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On Newton's First Law of Motion

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We present arguments in support of the view that Newton's first law of motion extends itself to *stochastic motions* as follows: Every entity perseveres in its state of *independent and stationary increments* except insofar as it is compelled to change its state by forces impressed. Some of the far-reaching consequences of the extended law are briefly touched upon as well.

1. Prelude on stochastic motions

Stochastic differential equations, whose solutions are *stochastic motions* (processes) describing the position of an entity, are currently used in a diversity of scientific areas to model various phenomena. It is a common practice in many of these models to derive a stochastic differential equation by ad hoc arguments rather than from the first principles. The classic concept of velocity is not applicable in these models (whenever the position process is driven by a Wiener process for instance), and consequently the traditional formulations of the laws of motion (originating in classical mechanics) are not applicable either. This raises the question of finding the '*axioms of motion*' needed to put '*stochastic modelling*' on a firmer basis. Put differently, the aim is not only to focus on '*kinematics*' (describing the stochastic motion) but also '*dynamics*' (explaining its cause).

2. Dual meaning of Newton's first law

Newton's first law of motion was stated in the first two editions of his *Principia* [3] (published in 1687 and 1713) as follows: '*Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare*'. An accurate translation of the Latin original into English given in [3, p. 416] reads: *Every body perseveres in its state of being at rest or of moving uniformly straight forward except insofar as it is compelled to change its state by forces impressed.* The accuracy of this translation is reached by converting the Latin word '*perseverare*' into the English word '*perseveres*' and the Latin phrase '*nisi quatenus*' into the English phrase '*except insofar as*' so that the Latin original is genuinely replicated in English.

Earlier translations of the Latin original into English, including the very first one by Motte in 1729, were somewhat inaccurate in this regard due to the natural branching of the Latin

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word '*perseverare*' into the English words '*continue/preserve*' combined with '*until/so-far-as-not*' (see [2] for a detailed historical and conceptual analysis). This gave rise to two forms of Newton's first law in English translations, one '*temporal*' and another '*quantitative*', both being contained in the Latin original. The *temporal* form addresses an interval of time $[0,T)$ when external forces are absent and the body retains its natural state of *rest* or *uniform rectilinear motion*. The *quantitative* form addresses an instant of time T when external forces are present but the body is still in its natural state (of *rest* or *uniform rectilinear motion*) which will be changed only in the very next instant of time. (See also [3, pp 109-111] for additional arguments which relate Newton's first law of motion to Newton's second law of motion.)

It appears that the dual meaning of Newton's first law in the Latin original was intentional. The temporal form embodies Galileo's vision of the 'uniform rectilinear motion' of a body as its natural state and thus departs from the view held by Aristotle (and Kepler) that the only natural state of a body is to be at 'rest' (implying the need of a 'mover' to maintain a 'uniform rectilinear motion' itself). The quantitative form embodies Galileo's perception of the 'force of inertia' exerted by the body during a change of its natural state and thus departs from the view held by Descartes (and Huygens) that such a 'force' did not exist. The dual meaning of Newton's first law of motion is obtained by addressing simultaneously (i) a body's maintenance of a natural state of motion (when external forces are absent) and (ii) a body's resistance to a change in the natural state of motion (when external forces are present). This synthesis has been recognised as the main power of Newton's first law (see [2] for further details).

3. The question of merger

Part (i) of the dual meaning addresses a highly idealised situation which does not occur in reality and thus could only be understood as a powerful thought experiment. Part (ii) of the dual meaning is far removed from part (i) because the presence of external forces excludes their absence. On deeper reflection, this raises the question whether it is possible to merge the (seemingly disjoint) parts (i) and (ii) of the dual meaning and obtain a single meaning of Newton's first law of motion. It appears to be evident that if such a merger is possible, then one should focus on the highly idealised part (i) and move it closer to its (seemingly distant) relative embodied in part (ii) of the dual meaning.

A starting point in realising the merger is to recall that the absence of external forces can also be thought of as if the resulting force were zero (both in magnitude and direction) when external forces are present. This realisation moves part (i) an infinitesimal step closer to part (ii) of the dual meaning. To move beyond the infinitesimal step, it is apparent that the 'nature' ought to exert the resulting forces in opposite directions so to cancel each other out, however, in such a way that the cancellation fails at infinitesimally small time scales that are beyond any reach. Denoting the position of an entity in this static/dynamic equilibrium by X_t at time $t \in [0,\infty)$, and realising the motion $X = (X_t)_{t \geq 0}$ under the same conditions twice, the observable outcome would be that X exhibits two *different* sample paths $t \mapsto X_t$ both satisfying the same initial condition $X_0 = x_0$. This means that the motion X is not only *chaotic* but also *stochastic*.

It is important to bear in mind that the infinitesimal cancellation of the resulting forces fails in every possible direction of the state space of X with no imposed bound on the total number of external forces including their magnitude and/or direction either. Clearly, part (i) of the dual meaning expanded in that way addresses a highly idealised situation as well, which however has the advantage of being closer to reality if one accounts for all forces in the 'nature'. Moreover, the static/dynamic equilibrium of the 'nature' just described evidently merges parts (i) and (ii) of the dual meaning and establishes a single meaning of Newton's first law of motion.

4. Stationary independent increments

The static/dynamic equilibrium can be fully described by focusing on increments

$$X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}} \tag{4.1}$$

for $0 \le t_0 < t_1 < \ldots < t_{n-1} < t_n$ and inferring that they are (a) *independent* and (b) *stationary* (i.e. equally distributed whenever $t_1 - t_0 = \ldots = t_n - t_{n-1}$) for any $n \ge 2$ given and fixed. Both properties embody a canonical meaning which is close to naive intuition that underpins the static/dynamic equilibrium.

The property (a) shows that the stochastic motion X has *no memory* and always renews itself completely afresh. Moreover, setting $M_t := X_t - \mathsf{E}(X_t)$ whenever the expected values are finite for $t \ge 0$, one can easily verify using (a) that the centred motion $M = (M_t)_{t \ge 0}$ is a *martingale* (i.e. the best prediction of the future position M_t given the observed positions M_r for $r \in [0, s]$ is the present position M_s for $s \in [0, t)$ given and fixed). On the other hand, setting $m(t) := \mathsf{E}(X_t) - \mathsf{E}(X_0)$ whenever the expected values are finite for $t \ge 0$, one can easily verify using (b) that m satisfies the *Cauchy functional equation* $m(s+t) = m(s) + m(t)$ for $s, t \ge 0$ so that $m(t) = at$ for $t \ge 0$ (whenever measurable) with some real constant a. Hence we see that $\mathsf{E}(X_t) = at + b$ for $t \ge 0$ with $a = \mathsf{E}(X_1)$ and $b = \mathsf{E}(X_0)$ showing that the stochastic motion X in the static/dynamic equilibrium behaves on average analogously to the natural motions of Aristotle ($a = 0$) and/or Galileo ($a \ne 0$).

Newton's first law of motion then extends as follows: Every entity perseveres in its state of *independent and stationary increments* except insofar as it is compelled to change its state by forces impressed. The term 'entity' applies generally to anything having real or distinct existence, including any dual nature depending on means of observation, for instance. The motion X in the latter case may be rather seen as a *motion of the probability law* of X_t for $t \ge 0$, while $t \mapsto X_t$ itself may be viewed as one of its sample path realisations.

In addition to the properties (a) and (b) in relation to (4.1) above, the static/dynamic equilibrium also includes the continuity property

$$X_s \to X_t \tag{4.2}$$

in probability (i.e. the probability that a distance from X_s to X_t is larger than any small positive number tends to zero) as $s \to t$ in $[0, \infty)$. It needs to be noted that this weak continuity property (as well as the properties (a) and (b) in relation to (4.1) above) is truly a property of the probability law of X and a sample path $t \mapsto X_t$ need not be continuous on $[0, \infty)$. It is known however that a stochastic motion X satisfying the properties (a) and (b) in relation to (4.1) above, together with the weak continuity property (4.2) itself, can always be realised through its sample paths which are *right-continuous* and have *left limits* as functions of time. The times at which jumps of the sample paths occur are *unpredictable*.

5. General description and examples

The structure of stochastic motions X having independent and stationary increments (4.1) and satisfying the weak continuity property (4.2) has been completely described firstly by Einstein and Wiener in the continuous sample path case and then by de Finetti, Kolmogorov, and Lévy in the general/discontinuous sample path case. The resulting stochastic motions (or processes) bear the name of Lévy in the modern literature (see e.g. [1]). Leaving aside the more familiar Wiener process, whose sample paths (although nowhere differentiable) are continuous, a key starting fact in the discontinuous sample path case is that

$$N_t(A) = \sum_{s \in (0,t]} I(\Delta X_s \in A) \tag{5.1}$$

is a Poisson process for $t \geq 0$ with intensity

$$\nu(A) = \mathsf{E}\Big(\sum_{s \in (0,1]} I(\Delta X_s \in A)\Big) \tag{5.2}$$

for any measurable set $A \subset I\!\!R$ not containing a (small) open ball with centre at 0, where we set $\Delta X_s = X_s - X_{s-}$ for $s \geq 0$. (In particular it means that the compensated process $\bar{N}(A) = (N_t(A) - t\nu(A))_{t \geq 0}$ is a martingale.) The measure ν defined by (5.2) satisfies $\int_{I\!\!R}(1 \wedge x^2)\, \nu(dx) < \infty$ and is called the *Lévy measure*. Extensions from $I\!\!R$ to $I\!\!R^d$ with $d \geq 2$ and more general state spaces of X are straightforward.

The *Lévy-Itô decomposition* of X states that

$$X_t = \sigma W_t + t\, \mathsf{E}\Big[X_1 - \textstyle\int_{|x|>c} x\, N_1(dx)\Big] + \displaystyle\int_{|x| \leq c} x\, \big[N_t(dx) - t\nu(dx)\big] + \int_{|x|>c} x\, N_t(dx) \tag{5.3}$$

where $\sigma > 0$ does not dependent on any chosen truncation level $c > 0$ and $W = (W_t)_{t \geq 0}$ is a standard Wiener process. The finite constant $\mu_c := \mathsf{E}[X_1 - \int_{|x|>c} x\, N_1(dx)]$ plays the role of a drift, the first integral on the right-hand side of (5.3) represents a mixture of compensated Poisson processes (with bounded jumps), and the second integral on the right-hand side of (5.3) represents a mixture of Poisson processes (with unbounded jumps). From (5.3) we see that a 'free particle' moves on a 'straight line' when $\sigma = 0$ and $\nu = 0$ (see [5]) and undertakes 'quantum jumps' (as the motion of probability laws) when $\nu \neq 0$ (see [6]).

The *Lévy-Khintchine formula* of X states that

$$\mathsf{E}\left[e^{i\lambda X_t}\right] = e^{t\psi(\lambda)} \tag{5.4}$$

for $t > 0$ where the exponent function ψ is explicitly given by

$$\psi(\lambda) = -\frac{\sigma^2}{2}\lambda^2 + i\mu_c\lambda + \int_{|x| \leq c} (e^{i\lambda x} - 1 - \lambda x)\, \nu(dx) + \int_{|x|>c} (e^{i\lambda x} - 1)\, \nu(dx) \tag{5.5}$$

for $\lambda \geq 0$. Recalling that X has independent and stationary increments, we see that (5.4)+(5.5) completely determine all possible probability laws of X.

Examples of stochastic motions X having independent and stationary increments (4.1) and satisfying the weak continuity property (4.2) include: (1) *Straight line* $X_t = x_0 + \mu t$

for $t \geq 0$ (Aristotle $\mu = 0$ & Galileo $\mu \in I\!R$); (2) *Wiener process* $X_t = x_0 + \mu t + \sigma W_t$ for $t \geq 0$ (Einstein & Wiener $\sigma > 0$); (3) *Poisson process* ($\nu = \lambda \delta_1$; the length of time between two jumps is $\text{Exp}(\lambda)$ distributed; the size of each jump is 1); (4) *Compound Poisson process* (ν is any finite measure on $I\!R \setminus \{0\}$; the length of time between two jumps is $\text{Exp}(\lambda)$ distributed with $\lambda := \nu(I\!R \setminus \{0\})$; the size of each jump is ν/λ distributed); (5) *Stable process* ($\nu(dx) = (c_1/(-x)^{1+\alpha} I(x<0) + c_2/x^{1+\alpha} I(x>0))\, dx$ where $c_1 \geq 0$, $c_2 \geq 0$ with $c_1 + c_2 > 0$ and $\alpha \in (0,2)$); (6) *Gamma process* ($\nu(dx) = e^{-x}/x\, I(x>0)\, dx$). For more details see [1] and the references therein.

6. Synopsis

Merging the dual meanings of Newton's first law of motion into a single/canoncial meaning, as embodied in independent and stationary increments of the stochastic motion, one unleashes a full power of the extended first law. The merger itself removes the conceptual barrier separating the traditional formulation of Newton's first law from its applicability to more general stochastic motions to which, among other things, the classic concept of velocity is not applicable (recall that the sample paths of a standard Wiener process are nowhere differentiable).

In accordance with the traditional interpretation, the extended first law implies that stochastic motions having independent and stationary increments represent inertial frames, which then can be used to account for the presence of external forces and derive the equation of motion using a simple principle of their superposition. The extended first law thus aids in deriving the equation of motion, and this appears to be its single most-useful practical implication, leaving aside any deeper/phylosophical ones.

For such an in-depth analysis, we may point out a derivation of the equation of motion for a Brownian particle, which is under the influence of an external force in the fluid with a non-constant temperature (see Sections 4-6 and the resulting equation (6.4) in [4]). This equation of motion is 'driven' by a standard Wiener process and its solution represents the position of the Brownian particle as a function of time. On more careful inspection of the derivation, one sees that the underlying inertial frame of a standard Wiener process plays a fundamental role in accounting for the external forces present, which in turn yields the equation of motion itself. The analogous arguments clearly apply in more general settings as well whenever the inertial frames are identified with Wiener processes.

Equations of motions 'driven' by more general Lévy processes have also been studied in the literature, though with no awareness or reference to inertial frames of the extended first law. The foundations laid down for a unifying view of the extended first law open up new avenues for research in these and related settings.

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