Five lectures

on the general theory of

Optimal Stopping

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Lecture 1. INTRODUCTION.

1. Connections of the **Optimal stopping theory** and the **Mathematical analysis** (especially PDE-theory) are as well illustrated by the

> **Dirichlet problem** for the Laplace equation: to find a harmonic function $u = u(x)$ in the class C^2 in the bounded open domain $C \subseteq \mathbb{R}^d$ i.e. to find a function $u \in C^2$ that satisfies to the equation
>
> $$\Delta u = 0, \quad x \in C \qquad (*)$$
>
> and the boundary condition
>
> $$u(x) = G(x), \qquad x \in \partial D, \quad \text{where } D = \mathbb{R}^d \setminus C. \qquad (**)$$

Let

$$\tau_D = \inf\{t : B_t^x \in D\},$$

where

$$B_t^x = x + B_t.$$

with a d-dimensional standard Brownian motion.

Then the probabilistic solution of the Dirichlet problem

$$\Delta u = 0, \quad x \in C$$

$$u(x) = G(x), \quad x \in \partial D$$

is given by the formula

$$u(x) = \mathsf{E} G(B^x_{\tau_D}), \quad x \in C \cup \partial D$$

$$\left(u(x) = \mathsf{E}_x G(B_{\tau_D}) \right)$$

The **optimal stopping theory** operates with the **optimization** problems when we have a **set of domains** $\mathcal{C} = \left\{C : C \subseteq \mathbb{R}^d\right\}$ and we want to find the function

$$U(x) = \sup_{\tau_D} \mathsf{E}_x G(B_{\tau_D}),$$

where $G = G(x)$ is given for all $x \in R^d$, $D \in \mathcal{D} = \{D = \bar{C} : C \in \mathcal{C}\}$ or, generally, to find the function

$$V(x) = \sup_{\tau} \mathsf{E}_x G(B_\tau),$$

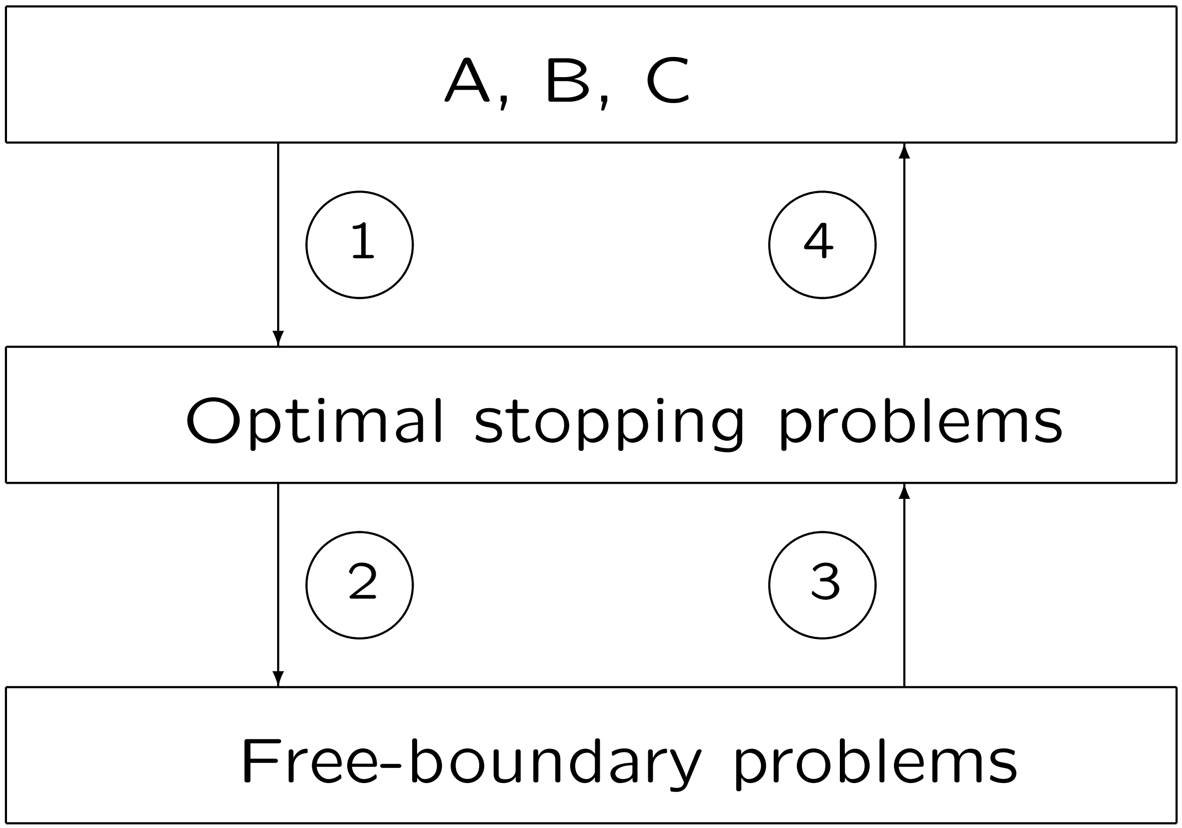
where τ is an **arbitrary finite stopping time** defined by the process B.

2. The following scheme illustrates the kind of **concrete** problems of **general interest** that will be studied in the courses of lectures:

A. Theory of probability sharp inequalities	B. Mathematical statistics sequential analysis

C. Financial mathematics stochastic equilibria

The solution method for problems **A, B, C** consists of **reformulation** to an optimal stopping problem and **reduction** to a free-boundary problem as stated in the diagram


A, B, C
1
4
Optimal stopping problems
2
3
Free-boundary problems

3. To get some idea of the character of problems **A, B, C** that will be studied, let us begin with the following remarks.

(A) Let $B = (B_t)_{t \geq 0}$ be a standard Brownian motion. Then (Wald identities)

$$\mathsf{E}B_T = 0 \quad \text{and} \quad \mathsf{E}B_\tau = 0 \quad \text{if} \quad \mathsf{E}\sqrt{\tau} < \infty,$$

$$\mathsf{E}B_T^2 = T \quad \text{and} \quad \mathsf{E}B_\tau^2 = \mathsf{E}\tau \quad \text{if} \quad \mathsf{E}\tau < \infty.$$

From Jensen's inequality and $\mathsf{E}|B_\tau|^2 = \mathsf{E}\tau$ we get

$$\mathsf{E}|B_\tau|^p \leq (\mathsf{E}\tau)^{p/2} \quad \text{for} \quad 0 < p \leq 2$$

$$(\mathsf{E}\tau)^{p/2} \leq \mathsf{E}|B_\tau|^p \quad \text{for} \quad 2 \leq p < \infty.$$

Davis B., 1976:

$$\mathsf{E}|B_\tau| \leq z_1^* \mathsf{E}\sqrt{\tau}, \qquad z_1^* = 1.30693\ldots$$

Now our main interest relates with the estimation of the expectations

$$\mathsf{E} \max_{t \le \tau} B_t \quad \text{and} \quad \mathsf{E} \max_{t \le \tau} |B_t|.$$

We have

$$\max B \overset{law}{=} |B|.$$

So,

$$\mathsf{E} \max_{t \le T} B_t = \mathsf{E}|B_T| = \sqrt{\frac{2}{\pi} T} \quad \Big(< \sqrt{\tau} \Big)$$

and

$$\mathsf{E} \max_{t \le \tau} B_t = \mathsf{E}|B_\tau| \le \begin{cases} \sqrt{\mathsf{E}\tau}, \\ \\ z_1^* \mathsf{E}\sqrt{\tau}, \ z_1^* = 1.30993\ldots \end{cases}$$

The case of $\max |B|$ is more difficult. We know that

$$\mathsf{P}\left(\max_{t\le T} |B_t| \le x\right) = \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \exp\left(-\frac{\pi^2(2n+1)^2}{8x^2}\right)$$

From here it is possible to obtain (but it's not easy!) that

$$\mathsf{E}\max_{t\le T} |B_t| = \sqrt{\frac{\pi}{2}T} \quad \Big(= 1.25331\ldots\Big)$$

(Recall that $\mathsf{E}|B_T| = \sqrt{\frac{2}{\pi}T} \quad (= 0.79788\ldots)$)

Simple proof:

$$(B_{at}; t \geq 0) \overset{law}{=} (\sqrt{a} B_t; t \geq 0).$$

Take $\sigma = \inf\{t > 0 : |B_t| = 1\}$. Then

$$\mathsf{P}\left(\sup_{0 \leq t \leq 1} |B_t| \leq x\right) = \mathsf{P}\left(\sup_{0 \leq t \leq 1} |B_{t/x^2}| \leq 1\right) =$$

$$= \mathsf{P}\left(\sup_{0 \leq t \leq 1/x^2} |B_t| \leq 1\right) = \mathsf{P}\left(\sigma \geq \frac{1}{x^2}\right) =$$

$$= \mathsf{P}\left(\frac{1}{\sqrt{\sigma} \leq x}\right), \text{ i.e. } \quad \sup_{0 \leq t \leq 1} |B_t| \overset{law}{=} \frac{1}{\sqrt{\sigma}}$$

The normal distribution property

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \mathsf{E} e^{-\frac{x^2}{2a^2}} dx = a, \quad a > 0.$$

So,

$$\mathsf{E}\sup_{0\le t\le 1}|B_t| = \mathsf{E}\frac{1}{\sqrt{\sigma}} = \sqrt{\frac{2}{\pi}}\int_0^\infty \mathsf{E}e^{-\frac{x^2\sigma}{2}}\,dx.$$

We have

$$\mathsf{E}e^{-\lambda\sigma} = \frac{1}{\cosh\sqrt{2\lambda}}.$$

Hence

$$\begin{aligned}\mathsf{E}\sup_{0\le t\le 1}|B_t| &= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{dx}{\cosh x} = \\ &= 2\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^x\,dx}{e^2x+1} = \sqrt{\frac{2}{\pi}}\int_1^\infty \frac{dy}{1+y^2} = \\ &= 2\sqrt{\frac{2}{\pi}}\arctan(x)\Big|_1^\infty = 2\sqrt{\frac{2}{\pi}}\cdot\frac{\pi}{4} = \sqrt{\frac{\pi}{2}}\end{aligned}$$

$$\mathsf{E} \sup_{0 \le t \le T} |B_t| = \sqrt{T}, \qquad \mathsf{E} \sup_{0 \le t \le 1} |B_t| = \sqrt{\frac{\pi}{2} T}.$$

In the connection with MAX the following may be interesting. In a speech delivered in 1856 before a grand meeting at the St.-Petersburg University the great mathematician

P.L. Chebyshev (1821-1894)

has formulated some statements about the "unity of theory and practice". In particular he underlined that

> "a large portion of the practical questions can be stated in the form of problems of MAXIMUM and MINIMUM... Only the solution of these problems can satisfy the requests of practice which is always in search of the best and the most efficient".

4. Suppose now that instead of $\max_{t \le T} |B_t|$, where as already known

$$\mathsf{E} \max_{t \le T} |B_t| = \sqrt{\frac{\pi}{2} T},$$

we give some **random** time τ and we want to find

$$\mathsf{E} \max_{0 \le t \le \tau} |B_t|.$$

It is clear that it is virtually impossible to compute this expectation for every stopping time τ of B. Thus, as the second best thing, one can try to bound the expectation with a quantity which is easier computed. A natural candidate for the latter is $\mathsf{E}\tau$ at least when finite. In this way a problem A has appeared.

This problem then leads to the following **maximal inequality**:

$$\mathsf{E}\Big(\max_{0\le t\le \tau}|B_t|\Big) \le C\sqrt{\mathsf{E}\tau} \tag{1}$$

which is valid for all stopping times τ of B with the best constant C equal to $\sqrt{2}$.

From our lecture we will see that the problem A just formulated can be solved in the form (1) by reformulation to the following **optimal stopping problem**:

$$V_* = \sup_{\tau} \mathsf{E}\Big(\max_{0\le t\le \tau}|B_t| - c\tau\Big) \tag{2}$$

where the supremum is taken over all stopping times τ of B satisfying $\mathsf{E}\tau < \infty$, and the constant $c > 0$ is given and fixed. It constitutes Step 1 in the diagram above.

If $V_* = V_*(c)$ can be computed, then from (2) we get

$$\mathsf{E}\Big(\max_{0 \le t \le \tau} |B_t| \Big) \le V_*(c) + c\, \mathsf{E}\tau \tag{3}$$

for all stopping times τ of B and all $c > 0$. Hence we find

$$\mathsf{E}\Big(\max_{0 \le t \le \tau} |B_t| \Big) \le \inf_{c > 0} \Big(V_*(c) + c\, \mathsf{E}\tau \Big) \tag{4}$$

for all stopping times τ of B. The right-hand side in (4) defines a function of $\mathsf{E}\tau$ that, in view of (2), provides a sharp bound of the left-hand side.

Our lectures demonstrate that the **optimal stopping problem** (2) can be reduced to a **free-boundary problem**. This constitutes Step 2 in the diagram above. Solving the free-boundary problem one finds that $V_*(c) = 1/2c$. Inserting this into (4) yields

$$\inf_{c > 0} \mathsf{E}\Big(V_*(c) + c\, \mathsf{E}\tau \Big) = \sqrt{2\, \mathsf{E}\tau} \tag{5}$$

so that the inequality (4) reads as follows:

$$\mathsf{E}\Big(\max_{0\le t\le\tau}|B_t|\Big)\le\sqrt{2\,\mathsf{E}\tau}\tag{6}$$

for all stopping times τ of B. This is exactly the inequality (1) above with $C=\sqrt{2}$. The constant $\sqrt{2}$ is, indeed, the best possible in (6). In the lectures we consider similar sharp inequalities for other stochastic processes using ramifications of the method just exposed. Apart from being able to derive sharp versions of known inequalities the method can also be used to derive some new inequalities.

(B) The classic example of a problem in **sequential analysis** is the Wald's problem ("Sequential analysis", 1947) of sequential testing of two statistical hypotheses

$$H_0\colon \mu=\mu_0 \quad \text{and} \quad H_1\colon \mu=\mu_1\tag{7}$$

about the drift parameter $\mu\in\mathbb{R}$ of the observed process

$$X_t=\mu t+B_t\tag{8}$$

for $t\ge 0$ where $B=(B_t)_{t\ge 0}$ is a standard Brownian motion.

Another classic example of a problem in **sequential analysis** is the problem of sequential testing of two statistical hypotheses

$$H_0\colon \lambda = \lambda_0 \quad \text{and} \quad H_1\colon \lambda = \lambda_1 \tag{9}$$

about the intensity parameter $\lambda > 0$ of the observed process

$$X_t = N_t^{\lambda} \tag{10}$$

for $t \geq 0$ where $N = (N_t)_{t \geq 0}$ is a standard Poisson process.

The basic problem in both cases seeks to find the **optimal decision rule** (τ_*, d_*) in the class $\Delta(\alpha, \beta)$ consisting of decision rules (d, τ), where τ is the time of stopping and accepting H_1 if $d = d_1$ or accepting H_0 if $d = d_0$, such that the probability errors of the first and second kind satisfy:

$$\mathsf{P}(\text{accept } H_1 \mid \text{true } H_0) \leq \alpha \tag{11}$$

$$\mathsf{P}(\text{accept } H_0 \mid \text{true } H_1) \leq \beta \tag{12}$$

and the mean times of observation $\mathsf{E}_0\tau$ and $\mathsf{E}_1\tau$ are as small as possible. It is assumed above that $\alpha > 0$ and $\beta > 0$ with $\alpha + \beta < 1$.

It turns out that with this (*variational*) problem one may associate an optimal stopping (*Bayesian*) problem which in turn can be reduced to a free-boundary problem. This constitutes Steps 1 and 2 in the diagram above. Solving the free-boundary problem leads to an optimal decision rule (τ_*, d_*) in the class $\Delta(\alpha, \beta)$ satisfying (11) and (12) as well as the following two identities:

$$\mathsf{E}_0 \tau = \inf_{(\tau,d)} \mathsf{E}_0 \tau \tag{13}$$

$$\mathsf{E}_1 \tau = \inf_{(\tau,d)} \mathsf{E}_1 \tau \tag{14}$$

where the infimum is taken over all decision rules (τ, d) in $\Delta(\alpha, \beta)$. This constitutes Steps 3 and 4 in the diagram above.

In our lectures we study these as well as closely related problems of **quickest detection**. (The story of the creating of the quickest detection problem of randomly appearing signal, it's mathematical formulation and our route of the solving of the problem (1961) will be given in my lecture at Monday January, 23 on the Symposium after the school).

Two of the prime findings, which also reflect the historical development of these ideas, are the **principles of smooth and continuous fit**, respectively.

C) One of the best-known specific problems of **mathematical finance**, that has a direct connection with optimal stopping problems, is the problem of determining the arbitrage-free price of the **American put option**.

Consider the Black–Scholes model where the stock price $X = (X_t)_{t \geq 0}$ is assumed to follow a geometric Brownian motion

$$X_t = x \text{ exp}\Big(\sigma B_t + (r - \sigma^2/2)\, t\Big) \tag{15}$$

where $x > 0$, $\sigma > 0$, $r > 0$ and $B = (B_t)_{t \geq 0}$ is a standard Brownian motion. By Itô's formula one finds that the process X solves

$$dX_t = r X_t \, dt + \sigma X_t \, dB_t \tag{16}$$

with $X_0 = x$.

General theory of financial mathematics makes it clear that the initial problem of determining the arbitrage-free price of the American put option can be reformulated as the following optimal stopping problem:

$$V_* = \sup_\tau \mathsf{E} e^{-r\tau}(K - X_\tau)^+ \tag{17}$$

where the supremum is taken over all stopping times τ of X. This constitutes Step 1 in the diagram above. The constant $K > 0$ is called the 'strike price'. It has a certain financial meaning which we set aside for now.

It turns out that the optimal stopping problem (17):

$$V_* = \sup_\tau \mathsf{E} e^{-r\tau}(K - X_\tau)^+$$

can be reduced again to a free-boundary problem which can be solved explicitly.

It yields the existence of a constant b_* such that the stopping time

$$\tau_* = \inf \{ \, t \geq 0 \mid X_t \leq b_* \, \} \tag{18}$$

is optimal in (17). This constitutes Steps 2 and 3 in the diagram above. Both the optimal stopping point b_* and the arbitrage-free price V_* can be expressed explicitly in terms of the other parameters in the problem. A financial interpretation of these expressions constitutes Step 4 in the diagram above.

In the formulation of the problem (17) above no restriction was imposed on the class of admissible stopping times, i.e. for certain reasons of simplicity it was assumed there that τ belongs to the class of stopping times

$$\mathfrak{M} = \{ \, \tau \mid 0 \leq \tau < \infty \, \} \tag{19}$$

without any restriction on their size.

A more realistic requirement on a stopping time in search for the arbitrage-free price leads to the following optimal stopping problem:

$$V_*^T = \sup_{\tau \in \mathfrak{M}^T} \mathsf{E} e^{-r\tau}(K - X_\tau)^+ \tag{20}$$

where the supremum is taken over all τ belonging to the class of stopping times

$$\mathfrak{M}^T = \{\, \tau \mid 0 \le \tau \le T \,\} \tag{21}$$

with the horizon T being finite.

The optimal stopping problem (20) can be also reduced to a free-boundary problem that apparently cannot be solved explicitly.

Its study yields that the stopping time

$$\tau_* = \inf \{ \, 0 \le t \le T \mid X_t \le b_*(t) \, \} \tag{22}$$

is optimal in (20), where $b_* \colon [0,T] \rightarrow \mathbb{R}$ is an increasing continuous function. A nonlinear Volterra integral equation can be derived which characterizes the optimal stopping boundary $t \mapsto b_*(t)$ and can be used to compute its values numerically as accurate as desired. The comments on Steps 1–4 in the diagram above made in the infinite horizon case carry over to the finite horizon case without any change.

In our lectures we study these and other similar problems that arise from various financial interpretations of options.

5. So far we have only discussed problems A, B, C and their reformulations as optimal stopping problems. Now we want to address the methods of solution of optimal stopping problems and their reduction to free-boundary problems.

There are essentially two equivalent approaches to finding a solution of the optimal stopping problem. The first one deals with the problem

$$V_* = \sup_{\tau \in \mathfrak{M}} \mathsf{E} G_\tau \tag{23}$$

in the case of **infinite horizon**, or the problem

$$V_*^T = \sup_{\tau \in \mathfrak{M}^T} \mathsf{E} G_\tau \tag{24}$$

in the case of **finite horizon**, where $\mathfrak{M}$ and $\mathfrak{M}^T$ are the classes of stopping times defined in (19) and (21), respectively.

In this formulation it is important to realize that $G = (G_t)_{t \geq 0}$ is an arbitrary stochastic process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathsf{P})$, where it is assumed that G is adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$ which in turn makes each τ from $\mathfrak{M}$ or $\mathfrak{M}^T$ a stopping time.

Since the method of solution to the problems (23) and (24) is based on results from the theory of martingales (**Snell's envelope**, 1952), the method itself is often referred to as the **martingale method**.

On the other hand, if we are to take a state space $(E, \mathcal{B})$ large enough, then one obtains the "Markov representation" $G_t = G(X_t)$ for some measurable function G, where $X = (X_t)_{t \geq 0}$ is a Markov process with values in E. Moreover, following the contemporary theory of Markov processes it is convenient to adopt the definition of a Markov process X as the **family** of Markov processes

$$((X_t)_{t \geq 0}, (\mathcal{F}_t)_{t \geq 0}, (\mathsf{P}_x)_{x \in E}) \tag{25}$$

where $\mathsf{P}_x(X_0 = x) = 1$ meaning that the process X starts at x under P_x. Such a point of view is convenient, for example, when dealing with the Kolmogorov forward or backward equations, which presuppose that the process can start at any point in the state space.

Likewise, it is a profound attempt, developed in stages, to study optimal stopping problems through functions of initial points in the state space.

In this way we have arrived to the second approach which deals with the problem

$$V(x) = \sup_{\tau} \mathsf{E}_x G(X_\tau) \tag{26}$$

where the supremum is taken over $\mathfrak{M}$ or $\mathfrak{M}^T$ as above (**Dynkin's formulation**, 1963). Thus, if the Markov representation of the initial problem is valid, we will refer to the **Markovian method** of solution.

6. To make the exposed facts more transparent, let us consider the optimal stopping problem

$$V_* = \sup_{\tau} \mathsf{E}\Big(\max_{0 \le t \le \tau} |B_t| - c\tau \Big)$$

in more detail.

Denote

$$X_t = |x + B_t| \tag{27}$$

for $x \geq 0$, and enable the maximum process to start at any point by setting for $s \geq x$

$$S_t = s \vee \Big(\max_{0 \leq r \leq t} X_r \Big) \tag{28}$$

The process $S = (S_t)_{t \geq 0}$ is not Markov, but the pair $(X, S) = (X_t, S_t)_{t \geq 0}$ forms a Markov process with the state space $E = \{\, (x,s) \in \mathbb{R}^2 \mid 0 \leq x \leq s \,\}$. The value V_* from (2) above coincides with the value function

$$V_*(x,s) = \sup_{\tau} \mathsf{E}_{x,s}\Big(S_\tau - c\tau \Big) \tag{29}$$

when $x = s = 0$. The problem thus needs to be solved in this more general form.

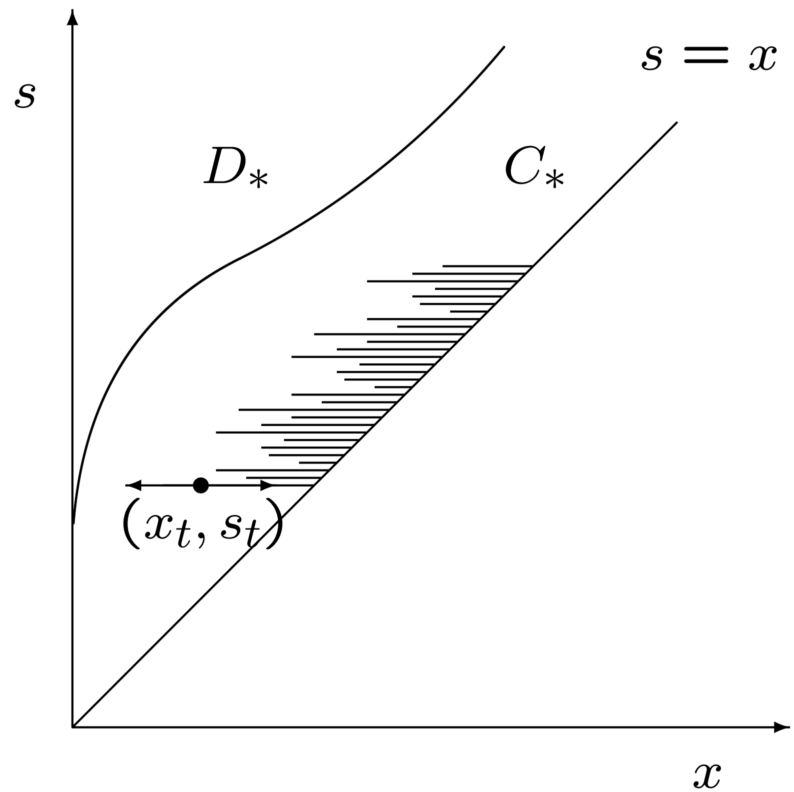
The general theory of optimal stopping for Markov processes makes it clear that the optimal stopping time in (29) can be written in the form

$$\tau_* = \inf\{\, t \geq 0 \mid (X_t, S_t) \in D_* \} \tag{30}$$

where D_* is a **stopping set**, and $C_* = E \setminus D_*$ is the **continuation set**. In other words, if the observation of X was not stopped before time t since $X_s \in C_*$ for all $0 \leq s < t$, and we have that $X_t \in D_*$, then it is optimal to stop the observation at time t. On the other hand, if it happens that $X_t \in C_*$ as well, then the observation of X should be continued.

Heuristic considerations about the shape of the sets C_* and D_* makes it plausible to guess that there exist a point $s_* \geq 0$ and a continuous increasing function $s \mapsto g_*(s)$ with $g_*(s_*) = 0$ such that

$$D_* = \{\, (x, s) \in \mathbb{R}^2 \mid 0 \leq x \leq g_*(s)\,,\ s \geq s_* \,\} \tag{31}$$



Note that such a guess about the shape of the set D_* can be made using the following intuitive arguments. If the process (X, S) starts from a point (x, s) with small x and large s, then it is reasonable to stop immediately because to increase the value s one needs a large time τ which in the formula (29) appears with a minus sign.

At the same time it is easy to see that if x is close or equal to s then it is reasonable to continue the observation, at least for small time Δ, because s will increase for the value $\sqrt{\Delta}$ while the cost for using this time will be $c\Delta$, and thus $\sqrt{\Delta} - c\Delta > 0$ if Δ is small enough.

Such an a priori analysis of the shape of the boundary between the stopping set C_* and the continuation set D_* is typical to the act of finding a solution to the optimal stopping problem. The art of **guessing** in this context very often plays a crucial role in solving the problem.

Having guessed that the stopping set D_* in the optimal stopping problem

$$V_*(x,s) = \sup_{\tau} \mathsf{E}_{x,s}\Big(S_\tau - c\tau\Big)$$

takes the form

$$D_* = \{\,(x,s) \in \mathbb{R}^2 \mid 0 \le x \le g_*(s)\,,\ s \ge s_*\,\},$$

it follows that τ_* attains the supremum i.e.

$$V_*(x,s) = \mathsf{E}_{x,s}\Big(S_{\tau_*} - c\tau_*\Big) \tag{32}$$

for all $(x,s) \in E$. Denote by $\mathbb{L}_X = (1/2)\,\partial^2/\partial x^2$ the infinitesimal operator of the process X and consider $V_*(x,s)$ as defined by the right-hand side of (32) for (x,s) in the continuation set

$$C_* = C_*^1 \cup C_*^2 \tag{33}$$

where the two subsets are defined as follows:

$$C_*^1 = \{\, (x,s) \in \mathbb{R}^2 \mid 0 \le x \le s < s_* \,\} \tag{34}$$

$$C_*^2 = \{\, (x,s) \in \mathbb{R}^2 \mid g_*(s) < x \le s\,,\ s \ge s_* \,\}. \tag{35}$$

By the strong Markov property one finds that V_* solves the following equation:

$$\mathbb{L}_X V_*(x,s) = c \tag{36}$$

for (x,s) in C_*. Note that if the process (X,S) starts at a point (x,s) with $x < s$ then during a positive time interval the second component S of the process does not change and remains equal to s.

This explains why the infinitesimal operator of the process (X, S) reduces to the infinitesimal operator of the process X in the interior of C_*. On the other hand, from the structure of the process (X, S) it follows that at the diagonal in $\mathbb{R}^2_+$ the condition of **normal reflection** holds

$$\frac{\partial V_*}{\partial s}(x, s)\Big|_{x=s-} = 0. \tag{37}$$

Moreover, it is clear that for $(x, s) \in D_*$ the condition of **instantaneous stopping** holds

$$V_*(x, s) = s. \tag{38}$$

Finally, either by guessing or providing rigorous arguments, it is found that at the optimal boundary g_* the condition of **smooth fit** holds

$$\frac{\partial V_*}{\partial x}(x, s)\Big|_{x=g_*(s)+} = 0. \tag{39}$$

This analysis indicates that the value function V_* and the optimal stopping boundary g_* can be obtained by searching for the **pair of functions** (V, g) solving the following **free-boundary problem**:

$$\mathbb{L}_X V(x,s) = c \quad \text{for } (x,s) \text{ in } C_g \tag{40}$$

$$\frac{\partial V}{\partial s}(x,s)\Big|_{x=s-} = 0 \quad \text{(normal reflection)} \tag{41}$$

$$V(x,s) = s \quad \text{for } (x,s) \text{ in } D_g \text{ (instantaneous stopping)} \tag{42}$$

$$\frac{\partial V}{\partial x}(x,s)\Big|_{x=g(s)+} = 0 \quad \text{(smooth fit)} \tag{43}$$

where the two sets are defined as follows:

$$C_g = \{\, (x,s) \in \mathbb{R}^2 \mid 0 \le x \le s < s_0 \;\text{ or }\; g(s) < x \le s \;\text{ for }\; s \ge s_0 \,\} \tag{44}$$

$$D_g = \{\, (x,s) \in \mathbb{R}^2 \mid 0 \le x \le g(s)\,,\; s \ge s_0 \,\} \tag{45}$$

with $g(s_0) = 0$. It turns out that this system does not have a unique solution so that an additional criterion is needed to make it unique in general.

Let us briefly show how to solve the free-boundary problem (40)–(43) by picking the right solution (more details will be given in the lectures).

From (40) one finds that for (x, s) in C_g we have

$$V(x, s) = cx^2 + A(s)\, x + B(s) \tag{46}$$

where A and B are some functions of s. To determine A and B as well as g we can use the three conditions

$$\frac{\partial V}{\partial s}(x, s)\Big|_{x=s-} = 0 \quad \text{(normal reflection)}$$

$$V(x, s) = s \quad \text{for} \quad (x, s) \quad \text{in} \quad D_g \quad \text{(instantaneous stopping)}$$

$$\frac{\partial V}{\partial x}(x, s)\Big|_{x=g(s)+} = 0 \quad \text{(smooth fit)}$$

which yield

$$g'(s) = \frac{1}{2(s - g(s))}, \quad \text{for} \quad s \geq s_0. \tag{47}$$

It is easily verified that the linear function

$$g(s) = s - \frac{1}{2c} \tag{48}$$

solves (47). In this way a candidate for the optimal stopping boundary g_* is obtained.

For all $(x, s) \in E$ with $s \geq 1/2c$ one can determine $V(x, s)$ explicitly using

$$V(x, s) = cx^2 + A(s)\, x + B(s)$$

and

$$g(s) = s - \frac{1}{2c}.$$

This in particular gives that $V(1/2c, 1/2c) = 3/4c$.

For other points $(x, s) \in E$ when $s < 1/2c$ one can determine $V(x, s)$ using that the observation must be continued. In particular for $x = s = 0$ this yields that

$$V(0, 0) = V(1/2c, 1/2c) - c\, \mathsf{E}_{0,0}(\sigma) \tag{49}$$

where σ is the first hitting time of the process (X, S) to the point $(1/2c, 1/2c)$.

Because $\mathsf{E}_{0,0}(\sigma) = \mathsf{E}_{0,0}(X_\sigma^2) = (1/2c)^2$ and $V(1/2c, 1/2c) = 3/4c$, we find that

$$V(0, 0) = \frac{1}{2c} \tag{50}$$

as already indicated prior to (5) above. In this way a candidate for the value function V_* is obtained.

The key role in the proof of the fact that $V = V_*$ and $g = g_*$ is played by **Itô's formula** (stochastic calculus) and the **optional sampling theorem** (martingale theory). This step forms a **verification theorem** that makes it clear that the solution of the free-boundary problem coincides with the solution of the optimal stopping problem.

7. The important point to be made in this context is that the verification theorem is usually not difficult to prove in the cases when a candidate solution to the free-boundary problem is obtained **explicitly**. This is quite typical for one-dimensional problems with **infinite horizon**, or some simpler two-dimensional problems, as the one just discussed. In the case of problems with **finite horizon**, however, or other multidimensional problems, the situation can be radically different. In these cases, in a manner quite opposite to the previous ones, the general results of optimal stopping can be used to prove the existence of a solution to the free-boundary problem, thus providing an alternative to analytic methods. Studies of this type will be also presented in the lectures of three courses on the School.

8. From the material exposed above it is clear that our basic interest concerns the case of **continuous** time. The theory of optimal stopping in the case of continuous time is considerably more complicated than in the case of **discrete** time. However, since the former theory uses many basic ideas from the latter, we have chosen to present the case of discrete time first, both in the **martingale** and **Markovian** setting, which is then likewise followed by the case of continuous time. The two theories form several my lectures.

Lecture 2-3. Theory of optimal stopping for discrete time.
A. Martingale approach.

1. Definitions

$$(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, \mathsf{P}), \quad \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n \subseteq \cdots \subseteq \mathcal{F}, \quad G = (G_n)_{n\geq 0}.$$

Gain G_n is $\mathcal{F}_n$-measurable

Stopping (Markov) time $\tau = \tau(\omega)$:

$$\tau\colon \Omega \to \{0, 1, \dots, \infty\}, \quad \{\tau \leq n\} \in \mathcal{F}_n \text{ for all } n \geq 0.$$

$\mathfrak{M}$ is the family of all *finite* stopping times

$\overline{\mathfrak{M}}$ is the family of all stopping times

$\mathfrak{M}_n^N = \{\tau \in \mathfrak{M} \,|\, n \leq \tau \leq N\}$

For simplicity we will set $\mathfrak{M}^N = \mathfrak{M}_0^N$ and $\mathfrak{M}_n = \mathfrak{M}_n^\infty$.

The *optimal stopping problem* to be studied seeks to solve

$$\boxed{V_* = \sup_{\tau} \mathsf{E} G_\tau} \tag{51}$$

For the existence of $\mathsf{E}G_\tau$ suppose (for simplicity) that

$$\mathsf{E} \sup_{0 \le k < \infty} |G_k| < \infty \tag{52}$$

then $\mathsf{E}G_\tau$ is well defined for all $\tau \in \mathfrak{M}_n^N$, $n \le N < \infty$.

In the class $\mathfrak{M}_n^N$ we consider

$$\boxed{V_n^N = \sup_{\tau \in \mathfrak{M}_n^N} \mathsf{E} G_\tau} \qquad 0 \le n \le N. \tag{53}$$

Sometimes it is also of interest to admit that τ in (51) takes the value ∞ ($\mathsf{P}(\tau = \infty) > 0$), so that $\tau \in \overline{\mathfrak{M}}$. We put $G_\tau = 0$ on $\{\tau = \infty\}$.

Sometimes it is useful to set $G_\infty = \limsup_{n \to \infty} G_n$.

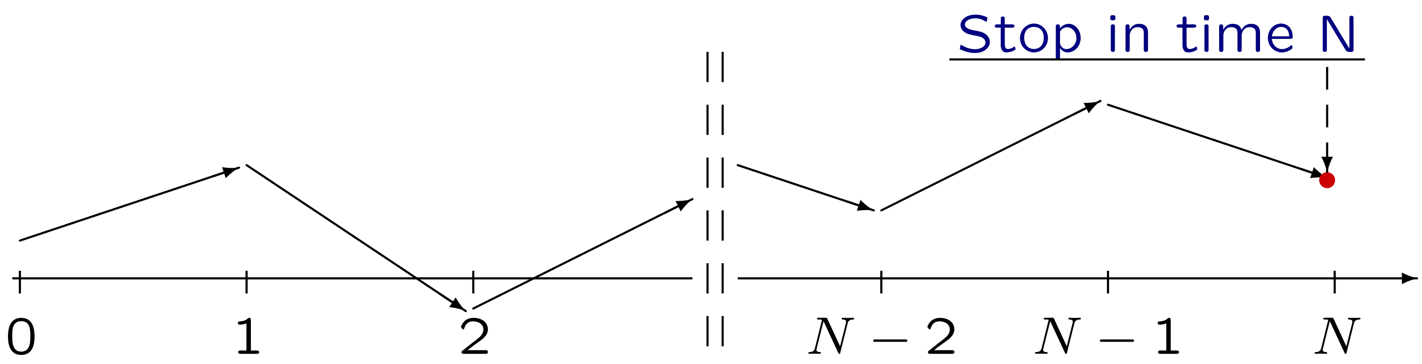
2. The method of backward induction.

$$V_n^N = \sup_{n \le \tau \le N} \mathsf{E} G_\tau$$

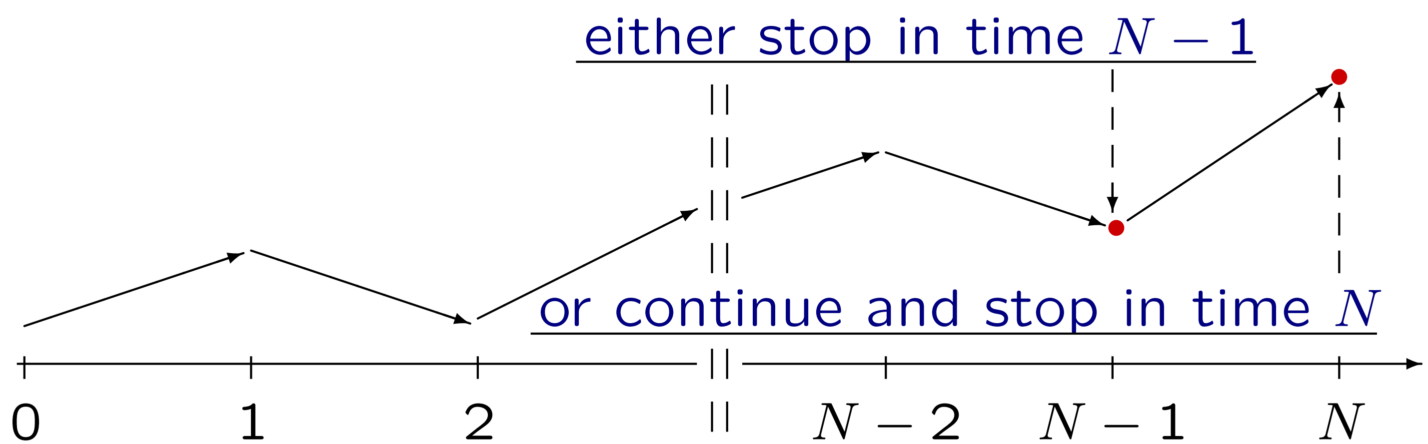
To solve this problem we introduce (by backward induction) a special stochastic sequence $S_N^N, S_{N-1}^N, \ldots, S_0^N$:

$$S_N^N = G_N, \quad S_n^N = \max\{G_n, \mathsf{E}(S_{n+1}^N \,|\, \mathcal{F}_n)\},$$
$$n = N-1, \ldots, 0.$$

If $n = N$ we have to stop and our stochastic gain S_N^N, equals G_N.



For $n = N-1$ we can either stop or continue. If we stop, our gain S^N_{N-1}, equals G_{N-1}, and if we continue our gain S^N_{N-1} will be equal to $\mathsf{E}(S^N_N \,|\, \mathcal{F}_{N-1})$.



So,

$$S^N_{N-1} = \max\{G_{N-1}, \mathsf{E}(S^N_N \,|\, \mathcal{F}_{N_1})\}$$

and optimal stopping time is

$$\tau^N_{N-1} = \min\{N-1 \le k \le N \colon S^N_k = G_k\}.$$

Define now a sequence $(S_n^N)_{0 \le n \le N}$ recursively as follows:

$$S_n^N = G_N, \quad n = N,$$
$$S_n^N = \max\{G_n, \mathsf{E}(S_{n+1}^N \,|\, \mathcal{F}_n)\}, \;\; n = N-1, \dots, 0.$$

The described method suggests to consider the following stopping time:

$$\tau_n^N = \inf\{n \le k \le N \colon S_k^N = G_k\}$$

for $0 \le n \le N$.

The first part of the following theorem shows that S_n^N and τ_n^N solve the problem in a stochastic sense.

The second part of the theorem shows that this leads also to a solution of the initial problem

$$V_n^N = \sup_{n \le \tau \le N} \mathsf{E} G_\tau \quad \text{for each} \quad n = 0, 1, \dots, N.$$

Theorem 1. (Finite horizon)

I. *For all* $0 \le n \le N$ *we have*:

$$\text{(a)} \quad S_n^N \ge \mathsf{E}(G_\tau \,|\, \mathcal{F}_n), \quad \forall \tau \in \mathfrak{M}_n^N;$$

$$\text{(b)} \quad S_n^N = \mathsf{E}(G_{\tau_n^N} \,|\, \mathcal{F}_n).$$

II. *Moreover, if* $0 \le n \le N$ *is given and fixed, then we have*:

(c) τ_n^N *is optimal in* $\quad V_n^N = \sup\limits_{n \le \tau \le N} \mathsf{E} G_\tau$;

(d) *if* τ_* *is also optimal then* $\tau_n^N \le \tau_*$;

(e) *the sequence* $(S_k^N)_{n \le k \le N}$ *is the smallest supermartingale which dominates* $(G_k)_{n \le k \le N}$ *(Snell's envelope)*

(f) *the stopped sequence* $(S_{k \wedge \tau_n^N}^N)_{n \le k \le N}$ *is a martingale.*

Proof of Theorem 1.

I. Induction over $n = N, N-1, \ldots, 0$.

Conditions

$$\text{(a)} \quad S_n^N \geq \mathsf{E}(G_\tau \,|\, \mathcal{F}_n), \quad \forall \tau \in \mathfrak{M}_n^N;$$

and

$$\text{(b)} \quad S_n^N = \mathsf{E}(G_{\tau_n^N} \,|\, \mathcal{F}_n).$$

are trivially satisfied for $n = N$.

Suppose that (a) and (b) are satisfied for $n = N, N-1, \ldots, k$ where $k \geq 1$, and let us show that they must then also hold for $n = k-1$.

(a) $\left(S_n^N \geq \mathsf{E}(G_\tau \,|\, \mathcal{F}_n), \quad \forall \tau \in \mathfrak{M}_n^N\right)$: Take $\tau \in \mathfrak{M}_{k-1}^N$ and set $\bar{\tau} = \tau \vee k$; then $\bar{\tau} \in \mathfrak{M}_k^N$ and since $\{\tau \geq k\} \in \mathcal{F}_{k-1}$ it follows that

$$\begin{aligned}
\mathsf{E}(G_\tau \,|\, \mathcal{F}_{k-1}) &= \mathsf{E}[I(\tau = k-1) G_{k-1} \,|\, \mathcal{F}_{k-1}] \\
&\qquad\qquad + \mathsf{E}[I(\tau \geq k) G_{\bar{\tau}} \,|\, \mathcal{F}_{k-1}] \\
&= I(\tau = k-1) G_{k-1} \\
&\qquad + I(\tau \geq k)\, \mathsf{E}[\mathsf{E}(G_{\bar{\tau}} \,|\, \mathcal{F}_k) \,|\, \mathcal{F}_{k-1}].
\end{aligned} \tag{54}$$

By the induction hypothesis, (a) holds for $n = k$. Since $\bar{\tau} \in \mathfrak{M}_k^N$ this implies that

$$\mathsf{E}(G_{\bar{\tau}} \,|\, \mathcal{F}_k) \leq S_k^N. \tag{55}$$

From $S_n^N = \max(G_n, \mathsf{E}(S_{n+1}^N \,|\, \mathcal{F}_n))$ for $n = k-1$ we have

$$G_{k-1} \leq S_{k-1}^N, \tag{56}$$

$$\mathsf{E}(S_k^N \,|\, \mathcal{F}_{k-1}) \leq S_{k-1}^N. \tag{57}$$

Using (55)–(57) in (54) we get

$$\begin{aligned} \mathsf{E}(G_\tau \,|\, \mathcal{F}_{k-1}) &\le I(\tau\!=\!k\!-\!1)\, S_{k-1}^N \\ &\quad + I(\tau\!\ge\!k)\, \mathsf{E}(S_k^N \,|\, \mathcal{F}_{k-1}) \\ &\le I(\tau\!=\!k\!-\!1)\, S_{k-1}^N \\ &\quad + I(\tau\!\ge\!k)\, S_{k-1}^N = S_{k-1}^N. \end{aligned} \tag{58}$$

This shows that

$$S_n^N \ge \mathsf{E}(G_\tau \,|\, \mathcal{F}_n), \quad \forall \tau \in \mathfrak{M}_n^N$$

holds for $n = k-1$ as claimed.

(b) $\left(S_n^N = \mathsf{E}(G_{\tau_n^N} \,|\, \mathcal{F}_n)\right)$: To prove (b) for $n = k-1$ it is enough to check that all inequalities in (54) and (58) remain equalities when $\tau = \tau_{k-1}^N$. For this, note that

$$\begin{aligned} &\tau_{k-1}^N = \tau_k^N && \text{on} \quad \{\tau_{k-1}^N \ge k\}; \\ &G_{k-1} = S_{k-1}^N && \text{on} \quad \{\tau_{k-1}^N = k-1\}; \\ &\mathsf{E}(S_k^N \,|\, \mathcal{F}_{k-1}) = S_{k-1}^N && \text{on} \quad \{\tau_{k-1}^N \ge k\}. \end{aligned}$$

Then we get

$$
\begin{aligned}
\mathsf{E}\Big[G_{\tau^N_{k-1}} \,|\, \mathcal{F}_{k-1}\Big] &= I(\tau^N_{k-1} = k-1)\, G_{k-1} \\
&\quad + I(\tau^N_{k-1} \geq k)\, \mathsf{E}\Big[\mathsf{E}(G_{\tau^N_k} \,|\, \mathcal{F}_k) \,|\, \mathcal{F}_{k-1}\Big] \\
&= I(\tau^N_{k-1} = k-1)\, G_{k-1} \\
&\quad + I(\tau^N_{k-1} \geq k)\, \mathsf{E}(S^N_k \,|\, \mathcal{F}_{k-1}) \\
&= I(\tau^N_{k-1} = k-1)\, S^N_{k-1} \\
&\quad + I(\tau^N_{k-1} \geq k)\, S^N_{k-1} = S^N_{k-1}.
\end{aligned}
$$

Thus

$$S^N_n = \mathsf{E}(G_{\tau^N_n} \,|\, \mathcal{F}_n)$$

holds for $n = k-1$. (We supposed by induction that (b) holds for $n = N, \dots, k$.)

(c) $\Big(\tau_n^N$ is optimal in $V_n^N = \sup\limits_{n\le\tau\le N} \mathsf{E}G_\tau\Big)$:

Take expectation E in

$$S_n^N \ge \mathsf{E}(G_\tau \,|\, \mathcal{F}_n), \quad \tau \in \mathfrak{M}_n^n.$$

Then we find that

$$\mathsf{E}S_n^N \ge \mathsf{E}G_\tau \quad \text{for all} \quad \tau \in \mathfrak{M}_n^N$$

and by taking the supremum over all $\tau \in \mathfrak{M}_n^N$ we see that

$$\mathsf{E}S_n^N \ge V_n^N \quad \Big(= \sup_{\tau\in\mathfrak{M}_n^N} \mathsf{E}G_\tau\Big).$$

On the other hand, taking the expectation in

$$S_n^N = \mathsf{E}(G_{\tau_n^N} \,|\, \mathcal{F}_n) \quad \text{we get} \quad \mathsf{E}S_n^N = \mathsf{E}G_{\tau_n^N}$$

which shows that

$$\mathsf{E}S_n^N \le V_n^N \quad \Big(= \sup_{\tau\in\mathfrak{M}_n^N} \mathsf{E}G_\tau\Big).$$

So,

$$\mathsf{E}S_n^N = V_n^N$$

and since $\mathsf{E}S_n^N = \mathsf{E}G_{\tau_n^N}$, we see that

$$V_n^N = \mathsf{E}G_{\tau_n^N}$$

implying the claim (c): "The stopping time τ_n^N is optimal".

(d) $\Big($ if τ_* is also optimal then $\tau_n^N \le \tau_*\Big)$:
If τ_* is also optimal then $\tau_n^N \le \tau_*$. We claim that the optimality of τ_* implies that $S_{\tau_*}^N = G_{\tau_*}$ (P-a.s.)".

Indeed, for all $n \le k \le N$,

$$S_k^N \ge G_k \qquad \text{thus,} \qquad S_{\tau_*}^N \ge G_{\tau_*}.$$

If $S_{\tau_*}^N \ne G_{\tau_*}$ (P-a.s.) then

$$\mathsf{P}(S_{\tau_*}^N > G_{\tau_*}) > 0.$$

It thus follows that

$$\mathsf{E}G_{\tau_*} < \mathsf{E}S^N_{\tau_*} \overset{(\alpha)}{\leq} \mathsf{E}S^N_n \overset{(\beta)}{=} V^N_n$$

where

> the second inequality (α) follows by the supermartingale property of $(S^N_k)_{n\le k\le N}$ (see (e)) and the optional sampling theorem, and
>
> the equality (β) was obtained in (c).

The strict inequality $\mathsf{E}G_{\tau_*} < V^N_n$, however, contradicts the fact that τ_* is optimal.

Hence $S^N_{\tau_*} = G_{\tau_*}$ (P-a.s.) and the fact that $\tau^N_n \le \tau_*$ (P-a.s.) follows from the definition

$$\tau^N_n = \inf\{n \le k \le N \colon S^N_k = G_k\}.$$

(e) $\Big($the sequence $(S_k^N)_{n\le k\le N}$ is the smallest supermartingale which dominates $(G_k)_{n\le k\le N}\Big)$:

The sequence $(S_k^N)_{n\le k\le N}$ is the smallest supermartingale which dominates $(G_k)_{n\le k\le N}$''.

From

$$S_k^N = \max\{G_k, \mathsf{E}(S_{k+1}^N \,|\, \mathcal{F}_k)\}, \quad k = N-1,\dots,n,$$

we see that $(S_k^N)_{n\le k\le N}$ is a supermartingale:

$$S_k^N \ge \mathsf{E}(S_{k+1}^N \,|\, \mathcal{F}_k).$$

Also we have $S_k^N \ge G_k$. It means that $(S_k^N)_{n\le k\le N}$ is a supermartingale which dominates $(G_k)_{n\le k\le N}$.

Suppose that $(\widetilde{S}_k)_{n\le k\le N}$ is another supermartingale which dominates $(G_k)_{n\le k\le N}$, then the claim that $\widetilde{S}_k \ge S_k^N$ (P-a.s.) can be verified by induction over $k = N, N-1, \dots, l$.

Indeed, if $k = N$ then the claim follows by $S_n^N = G_N$ for $n = N$.

Assuming that $\widetilde{S}_k \ge S_k^N$ for $k = N, N-1, \dots, l$ with $l \ge n+1$ it follows that

$$\begin{aligned} S_{l-1}^N &= \max(G_{l-1}, \mathsf{E}(S_l^N \,|\, \mathcal{F}_{l-1})) \\ &\le \max(G_{l-1}, \mathsf{E}(\widetilde{S}_l \,|\, \mathcal{F}_{l-1})) \le \widetilde{S}_{l-1} \quad \text{(P-a.s.)} \end{aligned}$$

using the supermartingale property of $(\tilde{S}_k)_{n\le k\le N}$. So, $(S_k^N)_{n\le k\le N}$ is the smallest supermartingale which dominates $(G_k)_{n\le k\le N}$ (Snell's envelop).

(f) $\Big($ the stopped sequence $(S^N_{k\wedge\tau_n^N})_{n\le k\le N}$ is a martingale$\Big)$:

To verify the martingale property

$$\mathsf{E}\Big[S^N_{(k+1)\wedge\tau_n^N}\,|\,\mathcal{F}_k\Big] = S^N_{k\wedge\tau_n^N}$$

with $n \le k \le N-1$ given and fixed, note that

$$\begin{aligned}
\mathsf{E}\Big[S^N_{(k+1)\wedge\tau_n^N}\,|\,\mathcal{F}_k\Big] &= \mathsf{E}\Big[I(\tau_n^N \le k)\, S^N_{k\wedge\tau_n^N}\,|\,\mathcal{F}_k\Big] \\
&\qquad + \mathsf{E}\Big[I(\tau_n^N \ge k+1)\, S^N_{k+1}\,|\,\mathcal{F}_k\Big] \\
&= I(\tau_n^N \le k) S^N_{k\wedge\tau_n^N} + I(\tau_n^N \ge k+1)\mathsf{E}(S^N_{k+1}\,|\,\mathcal{F}_k) \\
&= I(\tau_n^N \le k) S^N_{k\wedge\tau_n^N} + I(\tau_n^N \ge k+1) S^N_k = S^N_{k\wedge\tau_n^N}
\end{aligned}$$

where we used that

$$S^N_k = \mathsf{E}(S^N_{k+1}\,|\,\mathcal{F}_k) \quad \text{on } \{\,\tau_n^N \ge k+1\,\}$$

and $\{\,\tau_n^N \ge k+1\,\} \in \mathcal{F}_k$ since τ_n^N is a stopping time.

Summary

1) The optimal stopping problem

$$V_0^N = \sup_{\tau \in \mathfrak{M}_0^N} \mathsf{E} G_\tau$$

is solved inductively by solving the problems

$$V_n^N = \sup_{\tau \in \mathfrak{M}_n^N} \mathsf{E} G_\tau$$

for $n = N, N-1, \dots, 0$.

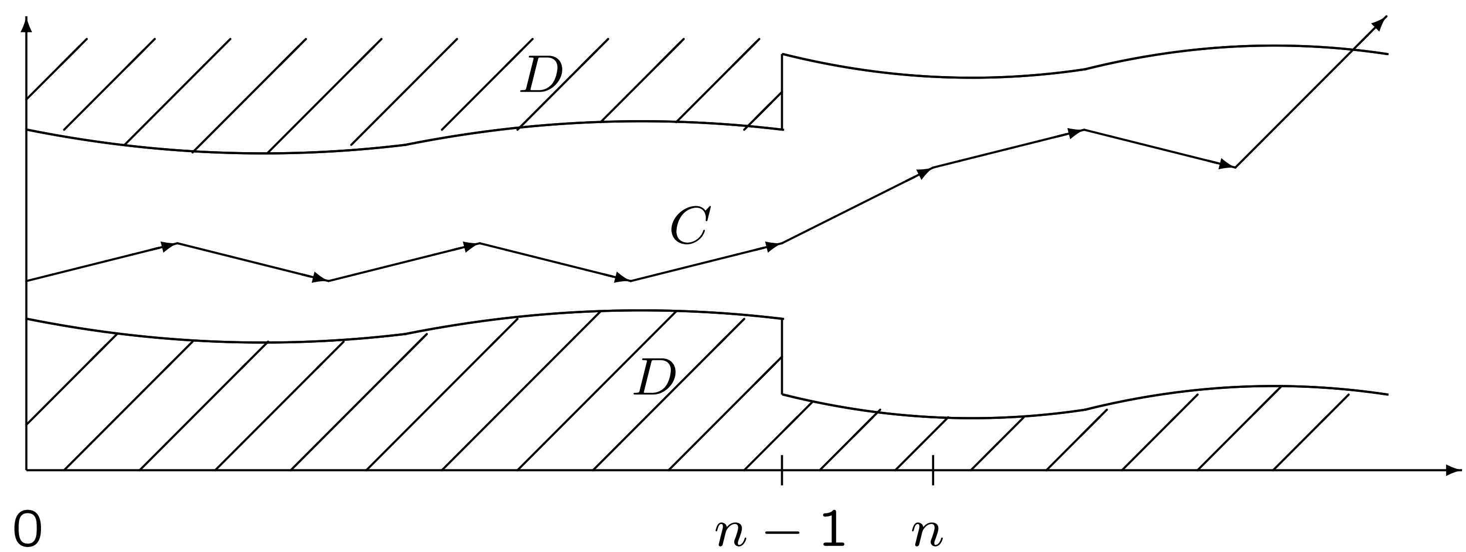
2) The optimal stopping rule τ_n^N for V_n^N satisfies

$$\tau_n^N = \tau_k^N \quad \text{on } \{\tau_n^N \geq k\}$$

for $0 \leq n \leq k \leq N$ when τ_k^N is the optimal stopping rule for V_k^N. In other words, this means that if it was not optimal to stop within the time set $\{n, n+1, \dots, k-1\}$ then the same optimality rule for V_n^N applies in the time set $\{k, k+1, \dots, N\}$.

3) In particular, when specialized to the problem V_0^N, the following general principle (of dynamic programming) is obtained:

> if the stopping rule τ_0^N is optimal for V_0^N and it was not optimal to stop within the time set $\{0, 1, \ldots, n-1\}$, then starting the observation at time n and being based on the information $\mathcal{F}_n$, the same stopping rule is still optimal for the problem V_n^N.



3. The method of essential supremum

The method of backward induction by its nature requires that the horizon N be FINITE so that the case of infinite horizon remains uncovered.

It turns out, however, that the random variables S_n^N defined by the recurrent relations

$$
\begin{aligned}
S_n^N &= G_N, \quad n = N, \\
S_n^N &= \max\{G_n, \mathsf{E}(S_{n+1}^N \,|\, \mathcal{F}_n)\}, \;\; n = N-1, \ldots, 0,
\end{aligned}
$$

admit a different characterization which can be directly extended to the case of infinite horizon N.

This characterization forms the base of the SECOND method that will now be presented.

Note that

$$\text{(a)} \quad S_n^N \geq \mathsf{E}(G_\tau \,|\, \mathcal{F}_n) \quad \forall \tau \in \mathfrak{M}_n^N;$$
$$\text{(b)} \quad S_n^N = \mathsf{E}(G_{\tau_n^N} \,|\, \mathcal{F}_n)$$

from Theorem 1 suggest that the following identity should hold:

$$S_n^N = \sup_{\tau \in \mathfrak{M}_n^N} \mathsf{E}(G_\tau \,|\, \mathcal{F}_n).$$

Difficulty: $\sup_{\tau \in \mathfrak{M}_n^N} \mathsf{E}(G_\tau \,|\, \mathcal{F}_n)$ need not define a measurable function.

To overcome this difficulty it turns out that the concept of

essential supremum

proves useful.

Lemma (about Essential Supremum).

Let $\{Z_\alpha, \alpha \in \mathfrak{A}\}$ *be a family of random variables defined on* $(\Omega, \mathcal{F}, \mathsf{P})$ *where the index set* $\mathfrak{A}$ *can be arbitrary.*

Then there exists a countable subset J *of* $\mathfrak{A}$ *such that the random variable* $Z^*\colon \Omega \to \overline{\mathbb{R}}$ *defined by*

$$Z^* = \sup_{\alpha \in J} Z_\alpha$$

satisfies the following two properties:

(a) $\mathsf{P}(Z_\alpha \leq Z^*) = 1$, $\forall \alpha \in \mathfrak{A}$;

(b) *If* $\tilde{Z}\colon \Omega \to \overline{\mathbb{R}}$ *is another random variable satisfying* $\mathsf{P}(Z_\alpha \leq Z^*) = 1$, $\forall \alpha \in \mathfrak{A}$, *then* $\mathsf{P}(Z^* \leq \tilde{Z}) = 1$.

Moreover, if the family $\{Z_\alpha, \alpha \in \mathfrak{A}\}$ is upwards directed in the sense that

$$\text{\textit{for any} } \alpha \text{ \textit{and} } \beta \text{ \textit{in} } \mathfrak{A} \text{ \textit{there exists} } \gamma \text{ \textit{in} } \mathfrak{A}$$
$$\text{\textit{such that} } \max(Z_\alpha, Z_\beta) \le Z_\gamma \text{ (P-\textit{a.s.})},$$

then the countable set $J = \{\alpha_n, n \ge 1\}$ can be chosen so that

$$Z^* = \lim_{n\to\infty} Z_{\alpha_n} \quad \text{(P-\textit{a.s.})}$$

where $Z_{\alpha_1} \le Z_{\alpha_2} \le \cdots$ (P-a.s.).

Proof. (1) Since $x \mapsto \frac{2}{\pi}\arctan(x)$ is a strictly increasing function from $\overline{\mathbb{R}}$ to $[-1, 1]$, it is no restriction to assume that $|Z_\alpha| \le 1$.

(2) Let $\mathcal{C}$ denote the family of all countable subsets C of $\mathfrak{A}$. Choose an increasing sequence $\{C_n, n \ge 1\}$ in $\mathcal{C}$ such that

$$a \overset{def}{=} \sup_{C\in\mathcal{C}} \mathsf{E}\Big(\sup_{\alpha\in C} Z_\alpha\Big) = \sup_{n\ge1} \mathsf{E}\Big(\sup_{\alpha\in C_n} Z_\alpha\Big).$$

Then $J \stackrel{def}{=} \bigcup_{n=1}^{\infty} C_n$ is a countable subset of $\mathfrak{A}$ and we claim that Z^* defined by

$$Z^* = \sup_{\alpha \in J} Z_\alpha$$

satisfies the properties (a) and (b).

(3) To verify these claims take $\alpha \in \mathfrak{A}$ arbitrarily and note the following. If $\alpha \in J$ then $Z_\alpha \leq Z^*$ so that (a) holds. If $\alpha \notin J$ and we assume that $\mathsf{P}(Z_\alpha > Z^*) > 0$, then

$$a < \mathsf{E}(Z^* \vee Z_\alpha) \leq a$$

since $a = \mathsf{E} Z^* \in [-1, 1]$ (by the monotone convergence theorem) and $J \cup \{\alpha\}$ belongs to $\mathcal{C}$. As the strict inequality is clearly impossible, we see that $\mathsf{P}(Z_\alpha \leq Z^*) = 1$ holds for all $\alpha \in \mathfrak{A}$ as claimed. Moreover, it is obvious that (b) follows from $Z^* = \sup_{\alpha \in J} Z_\alpha$ and (a): $\mathsf{P}(Z_\alpha \leq Z^*) = 1$, $\forall \alpha \in \mathfrak{A}$, since J is countable.

Finally, assume that the condition in (c) is satisfied. Then the initial countable set

$$J = \{\alpha_1, \alpha_2, \ldots\}$$

can be replaced by a new countable set $J^\circ = \{\alpha_1^\circ, \alpha_2^\circ, \ldots\}$ if we initially set $\alpha_1^\circ = \alpha_1$, and then inductively choose $\alpha_{n+1}^\circ \geq \alpha_n^\circ \vee \alpha_{n+1}$ for $n \geq 1$, where $\gamma \geq \alpha \vee \beta$ corresponds to Z_α, Z_β and Z_γ such that $Z_\gamma \geq Z_\alpha \vee Z_\beta$ (P-a.s.). The concluding claim $Z^* = \lim_{n\to\infty} Z_{\alpha_n}$ in (c) is then obvious, and the proof of the lemma is complete. $\square$

With the concept of essential supremum we may now rewrite

$$S_n^N \geq \mathsf{E}(G_\tau \,|\, \mathcal{F}_n) \quad \forall \tau \in \mathfrak{M}_n^N;$$
$$S_n^N = \mathsf{E}(G_{\tau_n^N} \,|\, \mathcal{F}_n)$$

in Theorem 51 above as follows:

$$S_n^N = \operatorname*{ess\,sup}_{n \leq \tau \leq N} \mathsf{E}(G_\tau \,|\, \mathcal{F}_n)$$

for all $0 \leq n \leq N$.

This ess sup identity provides an additional characterization of the sequence of r.v.'s $(S_n^N)_{0\le n\le N}$ introduced initially by means of the recurrent relations

$$
\begin{aligned}
&S_n^N = G_N, \quad n = N,\\
&S_n^N = \max\{G_n, \mathsf{E}(S_{n+1}^N \,|\, \mathcal{F}_n)\}, \;\; n = N-1, \dots, 0.
\end{aligned}
$$

Its advantage in comparison with these recurrent relations lies in the fact that the identity

$$S_n^N = \operatorname*{ess\,sup}_{n\le\tau\le N} \mathsf{E}(G_\tau \,|\, \mathcal{F}_n)$$

can naturally be extended to the case of INFINITE horizon N. This programme will now be described.

Consider (instead of $V_n^N = \sup_{\tau\in\mathfrak{M}_n^N} \mathsf{E}G_\tau$)

$$V_n = \sup_{\tau\in\mathfrak{M}_n^\infty} \mathsf{E}G_\tau.$$

To solve this problem we will consider the sequence of r.v.'s $(S_n)_{n\geq 0}$ defined as follows:

$$S_n = \operatorname*{ess\,sup}_{\tau \geq n} \mathsf{E}(G_\tau \,|\, \mathcal{F}_n)$$

as well as the following stopping time:

$$\tau_n = \inf\{k \geq n \,|\, S_k = G_k\}$$

for $n \geq 0$ where $\inf \varnothing = \infty$ by definition.

The first part (I) of the following theorem shows that $(S_n)_{n\geq 0}$ satisfies the same recurrent relations as $(S_n^N)_{0\leq n\leq N}$.

The second part (II) of the theorem shows that S_n and τ_n solve the problem in a stochastic sense.

The third part (III) shows that this leads to a solution of the initial problem $V_n = \sup_{\tau\geq n} \mathsf{E} G_\tau$.

The fourth part (IV) provides a supermartingale characterization of the solution.

Theorem 2 (Infinite horizon).

Consider the optimal stopping problems $V_n = \sup_{\tau \geq n} \mathsf{E} G_\tau$, $\tau \in \mathfrak{M}_n^\infty$, $n \geq 0$ *assuming that the condition* $\mathsf{E} \sup_{0 \leq k < \infty} |G_k| < \infty$ *holds.*

I. *The following recurrent relations hold*:

$$S_n = \max\{G_n, \mathsf{E}(S_{n+1} \,|\, \mathcal{F}_n)\}, \quad \forall n \geq 0.$$

II. *Assume moreover if required below that*

$$\mathsf{P}(\tau_n < \infty) = 1.$$

Then for all $n \geq 0$ *we have*:

$$S_n \geq \mathsf{E}(G_\tau \,|\, \mathcal{F}_n) \quad \forall \tau \in \mathfrak{M}_n,$$
$$S_n = \mathsf{E}(G_{\tau_n} \,|\, \mathcal{F}_n).$$

III. *Moreover, if* $n \geq 0$ *is given and fixed, then we have*:

The stopping time τ_n *is optimal in* $V_n = \sup_{\tau \geq n} \mathsf{E} G_\tau$.

If τ_* *is an optimal stopping time for* $V_n = \sup_{\tau \geq n} \mathsf{E} G_\tau$ *then* $\tau_n \leq \tau_*$ (P-*a.s.*).

IV. *The sequence* $(S_k)_{k \geq n}$ *is the smallest supermartingale which dominates* $(G_k)_{k \geq n}$ *(Snell's envelop).*

The stopped sequence $(S_{k \wedge \tau_n})_{k \geq n}$ *is a martingale.*

Finally, if the condition $\mathsf{P}(\tau_n < \infty) = 1$ *fails so that* $\mathsf{P}(\tau_n = \infty) > 0$, *then there is NO optimal stopping time in* $V_n = \sup_{\tau \geq n} \mathsf{E} G_\tau$.

Proof. I. We need prove the recurrent relations

$$S_n = \max\{G_n, \mathsf{E}(S_{n+1} \,|\, \mathcal{F}_k)\}, \quad n \geq 0.$$

Let us first show that

$$S_n \leq \max\{G_n, \mathsf{E}(S_{n+1} \,|\, \mathcal{F}_k)\}.$$

For this, take $\tau \in \mathfrak{M}_n$ and set $\bar{\tau} = \tau \vee (n+1)$.
Then $\bar{\tau} \in \mathfrak{M}_{n+1}$, and since $\{\tau \geq n+1\} \in \mathcal{F}_n$ we have

$$\begin{aligned}
\mathsf{E}(G_\tau \,|\, \mathcal{F}_n) &= \mathsf{E}[I(\tau = n)G_n \,|\, \mathcal{F}_n] + \mathsf{E}[I(\tau \geq n+1)G_{\bar{\tau}} \,|\, \mathcal{F}_n] \\
&= I(\tau = n)G_n + I(\tau \geq n+1)\mathsf{E}(G_{\bar{\tau}} \,|\, \mathcal{F}_n) \\
&= I(\tau = n)G_n + I(\tau \geq n+1)\mathsf{E}[\mathsf{E}(G_{\bar{\tau}} \,|\, \mathcal{F}_{n+1}) \,|\, \mathcal{F}_n] \\
&\leq I(\tau = n)G_n + I(\tau \geq n+1)\mathsf{E}S_{n+1} \,|\, \mathcal{F}_n) \\
&\leq \max\{G_n, \mathsf{E}(S_{n+1} \,|\, \mathcal{F}_n)\}.
\end{aligned}$$

From this inequality it follows that

$$S_n = \operatorname*{ess\,sup}_{\tau \geq n} \mathsf{E}(G_\tau \,|\, \mathcal{F}_n) \leq \max\{G_n, \mathsf{E}(S_{n+1} \,|\, \mathcal{F}_n)\}$$

which is the desired inequality.

For the reverse inequality, let us first note that

$$S_n \geq G_n \quad \text{(P-a.s.)}$$

by the definition of S_n, so that it is enough to show that

$$S_n \geq \mathsf{E}(S_{n+1} \,|\, \mathcal{F}_n)$$

which is the supermartingale property of $(S_n)_{n \geq 0}$. (It is the most difficult part of the proof.)

To verify this inequality, let us first show that the family $\{\mathsf{E}(G_\tau \,|\, \mathcal{F}_{n+1}); \tau \in \mathfrak{M}_{n+1}\}$ is upwards directed in the sense that

$$\begin{array}{l}\text{for any } \alpha \text{ and } \beta \text{ in } \mathfrak{A} \text{ there exists } \gamma \text{ in} \\ \mathfrak{A} \text{ such that } Z_\alpha \vee Z_\beta \leq Z_\gamma.\end{array} \qquad (*)$$

For this, note that if σ_1 and σ_2 are from $\mathfrak{M}_{n+1}$ and we set $\sigma_3 = \sigma_1 I_A + \sigma_2 I_{\bar{A}}$ where

$$A = \{\mathsf{E}(G_{\sigma_1} \,|\, \mathcal{F}_{n+1}) \geq \mathsf{E}(G_{\sigma_2} \,|\, \mathcal{F}_{n+1})\},$$

then $\sigma_3 \in \mathfrak{M}_{n+1}$ and we have

$$\begin{aligned} \mathsf{E}(G_{\sigma_3} \,|\, \mathcal{F}_{n+1}) &= \mathsf{E}(G_{\sigma_1} I_A + G_{\sigma_2} I_{\bar{A}} \,|\, \mathcal{F}_{n+1}) \\ &= I_A\, \mathsf{E}(G_{\sigma_1} \,|\, \mathcal{F}_{n+1}) + I_{\bar{A}}\, \mathsf{E}(G_{\sigma_2} \,|\, \mathcal{F}_{n+1}) \\ &= \mathsf{E}(G_{\sigma_1} \,|\, \mathcal{F}_{n+1}) \vee \mathsf{E}(G_{\sigma_2} \,|\, \mathcal{F}_{n+1}) \end{aligned}$$

implying $(*)$ as claimed. Hence by Lemma there exists a sequence $\{\sigma_k, k \geq 1\}$ in $\mathfrak{M}_{n+1}$ such that

$$\operatorname*{ess\,sup}_{\tau \geq n+1} \mathsf{E}(G_\tau \,|\, \mathcal{F}_{n+1}) = \lim_{k \to \infty} \mathsf{E}(G_{\sigma_k} \,|\, \mathcal{F}_{n+1})$$

where

$$\mathsf{E}(G_{\sigma_1} \,|\, \mathcal{F}_{n+1}) \leq \mathsf{E}(G_{\sigma_2} \,|\, \mathcal{F}_{n+1}) \leq \cdots \quad \text{(P-a.s.)}.$$

Since

$$S_{n+1} = \operatorname*{ess\,sup}_{\tau \geq n+1} \mathsf{E}(G_\tau \,|\, \mathcal{F}_{n+1}),$$

by the conditional monotone convergence theorem we get

$$\begin{aligned}
\mathsf{E}(S_{n+1} \,|\, \mathcal{F}_n) &= \mathsf{E}\Big[\lim_{k\to\infty} \mathsf{E}(G_{\sigma_k} \,|\, \mathcal{F}_{n+1}) \,|\, \mathcal{F}_n\Big] \\
&= \lim_{k\to\infty} \mathsf{E}\Big[\mathsf{E}(G_{\sigma_k} \,|\, \mathcal{F}_{n+1}) \,|\, \mathcal{F}_n\Big] \\
&= \lim_{k\to\infty} \mathsf{E}(G_{\sigma_k} \,|\, \mathcal{F}_n) \leq S_n.
\end{aligned}$$

So, $S_n = \max\{G_n, \mathsf{E}(S_{n+1} \,|\, \mathcal{F}_n)\}$ and the proof if I is complete.

II. The inequality $S_n \geq \mathsf{E}(G_\tau \,|\, \mathcal{F}_n)$, $\forall \tau \in \mathfrak{M}_n$, follows from the definition $S_n = \operatorname*{ess\,sup}_{\tau \geq n} \mathsf{E}(G_\tau \,|\, \mathcal{F}_n)$.

For the proof of the equality $S_n = \mathsf{E}(G_{\tau_n} \,|\, \mathcal{F}_n)$ we use the fact stated below in IV that the stopped sequence $(S_{k\wedge\tau_n})_{k\geq n}$ is a martingale.

Setting $G_n^* = \sup_{k \geq n} |G_k|$ we have

$$|S_k| \leq \operatorname*{ess\,sup}_{\tau \geq k} \mathsf{E}\Big(|G_\tau| \mid \mathcal{F}_k\Big) \leq \mathsf{E}(G_n^* \mid \mathcal{F}_k) \tag{$*$}$$

for all $k \geq n$. Since G_n^* is integrable due to $\mathsf{E}\sup_{k \geq n} |G_k| < \infty$, it follows from $(*)$ that $(S_k)_{k \geq n}$ is uniformly integrable.

Thus the optional sampling theorem can be applied to the martingale $(M_k)_{k \geq n} = (S_{k \wedge \tau_n})_{k \geq n}$ and we get

$$M_n = \mathsf{E}(M_{\tau_n} \mid \mathcal{F}_n). \tag{$**$}$$

Since $M_n = S_n$ and $M_{\tau_n} = S_{\tau_n}$ we see that $(**)$ is the same as $S_n = \mathsf{E}(G_{\tau_n} \mid \mathcal{F}_n)$.

III: "The stopping time τ_n is optimal in $V_n = \sup_{\tau \geq n} \mathsf{E} G_\tau$."

The proof uses II and is similar to the corresponding proof in Theorem 1 ($N < \infty$).

IV. "The sequence $(S_k)_{k\geq n}$ is the smallest supermartingale which dominates $(G_k)_{k\geq n}$ " (Snell's envelop).

We proved in I that $(S_k)_{k\geq n}$ is a supermartingale. Moreover, from the definition

$$S_n = \operatorname*{ess\,sup}_{\tau\geq n} \mathsf{E}(G_\tau \,|\, \mathcal{F}_n)$$

it follows that $S_k \geq G_k$, $k \geq n$, meaning that $(S_k)_{k\geq n}$ dominates $(G_k)_{k\geq n}$. Finally, if $(\tilde{S}_k)_{k\geq n}$ is another supermartingale which dominates $(G_k)_{k\geq n}$, then from $S_n = \mathsf{E}(G_{\tau_n} \,|\, \mathcal{F}_n)$ (Part II) we find

$$S_k = \mathsf{E}(G_{\tau_k} \,|\, \mathcal{F}_k) \leq \mathsf{E}(\tilde{S}_{\tau_k} \,|\, \mathcal{F}_k) \leq \tilde{S}_k, \quad \forall k \geq n.$$

(The final inequality follows by the optional sampling theorem being applicable since $\tilde{S}_k^- \leq G_k^- \leq G_n^*$ $(= \sup_{k\geq n} |G_k|)$ with G_n^* integrable.)

The statement

The stopped sequence $(S_{k\wedge\tau_n})_{k\geq n}$ is a martingale

is proved in exactly the same way as for case $N < \infty$.

Finally, note that the final claim

> If the condition $\mathsf{P}(\tau_n < \infty) = 1$ fails so that $\mathsf{P}(\tau_n = \infty) > 0$, then there is NO optimal stopping time in the problem $V_n = \sup_{\tau \geq n} \mathsf{E} G_\tau$

follows directly from III ("If τ_n is optimal stopping tome then $\tau_n \leq \tau_*$ (P-a.s.) for the problem $V_n = \sup_{\tau \geq n} \mathsf{E} G_\tau$").

Remark. From the definition

$$S_n = \operatorname*{ess\,sup}_{n \leq \tau \leq N} \mathsf{E}(G_\tau \,|\, \mathcal{F}_n)$$

it follows that

$$N \mapsto S_n^N \quad \text{and} \quad N \mapsto \tau_n^N$$

are increasing. So,

$$S_n^\infty = \lim_{N \to \infty} S_n^N \quad \text{and} \quad \tau_n^\infty = \lim_{N \to \infty} \tau_n^N$$

exist P-a.s. for each $n \geq 0$.

Note also that from

$$V_n^N = \sup_{n \le \tau \le N} \mathsf{E} G_\tau$$

it follows that $N \mapsto V_n^N$ is increasing, so that $V_n^\infty = \lim_{N\to\infty} V_n^N$ exists for each $n \ge 0$.

From $S_n^N = \operatorname{ess\,sup}_{n\le\tau\le n} \mathsf{E}(G_\tau \,|\, \mathcal{F}_n)$ and $S_n = \operatorname{ess\,sup}_{\tau\ge n} \mathsf{E}(G_\tau \,|\, \mathcal{F}_n)$ we see that

$$S_n^\infty \le S_n \quad \text{and} \quad \tau_n^\infty \le \tau_n. \tag{$*$}$$

Similarly,

$$V_n^\infty \le V_n \quad \Big(= \sup_{\tau\ge n} \mathsf{E} G_\tau \Big). \tag{$**$}$$

If condition $\mathsf{E} \sup_{n\le k<\infty} |G_k| < \infty$ does not hold then the inequalities in $(*)$ and $(**)$ can be strict.

Theorem 3 (From finite to infinite horizon).

If $\mathsf{E}\sup_{0\le k<\infty}|G_k| < \infty$ *then in* $S_n^\infty \le S_n$, $\tau_n^\infty \le \tau_n$ *and* $V_n^\infty \le V_n$ *we have equalities for all* $n \ge 0$.

Proof. From

$$S_n^N = \max\{G_n, \mathsf{E}(S_{n+1}^\infty \,|\, \mathcal{F}_n)\}, \quad n \ge 0,$$

we get

$$S_n^\infty = \max\{G_n, \mathsf{E}(S_{n+1}^\infty \,|\, \mathcal{F}_n)\}, \quad n \ge 0.$$

So, $(S_n^\infty)_{n\ge 0}$ is a supermartingale.

Since $S_n^\infty \ge G_n$ we see that

$$(S_n^\infty)^- \le G_n^- \le \sup_{n\ge 0} G_n^-, \quad n \ge 0.$$

So, $((S_n^\infty)^-)_{n\ge 0}$ is uniformly integrable.

Then by the optional sampling theorem we get

$$S_n^\infty \geq \mathsf{E}(S_\tau^\infty \,|\, \mathcal{F}_n) \tag{$*$}$$

for all $\tau \in \mathfrak{M}_n$.

Moreover, since $S_k^\infty \geq G_k$, $k \geq n$, it follows that $S_\tau^\infty \geq G_\tau$ for all $\tau \in \mathfrak{M}_n$, and hence

$$\mathsf{E}(S_\tau^\infty \,|\, \mathcal{F}_n) \geq \mathsf{E}(G_\tau \,|\, \mathcal{F}_n) \tag{$**$}$$

for all $\tau \in \mathfrak{M}_n$. From $(*)$, $(**)$ and

$$S_n = \operatorname*{ess\,sup}_{\tau \geq n} \mathsf{E}(G_\tau \,|\, \mathcal{F}_n)$$

we see that $S_n^\infty \geq S_n$.

Since the reverse inequality holds in general as shown above, this establishes that $S_n^\infty = S_n$ (P-a.s.) for all $n \geq 0$. From this it also follows that $\tau_n^\infty = \tau_n$ (P-a.s.), $n \geq 0$. Finally, the third identity $V_n^\infty = V_n$ follows by the monotone convergence theorem.

B. Markovian approach.

We will present basic results of optimal stopping when *the time is discrete* and *the process is Markovian*.

1. We consider a time-homogeneous Markov chain $X = (X_n)_{n \geq 0}$

    - defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, \mathsf{P}_x)$
    - taking values in a measurable space $(E, \mathcal{B})$

where for simplicity we will assume that

(a) $E = \mathbb{R}^d$ for some $d \geq 1$

(b) $\mathcal{B} = \mathcal{B}(\mathbb{R}^d)$ is the Borel σ-algebra on $\mathbb{R}^d$.

It is assumed that the chain X starts at x under P_x for $x \in E$.

It is also assumed that the mapping $x \mapsto \mathsf{P}_x(F)$ is measurable for each $F \in \mathcal{F}$.

It follows that the mapping $x \mapsto \mathsf{E}_x(Z)$ is measurable for each random variable Z.

Finally, without loss of generality we will assume that $(\Omega, \mathcal{F})$ equals the canonical space $(E^{\mathbb{N}_0}, \mathcal{B}^{\mathbb{N}_0})$ so that the shift operator $\theta_n \colon \Omega \to \Omega$ is well defined by

$$\theta_n(\omega)(k) = \omega(n+k) \quad \text{for} \quad \omega = (\omega(k))_{k \geq 0} \in \Omega \quad \text{and} \quad n, k \geq 0.$$

(Recall that $\mathbb{N}_0$ stands for $\mathbb{N} \cup \{0\}$.)

Given a measurable function $G\colon E \to \mathbb{R}$ satisfying the following condition (with $G(X_N) = 0$ if $N = \infty$):

$$\mathsf{E}_x\Big(\sup_{0 \le n \le N} |G(X_n)|\Big) < \infty$$

for all $x \in E$, we consider the optimal stopping problem

$$V^N(x) = \sup_{0 \le \tau \le N} \mathsf{E}_x G(X_\tau)$$

where $x \in E$ and the supremum is taken over all stopping times τ of X. The latter means that τ is a stopping time w.r.t. the natural filtration of X given by

$$\mathcal{F}_n^X = \sigma(X_k; 0 \le k \le n) \quad \text{for} \quad n \ge 0.$$

Since the same results remain valid if we take the supremum in

$$V^N(x) = \sup_{0 \le \tau \le N} \mathsf{E}_x G(X_\tau) \tag{$*$}$$

over stopping times τ w.r.t. $(\mathcal{F}_n)_{n \ge 0}$, and this assumption makes final conclusions more powerful (at least formally), we will assume in the sequel that the supremum in $(*)$ is taken over this larger class of stopping times.

Note also that in $(*)$ we admit that N can be $+\infty$ as well.

In this case, however, we still assume that the supremum is taken over stopping times τ, i.e. over Markov times τ satisfying $0 \le \tau < \infty$. In this way any specification of $G(X_\infty)$ becomes irrelevant for the problem $(*)$.

To solve

$$V^N(x) = \sup_{0 \le \tau \le N} \mathsf{E}_x G(X_\tau) \tag{$*$}$$

when $N < \infty$, we may note that by setting $G_n = G(X_n)$ for $n \ge 0$ the problem reduces to the problem

$$\boxed{V_n^N = \sup_{n \le \tau \le N} \mathsf{E}_x G_\tau}. \tag{$**$}$$

Having identified $(*)$ as $(**)$, we can apply the method of _backward induction_ which leads to a sequence of r.v.'s $(S_n^N)_{0 \le n \le N}$ and a stopping time $\tau_n^N = \inf\{n \le k \le N : S_k^N = G_k\}$.

The key identity is

$$\boxed{S_n^N = V^{N-n}(X_n)} \quad \text{for} \quad 0 \le n \le N. \tag{$***$}$$

Once $(***)$ is known to hold, the results of the Theorem 1 (finite horizon) from the Martingale theory translate immediately into the present setting and get a more transparent form.

To get formulation, let us define

$$C_n^N = \{\, x \in E : V^{N-n}(x) > G(x) \,\}$$
$$D_n^N = \{\, x \in E : V^{N-n}(x) = G(x) \,\}$$

for $0 \le n \le N$. We also define stopping time

$$\tau_D = \inf\{\, 0 \le n \le N \colon X_n \in D_n^N \,\}.$$

and the transition operator T of X

$$TF(x) = \mathsf{E}_x F(X_1)$$

for $x \in E$ whenever $F \colon E \to \mathbb{R}$ is a measurable function so that $F(X_1)$ is integrable w.r.t. P_x for all $x \in E$.

Theorem 4 (Finite horizon: The time-homogeneous case)

Consider the optimal stopping problems

$$V^n(x) = \sup_{0 \le \tau \le n} \mathsf{E}_x G(X_\tau) \qquad (*)$$

assuming that $\mathsf{E}_x \sup_{0 \le k \le N} |G(X_k)| < \infty$. *Then*

I. *Value functions* V^n *satisfy the "Wald–Bellman equation"*

$$V^n(x) = \max(G(x), TV^{n-1}(x)) \qquad (x \in E)$$

for $n = 1, \ldots, N$ *where* $V^0 = G$.

II. *The stopping time* $\tau_D = \inf\{0 \le n \le N : X_n \in D_n^N\}$ *is optimal in* $(*)$ *for* $n = N$.

III. *If* τ_* *is an optimal stopping time in* $(*)$ *then* $\tau_D \le \tau_*$ (P_x-*a.s.*) *for every* $x \in E$.

IV. *The sequence* $(V^{N-n}(X_n))_{0 \le n \le N}$ *is the smallest supermartingale which dominates* $(G(X_n))_{0 \le n \le N}$ *under* P_x *for* $x \in E$ *given and fixed.*

V. *The stopped sequence* $(V^{N-n}(X_{n \wedge \tau_D}))_{0 \le n \le N}$ *is a martingale under* P_x *for every* $x \in E$.

Proof. To verify the equality $S_n^N = V^{N-n}(X_n)$ recall that

$$S_n^N = \mathsf{E}_x(G(X_{\tau_n^N}) \,|\, \mathcal{F}_n) \tag{i}$$

for $0 \le n \le N$. Since $S_k^{N-n} \circ \theta_n = S_{n+k}^N$ we get that τ_n^N satisfies

$$\tau_n^N = \inf\{n \le k \le N : S_k^N = G(X_k)\} = n + \tau_0^{N-n} \circ \theta_n \tag{ii}$$

for $0 \le n \le N$.

Inserting (ii) into (i) and using the Markov property we obtain

$$\begin{aligned} S_n^N &= \mathsf{E}_x\Big[G(X_{n+\tau_0^{N-n}\circ\theta_n})\,|\,\mathcal{F}_n\Big] = \mathsf{E}_x\Big[G(X_{\tau_0^{N-n}})\circ\theta_n\,|\,\mathcal{F}_n\Big] \\ &= \mathsf{E}_{X_n} G(X_{\tau_0^{N-n}}) \overset{(\alpha)}{=} V^{N-n}(X_n) \end{aligned} \tag{iii}$$

where (α) follows by (i): $S_n^N = \mathsf{E}_x(G(X_{\tau_n^N})\,|\,\mathcal{F}_n)$, which imply

$$\mathsf{E}_x S_0^{N-n} = \mathsf{E}_x G(X_{\tau_0^{N-n}}) = \sup_{0\le\tau\le N-n} \mathsf{E}_x G(X_\tau) = V^{N-n}(x) \tag{iv}$$

for $0 \le n \le N$ and $x \in E$.

Thus $S_n^N = V^{N-n}(X_n)$ holds as claimed.

To verify the “Wald–Bellman equation”, note that the equality

$$S_n^N = \max\{G_n, \mathsf{E}(S_{n+1}^N \mid \mathcal{F}_n)\},$$

using the Markov property, reads as follows:

$$\begin{aligned}
V^{N-n}(X_n) &= \max\Big\{G(X_n), \mathsf{E}_x\Big[V^{N-n-1}(X_{n+1}) \mid \mathcal{F}_n\Big]\Big\} \\
&= \max\Big\{G(X_n), \mathsf{E}_x\Big[V^{N-n-1}(X_1) \circ \theta_n \mid \mathcal{F}_n\Big]\Big\} \\
&= \max\Big\{G(X_n), \mathsf{E}_{X_n} V^{N-n-1}(X_1)\Big\} \\
&= \max\Big\{G(X_n), TV^{N-n-1}(X_n)\Big\}
\end{aligned} \qquad (*)$$

for all $0 \le n \le N$. Letting $n = 0$ and using that $X_0 = x$ under P_x we see that $(*)$ yields $V^n(x) = \max\{G(x), TV^{n-1}(x)\}$.

The remaining statements of the theorem follow directly from the Martingale Theorem (1). The proof is complete. $\square$

The "Wald–Bellman equation" can be written in a more compact form as follows. Introduce the operator Q by setting

$$QF(x) = \max(G(x), TF(x))$$

for $x \in E$ where $F \colon E \to \mathbb{R}$ is a measurable function for which $F(X_1) \in L^1(\mathsf{P}_x)$ for $x \in E$. Then the "Wald–Bellman equation" reads as follows:

$$V^n(x) = Q^n G(x)$$

for $1 \leq n \leq N$ where Q^n denotes the n-th power of Q. These recursive relations form a constructive method for finding V^N when $\mathrm{Law}(X_1 \,|\, \mathsf{P}_x)$ is known for $x \in E$.

Time-inhomogeneous Markov chains $X = (X_n)_{n \ge 0}$

Put $Z_n = (n, X_n)$.

$Z = (Z_n)_{n \ge 0}$ is a time-homogeneous Markov chain.

Optimal stopping problem:

$$(*) \qquad \boxed{V^N(n,x) = \sup_{0 \le \tau \le N-n} \mathsf{E}_{n,x} G(n+\tau, X_{n+\tau})}, \quad 0 \le n \le N.$$

We assume

$$(**) \qquad \mathsf{E}_{n,x}\Big(\sup_{0 \le k \le N-n} |G(n+k, X_{n+k})| \Big) < \infty, \quad 0 \le n \le N.$$

Theorem 5 (Finite horizon: The time-inhomogeneous case)

Consider the optimal stopping problem $(*)$ *upon assuming that the condition* $(**)$ *holds. Then*:

I. *The function* V^n *satisfies the* <u>"Wald–Bellman equation"</u>

$$V^N(n,x) = \max(G(n,x), TV^N(n,x))$$

for $n = N-1, \ldots, 0$ *where*

$$TV^N(n,x) = \mathsf{E}_{n,x} V^N(n+1, X_{n+1}), \quad n = N-1, \ldots, 0,$$

and

$$TV^N(N-1,x) = \mathsf{E}_{N-1,x} G(N, X_N);$$

II. *The stopping time*

$$\tau_D^N = \inf\{n \le k \le N : (n+k, X_{n+k}) \in D\}$$

with

$$D = \Big\{(n,x) \in \{0,1,\dots,N\}\times E : V(n,x) = G(n,x)\Big\}$$

is optimal in the problem $(*)$:

$$V^N(n,x) = \sup_{0\le\tau\le N-n} \mathsf{E}_{n,x} G(n+\tau, X_{n+\tau});$$

III. *If* τ_*^N *is an optimal stopping time in* $(*)$ *then* $\tau_D^N \le \tau_*^N$ ($\mathsf{P}_{n,x}$-*a.s.*) *for every* $(n,x) \in \{0,1,\dots,N\}\times E$*;*

IV. *The value function* V^N *is the smallest superharmonic function which dominates the gain function* G *on* $\{0,\ldots,N\}\times E$,

$$TV^N(n,x) \le V^N(n,x), \quad V^N(n,x) \ge G(n,x);$$

V. *The stopped sequence*

$$\Big(V^N((n+k)\wedge\tau_D^N, X_{(n+k)\wedge\tau_D^N})\Big)_{0\le k\le N-n}$$

is a martingale under $\mathsf{P}_{n,x}$ *for every* $(n,x)\in\{0,1,\ldots,N\}\times E$;

The proof is carried out in exactly the same way as the proof of Theorem 4.

Optimal stopping for infinite horizon ($N = \infty$):

$$\boxed{V(x) = \sup_{\tau} \mathsf{E}_x G(X_\tau)}$$

Theorem 6

Assume $\mathsf{E}_x \sup_{n \geq 0} |G(X_n)| < \infty$, $x \in E$.

I. *The value function* V *satisfies the* "Wald–Bellman equation"

$$V(x) = \max(G(x), TV(x)), \quad x \in E.$$

II. *Assume moreover when required below that* $\mathsf{P}_x(\tau_D < \infty) = 1$ *for all* $x \in E$, *where*

$$\tau_D = \inf\{t \geq 0 : X_t \in D\}$$

with $D = \{x \in E : V(x) = G(x)\}$. *Then the stopping time* τ_D *is optimal.*

III. *If τ_* is an optimal stopping time then $\tau_D \le \tau_*$ (P_x-a.s. for every $x \in E$).*

IV. *The value function V is the smallest superharmonic function (Dynkin's characterization) ($TV \le V$) which dominates the gain function G on E, or, equivalently, $(V(X_n))_{n \ge 0}$ is the smallest supermartingale (under P_x, $x \in E$) which dominates $(G(X_n))_{n \ge 0}$.*

V. *The stopped sequence $(V(X_{n \wedge \tau_D}))_{n \ge 0}$ is a martingale under P_x for every $x \in E$.*

VI. *If the condition $\mathsf{P}_x(\tau_D < \infty) = 1$ fails so that $\mathsf{P}_x(\tau_D = \infty) > 0$ for some $x \in E$, then there is no optimal stopping time in the problem $V(x) = \sup_\tau \mathsf{E}_x G(X_\tau)$ for all $x \in E$.*

Corollary (Iterative method). *We have*

$$\boxed{V(x) = \lim_{n\to\infty} Q^n G(x)}$$

(*a constructive method for finding the value function* V).

Uniqueness in the Wald–Bellman equation

$$\boxed{F(x) = \max(G(x), TF(x))}$$

Suppose $\mathsf{E} \sup_{n\geq 0} F(X_n) < \infty$.

Then F equals the value function V if and only if the following "*boundary*
holds:

$$\limsup_{n\to\infty} F(X_n) = \limsup_{n\to\infty} G(X_n) \quad \mathsf{P}_x\text{-a.s.} \quad \forall x \in E.$$

2. Given $\alpha \in (0,1]$ and bounded $g\colon E \to \mathbb{R}$ and $c\colon E \to \mathbb{R}_+$, consider the optimal stopping problem

$$V(x) = \sup_\tau \mathsf{E}_x\Big(\alpha^\tau g(X_\tau) - \sum_{k=1}^{\tau} \alpha^{k-1} c(X_{k-1})\Big).$$

Let $\widetilde{X} = (\widetilde{X}_n)_{n \geq 0}$ denote the Markov chain X killed at rate α. It means that

$$\widetilde{T}F(x) = \alpha\, TF(x).$$

Then

$$V(x) = \sup_\tau \mathsf{E}_x\Big(g(\widetilde{X}_\tau) - \sum_{k=1}^{\tau} c(\widetilde{X}_{k-1})\Big).$$

The "Wald–Bellman equation" takes the following form:

$$V(x) = \max\Big\{g(x), \alpha TV(x) - c(x)\Big\}.$$

Lectures 4–5. Theory of optimal stopping for continuous time
A. Martingale approach

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ be a stochastic basis (a filtered probability space with right-continuous family $(\mathcal{F}_t)_{t\geq 0}$ where each $\mathcal{F}_t$ contains all P-null sets from $\mathcal{F}$.

Let $G = (G_t)_{t\geq 0}$ be a gain process. (We interpret G_t as the *gain* if the observation of G is stopped at time t.)

Definition. A random variable $\tau \colon \Omega \to [0, \infty]$ is called a *Markov time* if $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. A Markov time is called a *stopping time* if $\tau < \infty$ P-a.s.

We assume that $G = (G_t)_{t \geq 0}$ is right-continuous and left-continuous over stopping times (if $\tau_n \uparrow \tau$ then $G_{\tau_n} \to G_\tau$ P-a.s.).

We also assume that

$$\mathsf{E}\Big(\sup_{0 \leq t \leq T} |G_t|\Big) < \infty \qquad (G_T = 0 \;\text{ if }\; T = \infty).$$

Basic optimal stopping problem:

$$\boxed{V_t^T = \sup_{t \leq \tau \leq T} \mathsf{E}\, G_\tau}\;.$$

We shall admit that $T = \infty$. In this case the supremum is still taken over stopping times τ, i.e. over Markov times τ satisfying $t \leq \tau < \infty$.

Two ways to tackle the problem $V_t^T = \sup_{t \le \tau \le T} \mathsf{E}\, G_\tau$:

(1) Discrete time approximation

$[0,T] \longrightarrow \mathbb{T}^{(n)} = \left\{t_0^{(n)}, t_1^{(n)}, \dots, t_n^{(n)}\right\} \uparrow \mathbb{T}$ is a dense subset of $[0,T]$

$G \longrightarrow G^{(n)} = (G_{t_i^{(n)}})$

with applying previous discrete-time results and then passing to the limit $n \to \infty$;

(2) Straightforward extension of the method of essential supremum. This programme will now be addressed.

We denote for simplicity of the notation

$$V_t = V_t^T \qquad (T < \infty \quad \text{or} \quad T = \infty).$$

Consider the process $S = (S_t)_{t \geq 0}$ defined as follows:

$$S_t = \operatorname*{ess\,sup}_{\tau \geq t} \mathsf{E}(G_\tau | \mathcal{F}_t).$$

The process S is the *Snell's envelope* of G.

Introduce

$$\tau_t = \inf \{u \geq t \,|\, S_u = G_u\} \quad \text{where } \inf \varnothing = \infty \text{ by definition.}$$

We shall see below that

$$S_t \geq \max\{G_t, \mathsf{E}(S_u \,|\, \mathcal{F}_t)\} \quad \text{for} \quad u \geq t.$$

The reverse inequality is not true generally.

However,

$$S_t = \max\{G_t, \mathsf{E}(S_{\sigma \wedge \tau_t} \,|\, \mathcal{F}_t)\}$$

for every stopping time $\sigma \geq t$ and τ_t given above.

Theorem 1. *Consider the optimal stopping problem*

$$V_t = \sup_{\tau \geq t} \mathsf{E}\, G_\tau, \quad t \geq 0,$$

upon assuming $\mathsf{E} \sup_{t \geq 0} |G_t| < \infty$. *Assume moreover when required below that*

$$\mathsf{P}(\tau_t < \infty) = 1, \quad t \geq 0.$$

(*Note that this condition is automatically satisfied when the horizon* T *is finite.*) *Then*:

I. *For all* $t \geq 0$ *we have*

$$S_t \geq \mathsf{E}(G_\tau \,|\, \mathcal{F}_t) \quad \textit{for each } \tau \in \mathfrak{M}_t$$
$$S_t = \mathsf{E}(G_{\tau_t} \,|\, \mathcal{F}_t)$$

where $\mathfrak{M}_t = \{\tau : \tau \leq T\}$ *if* $T < \infty$,
$\mathfrak{M}_t = \{\tau : \tau < \infty\}$ *if* $T = \infty$.

II. *The stopping time* $\tau_t = \inf\{u \geq t : S_u = G_u\}$ *is optimal (for the problem* $V_t = \sup_{\tau \geq t} \mathsf{E} G_\tau$*).*

III. *If* τ_t^* *is an optimal stopping time as well then* $\tau_t \leq \tau_t^*$ P-*a.s.*

IV. *The process* $(S_u)_{u \geq t}$ *is the smallest right-continuous supermartingale which dominates* $(G_s)_{s \geq t}$.

V. *The stopped process* $(S_{u \wedge \tau_t})_{u \geq t}$ *is a right-continuous martingale.*

VI. *If the condition* $\mathsf{P}(\tau_t < \infty) = 1$ *fails so that* $\mathsf{P}(\tau_t = \infty) > 0$, *then there is no optimal stopping time.*

Proof. 1°. Let us first prove that $S = (S_t)_{t \geq 0}$ defined by

$$S_t = \operatorname*{ess\,sup}_{\tau \geq t} \mathsf{E}(G_\tau | \mathcal{F}_t)$$

is a supermartingale.

Show that the family $\{\mathsf{E}(G_\tau | \mathcal{F}_t) : \tau \in \mathfrak{M}_t\}$ is upwards directed in the sense that if σ_1 and σ_2 are from $\mathfrak{M}_t$ then there exists $\sigma_3 \in \mathfrak{M}_t$ such that

$$\mathsf{E}(G_{\sigma_1} | \mathcal{F}_t) \vee \mathsf{E}(G_{\sigma_2} | \mathcal{F}_t) \leq \mathsf{E}(G_{\sigma_3} | \mathcal{F}_t).$$

Put $\sigma_3 = \sigma_1 I_A + \sigma_2 I_{\bar{A}}$ where

$$A = \{\mathsf{E}(G_{\sigma_1} | \mathcal{F}_t) \geq \mathsf{E}(G_{\sigma_2} | \mathcal{F}_t)\}.$$

Then $\sigma_3 \in \mathfrak{M}_t$ and

$$\begin{aligned} \mathsf{E}(G_{\sigma_3} | \mathcal{F}_t) &= \mathsf{E}(G_{\sigma_1} I_A + G_{\sigma_2} I_{\bar{A}} | \mathcal{F}_t) = I_A \mathsf{E}(G_{\sigma_1} | \mathcal{F}_t) + I_{\bar{A}} \mathsf{E}(G_{\sigma_2} | \mathcal{F}_t) \\ &= \mathsf{E}(G_{\sigma_1} | \mathcal{F}_t) \vee \mathsf{E}(G_{\sigma_2} | \mathcal{F}_t). \end{aligned}$$

Hence there exists a sequence $\{\sigma_k; k \geq 1\}$ in $\mathfrak{M}_t$ such that

$$(*) \qquad \operatorname*{ess\,sup}_{\tau \in \mathfrak{M}_t} \mathsf{E}(G_\tau \,|\, \mathcal{F}_t) = \lim_{k\to\infty} \mathsf{E}(G_{\sigma_k} \,|\, \mathcal{F}_t)$$

where

$$\mathsf{E}(G_{\sigma_1} | \mathcal{F}_t) \leq \mathsf{E}(G_{\sigma_2} | \mathcal{F}_t) \leq \cdots \quad \mathsf{P}\text{-a.s.}$$

From $(*)$ and the conditional monotone convergence theorem (using $\mathsf{E}\sup_{t\geq 0} |G_t| < \infty$) we find that for $0 \leq s < t$

$$\begin{aligned}
\mathsf{E}(S_t \,|\, \mathcal{F}_s) &= \mathsf{E}\Big(\lim_{k\to\infty} \mathsf{E}(G_{\sigma_k} | \mathcal{F}_t) \,|\, \mathcal{F}_s \Big) \\
&= \lim_{k\to\infty} \mathsf{E}\big[\mathsf{E}(G_{\sigma_k} | \mathcal{F}_t) \,|\, \mathcal{F}_s\big] \\
&= \lim_{k\to\infty} \mathsf{E}(G_{\sigma_k} | \mathcal{F}_s) \leq S_s \quad \Big(= \operatorname*{ess\,sup}_{\tau \geq s} \mathsf{E}(G_\tau \,|\, \mathcal{F}_s) \Big).
\end{aligned}$$

Thus $(S_t)_{t\geq 0}$ is a supermartingale as claimed.

Note that from $\mathsf{E}\sup_{t\geq 0}|G_t| < \infty$ and

$$S_t = \operatorname*{ess\,sup}_{\tau\geq t} \mathsf{E}(G_\tau \,|\, \mathcal{F}_t),$$
$$\operatorname*{ess\,sup}_{\tau\geq t} \mathsf{E}(G_\tau \,|\, \mathcal{F}_t) = \lim_{k\to\infty} \mathsf{E}(G_{\sigma_k} \,|\, \mathcal{F}_t)$$

it follows that

$$\boxed{\mathsf{E}S_t = \sup_{\tau\geq t} \mathsf{E}G_\tau}\,.$$

2°. Let us next show that the supermartingale S admits a right-continuous modification $\widetilde{S} = (\widetilde{S}_t)_{t\geq 0}$.

From the general martingale theory it follows that it suffices to check that

$$t \rightsquigarrow \mathsf{E}S_t \quad \text{is right-continuous on } \mathbb{R}_+.$$

By the supermartingale property of S

$$\mathsf{E}S_t \geq \cdots \geq \mathsf{E}S_{t_2} \geq \mathsf{E}S_{t_1}, \qquad t_n \uparrow t.$$

So, $L := \lim_{n\to\infty} \mathsf{E}S_{t_n}$ exists and

$$\mathsf{E}S_t \geq L.$$

To prove the reverse inequality, fix $\varepsilon > 0$ and by means of $\mathsf{E}S_t = \sup_{\tau \geq t} \mathsf{E}G_\tau$ choose $\sigma \in \mathfrak{M}_t$ such that

$$\mathsf{E}G_\sigma \geq \mathsf{E}S_t - \varepsilon.$$

Fix $\delta > 0$ and note that there is no restriction to assume that $t_n \in [t, t+\delta]$ for all $n \geq 1$. Define

$$\sigma_n = \begin{cases} \sigma & \text{if } \ \sigma > t_n, \\ t+\sigma & \text{if } \ \sigma \leq t_n. \end{cases}$$

Then for all $n \geq 1$ we have

$$(*) \qquad \mathsf{E}G_{\sigma_n} = \mathsf{E}G_\sigma I(\sigma > t_n) + \mathsf{E}G_{t+\delta} I(\sigma \leq t_n) \leq \mathsf{E}S_{t_n}$$

since $\sigma_n \in \mathfrak{M}_{t_n}$ and $\mathsf{E}S_t = \sup_{\tau \geq t} \mathsf{E}G_\tau$. Letting $n \to \infty$ in $(*)$ and assuming that $\mathsf{E}\sup_{0 \leq t \leq T} |G_t| < \infty$ we get

$$\mathsf{E}G_\sigma I(\sigma > t) + \mathsf{E}G_{t+\delta} I(\sigma = t) \leq L \quad (= \lim_n \mathsf{E}S_{t_n}).$$

Letting now $\delta \downarrow 0$ and using that G is right-continuous we obtain

$$\mathsf{E}G_\sigma I(\sigma > t) + \mathsf{E}G_t I(\sigma = t) = \mathsf{E}G_\sigma \leq L.$$

From here and $\mathsf{E}G_\sigma \geq \mathsf{E}S_t - \varepsilon$ we see that $L \geq \mathsf{E}S_t - \varepsilon$ for all $\varepsilon > 0$. Hence $L \geq \mathsf{E}S_t$ and thus

$$\lim_{n\to\infty} \mathsf{E}S_{t_n} = L = \mathsf{E}S_t, \quad t_n \uparrow t,$$

showing that S admits a right-continuous modification $\widetilde{S} = (\widetilde{S}_t)_{t \geq 0}$ which we also denote by S throughout.

Let us prove property IV:

> The process $(S_u)_{u \geq t}$ is the smallest right-continuous supermartingale which dominates $(G_s)_{s \geq t}$.

For this, let $\widehat{S} = (\widehat{S}_u)_{u \geq t}$ be another right-continuous supermartingale which dominates $G = (G_u)_{u \geq t}$. Then by the optional sampling theorem (using $\mathsf{E} \sup_{t \geq 0} |G_t| < \infty$) we have

$$\widehat{S}_u \geq \mathsf{E}(\widehat{S}_\tau | \mathcal{F}_u) \geq \mathsf{E}(G_\tau | \mathcal{F}_u)$$

for all $\tau \in \mathfrak{M}_u$ when $u \geq t$. Hence by the definition $S_u = \operatorname{ess\,sup}_{\tau \geq u} \mathsf{E}(G_\tau | \mathcal{F}$ we find that $S_u \leq \widehat{S}_u$ (P-a.s.) for all $u \geq t$. By the right-continuity of S and $\widehat{S}$ this further implies that

$$\mathsf{P}(S_u \leq \widehat{S}_u \text{ for all } u \geq t) = 1$$

as claimed.

Property I: for all $t \geq 0$

$$(*) \qquad S_t \geq \mathsf{E}(G_\tau \,|\, \mathcal{F}_t) \quad \text{for each } \tau \in \mathfrak{M}_t,$$

$$(**) \qquad S_t = \mathsf{E}(G_{\tau_t} \,|\, \mathcal{F}_t).$$

The inequality $(*)$ follows from the definition $S_t = \operatorname*{ess\,sup}_{\tau \geq t} \mathsf{E}(G_\tau \,|\, \mathcal{F}_t)$.

The proof of $(**)$ is the most difficult part of the proof of the Theorem.

The sketch of the proof is as follows.

Assume that $G_t \geq 0$ for all $t \geq 0$.

(α) Introduce, for $\lambda \in (0,1)$, the stopping time

$$\tau_t^\lambda = \inf\{s \geq t : \lambda S_s \leq G_s\}$$

(Then $\lambda S_{\tau_t^\lambda} \leq G_{\tau_t^\lambda}$, $\tau_{t+}^\lambda = \tau_t$.)

(β) We show that

$$S_t = \mathsf{E}(S_{\tau_t^\lambda} \,|\, \mathcal{F}_t) \quad \text{for all} \quad \lambda \in (0,1).$$

So $S_t \leq (1/\lambda)\mathsf{E}(G_{\tau_t^\lambda} \,|\, \mathcal{F}_t)$ and letting $\lambda \uparrow 1$ we get

$$S_t \leq \mathsf{E}(G_{\tau_t^1} | \mathcal{F}_t)$$

where $\tau_t^1 = \lim_{\lambda \uparrow 1} \tau_t^\lambda$ ($\tau_t^\lambda \uparrow$ when $\lambda \uparrow$).

(γ) Verify that $\tau_t^1 = \tau_t$. Then $S_t \leq \mathsf{E}(G_{\tau_t} | \mathcal{F}_t)$ and evidently $S_t \geq \mathsf{E}(G_{\tau_t} | \mathcal{F}_t)$. Thus $S_t = \mathsf{E}(G_{\tau_t} | \mathcal{F}_t)$.

For the proof of property V:

> The stopped process $(S_{u\wedge\tau_t})_{u\geq t}$ is a right-continuous martingale

it is enough to prove that

$$\mathsf{E}\,S_{\sigma\wedge\tau_t} = \mathsf{E}\,S_t$$

for all bounded stopping times $\sigma \geq t$.

The optional sampling theorem implies

$$\mathsf{E}\,S_{\sigma\wedge\tau_t} \leq \mathsf{E}\,S_t. \tag{59}$$

On the other hand, from $S_t = \mathsf{E}(G_{\tau_t} \,|\, \mathcal{F}_t)$ and $S_{\tau_t} = G_{\tau_t}$ we see that

$$\mathsf{E}\,S_t = \mathsf{E}\,G_{\tau_t} = \mathsf{E}\,S_{\tau_t} \leq \mathsf{E}\,S_{\sigma\wedge\tau_t}.$$

Thus, $\mathsf{E}\,S_{\sigma\wedge\tau_t} = \mathsf{E}\,S_t$ and $(S_{u\wedge\tau_t})_{u\geq t}$ is a martingale. $\square$

B. Markovian approach

Let $X = (X_t)_{t \geq 0}$ be a strong Markov process defined on a filtered probability space

$$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathsf{P}_x)$$

where $\quad x \in E \ (= \mathbb{R}^d), \quad \mathsf{P}_x(X_0 = x) = 1,$
$\quad x \to \mathsf{P}_x(A)$ is measurable for each $A \in \mathcal{F}$.

Without loss of generality we will assume that

$$(\Omega, \mathcal{F}) = (E^{[0,\infty)}, \mathcal{B}^{[0,\infty)}) \quad \text{(canonical space)}$$

Shift operator $\theta_t = \theta_t(\omega) \colon \Omega \to \Omega$ is well defined by

$$\theta_t(\omega)(s) = \omega(t+s) \quad \text{for} \quad \omega = (\omega(s))_{s \geq 0} \in \Omega \quad \text{and} \quad t, s \geq 0.$$

We consider the optimal stopping problem

$$\boxed{V(x) = \sup_{0 \le \tau \le T} \mathsf{E}_x G(X_\tau)}$$

$$G(X_T) = 0 \quad \text{if} \quad T < \infty; \qquad \mathsf{E}_x \sup_{0 \le t \le T} |G(X_t)| < \infty.$$

Here $\tau = \tau(\omega)$ is a stopping time w.r.t.

$$(\mathcal{F}_t)_{t \ge 0} \quad (\mathcal{F}_t^X \subseteq \mathcal{F}_t,\ \ \mathcal{F}_t^X = \sigma(X_s; 0 \le s \le t)).$$

G is called the *gain function*,
V is called the *value function*.

Case $T = \infty$:

$$\boxed{\begin{gathered} V(x) = \sup_{\tau} \mathsf{E}_x G(X_\tau) \\ \mathsf{P}_x(X_0 = x) = 1 \end{gathered}}$$

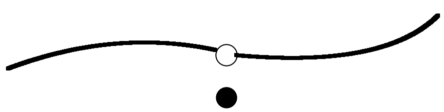
Introduce

the *continuation set* $C = \{x \in E : V(x) > G(x)\}$ and

the *stopping set* $D = \{x \in E : V(x) = G(x)\}$

NOTICE! If

V is lsc (lower semicontinuous) G is usc (upper semicontinuous)



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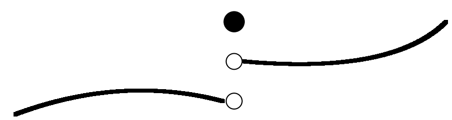

then

$$\boxed{C \text{ is open} \quad \text{and} \quad D \text{ is closed}}$$

The first entry time

$$\tau_D = \mathrm{inf}\{t \geq 0 : X_t \in D\}$$

for *closed* D is a stopping time since both X and $(\mathcal{F}_t)_{t \geq 0}$ are right-continuous.

Definition. A measurable function $F = F(x)$ is said to be *superharmonic* (for X) if

$$\mathsf{E}_x F(X_\sigma) \leq F(x)$$

for all stopping times σ and all $x \in E$. (It is assumed that $F(X_\sigma) \in L^1(\mathsf{P}_x)$ for all $x \in E$ whenever σ is a stopping time.)

We have:

F is superharmonic if and only if $(F(X_t))_{t \geq 0}$ is a supermartingale under P_x for every $x \in E$.

The following theorem presents

necessary conditions

for the existence of an optimal stopping time.

Theorem. *Let us assume that there exists an optimal stopping time* τ_* *in the problem*

$$V(x) = \sup_{\tau} \mathsf{E}_x G(X_\tau)$$

i.e. $V(x) = \mathsf{E}_x F(X_{\tau_*})$. *Then*

(I) *The value function* V *is the smallest superharmonic function (Dynkin's characterization) which dominates the gain function* G *on* E.

Let us in addition to "$V(x) = \mathsf{E}_x F(X_{\tau_*})$" assume that

$$V \text{ is lsc and } G \text{ is usc.}$$

Then

(II) *The stopping time* $\tau_D = \inf\{t \geq 0 : X_t \in D\}$ *satisfies*

$$\tau_D \leq \tau_* \quad (\mathsf{P}_x\text{-a.s.}, \;\; x \in E)$$

and is optimal;

(III) *The stopped process* $(V(X_{t \wedge \tau_D}))_{t \geq 0}$ *is a right-continuous martingale under* P_x *for every* $x \in E$.

Now we formulate

sufficient conditions

for the existence of an optimal stopping time.

Theorem. *Consider the optimal stopping problem*

$$V(x) = \sup_{\tau} \mathsf{E}_x G(X_\tau)$$

upon assuming that the condition

$$\mathsf{E}_x \sup_{t \geq 0} |G(X_t)| < \infty, \quad x \in E,$$

is satisfied.

Let us assume that there exists the smallest superharmonic function $\widehat{V}$ which dominates the gain function G on E.

Let us in addition assume that

$$\widehat{V} \text{ is lsc and } G \text{ is usc.}$$

Set $D = \{x \in E : \widehat{V}(x) = G(x)\}$ *and let* $\tau_D = \inf\{t : X_t \in D\}$.

We then have:

(a) *If* $\mathsf{P}_x(\tau_D < \infty) = 1$ *for all* $x \in E$, *then* $\widehat{V} = V$ *and* τ_D *is optimal in* $V(x) = \sup_\tau \mathsf{E}_x G(X_\tau)$;

(b) *If* $\mathsf{P}_x(\tau_D < \infty) < 1$ *for some* $x \in E$, *then there is no optimal stopping time in* $V(x) = \sup_\tau \mathsf{E}_x G(X_\tau)$.

Corollary (The existence of an optimal stopping time).

Infinite horizon ($T = \infty$). *Suppose that V is lsc and G is usc. If $\mathsf{P}_x(\tau_D < \infty) = 1$ for all $x \in E$, then τ_D is optimal. If $\mathsf{P}_x(\tau_D < \infty) < 1$ for some $x \in E$, then there is no optimal stopping time.*

Finite horizon ($T < \infty$). *Suppose that V is lsc and G is usc. Then τ_D is optimal.*

<u>Proof for $T = \infty$.</u> (The case $T < \infty$ can be proved in exactly the same way as the case $T = \infty$ if the process (X_t) is replaced by the process (t, X_t).)

The key is to show that

$$V \text{ is superharmonic.}$$

If so, then evidently V is the *smallest superharmonic function* which dominates G on E. Then the claims of the corollary follow directly from the Theorem (on sufficient conditions) above.

For this, note that V is measurable (since it is lsc) and thus so is the mapping

$$(*) \qquad V(X_\sigma) = \sup_\tau \mathsf{E}_{X_\sigma} G(X_\tau)$$

for any stopping time σ which is given and fixed.

On the other hand, by the strong Markov property we have

$$(**) \qquad \mathsf{E}_{X_\sigma} G(X_\tau) = \mathsf{E}_x \,[\, G(X_{\sigma+\tau\circ\theta_\sigma}) \,|\, \mathcal{F}_\sigma]$$

for every stopping time τ and $x \in E$. From $(*)$ and $(**)$ we see that

$$V(x_\sigma) = \operatorname*{ess\,sup}_\tau \mathsf{E}_x \,[\, G(X_{\sigma+\tau\circ\theta_\sigma}) \,|\, \mathcal{F}_\sigma]$$

under P_x where $x \in E$ is given and fixed.

We can show that the family

$$\Big\{ \mathsf{E}\,[X_{\sigma+\tau\circ\theta_\sigma}\,|\,\mathcal{F}_\sigma] : \tau \text{ is a stopping time} \Big\}$$

is upwards directed: if $\rho_1 = \sigma + \tau_1 \circ \theta_\sigma$ and $\rho_2 = \sigma + \tau_2 \circ \theta_\sigma$ then there is $\rho = \sigma + \tau \circ \theta_\sigma$ such that

$$\mathsf{E}\,[G(X_\rho)\,|\,\mathcal{F}_\sigma] = \mathsf{E}\,[G(X_{\rho_1})\,|\,\mathcal{F}_\sigma] \vee \mathsf{E}\,[G(X_{\rho_2})\,|\,\mathcal{F}_\sigma].$$

From here we can conclude that there exists a sequence of stopping times $\{\tau_n; n \geq 1\}$ such that

$$V(X_\sigma) = \lim_n \mathsf{E}_x\,[G(X_{\sigma+\tau_n\circ\theta_\sigma})\,|\,\mathcal{F}_n]$$

where the sequence $\{\mathsf{E}_x\,[\,G(X_{\sigma+\tau_n\circ\theta_\sigma})\,|\,\mathcal{F}_n]\}$ is *increasing* P_x-a.s.

By the monotone convergence theorem using $\mathsf{E}\sup_{t\geq 0}|G_t| < \infty$ we can conclude

$$\mathsf{E}_x V(X_\sigma) = \lim_n \mathsf{E}_x G(X_{\sigma+\tau_n\circ\theta_\sigma}) \leq V(x)$$

for all stopping times σ and all $x \in E$. This proves that V is superharmonic.

Remark 1. If the function

$$x \mapsto \mathsf{E}_x G(X_\tau)$$

is continuous (or lsc) for every stopping time τ, then $x \mapsto V(x)$ is lsc and the results of the Corollary are applicable. This yields a powerful existence result by simple means.

Remark 2. The above results have shown that the optimal stopping problem

$$V(x) = \sup_{\tau} \mathsf{E}_x G(X_\tau)$$

is equivalent to the problem of finding the *smallest superharmonic function* $\widehat{V}$ which dominates G on E. Once $\widehat{V}$ is found it follows that $V = \widehat{V}$ and $\tau_D = \inf\{t : G(X_t) = \widehat{V}(X_t)\}$ is optimal.

There are two traditional ways for finding $\widehat{V}$:

(i) *Iterative procedure* (constructive but non-explicit)

(ii) *Free-boundary problem* (explicit or non-explicit).

For (i), e.g., it is known that if G is lsc and

$$\mathsf{E}_x \inf_{t \geq 0} G(X_t) > -\infty \quad \text{for all} \quad x \in E,$$

then $\widehat{V}$ can be computed as follows:

$$\widehat{V}(x) = \lim_{n \to \infty} \lim_{N \to \infty} Q_n^N G(x)$$

where

$$Q_n G(x) := G(x) \vee \mathsf{E}_x G(X_{1/2^n})$$

and Q_n^N is the N-th power of Q_n.

The basic idea (ii) is that

$$\widehat{V} \quad \text{and} \quad C \text{ (or } D\text{)}$$

should solve the free-boundary problem:

$$\begin{aligned} (*) \quad & \mathbb{L}_X \widehat{V} \le 0 \\ (**) \quad & \widehat{V} \ge G \quad (\widehat{V} > G \text{ on } C \ \ \& \ \ \widehat{V} = G \text{ on } D) \end{aligned}$$

where $\mathbb{L}_X$ is the characteristic (infinitesimal) operator of X.

Assuming that G is smooth in a neighborhood of ∂C the following "rule of thumb" is valid.

If X after starting at ∂C enters immediately into int$\,(D)$ (e.g. when X is a diffusion process and ∂C is sufficiently nice) then the condition $\mathbb{L}_X \widehat{V} \le 0$ under $(**)$ splits into the two conditions:

$$\begin{aligned} &\mathbb{L}_X \widehat{V} = 0 \quad \text{in } C \\ &\frac{\partial \widehat{V}}{\partial x}\Big|_{\partial C} = \frac{\partial G}{\partial x}\Big|_{\partial C} \quad (\textit{smooth fit}). \end{aligned}$$

On the other hand, if X after starting at ∂C does not enter immediately into int$\,(D)$ (e.g. when X has jumps and no diffusion component while ∂C may still be sufficiently nice) then the condition $\mathbb{L}_X \widehat{V} \le 0$ (i.e. $(*)$) under $(**)$ splits into the two conditions:

$$\begin{aligned} &\mathbb{L}_X \widehat{V} = 0 \quad \text{in } C \\ &\widehat{V}\Big|_{\partial C} = G\Big|_{\partial C} \quad (\textit{continuous fit}). \end{aligned}$$

Proof of the Theorem on *necessary* conditions
Basic lines

(I) The value function V is the smallest superharmonic function which dominated the gain function G on E.

We have by the strong Markov property:

$$\begin{aligned} \mathsf{E}_x V(X_\sigma) &= \mathsf{E}_x \mathsf{E}_{X_\sigma} G(X_{\tau_*}) = \mathsf{E}_x \mathsf{E}_x [G(X_{\tau_*}) \circ \theta_\sigma \,|\, \mathcal{F}_\sigma] \\ &= \mathsf{E}_x G(X_{\sigma + \tau_* \circ \theta_\sigma}) \le \sup_\tau \mathsf{E}_x G(X_\tau) = V(x) \end{aligned}$$

for each stopping time σ and all $x \in E$.

Thus V is superharmonic.

Let F be a superharmonic function which dominates G on E. Then

$$\mathsf{E}_x G(X_\tau) \le \mathsf{E}_x F(X_\tau) \le F(x)$$

for each stopping time τ and all $x \in E$. Taking the supremum over all τ we find that $V(x) \le F(x)$ for all $x \in E$. Since V is superharmonic itself, this proves that V is the smallest superharmonic function which dominated G.

(II) Let us show that the stopping time

$$\tau_D = \inf\{t : V(X_t) = G(X_t)\}$$

is optimal (if V is lsc and G is usc).

We assume that there exists an optimal stopping time τ_*:

$$V(x) = \mathsf{E}_x G(X_{\tau_*}), \quad x \in E.$$

We claim that $V(X_{\tau_*}) = G(X_{\tau_*})$ P_x-a.s. for all $x \in E$.

Indeed, if $\mathsf{P}_x\{V(X_{\tau_*}) > G(X_{\tau_*})\} > 0$ for some $x \in E$, then

$$\mathsf{E}_x G(X_{\tau_*}) < \mathsf{E}_x V(X_{\tau_*}) \leq V(x)$$

since V is superharmonic, leading to a contradiction with the fact that τ_* is optimal. From the identity just verified it follows that

$$\tau_D \leq \tau_* \quad \mathsf{P}_x\text{-a.s.} \quad \text{for all} \quad x \in E.$$

By (I) the value function V is the superharmonic ($\mathsf{E}_x V(X_\sigma) \leq V(X)$ for all stopping time σ and $x \in E$). Setting $\sigma \equiv s$ and using the Markov property we get for all $t, s \geq 0$ and all $x \in E$

$$V(X_t) \geq \mathsf{E}_{X_t} V(X_s) = \mathsf{E}_x [V(X_{t+s}) \,|\, \mathcal{F}_t].$$

This shows that

> *The process $(V(X_t))_{t\geq 0}$ is a supermartingale under P_x for each $x \in E$.*

Suppose for the moment that V is *continuous*. Then obviously it follows that $(V(X_t))_{t\geq 0}$ is *right-continuous*. Thus, by the optional sampling theorem (using $\mathsf{E}\sup_{t\geq 0} |G(X_t)| < \infty$), we see that

$$\mathsf{E}_x V(X_\tau) \leq \mathsf{E}_x V(X_\sigma) \quad \text{for} \quad \sigma \leq \tau.$$

In particular, since $\tau_D \leq \tau_*$ we get

$$V(x) = \mathsf{E}_x G(X_{\tau_*}) = \mathsf{E}_x V(X_{\tau_*}) \leq \mathsf{E}_x V(X_{\tau_D}) = \mathsf{E}_x G(X_{\tau_D}) \leq V(x)$$

where we used that

$$V(X_{\tau_D}) = G(X_{\tau_D})$$

If $Y_t = V(X_t) - G(X_t)$ (≥ 0), then

This shows that τ_D is optimal if V is continuous.

If V is only lsc, then again (see the lemma below) the process $(V(X_t))_{t\geq 0}$ is right-continuous (P_x-a.s. for each $x \in E$), and the proof can be completed as above.

This shows that τ_D is optimal if V is lsc as claimed.

Lemma. *If a superharmonic function $F: E \to \mathbb{R}$ is lsc, then the supermartingale $(F(X_t))_{t\geq 0}$ is right-continuous (P_x-a.s. for each $x \in E$).*

We omit the proof.

(III) The stopped process $(V(X_{t\wedge\tau_D}))_{t\geq 0}$ is a right-continuous martingale under P_x for every $x \in E$.

Proof. By the strong Markov property we have

$$
\begin{aligned}
&\mathsf{E}_x[V(X_{t\wedge\tau_D}) \,|\, \mathcal{F}_{s\wedge\tau_D}] = \mathsf{E}_x\Big[\mathsf{E}_{X_{t\wedge\tau_D}} G(X_{\tau_D}) \,|\, \mathcal{F}_{s\wedge\tau_D}\Big] \\
&= \mathsf{E}_x\Big(\mathsf{E}_x[G(X_{\tau_D}) \circ \theta_{t\wedge\tau_D} \,|\, \mathcal{F}_{t\wedge\tau_D}] \,|\, \mathcal{F}_{s\wedge\tau_D}\Big) \\
&= \mathsf{E}_x\Big(\mathsf{E}_x[G(X_{\tau_D}) \,|\, \mathcal{F}_{t\wedge\tau_D}] \,|\, \mathcal{F}_{s\wedge\tau_D}\Big) = \mathsf{E}_x[G(X_{\tau_D}) \,|\, \mathcal{F}_{s\wedge\tau_D}] \\
&= \mathsf{E}_{X_{s\wedge\tau_D}} G(X_{\tau_D}) = V(X_{s\wedge\tau_D})
\end{aligned}
$$

for all $0 \leq s \leq t$ and all $x \in E$ proving the martingale property. The right-continuity of $\Big(V(X_{t\wedge\tau_D})\Big)_{t\geq 0}$ follows from the right-continuity of $(V(X_t))_{t\geq 0}$ that we proved above.

The proof of the theorem on necessary conditions is complete.

Remark. The result and proof of the Theorem extend in exactly the same form (by slightly changing the notation only) to the *finite* horizon problem

$$V_T(X) = \sup_{0 \le \tau \le T} \mathsf{E}_x G(X_\tau).$$

Now we formulate the theorem which provides

sufficient condition

for the existence of an optimal stopping time.

Theorem. *Consider the optimal stopping problem*

$$V(x) = \sup_{\tau} \mathsf{E}_x G(X_\tau)$$

upon assuming that $\mathsf{E}_x \sup_{t \geq 0} |G(X_t)| < \infty$, $x \in E$. *Let us assume that*

(a) *there exists the smallest superharmonic function* $\widehat{V}$ *which dominates the gain function* G *on* E;

(b) $\widehat{V}$ *is lsc and* G *is usc.*

Set $D = \{x \in E : \widehat{V}(x) = G(x)\}$ *and* $\tau_D = \inf\{t : X_t \in D\}$.

We then have:

(I) *If* $\mathsf{P}_x(\tau_D < \infty) = 1$ *for all* $x \in E$, *then* $\widehat{V} = V$ *and* τ_D *is optimal*;

(II) *If* $\mathsf{P}_x(\tau_D < \infty) < 1$ *for some* $x \in E$, *then there is no optimal stopping time.*

Sketch of the proof.

(I) Since $\widehat{V}$ is superharmonic majorant for G, we have

$$\mathsf{E}_x G(X_\tau) \le \mathsf{E}_x \widehat{V}(X_\tau) \le V(x)$$

for all stopping times τ and all $x \in E$. So

$$G(x) \le V(x) = \sup_\tau \mathsf{E}_x G(X_\tau) \le \widehat{V}(x)$$

for all $x \in E$.

Next step (difficult!): assuming that $\mathsf{P}_x(\tau_D < \infty) = 1$ for all $x \in E$, we prove the inequality

$$\widehat{V}(x) \le V(x)$$

and optimality of time τ_D.

(II) If $\mathsf{P}_x(\tau_D < \infty) < 1$ for some $x \in E$ then there is no optimal stopping time.

Indeed, by "necessary-condition theorem" if there exists optimal optimal τ_* then $\tau_D \le \tau_*$.

But τ_D takes value ∞ with positive probability for some $x \in E$.

So, for this state x we have $\mathsf{P}_x(\tau_* = \infty) > 0$ and τ_* cannot be optimal (in the class $\mathfrak{M} = \{\tau : \tau < \infty\}$). $\square$

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