that $f = \sum_{i=1}^{2^n} y_i 1_{c_{\pi_i}}$. Therefore all of the preceding results could be applied also in this general finite case, replacing the initial family $\mathcal{C} = \{ C_1, C_2, \ldots, C_n \}$ of arbitrary subsets of X by the family $\mathcal{C}_d = \{ C_{\pi_1}, C_{\pi_2}, \ldots, C_{\pi_{2^n}} \}$ of disjoint subsets of X . The only difference lies on different cardinalities of the given families. Let us also note that $\sigma(\mathcal{A} \cup \mathcal{C}) = \sigma(\mathcal{A} \cup \mathcal{C}_d)$. These facts justify our next attempts to work only with finite partitions of a given set X . We shall continue our study by considering some natural properties of the given finite partitions that will be of essential use later.

Let $(X, \mathcal{A}, \mu)$ be a finite measure space and let $C_1, C_2, \ldots, C_n$ be disjoint subsets of X. Then $C_1, C_2, \ldots, C_n$ are said to be *outer μ-separated*, if there exist $D_1, D_2, \ldots, D_n \in \mathcal{A}$, $C_1 \subset D_1,\ C_2 \subset D_2, \ldots, C_n \subset D_n$ satisfying

$$\mu\Big(\bigcup_{i \neq j} (D_i \cap D_j) \Big) = 0 \ .$$

It is easily verified that the following statements are equivalent:

(1.1) $\quad C_1, C_2, \ldots, C_n$ are outer μ-separated

(1.2) $\quad C_{i_1}, C_{i_2}, \ldots, C_{i_k}$ are outer μ-separated, $\forall \{ i_1, i_2, \ldots, i_k \} \subset \{ 1, 2, \ldots, n \}$

(1.3) $\quad C_i$ and C_j are outer μ-separated, $\forall i \neq j$ in $\{ 1, 2, \ldots, n \}$

(1.4) $$\mu^*(\bigcup_{i=1}^{n} C_i) = \sum_{i=1}^{n} \mu^*(C_i)$$

(1.5) $$\mu^*(\bigcup_{j=1}^{k} C_{i_j}) = \sum_{j=1}^{k} \mu^*(C_{i_j}) \ , \ \forall \{ i_1, i_2, \ldots, i_k \} \subset \{ 1, 2, \ldots, n \}$$

(1.6) $\quad \mu^*(C_i \cup C_j) = \mu^*(C_i) + \mu^*(C_j)$, $\forall i \neq j$ in $\{ 1, 2, \ldots, n \}$

(1.7) $\quad \mu\{\partial^*(C_1 \ldots C_n)\} = 0$, where $\partial^*(C_1 \ldots C_n) = \bigcup_{i \neq j} {C_i}^* \cap {C_j}^*$.

Further, the sets $C_1, C_2, \ldots, C_n$ are said to be *inner μ-separated*, if we have

$$\mu_*\Big((\bigcup_{i=1}^{n} C_i) \setminus (\bigcup_{i=1}^{n} (C_i)_*) \Big) = 0 \ .$$

It is easily verified that the following statements are equivalent:

(2.1) $\quad C_1, C_2, \ldots, C_n$ are inner μ-separated

(2.2) $\quad C_{i_1}, C_{i_2}, \ldots, C_{i_k}$ are inner μ-separated , $\forall \{ i_1, i_2, \ldots, i_k \} \subset \{ 1, 2, \ldots, n \}$

(2.3) $$\mu_*(\bigcup_{i=1}^{n} C_i) = \sum_{i=1}^{n} \mu_*(C_i)$$

(2.4) $$\mu_*(\bigcup_{j=1}^{k} C_{i_j}) = \sum_{j=1}^{k} \mu_*(C_{i_j}) \ , \ \forall \{ i_1, i_2, \ldots, i_k \} \subset \{ 1, 2, \ldots, n \}$$

(2.5) $\quad \mu\{\partial_*(C_1 \ldots C_n)\} = 0$

where as usual $\partial_*(C_1 \dots C_n)$ denotes the μ-kernel of $\big(\bigcup_{i=1}^n C_i\big) \setminus \big(\bigcup_{i=1}^n (C_i)_*\big)$. In particular, we may deduce:

(2.6) If $C_1, C_2, \dots, C_n$ are inner μ-separated, then C_i and C_j are inner μ-separated and we have

$$\mu_*(C_i \cup C_j) \;=\; \mu_*(C_i) + \mu_*(C_j)$$

for all $i \neq j$ in $\{\ 1, 2, \dots, n\ \}$.

However, let us point out that the converse to this last statement is not true in general, see Example 2 below.

Proposition 6.
Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and let $C_1, C_2, \dots, C_n$ be disjoint subsets of X . Let us put $sign^*(\mu, x) = \mu^*$, if $x \geq 0$ and $sign^*(\mu, x) = \mu_*$, if $x < 0$, as well as $sign_*(\mu, x) = \mu_*$, if $x \geq 0$ and $sign_*(\mu, x) = \mu^*$, if $x < 0$. Then the following statements are satisfied:

(1) *$C_1, C_2, \dots, C_n$ are outer μ-separated, if and only if*

$$\int^* \sum_{i=1}^n x_i \cdot 1_{C_i}\, d\mu = \sum_{i=1}^n x_i \cdot sign^*(\mu, x_i)\big(C_i\big)$$

for all $x_1, x_2, \dots, x_n \in \mathbf{R}$

(2) *$C_1, C_2, \dots, C_n$ are inner μ-separated, if and only if*

$$\int_* \sum_{i=1}^n x_i \cdot 1_{C_i}\, d\mu = \sum_{i=1}^n x_i \cdot sign_*(\mu, x_i)\big(C_i\big)$$

for all $x_1, x_2, \dots, x_n \in \mathbf{R}$

(3) *If $C_1, C_2, \dots, C_n$ are outer μ-separated, then they are inner μ-separated.*

Proof. The proof follows directly by Corollary 5.

□

Lemma 7.
Let $(X, \mathcal{A})$ be a measurable space and let $\mathcal{C} = \{\ C_i \mid i \in I\ \}$ be a finite or countable partition of X . Then we have

$$\sigma(\mathcal{A} \cup \mathcal{C}) \;=\; \Big\{\ \bigcup_{i \in I} (A_i \cap C_i)\ \Big|\ A_i \in \mathcal{A}\ ,\ i \in I\ \Big\}.$$

Proof. It is easy to verify that the right-hand family above is a σ-algebra. Using this fact the proof follows straightforwardly.

□

Theorem 8.
Let $(X, \mathcal{A}, \mu)$ *be a finite measure space, let* $\mathcal{C} = \{ C_1, C_2, \dots, C_k \}$ *be a finite partition of* X *, and let* ν *be a finitely additive measure on* $\sigma(\mathcal{A} \cup \mathcal{C})$ *such that* $r(\nu, \mathcal{A}) = \mu$ *. Then* ν *is countably additive.*

Proof. It is sufficient to show that for a given decreasing sequence $\{ B_n \ , \ n \geq 1 \} \subset \sigma(\mathcal{A} \cup \mathcal{C})$ satisfying $B_n \downarrow \emptyset$, as $n \to \infty$ we have $\lim_{n \to \infty} \nu(B_n) = 0$. Let us take such a sequence $\{ B_n \ , \ n \geq 1 \}$ in $\sigma(\mathcal{A} \cup \mathcal{C})$. Since $\nu(B_n) = \sum_{i=1}^{k} \nu(B_n \cap C_i)$, $\forall n \geq 1$, it is enough to show that $\lim_{n \to \infty} \nu(B_n \cap C_i) = 0$, for every $i = 1, \dots, k$. Since every B_n belongs to $\sigma(\mathcal{A} \cup \mathcal{C})$, by Lemma 7 we may deduce that $B_n = \bigcup_{i=1}^{k} {D_i}^n \cap C_i$, with some ${D_i}^n \in \mathcal{A}$, for $i = 1, \dots, k$. This shows that $B_n \cap C_i = {D_i}^n \cap C_i$ and hence $\nu(B_n \cap C_i) = \nu({D_i}^n \cap C_i) = \nu({C_i}^* \cap {D_i}^n \cap C_i)$. Therefore without loss of generality we can assume that ${D_i}^n \subset {C_i}^*$ for all $i = 1, \dots, k$. Since $B_n \downarrow \emptyset$, we have ${D_i}^n \downarrow E_i$, with some $E_i \in \mathcal{A}$ satisfying $E_i \subset {C_i}^* \setminus C_i$. Hence by definition of the μ-hull ${C_i}^*$ we may conclude $\nu(B_n \cap C_i) = \nu({D_i}^n \cap C_i) \leq \nu({D_i}^n \cap {C_i}^*) = \mu({D_i}^n \cap {C_i}^*) \downarrow \mu(E_i) = 0$, as $n \to \infty$, for every $i = 1, \dots, k$. These facts complete the proof.

□

Theorem 9.
Let $(X, \mathcal{A}, \mu)$ *be a finite measure space, let* $\mathcal{C} = \{ C_i \mid i \in \mathbf{N} \}$ *be a countable partition of* X *, and let* ν *be a finitely additive measure on* $\sigma(\mathcal{A} \cup \mathcal{C})$ *such that* $r(\nu, \mathcal{A}) = \mu$ *. Then* ν *is countably additive, if and only if the following condition is satisfied:*

$$\sum_{i=1}^{\infty} \nu(C_i) \ = \ \mu(X). \tag{1}$$

Proof. It is sufficient to show that for a given decreasing sequence $\{ B_n \ , \ n \geq 1 \} \subset \sigma(\mathcal{A} \cup \mathcal{C})$ satisfying $B_n \downarrow \emptyset$, as $n \to \infty$, we have $\lim_{n \to \infty} \nu(B_n) = 0$. Let us take such a sequence $\{ B_n \ , \ n \geq 1 \}$ in $\sigma(\mathcal{A} \cup \mathcal{C})$, and let us assume that (1) is satisfied. Since we have

$$\nu(X) \ = \ \sum_{i=1}^{n} \nu(C_i) \ + \ \nu(\bigcup_{i=n+1}^{\infty} C_i) \ = \ \mu(X)$$

by (1) we may conclude that $\nu(\cup_{i=n}^{\infty} C_i) \to 0$, for $n \to \infty$. In other words, for given $\varepsilon > 0$, there exists $n_\varepsilon \geq 1$ such that $\nu(\cup_{i=n}^{\infty} C_i) < \varepsilon$, whenever $n \geq n_\varepsilon$. Since every B_n belongs to $\sigma(\mathcal{A} \cup \mathcal{C})$, by Lemma 7 we may deduce that $B_n = \bigcup_{i=1}^{\infty} {D_i}^n \cap C_i$, with some ${D_i}^n \in \mathcal{A}$, for $i \in \mathbf{N}$. This shows that $B_n \cap C_i = {D_i}^n \cap C_i$ for all $n \geq 1$ and all $i \in \mathbf{N}$. Therefore we may conclude

$$\nu(B_n) \ = \ \sum_{i=1}^{n_\varepsilon} \nu(B_n \cap C_i) \ + \ \nu(\bigcup_{i=n_\varepsilon+1}^{\infty} (B_n \cap C_i) \ <$$

$$< \ \sum_{i=1}^{n_\varepsilon} \nu(B_n \cap C_i) \ + \ \varepsilon \ = \ \sum_{i=1}^{n_\varepsilon} \nu({D_i}^n \cap C_i) \ + \ \varepsilon$$

for all $n \geq 1$. Since $\mathcal{P}_\varepsilon(\mathcal{C}) = \{ C_1, C_2, \dots, C_{n_\varepsilon}, \cup_{i=n_\varepsilon+1}^\infty C_i \}$ is a finite partition of X and $\nu_\varepsilon = r(\nu, \sigma(\mathcal{A} \cup \mathcal{P}_\varepsilon(\mathcal{C})))$ is a finitely additive measure on $\sigma(\mathcal{A} \cup \mathcal{P}_\varepsilon(\mathcal{C}))$ such that $r(\nu_\varepsilon, \mathcal{A}) = r(\nu, \mathcal{A}) = \mu$, by Theorem 8 we may conclude that ν_ε is countably additive on $\sigma(\mathcal{A} \cup \mathcal{P}_\varepsilon(\mathcal{C}))$. Since $B_n \downarrow \emptyset$, as $n \to \infty$, then $B_n \cap C_i = {D_i}^n \cap C_i \downarrow \emptyset$, as $n \to \infty$, for all $i = 1, \dots, n_\varepsilon$. Therefore $\lim_{n\to\infty} \nu_\varepsilon({D_i}^n \cap C_i) = \lim_{n\to\infty} \nu({D_i}^n \cap C_i) = 0$, for all $i = 1, \dots, n_\varepsilon$, and by (2) we may conclude $\limsup_{n\to\infty} \nu(B_n) \leq \varepsilon$. Since $\varepsilon > 0$ was arbitrary, thus $\lim_{n\to\infty} \nu(B_n) = 0$, and the proof is complete.

□

Theorem 10.

Let $(X, \mathcal{A}, \mu)$ be a finite measure space, let $\mathcal{C}$ be a family of subsets of X , and let $\alpha : \mathcal{C} \to \mathbf{R}_+$ be a given function. Then the following three statements are equivalent:

(1) $$\sum_{i=1}^n \alpha(C_i) - \sum_{j=1}^m \alpha(D_j) \leq \int^* (\sum_{i=1}^n 1_{C_i} - \sum_{j=1}^m 1_{D_j})\, d\mu$$

for all not necessarily different $C_1, \dots, C_n, D_1, \dots, D_m \in \mathcal{C}$ with $n, m \geq 1$

(2) $$\sum_{i=1}^n r_i \alpha(C_i) \leq \int^* (\sum_{i=1}^n r_i 1_{C_i})\, d\mu$$

for all $r_1, \dots, r_n \in \mathbf{R}$ and all $C_1, \dots, C_n \in \mathcal{C}$ with $n \geq 1$

(3) *There exists $\nu \in ba_+(X)$ such that $r(\nu, \mathcal{A}) = \mu$ and $\nu(C) = \alpha(C)$ for all $C \in \mathcal{C}$.*

Proof. (1) $\Rightarrow$ (2) : Let $C_1, \dots, C_n$ be arbitrary members of $\mathcal{C}$, for some $n \geq 1$. Define a map $\alpha_{c_1 \dots c_n} : \mathbf{R}^n \to \mathbf{R}$ by

$$\alpha_{c_1 \dots c_n}(x) = \sum_{i=1}^n x_i \alpha(C_i)$$

for all $x = (x_1, \dots, x_n) \in \mathbf{R}^n$. Then $\alpha_{c_1 \dots c_n} \in (\mathbf{R}^n)^*$ and by (1) we have

(4) $$\alpha_{c_1 \dots c_n}(x) \leq {p^*}_{c_1 \dots c_n}(x)$$

for all $x \in \mathbf{Z}^n$. Since $\alpha_{c_1 \dots c_n}$ and ${p^*}_{c_1 \dots c_n}$ are positively homogeneous maps on $\mathbf{R}^n$, one can easily check that (4) remains true for all $x \in \mathbf{Q}^n$. But then the continuity of $\alpha_{c_1 \dots c_n}$ and ${p^*}_{c_1 \dots c_n}$ implies that (4) remains true for all $x \in \mathbf{R}^n$, and (2) is proved.

(2) $\Rightarrow$ (3) : Let $sp(\mathcal{C}) = sp(\{ 1_C \mid C \in \mathcal{C} \})$ be the smallest subspace of $\mathbf{R}^X$ generated by $\mathcal{C}$. Then we have

$$sp(\mathcal{C}) = \{ \sum_{i=1}^n \alpha_i 1_{C_i} \mid \alpha_i \in \mathbf{R} ,\ C_i \in \mathcal{C} ,\ n \in \mathbf{N} \}$$

and $sp(\mathcal{C}) < B_s(X, \sigma(\mathcal{C})) < B(X)$. Define a map $f : sp(\mathcal{C}) \to \mathbf{R}$ by

$$f(c) \;=\; \sum_{i=1}^{n} x_i \alpha(C_i)$$

for all $c = \sum_{i=1}^{n} x_i 1_{C_i} \in sp(\mathcal{C})$. Then $f \in sp(\mathcal{C})^+$ and by (2) we have

$$f(c) \leq p^*(c)$$

for all $c \in sp(\mathcal{C})$. Since p^* is positively homogeneous and subadditive on $B(X)$ then by the Hahn-Banach theorem there exists $F \in B(X)^+$ such that $F(c) = f(c)$ for all $c \in sp(\mathcal{C})$ and $F(b) \leq p^*(b)$ for all $b \in B(X)$. Since $p^*(-b) = -p_*(b)$ for all $b \in B(X)$, we have

(5) $$p_*(b) \;\leq\; F(b) \;\leq\; p^*(b) \;, \quad \forall b \in B(X).$$

For more informations in this direction we shall refer the reader to [18]. In particular we may conclude that $F(b) = p^*(b) = p_*(b) = \int b \; d\mu$ whenever $b \in B(X, \mathcal{A})$. Let us now define a map $\nu : 2^X \to \mathbf{R}$ by

$$\nu(A) = F(1_A)$$

for all $A \in 2^X$. Then $\nu \in ba(X)$ and by (5) and (2.1.4) we have

$$\mu_*(A) \;=\; \int_* 1_A \; d\mu \;=\; p_*(1_A) \;\leq\; F(1_A) \;=\; \nu(A) \;=$$

$$=\; F(1_A) \;\leq\; p^*(1_A) \;=\; \int^* 1_A \; d\mu \;=\; \mu^*(A)$$

for all $A \in 2^X$. Therefore we may conclude that $\nu \in ba_+(X)$, $r(\nu, \mathcal{A}) = \mu$ and $\nu(C) = F(1_C) = f(1_C) = \alpha(C)$ for all $C \in \mathcal{C}$. These facts complete the proof of (3). Moreover, let us note that using (5) we may easily deduce that $F \in B(X)^*$. For more details in this direction see [18].

(3) $\Rightarrow$ (1) : Let $C_1, \ldots, C_n$ and $D_1, \ldots, D_m$ be arbitrary but not necessarily different members of $\mathcal{C}$ with some $n, m \geq 1$, and let $g = \sum_{i=1}^{n} 1_{C_i} - \sum_{j=1}^{m} 1_{D_j}$. Then $g \in B_s(X, \sigma(\mathcal{C}))$ and if $f \in L^1(\mu)$ such that $g \leq f$, then it is not a restriction to assume that $f \in B(X, \mathcal{A})$, otherwise we can replace the starting f by $f \wedge \| \, g \, \|$. Since $\mu = r(\nu, \mathcal{A})$, then we have $\int g \; d\nu \leq \int f \; d\nu = \int f \; d\mu$, for more details see [18]. Now taking infimum over all functions $f \in L^1(\mu)$ satisfying $g \leq f$ we may conclude

$$\int g \; d\nu \;=\; \sum_{i=1}^{n} \nu(C_i) - \sum_{j=1}^{m} \nu(D_j) \;=\; \sum_{i=1}^{n} \alpha(C_i) - \sum_{j=1}^{m} \alpha(D_j) \leq$$

$$\leq\; \int^* g \; d\mu \;=\; \int^* (\sum_{i=1}^{n} 1_{C_i} - \sum_{j=1}^{m} 1_{D_j}) \; d\mu \;.$$

Thus (1) is satisfied, and the proof is complete.

$\square$

Corollary 11.

Let $(X, \mathcal{A}, \mu)$ *be a finite measure space, let* $\mathcal{C}$ *be a family of subsets of* X *, and let* $\alpha : \mathcal{C} \to \mathbf{R}_+$ *be a given function. Then the following statements are satisfied:*

(1) *There exists a finitely additive extension* ν *of* μ *to* $\sigma(\mathcal{A} \cup \mathcal{C})$ *such that* $\nu(C) = \alpha(C)$ *for all* $C \in \mathcal{C}$ *, if and only if the following condition is satisfied:*

$$\text{(1.1)} \qquad \sum_{i=1}^{n} \alpha(C_i) - \sum_{j=1}^{m} \alpha(D_j) \ \leq \ \int^{*} (\sum_{i=1}^{n} 1_{C_i} - \sum_{j=1}^{m} 1_{D_j})\, d\mu$$

for all not necessarily different $C_1, \ldots, C_n, D_1, \ldots, D_m \in \mathcal{C}$ *with* $n, m \geq 1$ *.*

(2) *If the members of* $\mathcal{C}$ *are disjoint, then there exists a finitely additive extension* ν *of* μ *to* $\sigma(\mathcal{A} \cup \mathcal{C})$ *such that* $\nu(C) = \alpha(C)$ *for all* $C \in \mathcal{C}$ *, if and only if either of the following two equivalent conditions is satisfied:*

$$\text{(2.1)} \qquad \int_{*} \sum_{i=1}^{n} 1_{C_i}\, d\mu \ \leq \ \sum_{i=1}^{n} \alpha(C_i) \ \leq \ \int^{*} \sum_{i=1}^{n} 1_{C_i}\, d\mu$$

for all not necessarily different $C_1, \ldots, C_n \in \mathcal{C}$ *,* $\forall n \geq 1$ *;*

$$\text{(2.2)} \qquad \mu_{*}(\bigcup_{i=1}^{n} C_i) \ \leq \ \sum_{i=1}^{n} \alpha(C_i) \ \leq \ \mu^{*}(\bigcup_{i=1}^{n} C_i)$$

for all mutually different $C_1, \ldots, C_n \in \mathcal{C}$ *with* $n \geq 1$ *.*

(3) *There exists (at least one) finitely additive extension* ν *of* μ *to* $\sigma(\mathcal{A} \cup \mathcal{C})$*.*

(4) *If there exists a finite or countable partition* $\mathcal{D}$ *of* X *such that* $\sigma(\mathcal{A} \cup \mathcal{C}) = \sigma(\mathcal{A} \cup \mathcal{D})$ *, then there exists at least one countably additive extension* ν *of* μ *to* $\sigma(\mathcal{A} \cup \mathcal{C})$ *. In particular, if* $\mathcal{C}$ *is a finite family, or a countable partition of* X *, or a monotone (increasing or decreasing) countable family, then there exists (at least one) countably additive extension* ν *of* μ *to* $\sigma(\mathcal{A} \cup \mathcal{C})$*.*

Proof. (1): Let ν be a finitely additive extension of μ to $\sigma(\mathcal{A} \cup \mathcal{C})$, and let $C_1, \ldots C_n, D_1, \ldots D_m$ be arbitrary but not necessarily different members of $\mathcal{C}$, for some $n, m \geq 1$. Let us consider a map $g \in B_s(X, \sigma(\mathcal{A} \cup \mathcal{C}))$ defined by $g = \sum_{i=1}^{n} 1_{C_i} - \sum_{j=1}^{m} 1_{D_j}$. If $f \in L^1(\mu)$ such that $g \leq f$, then as above it is not a restriction to assume that $f \in B(X, \mathcal{A})$, otherwise we can replace the starting f by $f \wedge \| g \|$. Since $r(\nu, \mathcal{A}) = \mu$, then we have $\int g\ d\nu \leq \int f\ d\nu = \int f\ d\mu$, and for more details see [18]. Now taking infimum over all functions $f \in L^1(\mu)$ satisfying $g \leq f$ we may conclude

$$\int g\ d\nu \ = \ \sum_{i=1}^{n} \nu(C_i) \ - \ \sum_{j=1}^{m} \nu(D_j) \ = \ \sum_{i=1}^{n} \alpha(C_i) \ - \ \sum_{j=1}^{m} \alpha(D_j) \ \leq$$

$$\leq \int^* g \, d\mu \; = \; \int^* (\sum_{i=1}^n 1_{C_i} - \sum_{j=1}^m 1_{D_j}) \, d\mu \; .$$

Hence we see that (1.1) is satisfied. Conversely, if (1.1) holds, then by the implication (3) $\Rightarrow$ (1) in Theorem 10 there exists a finitely additive extension $\tilde{\nu}$ of μ to 2^X such that $\tilde{\nu}(C) = \mu(C)$ for all $C \in \mathcal{C}$. But then $\nu = r\big(\tilde{\nu}, \sigma(\mathcal{A} \cup \mathcal{C})\big)$ is a finitely additive measure on $\sigma(\mathcal{A} \cup \mathcal{C})$ such that $r(\nu, \mathcal{A}) = \mu$, and $\nu(C) = \tilde{\nu}(C) = \alpha(C)$ for all $C \in \mathcal{C}$. These facts complete the proof of (1).

(2): We shall first show that (1.1) is equivalent to (2.1), and then that (2.1) is equivalent to (2.2). Since for given $D_1, \ldots, D_m \in \mathcal{C}$ with $m \geq 1$ we have $\int^*(-\sum_{j=1}^m 1_{D_j}) \, d\mu = -\int_*(\sum_{j=1}^m 1_{D_j}) \, d\mu$, the implication (1.1) $\Rightarrow$ (2.1) follows directly by Proposition 1. Conversely, suppose that (2.1) holds and take not necessarily different $C_1, \ldots, C_n, D_1, \ldots, D_m \in \mathcal{C}$, for some $n, m \geq 1$. Then we may write

$$g \; = \; \sum_{i=1}^n 1_{C_i} - \sum_{j=1}^m 1_{D_j} \; = \; \sum_{i=1}^k n_i 1_{E_i} - \sum_{j=1}^l m_j 1_{F_j}$$

where $E_1, \ldots, E_k, F_1, \ldots, F_l \in \mathcal{C}$ are mutually different sets such that $\{E_1, \ldots, E_k, F_1, \ldots, F_l\} = \{C_1, \ldots, C_n, D_1, \ldots, D_m\}$, and where $n_1, \ldots, n_k, m_1, \ldots, m_l \in \mathbf{N}$ are the numbers of their occurrences in the starting representation of the function g , respectively. By our assumption (2.1) we may deduce the following two inequalities:

$$\sum_{i=1}^k n_i \alpha(E_i) \; \leq \; \int^* \sum_{i=1}^k n_i 1_{E_i} \, d\mu$$

$$-\sum_{j=1}^l m_j \alpha(F_j) \; \leq \; -\int_* \sum_{j=1}^l m_j 1_{F_j} \, d\mu \; .$$

By Proposition 1 hence we may conclude

$$\begin{aligned} \sum_{i=1}^n \alpha(C_i) - \sum_{j=1}^m \alpha(D_j) \; &= \; \sum_{i=1}^k n_i \alpha(E_i) - \sum_{j=1}^l m_j \alpha(F_j) \; \leq \\ &\leq \; \int^* \sum_{i=1}^k n_i 1_{E_i} \, d\mu - \int_* \sum_{j=1}^l m_j 1_{F_j} \, d\mu \; = \\ &= \; \int^* (\sum_{i=1}^k n_i 1_{E_i} - \sum_{j=1}^l m_j 1_{F_j}) \, d\mu \; = \; \int^* (\sum_{i=1}^n 1_{C_i} - \sum_{j=1}^m 1_{D_j}) \, d\mu \; . \end{aligned}$$

This proves the implication (2.1) $\Rightarrow$ (1.1) and finishes the first part of the proof of (2).

Let us now pass to the equivalence between (2.1) and (2.2). Since the implication (2.1) $\Rightarrow$ (2.2) is obvious, it is enough to show that (2.2) implies (2.1). Thus suppose that (2.2) holds, and take $C_1, \ldots, C_n \in \mathcal{C}$, for some $n \geq 1$. It will be clear from our next arguments that it is not a restriction to assume that $C_1, \dots, C_n$ are mutually disjoint and that the given integers, which are equal to the numbers of their occurrences in the general case of (2.1) respectively, satisfy $m_1 \geq m_2 \geq \dots \geq m_n \geq 1$. Let us define:

$$D_1 = {C_1}^* \ , \ D_2 = {C_2}^* \setminus {C_1}^*, \dots, D_n = {C_n}^* \setminus \big(\bigcup_{i=1}^{n-1} {C_i}^* \big)$$

where ${C_i}^*$ is the μ-hull of C_i for all $i = 1, \dots, n$. Then by Proposition 3 and (2.2) we find

$$\begin{aligned}
\int^* \sum_{i=1}^n m_i 1_{C_i} \, d\mu \ &= \ \sum_{i=1}^n m_i \mu(D_i) \ = \ m_n \cdot \sum_{i=1}^n \mu(D_i) \ + \ (m_{n-1} - m_n) \cdot \sum_{i=1}^{n-1} \mu(D_i) \ + \ \dots \\
&\quad \dots \ + \ (m_2 - m_3) \cdot \sum_{i=1}^2 \mu(D_i) \ + \ (m_1 - m_2) \cdot \mu(D_1) \ = \\
&= \ m_n \cdot \mu^*(\bigcup_{i=1}^n C_i) + (m_{n-1} - m_n) \cdot \mu^*(\bigcup_{i=1}^{n-1} C_i) \ + \ \dots \\
&\quad \dots \ + \ (m_2 - m_3) \cdot \mu^*(C_1 \cup C_2) \ + \ (m_1 - m_2) \cdot \mu^*(C_1) \ \geq \\
&\geq \ m_n \cdot \sum_{i=1}^n \alpha(C_i) \ + \ (m_{n-1} - m_n) \cdot \sum_{i=1}^{n-1} \alpha(C_i) \ + \ \dots \\
&\quad \dots \ + \ (m_2 - m_3) \cdot \sum_{i=1}^2 \alpha(C_i) + (m_1 - m_2) \cdot \alpha(C_1) \ = \ \sum_{i=1}^n m_i \alpha(C_i) \ .
\end{aligned}$$

Similarly, put $\partial_*(C_1) = \emptyset$, let $\partial_*(C_1 \dots C_k)$ be the μ-kernel of the set $\big(\bigcup_{i=1}^k C_i\big) \setminus \big(\bigcup_{i=1}^k (C_i)_*\big)$ for all $k = 2, \dots, n$, and let us define:

$$\begin{aligned}
&D_1 = (C_1)_* \ , \ D_2 = (C_2)_* \cup \{\partial_*(C_1, C_2) \setminus \partial_*(C_1)\}, \dots, \\
&D_n = (C_n)_* \cup \{\partial_*(C_1 \dots C_n) \setminus \partial_*(C_1 \dots C_{n-1})\}
\end{aligned}$$

where $(C_i)_*$ is the μ-kernel of C_i for $i = 1, \dots, n$. Then by Proposition 4 and (2.2) we find

$$\begin{aligned}
\int_* \sum_{i=1}^n m_i 1_{C_i} \, d\mu \ &= \ \sum_{i=1}^n m_i \mu(D_i) \ = \\
&= \ m_1 \cdot \mu\{(C_1)_*\} \ + \ m_2 \cdot \big(\mu\{(C_2)_*\} + \mu\{\partial_*(C_1, C_2)\}\big) \ + \ \dots \\
&\quad \dots \ + \ m_3 \cdot \big(\mu\{(C_2)_*\} + \mu\{\partial_*(C_1, C_2, C_3) \setminus \partial_*(C_1, C_2)\}\big) \ + \ \dots \\
&\quad \dots \ + \ m_n \cdot (\mu\{(C_n)_*\} + \mu\{\partial_*(C_1, \dots, C_n) \setminus \partial_*(C_1, \dots, C_{n-1})\}) = \\
&= \ (m_1 - m_2) \cdot \mu\{(C_1)_*\} \ + \ (m_2 - m_3) \cdot \mu\{(C_1 \cup C_2)_*\} \ + \ \dots
\end{aligned}$$

$$\ldots + (m_{n-1}-m_n)\cdot\mu\{(\bigcup_{i=1}^{n-1} C_i)_*\} + m_n\cdot\mu\{(\bigcup_{i=1}^{n} C_i)_*\} =$$

$$= (m_1-m_2)\cdot\mu_*(C_1) + (m_2-m_3)\cdot\mu_*(C_1\cup C_2) + \ldots$$

$$\ldots + (m_{n-1}-m_n)\cdot\mu_*(\bigcup_{i=1}^{n-1} C_i) + m_n\cdot\mu_*(\bigcup_{i=1}^{n} C_i) \le (m_1-m_2)\cdot\alpha(C_1) +$$

$$+ (m_2-m_3)\cdot\sum_{i=1}^{2}\alpha(C_i) + \ldots + (m_{n-1}-m_n)\cdot\sum_{i=1}^{n-1}\alpha_i + m_n\cdot\sum_{i=1}^{n}\alpha_i =$$

$$= \sum_{i=1}^{n} m_i\alpha(C_i) \ .$$

Hence we see that (2.1) is satisfied, and by using (1) one can easily complete the proof of (2).

(3): Let us consider the setting of Theorem 10 with $\tilde{\mathcal{C}} = \{X\}$ and $\tilde{\alpha}(X) = \mu(X)$. Then (1) in Theorem 10 is obviously satisfied, and thus by (3) in Theorem 10 there exists $\tilde{\nu}\in ba_+(X)$ satisfying $r(\tilde{\nu},\mathcal{A}) = \mu$. Now let us denote $\nu = r(\tilde{\nu},\sigma(\mathcal{A}\cup\mathcal{C}))$, then ν is a finitely additive measure on $\sigma(\mathcal{A}\cup\mathcal{C}))$ such that $r(\nu,\mathcal{A}) = \mu$. This fact completes the proof of (3).

(4): Let $\mathcal{C} = \{\ C_1, C_2,\ \ldots\ \}$ be a countable partition of X . Let us define

$$D_1 = {C_1}^* \ ,\ D_2 = {C_2}^*\setminus {C_1}^*, \ldots, D_n = {C_n}^*\setminus(\bigcup_{i=1}^{n-1} {C_i}^*)$$

for all $n\ge 2$, and let us define a map $\alpha : \mathcal{C}\to \mathbf{R}_+$ by

$$\alpha(C_i) = \mu(D_i)$$

for all $i\ge 1$. Let $n\ge 1$ and $m_1,\ldots,m_n\in\mathbf{N}_0$, and let $f,g\in L^1(\mu)$ such that $g\le\sum_{i=1}^n m_i 1_{C_i}\le f$. Since $g\cdot 1_{C_i}\le m_i\le f\cdot 1_{C_i}$ and $D_i\subset {C_i}^*$, then we have $g\cdot 1_{D_i}\le m_i\cdot 1_{D_i}\le f\cdot 1_{D_i}$ μ-$a.s.$ for all $i = 1,\ldots,n$. It is not a restriction to assume that $f(x) = g(x) = 0$, $\forall x\in X\setminus(\bigcup_{i=1}^n D_i)$, and thus we may deduce

$$g = \sum_{i=1}^{n} g\cdot 1_{D_i} \le \sum_{i=1}^{n} m_i\cdot 1_{D_i} \le \sum_{i=1}^{n} f\cdot 1_{D_i} = f \ .$$

Therefore we have

$$\int g\ d\mu \le \sum_{i=1}^{n} m_i\mu(D_i) = \sum_{i=1}^{n} m_i\alpha(C_i) \le \int f\ d\mu$$

and by definition of the upper and lower μ-integrals we may conclude

$$\int_* \sum_{i=1}^{n} m_i 1_{C_i}\ d\mu \le \sum_{i=1}^{n} m_i\alpha(C_i) \le \int^* \sum_{i=1}^{n} m_i 1_{C_i}\ d\mu$$

whenever $m_1,\ldots,m_n\in\mathbf{N}_0$ with $n\ge 1$. This shows that (2.1) is satisfied and according to (2) there exists a finitely additive extension ν of μ to $\sigma(\mathcal{A}\cup\mathcal{C})$ satisfying $\nu(C_i) = \alpha(C_i) = \mu(D_i)$ for all $i \geq 1$. Since we obviously have

$$\sum_{i=1}^{\infty} \nu(C_i) = \sum_{i=1}^{\infty} \alpha(C_i) = \sum_{i=1}^{\infty} \mu(D_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(D_i) =$$

$$= \lim_{n\to\infty} \mu(\bigcup_{i=1}^{n} D_i) = \lim_{n\to\infty} \mu(\bigcup_{i=1}^{n} {C_i}^*) = \lim_{n\to\infty} \mu^*(\bigcup_{i=1}^{n} C_i) = \mu(X)$$

by Theorem 8 we may conclude that ν is actually countably additive. This fact proves the first part of (4). The remaining part of (4) follows now easily, and we shall leave its verification to the reader. These facts complete the proof.

$\square$

Combining results from Theorem 8, Theorem 9 and Corollary 11 we are now in position to establish a necessary and sufficient condition for the existence of a countably additive extension of a given finite measure in the case of any finite (as well as disjoint countable) perturbation of its σ-algebra.

Theorem 12.

Let $(X, \mathcal{A}, \mu)$ be a finite measure space, let $\mathcal{C}$ be a family of subsets of X , and let $\alpha : \mathcal{C} \to \mathbf{R}_+$ be a given function. Then the following statements are satisfied:

(1) *If $\mathcal{C} = \{ C_1, \dots, C_n \}$ is a finite family, let us set*

$$C_\pi = (\bigcap_{i\in\pi} C_i) \cap (\bigcap_{i\in I_n\setminus\pi} {C_i}^c)$$

for all $\pi \in 2^{I_n}$, where $I_n = \{ 1, 2, \dots, n \}$. Then $\mathcal{C}_d = \{ C_{\pi_i} \mid i \in 2^{I^n} \}$ is a finite partition of X . Moreover, there exists a countably additive extension ν of μ to $\sigma(\mathcal{A}\cup\mathcal{C})$ such that $\nu(C_i) = \alpha(C_i)$ for all $i = 1, \dots, n$, if and only if there exists an extension α_0 of α to $\mathcal{C} \cup \mathcal{C}_d$ such that either of the following two equivalent conditions holds:

(1.1) $$\int_* \sum_{i\in\rho} p_i 1_{C_{\pi_i}} \, d\mu \leq \sum_{i\in\rho} p_i \alpha_0(C_{\pi_i}) \leq \int^* \sum_{i\in\rho} p_i 1_{C_{\pi_i}} \, d\mu$$

for all $p_i \in \mathbf{N}$ and all non-empty $\rho \subset 2^{I^n}$;

(1.2) $$\mu_*(\bigcup_{i\in\rho} C_{\pi_i}) \leq \sum_{i\in\rho} \alpha_0(C_{\pi_i}) \leq \mu^*(\bigcup_{i\in\rho} C_{\pi_i})$$

for all non-empty $\rho \subset 2^{I^n}$.

In particular, if $\mathcal{C} = \{ \mathcal{C}_1, \dots, \mathcal{C}_n \}$ is a partition of X , then there exists a countably *additive extension* ν *of* μ *to* $\sigma(\mathcal{A} \cup \mathcal{C})$ *such that* $\nu(C_i) = \alpha(C_i)$ *for all* $i = 1, \ldots, n$ *, if and only if either of the following two equivalent conditions holds:*

$$\int_* \sum_{j=1}^{k} p_{i_j} 1_{C_{i_j}} d\mu \ \leq \ \sum_{j=1}^{k} p_{i_j} \alpha(C_{i_j}) \ \leq \ \int^* \sum_{j=1}^{k} p_{i_j} 1_{C_{i_j}} d\mu \tag{1.3}$$

for all $p_{i_j} \in \mathbf{N}$ *and all* $\{ i_1, \ldots, i_k \} \subset \{ 1, \ldots, n \}$ *;*

$$\mu_*(\bigcup_{j=1}^{k} C_{i_j}) \ \leq \ \sum_{j=1}^{k} \alpha(C_{i_j}) \ \leq \ \mu^*(\bigcup_{j=1}^{k} C_{i_j}) \tag{1.4}$$

for all $\{ i_1, \ldots, i_k \} \subset \{ 1, \ldots, n \}$.

(2) *If* $\mathcal{C} = \{ C_i \mid i \in \mathbf{N} \}$ *is a disjoint countable family, then there exists a countably additive extension* ν *of* μ *to* $\sigma(\mathcal{A} \cup \mathcal{C})$ *such that* $\nu(C_i) = \alpha(C_i)$ *for all* $i \in \mathbf{N}$ *, if and only if there exists an extension* α_0 *of* α *to* $\mathcal{C}_0 = \mathcal{C} \cup \{C_0\}$ *, where* $C_0 = \bigcap_{i=1}^{\infty} {C_i}^c$ *, such that the following condition is satisfied:*

$$\sum_{i=0}^{\infty} \alpha_0(C_i) \ = \ \mu(X) \tag{2.1}$$

and either of the following two equivalent conditions is satisfied:

$$\int_* \sum_{j=1}^{n} p_{i_j} 1_{C_{i_j}} d\mu \ \leq \ \sum_{j=1}^{n} p_{i_j} \alpha_0(C_{i_j}) \ \leq \ \int^* \sum_{j=1}^{n} p_{i_j} 1_{C_{i_j}} d\mu \tag{2.2}$$

for all $p_{i_j} \in \mathbf{N}$ *and all* $i_1, \ldots, i_n \in \mathbf{N}_0$ *with* $n \geq 1$ *;*

$$\mu_*(\bigcup_{j=1}^{n} C_{i_j}) \ \leq \ \sum_{j=1}^{n} \alpha_0(C_{i_j}) \ \leq \ \mu^*(\bigcup_{j=1}^{n} C_{i_j}) \tag{2.3}$$

for all $i_1, \ldots, i_n \in \mathbf{N}_0$ *with* $n \geq 1$.

In particular, if $\mathcal{C} = \{ C_i \mid i \in \mathbf{N} \}$ *is a countable partition of* X *, then there exists a countably additive extension* ν *of* μ *to* $\sigma(\mathcal{A} \cup \mathcal{C})$ *, if and only if*

$$\sum_{i=1}^{\infty} \alpha(C_i) \ = \ \mu(X) \tag{2.4}$$

and either of the following two equivalent conditions is satisfied:

$$\int_* \sum_{j=1}^{n} p_{i_j} 1_{C_{i_j}} d\mu \ \leq \ \sum_{j=1}^{n} p_{i_j} \alpha(C_{i_j}) \ \leq \ \int^* \sum_{j=1}^{n} p_{i_j} 1_{C_{i_j}} d\mu \ , \tag{2.5}$$

for all $p_{i_j} \in \mathbf{N}$ *and all* $i_1, \ldots, i_n \in \mathbf{N}$ *with* $n \geq 1$ *;*

$$\text{(2.6)} \qquad \mu_*(\bigcup_{j=1}^{n} C_{i_j}) \;\le\; \sum_{j=1}^{n} \alpha(C_{i_j}) \;\le\; \mu^*(\bigcup_{j=1}^{n} C_{i_j}) \; ,$$

for all mutually different $i_1, \ldots, i_n \in \mathbf{N}$ *with* $n \ge 1$.

Proof. All statements are straightforward consequences of Theorem 8, Theorem 9 and (2) in Corollary 11. We shall leave their verifications to the reader.

□

Remark 3. Let us point out that (4) in Corollary 11, together with its proof, show that under conditions in (2) in Theorem 12 there exist (at least one) function $\alpha : \mathcal{C} \to \mathbf{R}_+$ and an extension α_0 of α to $\mathcal{C}_0$ such that (2.1), (2.2) and (2.3) in Theorem 12 is satisfied. This fact indicates that these conditions cover real cases indeed. By Proposition 3.1 in [19] and (2) in Corollary 11 we may easily conclude that the same conclusion holds under conditions (1) in Theorem 12.

Corollary 13.
Let $(X, \mathcal{A}, \mu)$ *be a finite measure space, and let* $\mathcal{C}$ *be a (finite) family of subsets of* X. *Then we have:*

(1) *If the following condition is satisfied:*

$$\text{(1.1)} \qquad \int^* \sum_{i=1}^{n} 1_{C_i} \, d\mu \;=\; \sum_{i=1}^{n} \mu^*(C_i)$$

for all not necessarily different $C_1, C_2, \ldots, C_n \in \mathcal{C}$ *with* $n \ge 1$, *then there exists a finitely (countably) additive extension* ν *of* μ *to* $\sigma(\mathcal{A} \cup \mathcal{C})$ *satisfying* $\nu(C) = \mu^*(C)$ *for all* $C \in \mathcal{C}$. *Conversely, if there exists a finitely additive extension* ν *of* μ *to* $\sigma(\mathcal{A} \cup \mathcal{C})$ *satisfying* $\nu(C) = \mu^*(C)$ *for all* $C \in \mathcal{C}$, *then (1) holds. Moreover, if the members of* $\mathcal{C}$ *are disjoint, then (1.1) implies that for every function* $\alpha : \mathcal{C} \to \mathbf{R}_+$ *satisfying* $\alpha(C) \in [\mu_*(C), \mu^*(C)]$ *for all* $C \in \mathcal{C}$, *there exists a finitely (countably) additive extension* ν *of* μ *to* $\sigma(\mathcal{A} \cup \mathcal{C})$ *satisfying* $\nu(C) = \alpha(C)$ *for all* $C \in \mathcal{C}$.

(2) *If the following condition is satisfied:*

$$\text{(2.2)} \qquad \int_* \sum_{i=1}^{n} 1_{C_i} \, d\mu = \sum_{i=1}^{n} \mu_*(C_i)$$

for all not necessarily different $C_1, C_2, \ldots, C_n \in \mathcal{C}$ *with* $n \ge 1$, *then there exists a finitely (countably) additive extension* ν *of* μ *to* $\sigma(\mathcal{A} \cup \mathcal{C})$ *such that* $\nu(C) = \mu_*(C)$ *for all* $C \in \mathcal{C}$. *Conversely, if there exists a finitely additive extension* ν *of* μ *to* $\sigma(\mathcal{A} \cup \mathcal{C})$ *such that* $\nu(C) = \mu_*(C)$ *for all* $C \in \mathcal{C}$, *then (2) holds.*

Proof. (1): Let us define a map $\alpha : \mathcal{C} \to \mathbf{R}_+$ by $\alpha(C) = \mu^*(C)$ for all $C \in \mathcal{C}$. Then for given not necessarily different sets $C_1, \cdots, C_n, D_1, \cdots, D_m \in \mathcal{C}$ with $n, m \geq 1$ we have

$$\sum_{i=1}^{n} \alpha(C_i) - \sum_{j=1}^{m} \alpha(D_j) \ = \ \sum_{i=1}^{n} \mu^*(C_i) - \sum_{j=1}^{m} \mu^*(D_j) \ = \ \int^* \sum_{i=1}^{n} 1_{C_i} \ d\mu \ -$$

$$- \int^* \sum_{j=1}^{m} 1_{D_j} \ d\mu \ = \ \int^* \sum_{i=1}^{n} 1_{C_i} \ d\mu + \int_* - \sum_{j=1}^{m} 1_{D_j} \ d\mu \ \leq \ \int^* (\sum_{i=1}^{n} 1_{C_i} - \sum_{j=1}^{m} 1_{D_j}) \ d\mu \ .$$

Hence we see that (1.1) from Corollary 11 is satisfied and thus the first implication in (1) follows directly by Corollary 11. Conversely, since we have

$$\int^* \sum_{i=1}^{n} 1_{C_i} d\mu \ \leq \ \sum_{i=1}^{n} \int^* 1_{C_i} d\mu \ = \ \sum_{i=1}^{n} \mu^*(C_i)$$

it is enough to show the converse inequality. By (2.1.1) we may deduce

$$\int^* \sum_{i=1}^{n} 1_{C_i} d\mu \ = \ \int (\sum_{i=1}^{n} 1_{C_i})^* d\mu \ = \ \int (\sum_{i=1}^{n} 1_{C_i})^* \ d\nu \ \geq$$

$$\geq \ \int \sum_{i=1}^{n} 1_{C_i} \ d\nu \ = \ \sum_{i=1}^{n} \nu(C_i) \ = \ \sum_{i=1}^{n} \mu^*(C_i)$$

and the converse statement is proved. Moreover, let us suppose that the members of $\mathcal{C}$ are disjoint, and let $C_1, C_2, \ldots, C_n$ be mutually different elements from $\mathcal{C}$, for some $n \geq 1$. Then by (1.1) we have

$$\mu^*(\bigcup_{i=1}^{n} C_i) \ = \ \int^* \sum_{i=1}^{n} 1_{C_i} \ d\mu \ = \ \sum_{i=1}^{n} \mu^*(C_i)$$

and thus by (1.4) the sets $C_1, C_2, \ldots, C_n$ are outer μ-separated. Therefore by (3) in Proposition 6 we may deduce that $C_1, C_2, \ldots, C_n$ are inner μ-separated also, or in other words

$$\mu_*(\bigcup_{i=1}^{n} C_i) \ = \ \sum_{i=1}^{n} \mu_*(C_i) \ .$$

Thus we have

$$\mu_*(\bigcup_{i=1}^{n} C_i) \ = \ \sum_{i=1}^{n} \mu_*(C_i) \ \leq \ \sum_{i=1}^{n} \alpha(C_i) \ \leq \ \sum_{i=1}^{n} \mu^*(C_i) \ = \ \mu^*(\bigcup_{i=1}^{n} C_i)$$

and consequently (2.2) in Corollary 11 is satisfied. Therefore the last part of (1) follows directly by Corollary 11. Let us note that the corresponding countably additivity is a consequence of Theorem 8 in both cases. These facts complete the proof of (1).

(2): Let us define a map $\alpha : \mathcal{C} \to \mathbf{R}_+$ by $\alpha(C) = \mu_*(C)$ for all $C \in \mathcal{C}$. Then for given not necessarily different $C_1, \cdots, C_n, D_1, \cdots, D_m \in \mathcal{C}$ with $n, m \geq 1$ we have

$$\sum_{i=1}^{n} \alpha(C_i) - \sum_{j=1}^{m} \alpha(D_j) \;=\; \sum_{i=1}^{n} \mu_*(C_i) - \sum_{j=1}^{m} \mu_*(D_j) \;=\; \int_* \sum_{i=1}^{n} 1_{C_i}\; d\mu - \int_* \sum_{j=1}^{m} 1_{D_j}\; d\mu \;=$$

$$= \int_* \sum_{i=1}^{n} 1_{C_i}\; d\mu + \int^* - \sum_{j=1}^{m} 1_{D_j}\; d\mu \;\leq\; \int^* (\sum_{i=1}^{n} 1_{C_i} - \sum_{j=1}^{m} 1_{D_j})\; d\mu \;.$$

Hence we see that (1.1) in Corollary 11 is satisfied, and thus the first implication in (2) follows directly by Corollary 11. Let us note once again that the corresponding countably additivity is a consequence of Theorem 8. Conversely, since we have

$$\int_* \sum_{i=1}^{n} 1_{C_i} d\mu \;\geq\; \sum_{i=1}^{n} \int_* 1_{C_i} d\mu \;=\; \sum_{i=1}^{n} \mu_*(C_i)$$

it is enough to show the converse inequality. By (2.1.2) we may deduce

$$\int_* \sum_{i=1}^{n} 1_{C_i} d\mu \;=\; \int (\sum_{i=1}^{n} 1_{C_i})_* d\mu \;=\; \int (\sum_{i=1}^{n} 1_{C_i})_* d\nu \;\leq$$

$$= \int \sum_{i=1}^{n} 1_{C_i}\; d\nu \;=\; \sum_{i=1}^{n} \nu(C_i) \;=\; \sum_{i=1}^{n} \mu_*(C_i)$$

and the converse statement is proved. These facts complete the proof.

□

We shall now see how the preceding results can be used when looking for necessary and sufficient conditions for the existence of some maximal (or minimal) countably additive extensions of a given finite measure, taking the outer (or inner) measure on members from the given perturbation which is generated by a finite or countable (non-measurable) partition. The next two propositions (the first one for the maximal and the second one for the minimal extension) partially answer this question, and we shall leave their easy verifications to the reader as an illustration of the applicability of the preceding results. In Theorem 16 we will connect these results and show that the given extensions are unique actually.

Proposition 14.

Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and let $\mathcal{C} = \{\, C_i \mid i \in I \,\}$ be a finite or countable partition of X . Then the following seven statements are equivalent:

(1) $\mu\{\partial^*(C_1, C_2)\} = 0$ *for all mutually different* $C_1, C_2 \in \mathcal{C}$ *, where* $\partial^*(C_1, C_2) = {C_1}^* \cap {C_2}^*$

(2) $\mu^*(C_1 \cup C_2) = \mu^*(C_1) + \mu^*(C_2)$ *, for all mutually different* $C_1, C_2 \in \mathcal{C}$

(3) $\mu^*(\bigcup_{i=1}^{n} C_i) = \sum_{i=1}^{n} \mu^*(C_i)$ *, for all mutually different* $C_1, C_2, \ldots, C_n \in \mathcal{C}$ *with* $n \geq 1$ *, or in other words* $C_1, C_2, \ldots, C_n$ *are outer* μ*-separated*

(4) $\int^* \sum_{i=1}^{n} 1_{C_i} d\mu \;=\; \sum_{i=1}^{n} \mu^*(C_i)$ *, for all mutually different* $C_1, C_2, \ldots, C_n \in \mathcal{C}$ *with* $n \geq 1$

(5) $\displaystyle\int^{*} \sum_{i=1}^{n} x_i 1_{C_i} d\mu \;=\; \sum_{i=1}^{n} x_i \mu^{*}(C_i)$ *, for all mutually different* $C_1, C_2, \ldots, C_n \in \mathcal{C}$ *with* $n \geq 1$ *, and for all* $x_1, x_2, \ldots, x_n \in \mathbf{R}_+$

(6) $\displaystyle\sum_{C \in \mathcal{C}} \mu^{*}(C) = \mu(X)$

(7) *There is a countably additive extension* ν^{+} *of* μ *to* $\sigma(\mathcal{A} \cup \mathcal{C})$ *satisfying* $\nu^{+}(C) = \mu^{*}(C)$ *for all* $C \in \mathcal{C}$.

□

Proposition 15.

Let $(X, \mathcal{A}, \mu)$ *be a finite measure space, and let* $\mathcal{C} = \{\ C_i \ \mid \ i \in I\ \}$ *be a finite or countable partition of* X *. Let us consider the following statements:*

(1) $\mu\{\partial_{*}(C_1, C_2)\} = 0$ *, for all mutually different* $C_1, C_2 \in \mathcal{C}$ *, where* $\partial_{*}(C_1, C_2)$ *is the* μ*-kernel of* $(C_1 \cup C_2) \setminus \{(C_1)_{*} \cup (C_2)_{*}\}$

(2) $\mu_{*}(C_1 \cup C_2) = \mu_{*}(C_1) + \mu_{*}(C_2)$ *, for all mutually different* $C_1, C_2 \in \mathcal{C}$

(3) $\displaystyle\mu_{*}(\bigcup_{i=1}^{n} C_i) \;=\; \sum_{i=1}^{n} \mu_{*}(C_i)$ *, for all mutually different* $C_1, C_2, \ldots, C_n \in \mathcal{C}$ *with* $n \geq 1$ *, or in other words* $C_1, C_2, \ldots, C_n$ *are inner* μ*-separated*

(4) $\displaystyle\int_{*} \sum_{i=1}^{n} 1_{C_i} d\mu \;=\; \sum_{i=1}^{n} \mu_{*}(C_i)$ *, for all mutually different* $C_1, C_2, \ldots, C_n \in \mathcal{C}$ *with* $n \geq 1$

(5) $\displaystyle\int_{*} \sum_{i=1}^{n} x_i 1_{C_i} d \;=\; \sum_{i=1}^{n} x_i \mu_{*}(C_i)$ *, for all mutually different* $C_1, C_2, \ldots, C_n \in \mathcal{C}$ *with* $n \geq 1$ *, and for all* $x_1, x_2, \ldots, x_n \in \mathbf{R}_+$

(6) $\displaystyle\sum_{C \in \mathcal{C}} \mu_{*}(C) \;=\; \mu(X)$

(7) *There is a countably additive extension* ν_{+} *of* μ *to* $\sigma(\mathcal{A} \cup \mathcal{C})$ *such that* $\nu_{+}(C) = \mu_{*}(C)$ *for all* $C \in \mathcal{C}$.

Then the following relations are satisfied:

$$\{\ (6) \Leftrightarrow (7)\ \} \Rightarrow \{\ (3) \Leftrightarrow (4) \Leftrightarrow (5)\ \} \Rightarrow \{\ (1) \Leftrightarrow (2)\ \}.$$

Furthermore, if (6)+(7) is satisfied, then $\sigma(\mathcal{A} \cup \mathcal{C}) \subset \mathcal{A}^{\mu}$ *and* $\nu_{+} = r(\bar{\mu}, \sigma(\mathcal{A} \cup \mathcal{C}))$ *is the unique extension of* μ *to* $\sigma(\mathcal{A} \cup \mathcal{C})$ *. In particular, in this case we have* $\nu_{+}(C) = \mu_{*}(C) = \mu^{*}(C)$ *for all* $C \in \mathcal{C}$ *, and all statements (1)-(7) in Proposition 14 are fulfilled too.*

□

Remark 4. The last statement in the preceding proposition shows that (6) and (7) are very restrictive conditions on a given finite or countable partition, since they imply μ-measurability of its members, and moreover all statements from Proposition 14 are satisfied in this case too. We shall now show that the converse relation is also true.

Theorem 16.
Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and let $\mathcal{C}$ be a finite or countable partition of X. Then there exists a countably additive extension ν^+ of μ to $\sigma(\mathcal{A} \cup \mathcal{C})$ satisfying $\nu^+(C) = \mu^*(C)$ for all $C \in \mathcal{C}$, if and only if there exists a countably additive extension ν_+ of μ to $\sigma(\mathcal{A} \cup \mathcal{C})$ satisfying $\nu_+(C) = \mu_*(C)$ for all $C \in \mathcal{C}$. In this case $r\big(\bar{\mu}, \sigma(\mathcal{A} \cup \mathcal{C})\big)$ is the unique countably additive extension of μ to $\sigma(\mathcal{A} \cup \mathcal{C})$.

Proof. According to Proposition 14, Proposition 15 and Remark 2, it is enough to show that (1) in Proposition 14 implies (6) in Proposition 15. Since we have

$$\begin{aligned}\mu(X) \;&=\; \mu(\bigcup_{i=1}^{\infty} C_i) \;=\; \sum_{i=1}^{\infty} \mu\{(C_i)_*\} \;+\; \mu\{\partial_*(C_1, C_2, \ldots)\} \;= \\ &=\; \sum_{i=1}^{\infty} \mu_*(C_i) \;+\; \mu\{\partial_*(C_1, C_2, \ldots)\}\end{aligned}$$

with $\partial_*(C_1, C_2, \ldots) = X \backslash \big(\cup_{i=1}^{\infty} (C_i)_*\big)$, for this it is enough to show that $\mu\{\partial_*(C_1, C_2, \ldots)\} = 0$. Since $\partial_*(C_1, C_2, \ldots) \subset \cup_{n=1}^{\infty} \big(\partial_*(C_1, C_2, \ldots) \cap {C_n}^*\big)$, for this it is enough to show that $\mu(A_n) = 0$ for all $n \in \mathbf{N}$, where $A_n = \partial_*(C_1, C_2, \ldots) \cap {C_n}^*$ for $n \in \mathbf{N}$. Since we have

$$\begin{aligned}A_n &= \Big(A_n \setminus (\bigcup_{k \neq n} {C_k}^*)\Big) \cup \Big(A_n \cap (\bigcup_{k \neq n} {C_k}^*)\Big) \subset \\ &\subset \Big(A_n \setminus (\bigcup_{k \neq n} {C_k}^*)\Big) \cup \Big(\bigcup_{k \neq n} {C_n}^* \cap {C_k}^*\Big)\end{aligned}$$

and by (1) from Proposition 14 we may deduce that $\mu\{\cup_{k \neq n} ({C_n}^* \cap {C_k}^*)\} = 0$, it is enough to show that $\mu\{A_n \backslash (\cup_{k \neq n} {C_k}^*)\} = 0$. But since $A_n \backslash (\cup_{k \neq n} {C_k}^*) \in \mathcal{A}$ and $A_n \backslash (\cup_{k \neq n} {C_k}^*) \subset C_n \backslash (C_n)_*$, thus by definition of the μ-kernel $(C_n)_*$ of C_n we may conclude that $\mu\{A_n \backslash (\cup_{k \neq n} {C_k}^*)\} = 0$, and the proof is complete.

□

The next two examples show that the converse implications in Proposition 15 do not hold in general.

Example 1.
Let $X = \mathbf{R}^2$, $\mathcal{A} = \sigma(\{\, [u, u+1[\,\times\, [v, v+1[\;\mid\; u, v \in \mathbf{Z} \,\})$, and let $\mu = r(\lambda^2, \mathcal{A})$, where λ^2 denotes the two-dimensional Lebesgue measure. Let $C_1 = [1/2, 5/2] \times [1/2, 5/2]$, and let $C_2 = \{\, (x{+}2, y) \mid (x, y) \in C_1 \,\}$, $C_3 = \{\, (x, y{+}2) \mid (x, y) \in C_1 \,\}$, $C_4 = \{\, (x{+}2, y{+}2) \mid (x, y) \in C_1 \,\}$. Put $S = \{\, 1, 2, 3, 4 \,\}$, then it is easy to verify that the following statements are satisfied:

(1) $\qquad \mu_*(C_i) \; = \; 1 \;\; , \;\; \forall i \, \in \, S$

(2) $\qquad \mu\{\partial_*(C_i,C_j)\} = \mu\{\partial_*(C_i,C_j,C_k)\} = 0 \;\; , \quad \forall i \neq j \neq k \;$ in $\; S$

(3) $\qquad \mu\{\partial_*(C_1,C_2,C_3,C_4)\} \; = \; 1$

(4) $\qquad \mu^*(C_i) \; = \; 9 \;\; , \quad \forall i \, \in \, S$

(5) $\qquad \mu\{\partial^*(C_i,C_j)\} \; = \; 3 \;\; , \quad \forall i \neq j \;$ in $\; S$

(6) $\qquad \mu\{\partial^*(C_i,C_j,C_k)\} \; = \; 7 \;\; , \quad \forall i \neq j \neq k \;$ in $\; S$

(7) $\qquad \mu\{\partial^*(C_1,C_2,C_3,C_4)\} \; = \; 9 \;\; .$

Let us note that (2) and (3) show that (1) in Proposition 15 does not imply (3) in Proposition 15 in general, or in other words, that the inner μ-separability can not be characterized by the inner μ-separability of the two-dimensional (or even all proper) subfamilies.

Example 2.
Let $X = \,]0,1]$, $\mathcal{A} = \{\emptyset, X\}$, $\mu(X) = 1$, $\mu(\emptyset) = 0$, and let $C_1 =]1/2,1]$, $C_2 =]1/3,1/2]$, $C_3 =]1/4,1/3], \ldots$. Then $(X,\mathcal{A},\mu)$ is a finite measure space and $\mathcal{C} = \{ \, C_n \mid n \geq 1 \, \}$ is a countable partition of X . Furthermore we have $\mu_*(C_n) = \mu_*(\cup_{i=1}^n C_i) = 0$ for all $n \geq 1$, and thus we may conclude

$$\mu_*(\bigcup_{j=1}^n C_{i_j}) \; = \; \sum_{j=1}^n \mu_*(C_{i_j}) \; = \; 0$$

for all mutually different $C_{i_1}, C_{i_2}, \ldots, C_{i_n} \in \mathcal{C}$ with $n \geq 1$. But on the other hand we have

$$\sum_{C \in \mathcal{C}} \mu_*(C) = 0 \neq 1 = \mu(X).$$

This example shows that (3) in Proposition 15 does not imply (6) in Proposition 15 in general, and the finitely additive extension ν from (2) in Corollary 13 is not countably additive in general.

Acknowledgment. Thanks are due J. Hoffmann-Jørgensen for many interesting discussions, and to Z. Lipecki for telling me about some relevant references on the subject.

Note. After the completion of the first draft of this paper about seven years ago, we learned from Z. Lipecki several facts (listed below and in the footnotes of the main text) which relate the present work to the work already done in this area. As this work has been carried out completely independently from any such known result, we want to make it clear that no attempt was made in the text above to compare the results presented with the other known results on the subject; thus no novelty of the results is claimed either. We find it convenient, however, to mention a few facts in this direction.

In the process of solving the main problem by the method above, several natural questions have appeared and been answered (Proposition 1, Corollary 2, Proposition 3, Proposition 4, Corollary 5 and Proposition 6). These results are aimed to develop a machinery (relying upon the upper and lower integral) needed for answering the main problem later on in the text. We are unaware of references where these results could have been possibly written down earlier. The results of Theorem 8 and Theorem 9 seem to have an origin in [13] (see Lemma 4 and references therein). The results of Theorem 10, Corollary 11 and Theorem 12 seem to be essentially related to the fundamental papers of D. L. Guy [8] and D. Bierlein [3]. It may be quite possible that these results can be deduced from the results in these papers. (Observe, however, as noted by Schmidt and Waldschaks in [20], that Guy's original proof in [8] is incorrect, while the proof given by Bhaskara Rao and Bhaskara Rao in [2] is rather involved.) The results in Corollary 13, Proposition 14, Proposition 15 and Theorem 16 turn out to belong to a separate and well-established class of results on extreme extensions given for instance in [4], [12]-[15], [17], etc. We have no precise information on what is known of this material.

Certainly, one of the challenges left to an interested reader would be to show that everything presented in the text above either is well-known or can easily be deduced from the known results.

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