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Dedicated to the tercentenary of the birth of R. J. Boscovich (1711-1789)

Optimal Detection of a Hidden Target: The Median Rule

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We show that in the absence of any information about the 'hidden' target in terms of the observed sample path, and irrespectively of the distribution law of the observed process, the 'median' rule is optimal in both the space domain and the time domain. While the fact that the median rule minimises the spatial expectation can be seen as a direct extension of the well-known median characterisation dating back to R. J. Boscovich, the fact that this also holds for the temporal expectation seems to have stayed unnoticed until now. Building on this observation we derive new classes of median/quantile rules having a dynamic character.

1. Introduction

Imagine that you are observing a sample path $t \mapsto X_t$ of the continuous process X started at 0 and that you wish to detect when this sample path reaches a non-negative level ℓ that is not directly observable. Situations of this type occur naturally in many applied problems (such as the breakage of atomic clocks in satellites that stimulated the present exposition) and there is a whole range of hypotheses that can be introduced to study various particular aspects of the problem. In this paper we study the simplest and the most extreme case in which X and ℓ are assumed to be independent. This corresponds to the 'black box' situation (of the most uncertain nature) that plays a canonical role in many other problems of this type. The central question we want to examine is whether there is an optimal stopping time that plays a similar canonical role among all other stopping times.

The problem of detecting ℓ through the sequential observation of X admits two distinct formulations in this context (see Figure 1). The first formulation belongs to the space domain where we want to minimise the expected distance from X_τ to ℓ over all stopping times τ of X. Naturally the optimal stopping time will depend on the chosen distance function and it is not clear a priori how to select the most natural distance function in this respect. The second formulation belongs to the time domain where τ_ℓ denotes the first entry time of X at the level ℓ and we want to minimise the expected distance from τ to τ_ℓ over all stopping times τ of X (note that τ_ℓ is not directly observable either). In this case the optimal stopping time will depend on the chosen distance function as well and it is not clear a priori how to select the most natural distance function in this case either.

The main observation of the present paper is that there exists a single stopping time of X that is optimal in *both* the space domain and the time domain with respect to the expected

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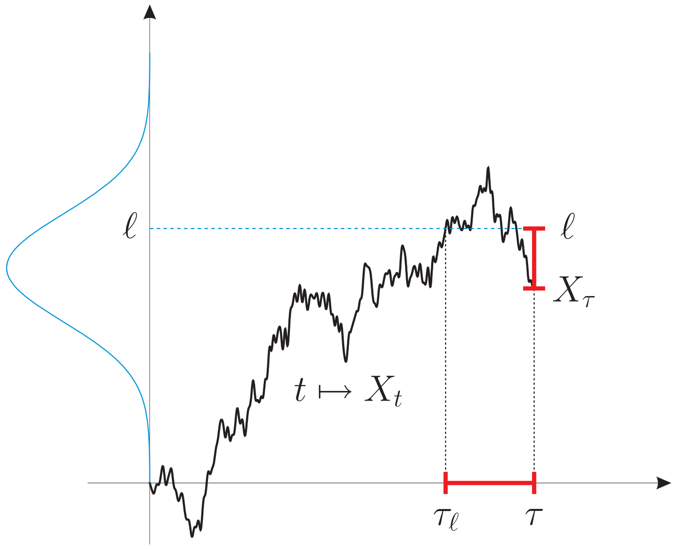

Figure 1. Optimal detection of a hidden target (one-sided case).

Euclidean distance on the real line. More precisely, assuming that $\mathsf{E}\ell < \infty$ and $\mathsf{E}\tau_\ell < \infty$, and denoting by $\mathfrak{m}$ the (lowest) median of ℓ, we show that the stopping time

$$\tau_{\mathfrak{m}} = \inf \{ \, t \geq 0 \mid X_t = \mathfrak{m} \, \} \tag{1.1}$$

minimises simultaneously both

$$\mathsf{E}|X_\tau - \ell| \quad \text{and} \quad \mathsf{E}|\tau - \tau_\ell| \tag{1.2}$$

over all stopping times τ of X. Thus, in the absence of any information about the 'hidden' target ℓ in terms of the observed sample path $t \mapsto X_t$, and irrespectively of the distribution law of X, the 'median' rule $\tau_{\mathfrak{m}}$ is optimal in both the space domain ($X_{\tau_{\mathfrak{m}}}$ is as 'close' as possible to ℓ) and the time domain ($\tau_{\mathfrak{m}}$ is as 'close' as possible to τ_ℓ). We then show that two-sided versions of the same problem when ℓ takes negative values as well (see Figure 2) and variational refinements of both problems (when the expected errors of early or late stopping are bounded from above) lead to new classes of median/quantile rules having a dynamic character. The latter analysis is based on the method of Lagrange multipliers in optimal stopping.

Significance of the main observation above is twofold. Firstly, while the fact that the median rule (1.1) minimises the first (spatial) expectation in (1.2) can be seen as a direct extension of the well-known median characterisation dating back to R. J. Boscovich (see [11] and the references therein), the fact that this also holds for the second (temporal) expectation in (1.2) seems to have stayed unnoticed until now. Secondly, stopping times with the power of being optimal in two distinct problems are both rare and remarkable. Apart from the sequential probability ratio test in Wald's sequential analysis where the optimality is obtained under each of the two probability measures (see e.g. [13, Chapter VI]), stopping Brownian motion as close as possible to its ultimate maximum seems to be the only other known example where this happens simultaneously in the space domain and the time domain (see [13, Section 30]).

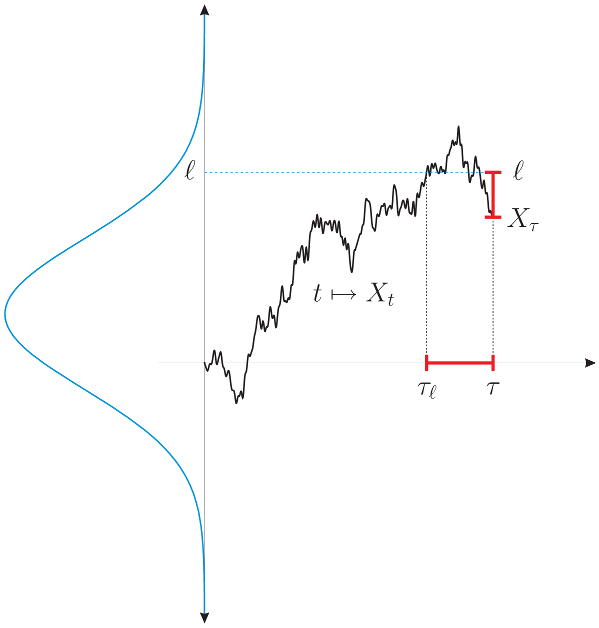

Figure 2. Optimal detection of a hidden target (two-sided case).

Denoting by F_ℓ the distribution function of ℓ and using that X and ℓ are independent, we see that a possible way of interpreting the problems (1.2) is as follows:

$$\inf_\tau \int_0^\infty \mathsf{E}|X_\tau - x|\, F_\ell(dx) \quad \text{and} \quad \inf_\tau \int_0^\infty \mathsf{E}|\tau - \tau_x|\, F_\ell(dx) \tag{1.3}$$

where the infima are taken over all stopping times τ of X and τ_x denotes the first entry time of X at the level x. From these representations we see that these optimal stopping problems seemingly go beyond the scope of the general optimal stopping theory (see e.g. [13]) in the sense that the loss functions take a rather particular form. We may also note that the problems of minimising the expectations in (1.2) over all stopping times τ of X belong to the class of 'optimal prediction' problems (within optimal stopping) since the underlying loss processes $t \mapsto |X_t - \ell|$ and $t \mapsto |t - \tau_\ell|$ are not adapted to the natural filtration generated by X (or its usual augmentation). Similar optimal prediction problems have been studied in recent years by a number of authors (see e.g. [1]-[9], [12]-[13], [14]-[16]). It may be noted in this context that the non-adapted factors ℓ and τ_ℓ in the optimal prediction problems (1.2) are not revealed at the 'end' of time (i.e. they are not measurable with respect to the σ-algebra generated by the process X). Obtaining fuller understanding of the structure of the solution to these/related problems in general and applicable solution techniques in particular appears to be worthy of further consideration.

2. Spatial problems

In this section we consider the space domain problems addressed in the introduction. Let $X = (X_t)_{t \geq 0}$ be a continuous stochastic process started at 0 and let ℓ be an independent real-valued random variable defined on the same probability space $(\Omega, \mathcal{F}, \mathsf{P})$.

Consider the optimal prediction problem

$$\inf_{\tau} \mathsf{E}|X_\tau - \ell|^p \tag{2.1}$$

where the infimum is taken over all stopping times τ of X (i.e. with respect to the natural filtration generated by X) and $\mathsf{E}\ell^p < \infty$ for some $p \geq 1$ given and fixed. Denoting by F_ℓ the distribution function of ℓ and using that X and ℓ are independent, we see that the optimal prediction problem (2.1) can be rewritten as follows:

$$\inf_{\tau} \int_{-\infty}^{\infty} \mathsf{E}|X_\tau - x|^p \, F_\ell(dx) \tag{2.2}$$

where the infimum is taken over all stopping times τ of X . This problem seemingly goes beyond the scope of the general optimal stopping theory (see e.g. [13]) in the sense that the loss function takes a rather particular form. There is a simple trick, however, that makes it possible to solve the optimal stopping problem (2.2) in infinite horizon (i.e. when τ is not bounded from above). For this, note that the independence of X and ℓ also implies that the optimal prediction problem (2.1) can be rewritten as follows:

$$\inf_{\tau} \int_{-\infty}^{\infty} \mathsf{E}|x - \ell|^p \, \mathsf{P}_{X_\tau}(dx) \tag{2.3}$$

where the infimum is taken over all stopping times τ of X and P_{X_τ} is the distribution law of X_τ . As long as the family of all distribution laws P_{X_τ} includes the Dirac measures at points in $I\!R$ when τ runs over all stopping times (meaning that X hits points in $I\!R$ almost surely), we see that (2.3) naturally leads to considering the auxiliary optimisation problem

$$\inf_{x} \mathsf{E}|x - \ell|^p \tag{2.4}$$

where the infimum is taken over all x in $I\!R$. The solution to the problem (2.4) is well known (see e.g. [10, pp. 313-316]). Indeed, one knows that there exists $x_p^* \in I\!R$ at which the infimum in (2.4) is attained (when $p = 1$ these x_p^* can form a bounded and closed interval and when $p > 1$ this x_p^* is unique). From (2.3) and (2.4) it follows that

$$\mathsf{E}|X_\tau - \ell|^p = \int_{-\infty}^{\infty} \mathsf{E}|x - \ell|^p \, \mathsf{P}_{X_\tau}(dx) \geq \mathsf{E}|x_p^* - \ell|^p \tag{2.5}$$

for all stopping (or random) times τ of X . Moreover, the final equality in (2.5) is attained at $\mathsf{P}_{X_{\tau_*}} = \delta_{x_p^*}$ where $\delta_{x_p^*}$ is the Dirac measure at x_p^* . This shows that the stopping time

$$\tau_* = \inf \{ \, t \geq 0 \mid X_t = x_p^* \, \} \tag{2.6}$$

is optimal in (2.1) whenever finite. The same argument shows that any other stopping time τ of X such that $X_\tau = x_p^*$ is also optimal in (2.1). Note also that the stopping time (2.6) is the least possible among all optimal stopping times (as long as x_1^* is taken to be as close as possible to 0 if the infimum in (2.4) is attained at a non-trivial interval when $p = 1$).

To describe the optimal level x_p^* more closely in some particular cases, recall that a number $\mathfrak{m}$ in $I\!R$ is called a *median* of ℓ if $\mathsf{P}(\ell < \mathfrak{m}) \leq \frac{1}{2} \leq P(\ell \leq \mathfrak{m})$. The set of all medians of ℓ

is a bounded and closed interval $M_\ell = [\mathfrak{m}, \mathfrak{M}]$ where $\mathfrak{m}$ is the lowest median of ℓ and $\mathfrak{M}$ is the highest median of ℓ (they can also be equal). If $p=1$ in (2.1) then x_1^* in (2.6) can be taken to be any median of ℓ, i.e.

$$x_1^* \in M_\ell \tag{2.7}$$

implying also that any stopping time τ_* of X such that $\mathsf{P}(X_{\tau_*} \in M_\ell) = 1$ is optimal in (2.1). If $p=2$ in (2.1) then x_2^* in (2.6) equals the (quadratic) *mean* of ℓ given by

$$x_2^* = \mu \tag{2.8}$$

where $\mu = \mathsf{E}\ell$. If $p=3$ then x_3^* in (2.6) equals the *cubic mean* given by

$$x_3^* = \nu \tag{2.9}$$

where ν is the unique real number satisfying

$$\mathsf{E}[(\ell - \nu)^2 I(\ell > \nu)] = \mathsf{E}[(\ell - \nu)^2 I(\ell < \nu)]\,. \tag{2.10}$$

If $p=4$ in (2.1) then x_4^* in (2.6) equals the *biquadratic mean* of ℓ given by

$$x_4^* = \mu + \sqrt[3]{\tfrac{\lambda}{2} + \sqrt{\tfrac{\lambda^2}{4} + \sigma^6}} + \sqrt[3]{\tfrac{\lambda}{2} - \sqrt{\tfrac{\lambda^2}{4} + \sigma^6}} \tag{2.11}$$

where $\mu = \mathsf{E}\ell$, $\sigma^2 = \mathsf{E}(\ell - \mu)^2$ and $\lambda = \mathsf{E}(\ell - \mu)^3$.

In addition to the power functions one can also consider other distance functions in the optimal prediction problem (2.1). For instance, looking at

$$\inf_\tau \mathsf{E}\Big(\log \tfrac{X_\tau}{\ell}\Big)^2 \tag{2.12}$$

and applying the same arguments as above we find that the following stopping time is optimal

$$\tau_* = \inf\,\{\, t \geq 0 \mid X_t = x_g^* \,\} \tag{2.13}$$

where x_g^* is the *geometric mean* of ℓ given by

$$x_g^* = e^{\mathsf{E}\log \ell} \tag{2.14}$$

whenever ℓ is non-negative.

It is possible to continue this list of examples and it is not clear a priori how to select the most natural distance function. One may observe, however, that if one is interested in the expected value of a given distance from X_τ to ℓ in $I\!R$, then the case $p=1$ in (2.1) plays a special role (as the expected value of the Euclidean distance in $I\!R$). It may also be noted that the optimal prediction problem (2.1) can have a trivial solution in some cases when the horizon is finite (or stopping times have finite means). For example, if X is a martingale (standard Brownian motion) we see from (2.2) using Jensen's inequality and the optional sampling theorem that it is optimal to stop at once.

3. Temporal problems

In this section we consider the time domain problems addressed in the introduction. Let $X = (X_t)_{t \ge 0}$ be a continuous stochastic process started at 0, let ℓ be an independent real-valued random variable, and let $\tau_\ell = \inf\{\, t \ge 0 \mid X_t = \ell \,\}$ be the first entry of X at the level ℓ. We assume that X and ℓ are defined on the same probability space $(\Omega, \mathcal{F}, \mathsf{P})$. The natural filtration generated by X is denoted by $(\mathcal{F}_t^X)_{t \ge 0}$. Stopping times with respect to the natural filtration $(\mathcal{F}_t^X)_{t \ge 0}$ are referred to as stopping times of X. Let F_ℓ denote the distribution function of ℓ and let $M_\ell = [\mathfrak{m}, \mathfrak{M}]$ denote the set of all medians of ℓ where $\mathfrak{m}$ is the lowest median of ℓ and $\mathfrak{M}$ is the highest median of ℓ (they can also be equal). We also assume that $\mathsf{E}\ell < \infty$ and $\mathsf{E}\tau_\ell < \infty$. Setting $S_t = \sup_{0 \le s \le t} X_s$ for $t \ge 0$, and denoting by F_{S_t} the distribution function of S_t, these two conditions can be rewritten as

$$\int_{-\infty}^{\infty} |x|\, F_\ell(dx) < \infty \quad \text{and} \quad \int_0^\infty \int_0^\infty F_{S_t}(x-)\, F_\ell(dx)\, dt < \infty \,. \tag{3.1}$$

Since $\mathsf{E}\tau_\ell = \int_{-\infty}^{\infty} \mathsf{E}\tau_x\, F_\ell(dx)$ note that $\mathsf{E}\tau_\ell < \infty$ implies that $\mathsf{P}(\tau_{\mathfrak{m}} < \infty) = 1$ and $\mathsf{P}(\tau_{\mathfrak{M}} < \infty) = 1$ where $\tau_{\mathfrak{m}}$ and $\tau_{\mathfrak{M}}$ denote the first entry times of X at $\mathfrak{m}$ and $\mathfrak{M}$ respectively. This is due to $\mu_{F_\ell}(\mathrm{supp}(\ell) \cap (-\infty, \mathfrak{m}]) > 0$ and $\mu_{F_\ell}(\mathrm{supp}(\ell) \cap [\mathfrak{M}, \infty)) > 0$ where μ_{F_ℓ} denotes the Lebesgue-Stieltjes measure associated with F_ℓ and $\mathrm{supp}(\ell)$ denotes the support of ℓ (for each x in its complement there exists an $\varepsilon > 0$ such that $\mu_{F_\ell}((x-\varepsilon, x+\varepsilon)) = 0$). We first consider the case when ℓ does not take negative values (see Figure 1).

Theorem 3.1 (The median rule). *Under the hypotheses stated above, let us assume that ℓ is non-negative. Then the stopping time*

$$\tau_{\mathfrak{m}} = \inf\{\, t \ge 0 \mid X_t = \mathfrak{m} \,\} \tag{3.2}$$

minimises simultaneously both

$$\mathsf{E}|X_\tau - \ell| \quad \textit{and} \quad \mathsf{E}|\tau - \tau_\ell| \tag{3.3}$$

over all stopping times τ of X.

Proof. The fact that $\tau_{\mathfrak{m}}$ minimises $\mathsf{E}|X_\tau - \ell|$ over all stopping times τ of X was derived in Section 2 above. We now show that this is also true for the second expectation in (3.3).

Let τ be any stopping time of X with $\mathsf{E}\tau < \infty$ (clearly the latter condition can be assumed without loss of generality). In the first step (to be implicit below) we will project the unknown information about τ_ℓ to the known information $\mathcal{F}_\tau$ at time τ and note that

$$\mathsf{E}|\tau - \tau_\ell| = \mathsf{E}\big[\mathsf{E}(|\tau - \tau_\ell| \mid \mathcal{F}_\tau)\big] \,. \tag{3.4}$$

To compute the conditional expectation we will express the argument $|\tau - \tau_\ell|$ in terms of an integral over $[0, \tau]$ using a well-known argument (see e.g. [13, p. 450]). For this, note that

$$\begin{aligned} |\tau - \tau_\ell| &= (\tau - \tau_\ell)^+ + (\tau_\ell - \tau)^+ = \int_0^\tau I(\tau_\ell \le t)\, dt + \int_0^{\tau_\ell} I(\tau \le t)\, dt \\ &= \int_0^\tau I(\tau_\ell \le t)\, dt + \int_0^{\tau_\ell} (1 - I(\tau > t))\, dt \end{aligned} \tag{3.5}$$

$$
\begin{aligned}
&= \int_0^\tau I(\tau_\ell \le t)\, dt + \tau_\ell - \int_0^{\tau_\ell} I(\tau > t)\, dt \\
&= \int_0^\tau I(\tau_\ell \le t)\, dt + \tau_\ell - \int_0^{\tau} I(\tau_\ell > t)\, dt \\
&= \int_0^\tau I(\tau_\ell \le t)\, dt + \tau_\ell - \int_0^{\tau} (1 - I(\tau_\ell \le t))\, dt \\
&= \int_0^\tau (2I(\tau_\ell \le t) - 1)\, dt + \tau_\ell\, .
\end{aligned}
$$

Taking E on both sides we get the well-known identity (cf. [13, p. 450])

$$
\begin{aligned}
\mathsf{E}|\tau - \tau_\ell| &= \mathsf{E} \int_0^\tau (2I(\tau_\ell \le t) - 1)\, dt + \mathsf{E}\tau_\ell \\
&= \mathsf{E} \int_0^\infty (2I(\tau_\ell \le t) - 1)\, I(t < \tau)\, dt + \mathsf{E}\tau_\ell \\
&= \int_0^\infty \mathsf{E}\big[(2I(\tau_\ell \le t) - 1)\, I(t < \tau)\big]\, dt + \mathsf{E}\tau_\ell \\
&= \int_0^\infty \mathsf{E}\big[\mathsf{E}\big((2I(\tau_\ell \le t) - 1)\, I(t < \tau)\,|\, \mathcal{F}_t^X\big)\big]\, dt + \mathsf{E}\tau_\ell \\
&= \int_0^\infty \mathsf{E}\big[I(t < \tau)\, \mathsf{E}\big(2I(\tau_\ell \le t) - 1\,|\, \mathcal{F}_t^X\big)\big]\, dt + \mathsf{E}\tau_\ell \\
&= \mathsf{E} \int_0^\infty I(t < \tau)\, \mathsf{E}\big(2I(\tau_\ell \le t) - 1\,|\, \mathcal{F}_t^X\big)\, dt + \mathsf{E}\tau_\ell \\
&= \mathsf{E} \int_0^\tau \big(2\,\mathsf{P}(\tau_\ell \le t\,|\, \mathcal{F}_t^X) - 1\big)\, dt + \mathsf{E}\tau_\ell\, .
\end{aligned}
\tag{3.6}
$$

Since X and ℓ are independent we find that

$$
\mathsf{P}(\tau_\ell \le t\,|\, \mathcal{F}_t^X) = \mathsf{P}(S_t \ge \ell\,|\, \mathcal{F}_t^X) = \mathsf{P}(s \ge \ell)\big|_{s = S_t} = F_\ell(S_t)
\tag{3.7}
$$

for all $t \ge 0$. Inserting (3.7) into (3.6) it follows that

$$
\mathsf{E}|\tau - \tau_\ell| = \mathsf{E} \int_0^\tau \big(2\, F_\ell(S_t) - 1\big)\, dt + \mathsf{E}\tau_\ell\, .
\tag{3.8}
$$

We now claim that $\tau_{\mathfrak{m}}$ from (3.2) minimises the right-hand side of (3.8) over all stopping times τ of X . Intuitively this is clear since the integrand in (3.8) is negative for $S_t < \mathfrak{m}$ and positive for $S_t > \mathfrak{M}$ while the sample path $t \mapsto S_t$ is increasing. Formally this can be verified as follows. If a stopping time τ of X with $\mathsf{E}\tau < \infty$ is given and fixed, define a new stopping time $\tilde{\tau}$ of X by setting $\tilde{\tau} = \tau_{\mathfrak{m}}\, I(\tau < \tau_{\mathfrak{m}}) + \tau\, I(\tau_{\mathfrak{m}} \le \tau \le \tau_{\mathfrak{M}}) + \tau_{\mathfrak{M}}\, I(\tau > \tau_{\mathfrak{M}})$ where we recall that $\tau_{\mathfrak{M}}$ denotes the first entry time of X at $\mathfrak{M}$. Using (3.8) we then have

$$
\begin{aligned}
\mathsf{E}|\tau - \tau_\ell| &= \mathsf{E} \int_0^\tau \big(2\, F_\ell(S_t) - 1\big)\, dt + \mathsf{E}\tau_\ell \\
&= \mathsf{E} \int_0^\tau \big(2\, F_\ell(S_t) - 1\big)\, I(\tau < \tau_{\mathfrak{m}})\, dt
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
&+ \mathsf{E}\int_0^{\tau} \big(2\,F_\ell(S_t)-1\big)\, I(\tau_{\mathfrak{m}} \le \tau \le \tau_{\mathfrak{M}})\, dt \\
&+ \mathsf{E}\int_0^{\tau} \big(2\,F_\ell(S_t)-1\big)\, I(\tau > \tau_{\mathfrak{M}})\, dt + \mathsf{E}\tau_\ell \\
\ge\; & \mathsf{E}\int_0^{\tau_{\mathfrak{m}}} \big(2\,F_\ell(S_t)-1\big)\, I(\tau < \tau_{\mathfrak{m}})\, dt \\
&+ \mathsf{E}\int_0^{\tau} \big(2\,F_\ell(S_t)-1\big)\, I(\tau_{\mathfrak{m}} \le \tau \le \tau_{\mathfrak{M}})\, dt \\
&+ \mathsf{E}\int_0^{\tau_{\mathfrak{M}}} \big(2\,F_\ell(S_t)-1\big)\, I(\tau > \tau_{\mathfrak{M}})\, dt + \mathsf{E}\tau_\ell \\
=\; & \mathsf{E}\int_0^{\tilde{\tau}} \big(2\,F_\ell(S_t)-1\big)\, dt + \mathsf{E}\tau_\ell = \mathsf{E}|\tilde{\tau}-\tau_\ell|
\end{aligned}
$$

where we use that $2\,F_\ell(S_t)-1 < 0$ for $t \in [\tau, \tau_{\mathfrak{m}})$ and $2\,F_\ell(S_t)-1 \ge 0$ for $t \in (\tau_{\mathfrak{M}}, \tau]$. The argument above also shows that the inequality in (3.9) is strict if and only if $\mathsf{P}(S_\tau < \mathfrak{m}) > 0$ or $\mathsf{P}(S_\tau > \mathfrak{M}) > 0$. Combining these facts we see that any stopping time τ_* of X satisfying $\mathsf{P}(S_{\tau_*} \in [\mathfrak{m}, \mathfrak{M}]) = 1$ minimises $\mathsf{E}|\tau-\tau_\ell|$ over all stopping times τ of X . This is true for $\tau_{\mathfrak{m}}$ in particular and the proof is complete. □

Remark 3.2. The generality of Theorem 3.1 makes it applicable to the case when the 'target' is not constant but a known continuous trajectory starting at a level ℓ which is not directly observable. This follows by subtracting the trajectory from the observed process and applying Theorem 3.1 to this new process. In this case ℓ plays the role of a random initial condition and the median rule states that one should *stop as soon as the distance from the observed process to the trajectory exceeds the (lowest) median of the initial condition.*

Remark 3.3. The fact that $\tau_{\mathfrak{m}}$ minimises $\mathsf{E}|\tau-\tau_\ell|$ over all stopping times τ of X can also be verified using the characterisation of the median described in (2.4) and (2.7) above. We are grateful to a referee for communicating the basic idea of this verification (which is to condition on a given sample path of the observed process). For this, suppose that $(\Omega, \mathcal{F}, \mathsf{P})$ equals $(\Omega' \otimes \Omega'', \mathcal{F}' \otimes \mathcal{F}'', \mathsf{P}' \otimes \mathsf{P}'')$ where $(\Omega', \mathcal{F}', \mathsf{P}')$ is a probability space on which X is defined, and $(\Omega'', \mathcal{F}'', \mathsf{P}'')$ is a probability space on which ℓ is defined. We extend X and ℓ to Ω by setting $X(\omega', \omega'') = X(\omega')$ and $\ell(\omega', \omega'') = \ell(\omega'')$ for $(\omega', \omega'') \in \Omega$. Recall that for $x \in I\!R_+$ we define $\tau_x(\omega') = \inf\{\, t \ge 0 \mid X_t(\omega') = x \,\}$ for $\omega' \in \Omega'$ and we extend τ_x to Ω by setting $\tau_x(\omega', \omega'') = \tau_x(\omega')$ for $(\omega', \omega'') \in \Omega$. Fix any $\omega' \in \Omega'$ and let $\mathfrak{m}$ be a median of ℓ . Due to the fact that $t \mapsto X_t$ is continuous and $\mathfrak{m} \ge X_0$ we see that

$$
\{\, \omega'' \in \Omega'' \mid \ell(\omega'') \le \mathfrak{m} \,\} = \{\omega'' \in \Omega'' \mid \tau_{\ell(\omega'')}(\omega') \le \tau_{\mathfrak{m}}(\omega') \,\} \tag{3.10}
$$

$$
\{\omega'' \in \Omega'' \mid \ell(\omega'') < \mathfrak{m} \,\} = \{\, \omega'' \in \Omega'' \mid \tau_{\ell(\omega'')}(\omega') < \tau_{\mathfrak{m}}(\omega') \,\}\,. \tag{3.11}
$$

Taking P'' on both sides in (3.10) and (3.11), and using that $\mathsf{P}''(\ell < \mathfrak{m}) \le \frac{1}{2} \le P''(\ell \le \mathfrak{m})$, we find that $\tau_{\mathfrak{m}}(\omega')$ is a median of the random variable $\omega'' \mapsto \tau_{\ell(\omega'')}(\omega')$ defined on $(\Omega'', \mathcal{F}'', \mathsf{P}'')$. Hence by (2.4) and (2.7) it follows that

$$
\int_{\Omega''} |\tau(\omega')-\tau_{\ell(\omega'')}(\omega')|\, d\mathsf{P}''(\omega'') \ge \int_{\Omega''} |\tau_{\mathfrak{m}}(\omega')-\tau_{\ell(\omega'')}(\omega')|\, d\mathsf{P}''(\omega'') \tag{3.12}
$$

where $\tau : \Omega' \rightarrow [0,\infty]$ is any stopping (or random) time of X . Integrating (3.12) with respect to P' we obtain

$$\int_{\Omega'}\int_{\Omega''} |\tau(\omega') - \tau_{\ell(\omega'')}(\omega')| \, d\mathsf{P}'(\omega') \, d\mathsf{P}''(\omega'') \geq \int_{\Omega'}\int_{\Omega''} |\tau_{\mathfrak{m}}(\omega') - \tau_{\ell(\omega'')}(\omega')| \, d\mathsf{P}'(\omega') \, d\mathsf{P}''(\omega'') \tag{3.13}$$

and this is the same as $\mathsf{E}|\tau - \tau_\ell| \geq \mathsf{E}|\tau_{\mathfrak{m}} - \tau_\ell|$ which completes the proof. Despite its revealing character it seems that this verification cannot be easily extended to the case when ℓ can take negative values as well (note that (3.10) and (3.11) fail in this case).

We next consider the case when ℓ can take both positive and negative values (see Figure 2). Recall that $S_t = \sup_{0 \leq s \leq t} X_s$ and set $I_t = \inf_{0 \leq s \leq t} X_s$ for $t \geq 0$.

Theorem 3.4 (The rolling median rule). *Under the hypotheses stated above, let us assume that ℓ is real-valued. Then the stopping time*

$$\sigma = \inf \{ \, t \geq 0 \mid F_\ell(S_t) - F_\ell(I_t-) \geq 1/2 \, \} \tag{3.14}$$

minimises $\mathsf{E}|\tau - \tau_\ell|$ over all stopping times τ of X .

Proof. Let τ be a stopping time of X with $\mathsf{E}\tau < \infty$. By (3.6) above we know that

$$\mathsf{E}|\tau - \tau_\ell| = \mathsf{E} \int_0^\tau \big(2\,\mathsf{P}(\tau_\ell \leq t \,|\, \mathcal{F}_t^X) - 1 \big) \, dt + \mathsf{E}\tau_\ell \, . \tag{3.15}$$

Since X and ℓ are independent we see that the analogue of (3.7) above becomes

$$\begin{aligned} \mathsf{P}(\tau_\ell \leq t \,|\, \mathcal{F}_t^X) &= \mathsf{P}(\tau_\ell \leq t, \ell > 0 \,|\, \mathcal{F}_t^X) + \mathsf{P}(\tau_\ell \leq t, \ell \leq 0 \,|\, \mathcal{F}_t^X) \\ &= \mathsf{P}(S_t \geq \ell > 0 \,|\, \mathcal{F}_t^X) + \mathsf{P}(I_t \leq \ell \leq 0 \,|\, \mathcal{F}_t^X) \\ &= \big(F_\ell(s) - F_\ell(0) \big)\big|_{s=S_t} + \big(F_\ell(0) - F_\ell(i-) \big)\big|_{i=I_t} \\ &= F_\ell(S_t) - F_\ell(I_t-) \end{aligned} \tag{3.16}$$

for all $t \geq 0$. Inserting (3.16) into (3.15) we find that

$$\mathsf{E}|\tau - \tau_\ell| = \mathsf{E} \int_0^\tau \big(2\big[F_\ell(S_t) - F_\ell(I_t-) \big] - 1 \big) \, dt + \mathsf{E}\tau_\ell \, . \tag{3.17}$$

Noting that $t \mapsto F_\ell(S_t) - F_\ell(I_t-)$ is increasing in $[0,1]$ the proof can be completed using the same arguments as in the proof of Theorem 3.1 above. □

Note that the stopping time (3.14) coincides with the stopping time (3.2) when ℓ is non-negative (since $F_\ell(0-) = 0$ in this case).

Remark 3.5. It should be noted that the stopping time (3.14) may no longer minimise $\mathsf{E}|X_\tau - \ell|$ over all stopping times τ of X when ℓ takes both positive and negative values (recall from Section 2 that the first entry time of X into the set of all medians of ℓ remains optimal in this case as well). Indeed, if the continuous distribution function F_ℓ of ℓ satisfies $F_\ell(-1) = F_\ell(1) = 1/2$ with $F_\ell(x) < 1/2$ for $x < -1$ and $F_\ell(x) > 1/2$ for $x > 1$, then the set of all medians of ℓ equals $[-1,1]$ and we know from Section 2 that a stopping time τ_* of X minimises $\mathsf{E}|X_\tau - \ell|$ over all stopping times τ of X if and only if $\mathsf{P}(X_{\tau_*} \in [-1,1]) = 1$. The stopping time (3.14) may fail to satisfy the latter condition since according to its rule we need to wait until the spread $F_\ell(S_t) - F_\ell(I_t-)$ reaches $1/2$ after starting at 0 and at that time X_t may/will no longer be in $[-1,1]$. Since ℓ takes no values in $[-1,1]$ with positive probability we see that this temporal rule is somewhat more realistic than the spatial rule of stopping in $[-1,1]$. This just adds to the value of having both criteria satisfied at once like in Theorem 3.1 above.

Remark 3.6. It is interesting to note that the stopping time (3.14) remains optimal in the same problem when the horizon is finite, i.e. the stopping time

$$\sigma = \inf \{ t \in [0,T) \mid F_\ell(S_t) - F_\ell(I_t-) \geq 1/2 \} \tag{3.18}$$

minimises $\mathsf{E}|\tau - \tau_\ell|$ over all stopping times τ of X satisfying $\tau \leq T$ where $T > 0$ (horizon) is given and fixed (and where we formally set $\inf \emptyset = T$). This can be established using the same arguments as in the proofs of Theorem 3.1 and Theorem 3.4 above.

Remark 3.7. The result of Theorem 3.4 remains valid if the process X is right-continuous. The proof can be carried out if we firstly redefine $\tau_{\mathfrak{m}}$ and $\tau_{\mathfrak{M}}$ in the proof of Theorem 3.1 by setting $\tau_{\mathfrak{m}} = \inf \{ t \geq 0 \,|\, X_t \geq \mathfrak{m} \}$ and $\tau_{\mathfrak{M}} = \inf \{ t \geq 0 \,|\, X_t \geq \mathfrak{M} \}$, and then extend the resulting argument as in the proof of Theorem 3.4.

4. Variational problems

1. In the spatial and temporal problems discussed in the previous two sections one often wants to give different weights to the probability errors of early or late stopping. The optimal prediction problem (2.1) then reads

$$\inf_\tau \mathsf{E}\big[c_1(\ell - X_\tau)^+ + c_2(X_\tau - \ell)^+ \big] \tag{4.1}$$

where the infimum is taken over all stopping times τ of X , and $c_1 > 0$ and $c_2 > 0$ are given and fixed constants (weights). Setting

$$p = \frac{c_1}{c_1 + c_2} \tag{4.2}$$

and noting that $p \in (0,1)$, recall that a number $\mathfrak{q}_p$ in $I\!R$ is called a *p-quantile* of ℓ if $\mathsf{P}(\ell < \mathfrak{q}_p) \leq p \leq P(\ell \leq \mathfrak{q}_p)$. The set of all p-quantiles of ℓ is a bounded and closed interval $Q_\ell^p = [\mathfrak{q}_p, \mathfrak{Q}_p]$ where $\mathfrak{q}_p$ is the lowest p-quantile of ℓ and $\mathfrak{Q}_p$ is the highest p-quantile of ℓ (they can also be equal). Note that $\mathfrak{q}_{1/2}$ coincides with the (lowest) median $\mathfrak{m}$ of ℓ . In the same way as in Section 2 one then finds that any stopping time τ_* of X such that $\mathsf{P}(X_{\tau_*} \in Q_\ell^p) = 1$ is optimal in (4.1). Moreover, the results of Theorem 3.1 (with Remarks 3.2+3.3) and Theorem 3.4 (with Remarks 3.5-3.7) extend as follows.

Theorem 4.1. *Under the hypotheses stated in the first paragraph of Section 3 above, let us assume that ℓ is non-negative. Then the stopping time*

$$\tau_{\mathfrak{q}_p} = \inf \{ t \geq 0 \mid X_t = \mathfrak{q}_p \} \tag{4.3}$$

minimises simultaneously both

$$\mathsf{E}\big[\,c_1(\ell - X_\tau)^+ + c_2(X_\tau - \ell)^+\big] \quad \textit{and} \quad \mathsf{E}\big[\,c_1(\tau_\ell - \tau)^+ + c_2(\tau - \tau_\ell)^+\big] \tag{4.4}$$

over all stopping times τ *of* X *.*

Proof. This can be derived in the same way as Theorem 3.1 above. □

Theorem 4.2. *Under the hypotheses stated in the first paragraph of Section 3 above, let us assume that* ℓ *is real-valued. Then the stopping time*

$$\sigma_p = \inf\{\, t \ge 0 \mid F_\ell(S_t) - F_\ell(I_t -) \ge p\,\} \tag{4.5}$$

minimises $\mathsf{E}\big[\,c_1(\tau_\ell - \tau)^+ + c_2(\tau - \tau_\ell)^+\big]$ *over all stopping times* τ *of* X *.*

Proof. This can be derived in the same way as Theorem 3.4 above. □

2. The preceding results can be used to solve the constrained problems where the expected errors of early or late stopping are bounded from above. Under the hypotheses stated in the first paragraph of Section 3 above, consider the constrained optimal prediction problems

$$V_1 = \inf_{\tau \in S_c} \mathsf{E}(\tau_\ell - \tau)^+ \tag{4.6}$$

$$V_2 = \inf_{\tau \in S^c} \mathsf{E}(\tau - \tau_\ell)^+ \tag{4.7}$$

where S_c denotes the family of all stopping times τ of X satisfying $\mathsf{E}(\tau - \tau_\ell)^+ \le c$ and S^c denotes the family of all stopping times τ of X satisfying $\mathsf{E}(\tau_\ell - \tau)^+ \le c$ for some $c > 0$ given and fixed. It is well known that the method of Lagrange multipliers provides a strategy for finding maxima or minima of functions subject to constraints. In analogy with the classic method of Lagrange multipliers let us form the Lagrange functions

$$L_1(\tau, \lambda) = \mathsf{E}(\tau_\ell - \tau)^+ + \lambda\big[\mathsf{E}(\tau - \tau_\ell)^+ - c\big] \tag{4.8}$$

$$L_2(\tau, \lambda) = \mathsf{E}(\tau - \tau_\ell)^+ + \lambda\big[\mathsf{E}(\tau_\ell - \tau)^+ - c\big] \tag{4.9}$$

where τ is a stopping time of X and $\lambda > 0$ is a given constant. The classic method of Lagrange multipliers then suggests that if τ_* is an optimal stopping time in (4.6) or (4.7) then there exists $\lambda_* > 0$ such that (τ_*, λ_*) is a stationary point of L_1 or L_2 respectively.

Motivated by these general facts let us assume that stopping times τ_λ^1 and τ_λ^2 are optimal in the unconstrained problems (4.8) and (4.9) respectively, i.e.

$$L_1(\tau_\lambda^1, \lambda) = \inf_\tau L_1(\tau, \lambda) \tag{4.10}$$

$$L_2(\tau_\lambda^2, \lambda) = \inf_\tau L_2(\tau, \lambda) \tag{4.11}$$

where the infima are taken over all stopping times τ of X and $\lambda > 0$ is given and fixed. Suppose moreover that there exist $\lambda_1^c > 0$ and $\lambda_2^c > 0$ such that

$$\mathsf{E}(\tau^1_{\lambda_1^c} - \tau_\ell)^+ = c \tag{4.12}$$

$$\mathsf{E}(\tau_\ell - \tau^2_{\lambda^c_2})^+ = c\,. \tag{4.13}$$

Then the stopping time $\tau^1_{\lambda^c_1}$ is optimal in the constrained problem (4.6) and the stopping time $\tau^2_{\lambda^c_2}$ is optimal in the constrained problem (4.7).

Indeed, if $\tau \in S_c$ then $\mathsf{E}(\tau - \tau_\ell)^+ \le c$ so that by (4.10)+(4.12) we see that

$$\begin{aligned}\mathsf{E}(\tau_\ell - \tau^1_{\lambda^c_1})^+ &= L_1(\tau^1_{\lambda^c_1}, \lambda^c_1) \le L_1(\tau, \lambda^c_1) \\ &= \mathsf{E}(\tau_\ell - \tau)^+ + \lambda^c_1 \big[\mathsf{E}(\tau - \tau_\ell)^+ - c \big] \le \mathsf{E}(\tau_\ell - \tau)^+\end{aligned} \tag{4.14}$$

which shows that $\tau^1_{\lambda^c_1}$ is optimal in (4.6) as claimed. Similarly, if $\tau \in S^c$ then $\mathsf{E}(\tau_\ell - \tau)^+ \le c$ so that by (4.11)+(4.13) we see that

$$\begin{aligned}\mathsf{E}(\tau^2_{\lambda^c_2} - \tau_\ell)^+ &= L_2(\tau^2_{\lambda^c_2}, \lambda^c_2) \le L_2(\tau, \lambda^c_2) \\ &= \mathsf{E}(\tau - \tau_\ell)^+ + \lambda^c_2 \big[\mathsf{E}(\tau_\ell - \tau)^+ - c \big] \le \mathsf{E}(\tau - \tau_\ell)^+\end{aligned} \tag{4.15}$$

which shows that $\tau^2_{\lambda^c_2}$ is optimal in (4.7) as claimed.

3. The unconstrained problems (4.10) and (4.11) were solved in Theorem 4.1 and Theorem 4.2 with $\lambda = c_2/c_1$ and $\lambda = c_1/c_2$ so that $p = 1/(1+\lambda)$ and $p = 1/(1+1/\lambda)$ respectively. Let us first consider the case when ℓ is non-negative. Then $\tau^1_{\lambda^c_1}$ and $\tau^2_{\lambda^c_2}$ are equal to $\tau_{\mathfrak{q}_p}$ from (4.3) with $p = p^c_1$ and $p = p^c_2$ in $[0,1]$ to be found so that (4.12) and (4.13) hold respectively. To examine (4.12) let us note that

$$\begin{aligned}\mathsf{E}(\tau_x - \tau_\ell)^+ &= \mathsf{E} \int_0^{\tau_x} F_\ell(S_t)\, dt = \mathsf{E} \int_0^\infty F_\ell(S_t)\, I(t < \tau_x)\, dt \\ &= \int_0^\infty \mathsf{E}\big[F_\ell(S_t)\, I(S_t < x) \big]\, dt = \int_0^\infty \int_0^{x-} F_\ell(s)\, F_{S_t}(ds)\, dt =: G(x)\end{aligned} \tag{4.16}$$

for $x \ge 0$ where F_{S_t} is the distribution function of S_t and we have $G(0) = 0$. Then $x \mapsto G(x)$ is strictly increasing (where strictly positive) and left-continuous on $(0, \alpha]$ where $\alpha = \sup[\mathrm{supp}(\ell)] \in [0, \infty]$ and $G(\alpha) = \infty$ if $\alpha = \infty$. Setting $\mathrm{Im}(G) = \{\, G(x) \mid x \in [0, \alpha] \,\}$ we see that for any $c \in \mathrm{Im}(G) \cap (0, \infty)$ there exists $x^c_1 \in [0, \alpha] \cap (0, \infty)$ such that $G(x^c_1) = c$. Let $p^c_1 = F_\ell(x^c_1)$ and define $\lambda^c_1 = (1 - p^c_1)/p^c_1$ so that $p^c_1 = 1/(1+\lambda^c_1)$ as needed. Then (4.12) holds with $\tau^1_{\lambda^c_1} = \tau_{\mathfrak{q}_p}$ where $p = p^c_1$ and hence by the result of Theorem 4.1 and (4.14) we see that this stopping time is optimal in (4.6). Similarly, to examine (4.13) let us note that

$$\begin{aligned}\mathsf{E}(\tau_\ell - \tau_x)^+ &= \mathsf{E}\tau_\ell - \mathsf{E} \int_0^{\tau_x} (1 - F_\ell(S_t))\, dt \\ &= \mathsf{E}\tau_\ell - \mathsf{E} \int_0^\infty (1 - F_\ell(S_t))\, I(t < \tau_x)\, dt \\ &= \mathsf{E}\tau_\ell - \int_0^\infty \mathsf{E}\big[(1 - F_\ell(S_t))\, I(S_t < x) \big]\, dt \\ &= \mathsf{E}\tau_\ell - \int_0^\infty \int_0^{x-} (1 - F_\ell(s))\, F_{S_t}(ds)\, dt =: \mathsf{E}\tau_\ell - H(x)\end{aligned} \tag{4.17}$$

for $x \ge 0$ with $H(0) = 0$. Then $x \mapsto H(x)$ is strictly increasing (where strictly in $(0, \mathsf{E}\tau_\ell)$) and left-continuous on $(0, \alpha]$ with $\alpha = \sup[\mathrm{supp}(\ell)] \in [0, \infty]$ and $H(\alpha) = \mathsf{E}\tau_\ell$ if $\alpha = \infty$.

Setting $\mathrm{Im}(\mathsf{E}\tau_\ell - H) = \{\, \mathsf{E}\tau_\ell - H(x) \mid x \in [0,\alpha]\,\}$ we see that for any $c \in \mathrm{Im}(\mathsf{E}\tau_\ell - H) \cap (0, \mathsf{E}\tau_\ell)$ there exists $x_2^c \in [0,\alpha] \cap (0,\infty)$ such that $H(x_2^c) = \mathsf{E}\tau_\ell - c$. Let $p_2^c = F_\ell(x_2^c)$ and define $\lambda_2^c = p_2^c/(1-p_2^c)$ so that $p_2^c = 1/(1+1/\lambda_2^c)$ as needed. Then (4.13) holds with $\tau^2_{\lambda_2^c} = \tau_{\mathfrak{q}_p}$ where $p = p_2^c$ and hence by the result of Theorem 4.1 and (4.15) we see that this stopping time is optimal in (4.7). In this way we have derived the following result.

Theorem 4.3. *Under the hypotheses stated in the first paragraph of Section 3 above, let us assume that* ℓ *is non-negative.*

(I): *For* $c \in Im(G) \cap (0,\infty)$ *let* $p_1^c = F_\ell(G^{-1}(c))$. *Then the stopping time*

$$\tau_{\mathfrak{q}_{p_1^c}} = \inf\{\, t \geq 0 \mid X_t = \mathfrak{q}_{p_1^c} \,\} \tag{4.18}$$

is optimal in (4.6). *Moreover, any stopping time* τ_*^1 *satisfying* $\mathsf{P}(S_{\tau_*^1} \in [\mathfrak{q}_{p_1^c}, \mathfrak{Q}_{p_1^c}]) = 1$ *is optimal in* (4.6).

(II): *For* $c \in Im(\mathsf{E}\tau_\ell - H) \cap (0, \mathsf{E}\tau_\ell)$ *let* $p_2^c = F_\ell(H^{-1}(\mathsf{E}\tau_\ell - c))$. *Then the stopping time*

$$\tau_{\mathfrak{q}_{p_2^c}} = \inf\{\, t \geq 0 \mid X_t = \mathfrak{q}_{p_2^c} \,\} \tag{4.19}$$

is optimal in (4.7). *Moreover, any stopping time* τ_*^2 *satisfying* $\mathsf{P}(S_{\tau_*^2} \in [\mathfrak{q}_{p_2^c}, \mathfrak{Q}_{p_2^c}]) = 1$ *is optimal in* (4.7).

Note that if the distribution function F_{S_t} is continuous for $t > 0$ then each of the two sets $\mathrm{Im}(G)$ and $\mathrm{Im}(\mathsf{E}\tau_\ell - H)$ is connected (in the sense that it cannot be represented as the union of two or more disjoint open subsets).

4. Let us now consider the case when ℓ is real-valued. Set $Z_t = F_\ell(S_t) - F_\ell(I_t-)$ for $t \geq 0$ and let $\sigma_p = \inf\{\, t \geq 0 \mid Z_t \geq p \,\}$ be the first entry time of the process Z into $[p,\infty)$ for $p \in [0,1]$. Then $\tau^1_{\lambda_1^c}$ and $\tau^2_{\lambda_2^c}$ are equal to σ_p from (4.5) with $p = p_1^c$ and $p = p_2^c$ in $[0,1]$ to be found so that (4.12) and (4.13) hold respectively. To examine (4.12) let us note that

$$\begin{aligned} \mathsf{E}(\sigma_z - \tau_\ell)^+ &= \mathsf{E} \int_0^{\sigma_z} \big[F_\ell(S_t) - F_\ell(I_t-)\big]\, dt \\ &= \mathsf{E} \int_0^{\infty} \big[F_\ell(S_t) - F_\ell(I_t-)\big]\, I(t < \sigma_z)\, dt \\ &= \int_0^\infty \mathsf{E}\big[Z_t\, I(Z_t < z)\big]\, dt = \int_0^\infty \int_0^{z-} y\, F_{Z_t}(dy)\, dt =: K(z) \end{aligned} \tag{4.20}$$

for $z \geq 0$ where F_{Z_t} is the distribution function of Z_t and we have $K(0) = 0$. Then $z \mapsto K(z)$ is strictly increasing (where strictly positive) and left-continuous on $(0,1]$. Setting $\mathrm{Im}(K) = \{\, K(z) \mid z \in [0,1]\,\}$ we see that for any $c \in \mathrm{Im}(K) \cap (0,\infty)$ there exists $z_1^c \in (0,1]$ such that $K(z_1^c) = c$. Let $p_1^c = z_1^c$ and define $\lambda_1^c = (1-p_1^c)/p_1^c$ so that $p_1^c = 1/(1+\lambda_1^c)$ as needed. Then (4.12) holds with $\tau^1_{\lambda_1^c} = \sigma_{p_1^c}$ and hence by the result of Theorem 4.2 and (4.14) we see that this stopping time is optimal in (4.6). Similarly, to examine (4.13) let us note that

$$\begin{aligned} \mathsf{E}(\tau_\ell - \sigma_z)^+ &= \mathsf{E}\tau_\ell - \mathsf{E} \int_0^{\tau_z} \big(1 - [F_\ell(S_t) - F_\ell(I_t-)]\big)\, dt \\ &= \mathsf{E}\tau_\ell - \mathsf{E} \int_0^{\infty} \big(1 - [F_\ell(S_t) - F_\ell(I_t-)]\big)\, I(t < \sigma_z)\, dt \end{aligned} \tag{4.21}$$

$$= \mathsf{E}\tau_\ell - \int_0^\infty \mathsf{E}\big[(1-Z_t)\, I(Z_t < z)\big]\, dt$$
$$= \mathsf{E}\tau_\ell - \int_0^\infty \int_0^{z-} (1-y)\, F_{Z_t}(dy)\, dt =: \mathsf{E}\tau_\ell - L(z)$$

for $z \ge 0$ with $L(0) = 0$. Then $z \mapsto L(z)$ is strictly increasing (where strictly in $(0, \mathsf{E}\tau_\ell)$) and left-continuous on $(0,1]$. Setting $\mathrm{Im}(\mathsf{E}\tau_\ell - L) = \{\, \mathsf{E}\tau_\ell - L(z) \mid z \in [0,1] \,\}$ we see that for any $c \in \mathrm{Im}(\mathsf{E}\tau_\ell - L) \cap (0, \mathsf{E}\tau_\ell)$ there exists $z_2^c \in (0,1]$ such that $L(z_2^c) = \mathsf{E}\tau_\ell - c$. Let $p_2^c = z_2^c$ and define $\lambda_2^c = p_2^c/(1-p_2^c)$ so that $p_2^c = 1/(1+1/\lambda_2^c)$ as needed. Then (4.13) holds with $\tau^2_{\lambda_2^c} = \sigma_{p_2^c}$ and hence by the result of Theorem 4.2 and (4.15) we see that this stopping time is optimal in (4.7). In this way we have derived the following result.

Theorem 4.4. *Under the hypotheses stated in the first paragraph of Section 3 above, let us assume that* ℓ *is real-valued.*

(I): *For* $c \in Im(K) \cap (0,\infty)$ *let* $p_1^c = K^{-1}(c)$. *Then the stopping time*

$$\sigma_{p_1^c} = \inf\{\, t \ge 0 \mid F_\ell(S_t) - F_\ell(I_t-) \ge p_1^c \,\} \tag{4.22}$$

is optimal in (4.6). *Moreover, any stopping time* σ_*^1 *satisfying* $\mathsf{P}(\sigma_{p_1^c} \le \sigma_*^1 \le \sigma^{p_1^c}) = 1$ *is optimal in* (4.6) *where* $\sigma^{p_1^c} = \inf\{\, t \ge 0 \mid F_\ell(S_t) - F_\ell(I_t-) > p_1^c \,\}$.

(II): *For* $c \in Im(\mathsf{E}\tau_\ell - L) \cap (0, \mathsf{E}\tau_\ell)$ *let* $p_2^c = L^{-1}(\mathsf{E}\tau_\ell - c)$. *Then the stopping time*

$$\sigma_{p_2^c} = \inf\{\, t \ge 0 \mid F_\ell(S_t) - F_\ell(I_t-) \ge p_2^c \,\} \tag{4.23}$$

is optimal in (4.7). *Moreover, any stopping time* σ_*^2 *satisfying* $\mathsf{P}(\sigma_{p_2^c} \le \sigma_*^2 \le \sigma^{p_2^c}) = 1$ *is optimal in* (4.7) *where* $\sigma^{p_2^c} = \inf\{\, t \ge 0 \mid F_\ell(S_t) - F_\ell(I_t-) > p_2^c \,\}$.

Note that if the distribution function F_{Z_t} is continuous for $t > 0$ then each of the two sets $\mathrm{Im}(K)$ and $\mathrm{Im}(\mathsf{E}\tau_\ell - L)$ is connected. Note also that many results in this section can be extended to the cases where the process X is right-continuous.

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