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Maximal Inequalities of Kahane-Khintchine's Type in Orlicz Spaces

GORAN PESKIR

Several maximal inequalities of Kahane-Khintchine's type in certain Orlicz spaces are proved. The method relies upon Lévy's inequality and the technique established in [14] which is obtained by Haagerup-Young-Stechkin's best possible constants in the classical Khintchine inequalities. Moreover by using Donsker's invariance principle it is shown that the numerical constant in the inequality deduced by the presented method is near to be as optimal as possible: *If* $\{ \varepsilon_i \mid i \geq 1 \}$ *is a Bernoulli sequence, and* $\| \cdot \|_\psi$ *denotes the Orlicz norm induced by the function* $\psi(x) = e^{x^2} - 1$ *for* $x \in \mathbf{R}$ *, then the following inequality is satisfied:*

$$\| \max_{1 \leq j \leq n} | \sum_{i=1}^{j} a_i \varepsilon_i | \|_\psi \leq \sqrt{\frac{18}{5}} \cdot \Big(\sum_{i=1}^{n} | a_i |^2 \Big)^{1/2}$$

for all $a_1, \ldots, a_n \in \mathbf{R}$ *and all* $n \geq 1$ *, and the best possible numerical constant which can take the place of* $\sqrt{18/5}$ *belongs to the interval* $]\sqrt{8/3}, \sqrt{18/5}\,]$. Sharp estimates of that type are also deduced for some other maximal inequalities in Orlicz spaces which are discovered in this paper.

1. Introduction

Let $\{ \varepsilon_i \mid i \geq 1 \}$ be a Bernoulli sequence defined on a probability space $(\Omega, \mathcal{F}, P)$, let $\psi(x) = e^{x^2} - 1$ for $x \in \mathbf{R}$, and let $\| \cdot \|_\psi$ denote the gauge norm on $(\Omega, \mathcal{F}, P)$, that is:

$$\| X \|_\psi = \inf \{ a > 0 \mid E[\psi(X/a)] \leq 1 \}$$

whenever X is a real valued random variable on $(\Omega, \mathcal{F}, P)$, with $\inf \emptyset = \infty$. Then it is well-known that the following inequality is satisfied:

$$\| \sum_{i=1}^{n} a_i \varepsilon_i \|_\psi \leq C \cdot \Big(\sum_{i=1}^{n} | a_i |^2 \Big)^{1/2} \tag{1}$$

for all $a_1, \ldots, a_n \in \mathbf{R}$ and all $n \geq 1$, where C is a numerical constant. Moreover, it is recently shown in [14] that the best possible numerical constant which can take the place of C in (1) is equal to $\sqrt{8/3}$. Let us in addition consider real valued random variables $\xi_1, \ldots, \xi_n$ defined on $(\Omega, \mathcal{F}, P)$, and let $S_j = \sum_{i=1}^{j} \xi_i$ for $j = 1, \ldots, n$. Then Lévy's inequality may be formulated as follows, see [8]: If for every $1 \leq j < n$ the random vector $(\xi_1, \ldots, \xi_n)$ has the same distribution

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as the random vector $(\xi_1, \dots, \xi_j, -\xi_{j+1}, \dots, -\xi_n)$, then we have:

$$P\{ \max_{1 \le j \le n} | S_j | > t \} \le 2 \cdot P\{ | S_n | > t \} \tag{2}$$

for all $t \ge 0$. In particular, if $\xi_1, \dots, \xi_n$ are independent and symmetric, then (2) is valid. In other words, the maximum of a finite number of partial sums is stochastically controlled by the last partial sum. This principle is indeed well-known and is established in many different forms, see [8] for a unifiable general approach and [4] for some close related facts in the operator theory. Consequently, having (2) in mind one might very naturally guess that the following maximal inequality corresponding to (1) should be satisfied:

$$\| \max_{1 \le j \le n} | \sum_{i=1}^{j} a_i \varepsilon_i | \|_{\psi} \le D \cdot \Big(\sum_{i=1}^{n} | a_i |^2 \Big)^{1/2} \tag{3}$$

for all $a_1, \dots, a_n \in \mathbf{R}$ and all $n \ge 1$, where D is a numerical constant. Indeed, it is easily verified by using the integration by parts formula and the fact that $\sqrt{2}\ \psi'(x/\sqrt{2}) \le \psi'(x)$ for $x \ge 0$, that this inequality follows immediately from (1) and (2) with $D = \sqrt{2}\ C$. However, after a quick look on (3) it is not quite clear what is the best possible value for D . And this paper is devoted to the study of these questions. In addition, we shall also establish the related maximal inequalities involving some other Orlicz norms, which correspond to those with a single partial sum given in [14]. Our main aim is to find the sharp estimates for the best possible constants appearing in these inequalities and in that way to show that many of the deduced estimates themselves are near to be as optimal as possible. For instance, we shall prove (3) by establishing the estimate which will provide to deduce that the best possible numerical constant which can take the place of D in (3) belongs to the interval $]\sqrt{8/3}, \sqrt{18/5}\,]$. The method used in the proofs relies upon Lévy's inequality and the technique established in [14] which is obtained by Haagerup-Young-Stechkin's best possible constants in the classical Khintchine inequalities. The final conclusions on the best possible constants are provided by using Donsker's invariance principle. Moreover by using the classical symmetrization technique the given inequalities will be extended in an appropriate way from the Bernoulli case to the case of more general real valued random variables.

2. Preliminary facts

In this paper we work within the following Orlicz norms and spaces:

$$\| X \|_{\psi} = \inf \{ a > 0 \mid E[\psi(X/a)] \le 1 \}$$

$$L^{\psi}(P) = \{ X \in M(P) \mid \lim_{\varepsilon \downarrow 0} \| \varepsilon X \|_{\psi} = 0 \}$$

$$\| X \|_{T_\psi} = \inf \{ a > 0 \mid E[\psi(X/a)] \le a \}$$

$$L^{T_\psi}(P) = \{ X \in M(P) \mid \lim_{\varepsilon \downarrow 0} \| \varepsilon X \|_{T_\psi} = 0 \}$$

$$\| X \|_{\Upsilon_\psi} = E[\psi(X)]$$

$$L^{\Upsilon_\psi}(P) = \{ X \in M(P) \mid \lim_{\varepsilon \downarrow 0} \| \varepsilon X \|_{\Upsilon_\psi} = 0 \}$$

where $M(P)$ denotes the set of all real valued random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, and $\psi(x) = e^{x^2} - 1$ for $x \in \mathbf{R}$. Recall that the Orlicz space $(L^{\psi}(P)\ ,\ \|\cdot\|_{\psi})$ is called *the gauge space*, and the Orlicz norm $\|\cdot\|_{\psi}$ is called *the gauge norm*. We remark that the quantity $\| X \|_{T_{\psi}}$ has been emerged in the study [6]. Its interest relies upon the fact that for more general functions ψ , the map $\|\cdot\|_{\psi}$ need not to be an Frechet norm, but $\|\cdot\|_{T_{\psi}}$ is so. For more details see [6] (p.17,18). For more informations in this direction in general we shall refer the reader to [6], [14] and [16]. Let us in addition remind that a real valued random variable X defined on a probability space $(\Omega, \mathcal{F}, P)$ is said to be *subnormal*, if its Laplace transform L_X is dominated on the real line by the Laplace transform of some normally distributed random variable. In other words X is subnormal, if there exist $\mu \in \mathbf{R}$ and $\sigma^2 > 0$ such that:

(1) $$L_X(t) \ \leq \ \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2 \right)$$

for all $t \in \mathbf{R}$. If (1) is fulfilled with $\mu = 0$ and $\sigma^2 = 1$, then X is said to be a *standard subnormal* random variable. A sequence of (standard) subnormal random variables will be called a (standard) subnormal sequence. If X is a subnormal random variable satisfying (1), then using Markov's inequality one can easily obtain, see [14]:

(2) $$P\{\ X \geq t\ \} \leq \ \exp\Big(- \frac{(t-\mu)^2}{2\sigma^2} \Big)$$

for all $t \geq 0$. In particular, if X is subnormal and symmetric satisfying (1), then we get:

(3) $$P\{\ |X| \geq t\ \} \leq \ 2 \cdot \exp\Big(- \frac{(t-\mu)^2}{2\sigma^2} \Big)$$

for all $t \geq 0$. Inequalities (2) and (3) form a part of so-called *classical Kahane-Khintchine inequalities for subnormal random variables*, see [10] (p.62). A finite or infinite sequence of independent and identically distributed random variables $\varepsilon_1, \varepsilon_2, \dots$ taking values ± 1 with the same probability $1/2$ is called *a Bernoulli sequence*. Let $\{\ \varepsilon_i \ |\ i \geq 1\ \}$ be a Bernoulli sequence, and let $\{\ a_i \ |\ i \geq 1\ \}$ be a sequence of real numbers. Put $S_n = \sum_{i=1}^n a_i \varepsilon_i$ and $A_n = \sum_{i=1}^n |\, a_i\,|^2$ for $n \geq 1$, then we have, see [14]:

(4) $$\{\ \frac{S_n}{\sqrt{A_n}}\ \ |\ \ n \geq 1\ \}\ \text{ is a standard subnormal sequence.}$$

Let $\{\ \xi_i \ |\ i \geq 1\ \}$ be a sequence of independent and identically distributed random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ with mean 0 and variance $\sigma^2 > 0$. Let us put $S_j = \sum_{i=1}^j \xi_i$ for $j \geq 1$, and let us define a random function $X_n : \Omega \to (\, C[0,1]\ ,\ \|\cdot\|_{\infty})$ by:

$$X_n\,(t,\omega) \ = \ \frac{1}{\sigma\sqrt{n}}\, S_{\,[nt]}(\omega) \ + \ (nt - [nt]) \cdot \frac{1}{\sigma\sqrt{n}}\, \xi_{\,[nt]+1}(\omega)$$

for $t \in [0,1]$, $\omega \in \Omega$ and $n \geq 1$, where $[nt]$ denotes the integer part of nt . Then *Donsker's invariance principle* states, see [1] (p.68):

(5) $$X_n \overset{\sim}{\longrightarrow} W$$

as $n \to \infty$, where $W = \{ W_t \mid t \in [0,1] \}$ is the Wiener process. Since $x \mapsto \sup_{t\in[0,1]} |x(t)|$ is continuous on $C[0,1]$, then we have:

(6)
$$\sup_{t\in[0,1]} |X_n(t)| \xrightarrow{\sim} \sup_{t\in[0,1]} |W_t|$$

as $n \to \infty$. Hence we easily get:

(7)
$$\frac{1}{\sigma\sqrt{n}} |\max_{1\le j\le n} S_j| \xrightarrow{\sim} \sup_{t\in[0,1]} |W_t|$$

as $n \to \infty$. Moreover we have:

(8)
$$P\{ \sup_{t\in[0,1]} W_t \le x \} = \frac{2}{\sqrt{2\pi}} \int_0^x e^{-t^2/2}\, dt$$

for all $x \ge 0$, or in other words:

(9)
$$P\{ \sup_{t\in[0,1]} W_t \ge x \} = 2 \cdot P\{ N \ge x \} = P\{ |N| \ge x \}$$

for all $x \ge 0$, where $N \sim N(0,1)$ is a standard normal random variable. By the symmetry of W under reflection through zero hence we easily find:

(10)
$$P\{ \sup_{t\in[0,1]} |W_t| \ge x \} < 2 \cdot P\{ |N| \ge x \}$$

for all $x \ge 0$. These facts are well-known, see [1] (p.70-72), and their use will be essential in the final conclusions on the best possible constants in the sequel.

3. Maximal inequalities in the gauge space $L^\psi(P)$

In this section we prove a maximal inequality of Kahane-Khintchine's type in the gauge space $(L^\psi(P), \|\cdot\|_\psi)$, see (1) in theorem 3.1. The method relies upon Lévy's inequality and Haagerup-Young-Stechkin's best possible constants in the classical Khintchine inequalities, see [14]. By using Donsker's invariance principle we show that the constant appearing in the deduced inequality is near to be as optimal as possible, see corollary 3.4. Using the classical symmetrization technique we extend the given results from the Bernoulli case to more general cases, see theorem 3.6.

Theorem 3.1. ***(A maximal inequality in the gauge space)***

Let $\{ \varepsilon_i \mid i \ge 1 \}$ be a Bernoulli sequence defined on a probability space $(\Omega, \mathcal{F}, P)$, and let $\|\cdot\|_\psi$ denote the gauge norm on $(\Omega, \mathcal{F}, P)$. Then the following maximal inequality is satisfied:

(1)
$$\| \max_{1\le j\le n} |\sum_{i=1}^{j} a_i\varepsilon_i| \|_\psi \le \sqrt{\frac{18}{5}} \cdot \Big(\sum_{i=1}^{n} |a_i|^2 \Big)^{1/2}$$

for all $a_1, \dots, a_n \in \mathbf{R}$ and all $n \ge 1$.

Proof. Given $a_1, \dots, a_n \in \mathbf{R}$ for some $n \geq 1$, we denote $A_n = \sum_{i=1}^n |a_i|^2$ and $M_n = \max_{1 \leq j \leq n} | S_j |$ with $S_j = \sum_{i=1}^j a_i \varepsilon_i$ for $1 \leq j \leq n$. Then by the definition of the gauge norm $\| \cdot \|_\psi$ it is enough to establish the following inequality:

$$\int_\Omega \exp \Big(\frac{1}{C^2 A_n} \cdot (M_n)^2 \Big) \, dP \leq 2 \tag{2}$$

with $C = \sqrt{18/5}$. In order to obtain an appropriate estimate for the left side in (2) we shall expand the integrand into Taylor's series, and then we shall apply the classical Khintchine inequalities (2.11) with (2.17) in [14]. First note that by Lévy's inequality (1.2) we have:

$$\begin{aligned} E(M_n)^{2k} &= \int_0^\infty P\{ (M_n)^{2k} > t \} \, dt = \int_0^\infty P\{ M_n > t^{1/2k} \} \, dt \\ &\leq 2 \int_0^\infty P\{ |S_n| > t^{1/2k} \} \, dt = 2 \int_0^\infty P\{ (S_n)^{2k} > t \} \, dt = 2 \cdot E(S_n)^{2k} \end{aligned} \tag{3}$$

for all $k \geq 1$. By the classical Khintchine inequalities (2.11) with (2.17) in [14] we have:

$$E(S_n)^{2k} \leq K(2k, 2) \cdot (A_n)^k \tag{4}$$

with $K(2k, 2) = 2^k \cdot \Gamma(k + 1/2)/\sqrt{\pi}$ for $k \geq 1$. Since $\Gamma(k + 1/2) = (2k-1)\,!! \cdot \sqrt{\pi}/2^k$ where $(2k-1)\,!! = (2k-1) \cdot (2k-3) \cdot \ldots \cdot 3 \cdot 1$ for $k \geq 1$ and $|\, 2/C^2 | < 1$, then by (3) and (4) we may conclude:

$$\begin{aligned} \int_\Omega \exp \Big(\frac{1}{C^2 A_n} \cdot (M_n)^2 \Big) \, dP &= E \Big[\exp \Big(\frac{1}{C^2 A_n} \cdot (M_n)^2 \Big) \Big] = \\ &= \sum_{k=0}^\infty \frac{1}{k!} \cdot \frac{1}{(C^2 A_n)^k} \cdot E\,(M_n)^{2k} \leq 1 + 2 \cdot \sum_{k=1}^\infty \frac{1}{k!} \cdot \frac{1}{(C^2 A_n)^k} \cdot E\,(S_n)^{2k} \\ &\leq 1 + 2 \cdot \sum_{k=1}^\infty \frac{1}{k!} \cdot \frac{1}{(C^2 A_n)^k} \cdot \frac{2^k \cdot \Gamma(k+1/2)}{\sqrt{\pi}} \cdot (A_n)^k \\ &= 1 + \frac{2}{\sqrt{\pi}} \cdot \sum_{k=1}^\infty \frac{1}{k!} \cdot \Big(\frac{2}{C^2} \Big)^k \cdot \Gamma(k+1/2) \\ &= 1 + 2 \cdot \sum_{k=1}^\infty \frac{(2k-1)\,!!}{2^k \cdot k!} \cdot \Big(\frac{2}{C^2} \Big)^k = 2 \cdot \Big(1 - \frac{2}{C^2} \Big)^{-1/2} - 1 = 2 \end{aligned} \tag{5}$$

Thus (2) is satisfied and the proof is complete.

$\square$

In order to show that the upper bound appearing in (1) in theorem 3.1 is sharp, we shall first turn out two preliminary results which are also of interest in themselves.

Lemma 3.2.

Let $\{ X_i \mid i \geq 1 \}$ *be a sequence of independent identically distributed random variables* *defined on a probability space* $(\Omega, \mathcal{F}, P)$ *with finite variance* $\sigma^2 > 0$ *, let* $S_n = \sum_{i=1}^n X_i$ *and* $Z_n = (1/\sigma\sqrt{n})\,(S_n - ES_n)$ *, and let* $M_n = (1/\sigma\sqrt{n}) \max_{1 \le j \le n} |S_j - ES_j|$ *for* $n \ge 1$ *. Suppose that* $\{ Z_n \mid n \ge 1 \}$ *is symmetric standard subnormal sequence, that is,* Z_n *is symmetric and we have* $L_{Z_n}(t) \le \exp(t^2/2)$ *for all* $t \in \mathbf{R}$ *and all* $n \ge 1$ *. Then for every* $C > \sqrt{2}$ *the sequence of random variables* $\{ \exp(M_n/C)^2 \mid n \ge 1 \}$ *is uniformly integrable.*

Proof. It might be proved in the same way as lemma 3.2 in [14] by using Lévy's inequality (1.2) and Kahane-Khintchine's inequality for subnormal random variables (2.3). We shall leave the details to the reader.

□

Proposition 3.3.

Let $\{ X_i \mid i \ge 1 \}$ *be a sequence of independent identically distributed random variables defined on a probability space* $(\Omega, \mathcal{F}, P)$ *with finite variance* $\sigma^2 > 0$ *, let* $S_n = \sum_{i=1}^n X_i$ *and* $Z_n = (1/\sigma\sqrt{n})\,(S_n - ES_n)$ *for* $n \ge 1$ *, let* $W = \{ W_t \mid t \in [0,1] \}$ *be the Wiener process, and let* $\| \cdot \|_\psi$ *denote the gauge norm on* $(\Omega, \mathcal{F}, P)$ *. If* $\{ Z_n \mid n \ge 1 \}$ *is a symmetric standard subnormal sequence, then we have:*

(1)
$$\Big\| \max_{1 \le j \le n} \frac{1}{\sigma\sqrt{n}} \mid S_j - ES_j \mid \Big\|_\psi \longrightarrow \Big\| \sup_{t \in [0,1]} \mid W_t \mid \Big\|_\psi$$

as $n \to \infty$ *, where* $\sqrt{8/3} < \| \sup_{t \in [0,1]} \mid W_t \mid \|_\psi < \sqrt{18/5}$ *.*

Proof. Statement (1) might be proved in exactly the same way as statement (1) in proposition 3.3 in [14] by using (2.7) and lemma 3.2. We shall leave the details to the reader. Let us in addition denote $Z = \sup_{t \in [0,1]} |W_t|$, and put $C = \| Z \|_\psi$. We must show that $\sqrt{8/3} < C < \sqrt{18/5}$, see [1] (p.79-80). The first inequality follows easily by (2.9) and the fact that for a standard normal random variable $N \sim N(0,1)$ we have $\| N \|_\psi = \sqrt{8/3}$. For the second inequality put $D = \sqrt{18/5}$, then by (2.10) we have:

(2)
$$\begin{aligned} \int_\Omega \exp\Big(\frac{Z}{D}\Big)^2 dP &= \int_0^\infty P\{ \exp(Z/D)^2 > t \}\, dt \\ &= 1 + \int_1^\infty P\{ Z > D \cdot \sqrt{\log t} \}\, dt \\ &< 1 + 2\int_1^\infty P\{ |N| > D \cdot \sqrt{\log t} \}\, dt \\ &= 1 + 2\int_1^\infty P\{ \exp(N/D)^2 > t \}\, dt \\ &= 2\int_\Omega \exp\Big(\frac{Z}{D}\Big)^2 dP - 1 = 2 \cdot \Big(1 - \frac{2}{D^2} \Big)^{-1/2} - 1 = 2 . \end{aligned}$$

Hence $C < D$ follows easily by the definition of the gauge norm $\| \cdot \|_\psi$, and the proof is complete.

□

For results and problems related to those presented in lemma 3.2 and proposition 3.3 we shall refer the reader to [14], see problem 3.5.

Corollary 3.4.

The best possible numerical constant which can take the place of $\sqrt{18/5}$ *in inequality (1) in theorem 3.1 belongs to the interval* $]\sqrt{8/3}, \sqrt{18/5}]$. *Moreover the given constant is not less than* $\| \sup_{t \in [0,1]} \mid W_t \mid \|_\psi$, *where* $W = \{ W_t \mid t \in [0,1] \}$ *is the Wiener process.* (*According to the referee's remark, computer calculations, considering a simple random walk with* n *steps, show that we have* $\| M_{60} \|_\psi > 1.807$ *with* M_n *as in the proof of theorem 3.1 for* $n \geq 1$. *This gives a better lower bound than* $\sqrt{8/3} = 1.633$) .

Proof. Let C be such a constant, then obviously $C \leq \sqrt{18/5}$. Taking $a_1 = \ldots = a_n = 1/\sqrt{n}$ in inequality (1) in theorem 3.1 with $\sqrt{18/5}$ replaced by C we get:

$$\text{(1)} \qquad \| \max_{1 \leq j \leq n} \frac{1}{\sqrt{n}} \mid \sum_{i=1}^{j} \varepsilon_i \mid \|_\psi \leq C$$

being valid for all $n \geq 1$. According to (2.4) the sequence $\{ (1/\sqrt{n}) \sum_{i=1}^{n} \varepsilon_i \mid n \geq 1 \}$ is a symmetric standard subnormal sequence. Thus letting $n \to \infty$ in (1) we may easily complete the proof by using the result of proposition 3.3. $\square$

Conjecture 3.5.

The best possible numerical constant which can take the place of $\sqrt{18/5}$ *in inequality (1) in theorem 3.1 is equal to* $\| \sup_{t \in [0,1]} |W_t| \|_\psi$, *where* $W = \{ W_t \mid t \in [0,1] \}$ *is the Wiener process.*

Using the classical symmetrization technique we shall extend the result of theorem 3.1 from the Bernoulli case to the case of more general real valued random variables. This procedure has several steps and the final result may be stated as follows.

Theorem 3.6.

Let $\{ X_i \mid i \geq 1 \}$ *be a sequence of independent a.s. bounded real valued random variables defined on a probability space* $(\Omega, \mathcal{F}, P)$, *let* $\| \cdot \|_\psi$ *denote the gauge norm, and let* $\| \cdot \|_\infty$ *denote the usual sup-norm on* $(\Omega, \mathcal{F}, P)$. *Then for every* $\alpha > 0$ *and every* $n \geq 1$ *we have:*

$$\text{(1)} \qquad \| \max_{1 \leq j \leq n} \mid \sum_{i=1}^{j} (X_i - EX_i) \mid \|_\psi \leq C_n(\alpha) \cdot \Big(\sum_{i=1}^{n} \| X_i - EX_i \|_\infty^\alpha \Big)^{1/\alpha}$$

where $C_n(\alpha)$ *is given by:*

$$C_n(\alpha) = \begin{cases} \sqrt{72/5} & , \text{ if } 0 < \alpha \leq 2 \\ \sqrt{72/5} \cdot n^{\frac{1}{2} - \frac{1}{\alpha}} & , \text{ if } 2 < \alpha < \infty . \end{cases}$$

Moreover, if $X_1, X_2, \ldots$ *are symmetric, then for every* $\alpha > 0$ *and every* $n \geq 1$ *we have:*

$$\text{(2)} \qquad \| \max_{1 \leq j \leq n} \mid \sum_{i=1}^{j} X_i \mid \|_\psi \leq D_n(\alpha) \cdot \Big(\sum_{i=1}^{n} \| X_i \|_\infty^\alpha \Big)^{1/\alpha}$$

where $D_n(\alpha)$ *is given by:*

$$D_n(\alpha) \;=\; \begin{cases} \sqrt{18/5} & , \;\; \textit{if} \;\; 0 < \alpha \le 2 \\ \sqrt{18/5} \;\cdot\; n^{\,\frac{1}{2}-\frac{1}{\alpha}} & , \;\; \textit{if} \;\; 2 < \alpha < \infty \;. \end{cases}$$

Finally, if $\{ \, Y_i \mid i \ge 1 \, \}$ *is a sequence of independent and symmetric real valued random variables defined on* $(\Omega, \mathcal{F}, P)$ *, then we have:*

(3)
$$\Big\| \, \frac{1}{\Big(\sum\limits_{i=1}^{n} \mid Y_i \mid^2 \Big)^{1/2}} \cdot \max_{1 \le j \le n} \Big| \sum_{i=1}^{j} Y_i \Big| \, \Big\|_{\psi} \le \sqrt{\frac{18}{5}}$$

for all $n \ge 1$.

Proof. Inequality (2) for $\alpha = 2$ might be proved in exactly the same way as inequality (1) in theorem 3.6 in [14] by using theorem 3.1 and working with the function f from $\mathbf{R}^n \times \mathbf{R}^n$ into $\mathbf{R}$ defined by:

$$f(x_1, \ldots, x_n, \delta_1, \ldots, \delta_n) = \exp\Big(\frac{1}{C_0 \cdot \sum_{i=1}^{n} |\, x_i \,|^2} \cdot \max_{1 \le j \le n} \Big| \sum_{i=1}^{j} x_i \delta_i \Big|^2 \Big)$$

for $(x_1, \ldots, x_n, \delta_1, \ldots, \delta_n) \in \mathbf{R}^n \times \mathbf{R}^n$ with $C_0 = 18/5$. Moreover in the course of this proof inequality (3) could be also established in the same manner as in the proof of theorem 3.6 in [14]. We shall leave the details to the reader. Inequality (2) for $\alpha \ne 2$ follows easily from inequality (2) with $\alpha = 2$ by using inequalities (2.20) and (2.21) in [14]. Similarly, inequality (1) for $\alpha \ne 2$ follows easily from inequality (1) with $\alpha = 2$ by using exactly the same argument. Therefore to complete the proof it is enough to deduce inequality (1) with $\alpha = 2$. Let $n \ge 1$ be given and fixed, put $X = (X_1, \ldots, X_n)$, and let $Y = (Y_1, \ldots, Y_n)$ be a random vector such that X and Y are independent and identically distributed. There is no restriction to assume that both X and Y are defined on $(\Omega, \mathcal{F}, P)$, and that we have $EX_i = 0$ for $i = 1, \ldots, n$. Put $S_j = \sum_{i=1}^{j} X_i$ and $T_j = \sum_{i=1}^{j} Y_i$ for $j = 1, \ldots, n$, and define $M_n = \max_{1 \le j \le n} |\, S_j \,|$ and $\hat{M}_n = \max_{1 \le j \le n} |\, S_j - T_j \,|$. We shall begin the proof by verifying the following inequality:

(4)
$$\| \, M_n \, \|_{\psi} \le \| \, \hat{M}_n \, \|_{\psi} \;.$$

Put $C(n) = \| \, \hat{M}_n \, \|_{\psi}$, then (4) will be satisfied as far as we have the following inequality:

(5)
$$E\Big(\exp\Big(\frac{M_n}{C(n)} \Big)^2 \Big) \le \; 2 \;.$$

In order to establish (5) we shall define a function g from $\mathbf{R}^n \times \mathbf{R}^n$ into $\mathbf{R}$ by:

$$g(s_1, \ldots, s_n, t_1, \ldots, t_n) = \exp\Big(\frac{1}{C(n)} \cdot \max_{1 \le j \le n} |\, s_j - t_j \,| \Big)^2$$

for $(s_1, \ldots, s_n, t_1, \ldots, t_n) \in \mathbf{R}^n \times \mathbf{R}^n$. Then for any fixed $(s_1, \ldots, s_n) \in \mathbf{R}^n$ the function $(t_1, \ldots, t_n) \mapsto g(s_1, \ldots, s_n, t_1, \ldots, t_n)$ is obviously convex from $\mathbf{R}^n$ into $\mathbf{R}$. Furthermore by our assumptions we have $T_j \in L^1(P)$ with $ET_j = 0$ for $j = 1, \ldots, n$. Therefore by Fubini's theorem and Jensen's inequality we may obtain:

$$Eg(S_1,\dots,S_n,ET_1,\dots,ET_n) \le Eg(S_1,\dots,S_n,T_1,\dots,T_n) \ .$$

Since $ET_j = 0$ for $j = 1,\dots,n$, then by the definition of the Orlicz norm $\| \cdot \|_\psi$ we get:

$$\begin{aligned} E\Big(\exp\Big(\frac{M_n}{C(n)}\Big)^2\Big) &= Eg(S_1,\dots,S_n,ET_1,\dots,ET_n) \\ &\le Eg(S_1,\dots,S_n,T_1,\dots,T_n) = E\Big(\exp\Big(\frac{\hat{M}_n}{C(n)}\Big)^2\Big) \le 2 \ . \end{aligned}$$

This fact proves (5), and thus (4) follows. Since X and Y are independent and identically distributed, and $X_1,\dots,X_n$ are by assumption independent, then $X-Y = (X_1-Y_1,\dots,X_n-Y_n)$ is sign-symmetric. Therefore by (4) and inequality (2) with $\alpha = 2$ we may conclude:

$$\begin{aligned} \| M_n \|_\psi \le \| \hat{M}_n \|_\psi &= \Big\| \max_{1\le j\le n} \Big| \sum_{i=1}^{j} (X_i-Y_i) \Big| \Big\|_\psi \le \\ &\le \sqrt{18/5} \cdot \Big(\sum_{i=1}^{n} \| X_i-Y_i \|_\infty^2 \Big)^{1/2} \\ &\le \sqrt{18/5} \cdot \Big(\sum_{i=1}^{n} 2 \cdot (\| X_i \|_\infty^2 + \| Y_i \|_\infty^2) \Big)^{1/2} \\ &\le \sqrt{72/5} \cdot \Big(\sum_{i=1}^{n} \| X_i \|_\infty^2 \Big)^{1/2} \end{aligned}$$

These facts complete the proof. □

4. Maximal inequalities in the Orlicz space $L^{T_\psi}(P)$

In this section we prove several maximal inequalities of Kahane-Khintchine's type corresponding to those from the previous section but this time involving the Orlicz norm $\| \cdot \|_{T_\psi}$ as defined in section 2, see (1) in theorem 4.1, (1) in theorem 4.6, (1) in theorem 4.7, and (1)+(2) in theorem 4.9. The method relies upon the facts obtained in the previous section and the procedure that is established in [14] for similar questions concerned with single partial sums. The estimates appearing through the whole section are sharp and near to be as optimal as possible. Despite the fact that the Orlicz norm $\| \cdot \|_{T_\psi}$ is not homogeneous, see [14], we may begin by establishing the following analogue of inequality (1) in theorem 3.1 for this norm.

Theorem 4.1.

Let $\{ \varepsilon_i \mid i \ge 1 \}$ be a Bernoulli sequence defined on a probability space $(\Omega,\mathcal{F},P)$, and let $\| \cdot \|_{T_\psi}$ denote the Orlicz norm on $(\Omega,\mathcal{F},P)$ as defined in section 2. Then the following maximal inequality is satisfied:

(1) $$\Big\| \frac{1}{\Big(\sum_{i=1}^{n} | a_i |^2\Big)^{1/2}} \cdot \max_{1\le j\le n} \Big| \sum_{i=1}^{j} a_i \varepsilon_i \Big| \Big\|_{T_\psi} \le C$$

for all $a_1,\dots,a_n \in \mathbf{R}$ and all $n \ge 1$, where C is the unique root of the algebraic equation

$x^4 + 4x^3 - 2x^2 - 8x - 8 = 0$ *for* $x > \sqrt{2}$.

Proof. Given $a_1, \dots, a_n \in \mathbf{R}$ for some $n \geq 1$, we denote $A_n = \sum_{i=1}^n |a_i|^2$ and $M_n = \max_{1 \leq j \leq n} | S_j |$ with $S_j = \sum_{i=1}^j a_i \varepsilon_i$ for $1 \leq j \leq n$. Then by the definition of the Orlicz norm $\| \cdot \|_{T_\psi}$ it is enough to establish the following inequality:

$$\int_\Omega \exp \left(\frac{1}{C^2 A_n} \cdot (M_n)^2 \right) dP \leq 1 + C \ . \tag{2}$$

In order to deduce (2) we shall use the estimate for its left side that is established in the proof of theorem 3.1. Namely, by (5) in the proof of theorem 3.1 we have:

$$\int_\Omega \exp \left(\frac{1}{x^2 A_n} \cdot (M_n)^2 \right) dP \leq 2 \cdot \left(1 - \frac{2}{x^2} \right)^{-1/2} - 1 \tag{3}$$

for all $x > \sqrt{2}$. Put $\alpha(x) = 2\,(1-2/x^2)^{-1/2} - 1$ and $\beta(x) = 1 + x$ for $x > \sqrt{2}$, then one can easily verify that there exists a unique number $C > \sqrt{2}$ such that $\alpha > \beta$ on $]\sqrt{2}, C\,[$, $\alpha < \beta$ on $]\,C, \infty[$, and $\alpha(C) = \beta(C)$. The given C satisfies the following algebraic equation $C^4 + 4C^3 - 2C^2 - 8C - 8 = 0$. Hence (2) follows directly by (3). These facts complete the proof. $\square$

Remark 4.2.

We have seen in the last proof that the numerical constant C appearing in (1) in theorem 4.1 is the unique solution of the algebraic equation $x^4 + 4x^3 - 2x^2 - 8x - 8 = 0$ for $x > \sqrt{2}$. By the well-known criterion for rational solutions for algebraic equations with rational coefficients, see [21], each rational solution of the above equation belongs to the set $\{ \pm 1, \pm 2, \pm 4, \pm 8 \}$. Hence one can easily verify that the above equation has no rational solutions at all. Therefore the numerical constant C appearing in (1) in theorem 4.1 is not a rational number. However one can easily verify that we have $C = 1.683981945\ldots \approx 357/212$ with $357/212 - C = 0.000196809\ldots$. Thus inequality (1) in theorem 4.1 is satisfied with $C = 357/212$.

Proposition 4.3.

Let $\{ X_i \mid i \geq 1 \}$ *be a sequence of independent identically distributed random variables defined on a probability space* $(\Omega, \mathcal{F}, P)$ *with finite variance* $\sigma^2 > 0$ *, let* $S_n = \sum_{i=1}^n X_i$ *and* $Z_n = (1/\sigma\sqrt{n})\,(S_n - ES_n)$ *for* $n \geq 1$ *, let* $W = \{ W_t \mid t \in [0,1] \}$ *be the Wiener process, and let* $\| \cdot \|_{T_\psi}$ *denote the Orlicz norm on* $(\Omega, \mathcal{F}, P)$ *as defined in section 2. If* $\{ Z_n \mid n \geq 1 \}$ *is a symmetric standard subnormal sequence, then we have:*

$$\Big\| \max_{1 \leq j \leq n} \frac{1}{\sigma\sqrt{n}} \mid S_j - ES_j \mid \Big\|_{T_\psi} \longrightarrow \Big\| \sup_{t \in [0,1]} \mid W_t \mid \Big\|_{T_\psi} \tag{1}$$

as $n \to \infty$ *. Moreover, let* C_s *denote the numerical constant given by (2) in theorem 4.1 in [14], and let* C_m *denote the numerical constant appearing in (1) in theorem 4.1 above. Then we have:*

$$C_s < \| \sup_{t \in [0,1]} |W_t| \, \|_{T_\psi} < C_m \tag{2}$$

where $C_s = 1.538615763\ldots$ *, and* $C_m = 1.683981945\ldots$.

Proof. Statement (1) might be proved in exactly the same way as statement (1) in proposition 4.3 in [14] by using (2.7) and lemma 3.2. We shall leave the details to the reader. Let us in addition denote $Z = \sup_{t \in [0,1]} |W_t|$, and put $C = \| Z \|_{T_\psi}$. We must show that $C_s < C < C_m$, see [1] (p.79-80). The first inequality follows easily by (2.9) and the fact that $C_s = \| N \|_{T_\psi}$, where $N \sim N(0,1)$ is a standard normal random variable, see proposition 4.3 in [14]. For the second inequality note that by (2) in the proof of proposition 3.3 and the definition of C_m we have:

$$\int_\Omega \exp\Big(\frac{Z}{C_m}\Big)^2 dP \;<\; 2\cdot\Big(1-\frac{2}{C_m^2}\Big)^{-1/2} - 1 \;=\; 1 + C_m \;.$$

Hence $C < C_m$ follows easily by the definition of the Orlicz norm $\|\cdot\|_{T_\psi}$, and the proof is complete. □

Corollary 4.4.

The best possible numerical constant which can take the place of C in inequality (1) in theorem 4.1 belongs to the interval $]C_s, C_m]$ with C_s and C_m as in proposition 4.3. Moreover the given constant is not less than $\| \sup_{t\in[0,1]} |W_t| \|_{T_\psi}$, where $W = \{ W_t \mid t \in [0,1] \}$ is the Wiener process. (Again, according to the referee's remark, computer calculations show that the lower bound can be replaced by 1.625) .

Proof. It might be proved in exactly the same way as statement (1) in corollary 3.4 by using (2.4) and the result of proposition 4.3. We shall leave the details to the reader. □

Conjecture 4.5.

The best possible numerical constant which can take the place of C in inequality (1) in theorem 4.1 is equal to $\| \sup_{t\in[0,1]} |W_t| \|_{T_\psi}$, where $W = \{ W_t \mid t \in [0,1] \}$ is the Wiener process.

Using the classical symmetrization technique we shall extend the result of theorem 4.1 from the Bernoulli case to the case of general symmetric real valued random variables.

Theorem 4.6.

Let $\{ X_i \,|\, i \geq 1 \}$ be a sequence of independent symmetric real valued random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, and let $\|\cdot\|_{T_\psi}$ denote the Orlicz norm on $(\Omega, \mathcal{F}, P)$ as defined in section 2. Then the following maximal inequality is satisfied:

$$\text{(1)} \qquad \Big\| \frac{1}{\Big(\sum_{i=1}^n |X_i|^2\Big)^{1/2}} \cdot \max_{1\leq j\leq n} \Big|\sum_{i=1}^j X_i\Big| \Big\|_{T_\psi} \leq C$$

where C is the numerical constant appearing in (1) in theorem 4.1.

Proof. It might be proved by using theorem 4.1 in exactly the same way as it has been suggested for the proof of inequality (3) in the course of the proof of inequality (2) for $\alpha = 2$ in the beginning of the proof of theorem 3.6 with $C_0 = \sqrt{C}$. We shall leave the details to the reader. □

We shall continue our considerations by trying to move the expression $(\sum_{i=1}^n |a_i|^2)^{1/2}$ in inequality (1) in theorem 4.1 from the left side of that inequality to the right one, see (1) in theorem 3.1. Let $\{ \varepsilon_i \mid i \geq 1 \}$ be a Bernoulli sequence defined on a probability space $(\Omega, \mathcal{F}, P)$, and let $\| \cdot \|_{T_\psi}$ denote the Orlicz norm on $(\Omega, \mathcal{F}, P)$ as defined in section 2. Then by (4.2) in [14] and (1) in theorem 4.1 we have:

$$\text{(1)} \qquad \| \max_{1 \leq j \leq n} | \sum_{i=1}^{j} a_i \varepsilon_i | \|_{T_\psi} \leq C \cdot \Big(\sum_{i=1}^{n} |a_i|^2 \Big)^{1/2}$$

being valid for all $a_1, \dots, a_n \in \mathbf{R}$ and all $n \geq 1$ for which $\sum_{i=1}^n |a_i|^2 \geq 1$, where C is the numerical constant appearing in (1) in theorem 4.1. Moreover putting $a_1 = \dots = a_n = 1/n$ for $n \geq 1$ and using (2.7) one can easily verify that (1) does not hold in general. Note that in this case we have $\sum_{i=1}^n |a_i|^2 = 1/n \to 0$ for $n \to \infty$. However by (2.4) in [14] and (1) in theorem 3.1 we easily find:

$$\text{(2)} \qquad \| \max_{1 \leq j \leq n} | \sum_{i=1}^{j} a_i \varepsilon_i | \|_{T_\psi} \leq \sqrt{18/5} \cdot \Big[\Big(\sum_{i=1}^{n} |a_i|^2 \Big)^{1/2} \vee \Big(\sum_{i=1}^{n} |a_i|^2 \Big)^{1/4} \Big]$$

being valid for all $a_1, \dots, a_n \in \mathbf{R}$ and all $n \geq 1$. In particular, we have:

$$\text{(3)} \qquad \| \max_{1 \leq j \leq n} | \sum_{i=1}^{j} a_i \varepsilon_i | \|_{T_\psi} \leq \sqrt{18/5} \cdot \Big(\sum_{i=1}^{n} |a_i|^2 \Big)^{1/4}$$

for all $a_1, \dots, a_n \in \mathbf{R}$ and all $n \geq 1$ for which $\sum_{i=1}^n |a_i|^2 \leq 1$. Moreover, given $a_1, \dots, a_n \in \mathbf{R}$ for some $n \geq 1$, put $A_n = \sum_{i=1}^n |a_i|^2$ and $M_n = \max_{1 \leq j \leq n} | S_j |$ with $S_j = \sum_{i=1}^j a_i \varepsilon_i$ for $1 \leq j \leq n$. Then by (5) in the proof of theorem 3.1 we have:

$$\text{(4)} \qquad \int_\Omega \exp \Big(\frac{1}{C^2 \sqrt{A_n}} \cdot (M_n)^2 \Big) \, dP = \int_\Omega \exp \Big(\frac{\sqrt{A_n}}{C^2 A_n} \cdot (M_n)^2 \Big) \, dP$$
$$\leq 2 \cdot \Big(1 - \frac{2 \sqrt{A_n}}{C^2} \Big)^{-1/2} - 1$$

whenever $2\sqrt{A_n}/C^2 < 1$. Since $C > \sqrt{2}$, then the last inequality is valid in the case where $A_n \leq 1$. Moreover one can easily verify that we have:

$$2 \cdot x^3 + \frac{8}{C} \cdot x^2 + \Big(\frac{8}{C^2} - C^2 \Big) \cdot x - 4C \leq 0$$

for all $0 \leq x \leq 1$, and thus the following inequality is satisfied:

$$\text{(5)} \qquad 2 \cdot \Big(1 - \frac{2}{C^2} \, x \Big)^{-1/2} - 1 \leq 1 + C \cdot \sqrt{x}$$

for all $0 \leq x \leq 1$. By (4) and (5) we get:

$$\int_\Omega \exp \Big(\frac{1}{C^2 \sqrt{A_n}} \cdot (M_n)^2 \Big) \, dP \leq 1 + C \cdot \sqrt[4]{A_n} \, .$$

Hence by the definition of the Orlicz norm $\| \cdot \|_{T_\psi}$ we may deduce:

$$\text{(6)} \qquad \| \max_{1 \leq j \leq n} | \sum_{i=1}^{j} a_i \varepsilon_i | \|_{T_\psi} \leq C \cdot \Big(\sum_{i=1}^{n} |a_i|^2 \Big)^{1/4}$$

being valid for all $a_1, \dots, a_n \in \mathbf{R}$ and all $n \geq 1$ for which $\sum_{i=1}^n |a_i|^2 \leq 1$. Now by (1) and (6) we may conclude:

$$\| \max_{1 \leq j \leq n} | \sum_{i=1}^j a_i \varepsilon_i | \|_{T_\psi} \leq C \cdot \Big[\Big(\sum_{i=1}^n |a_i|^2 \Big)^{1/2} \vee \Big(\sum_{i=1}^n |a_i|^2 \Big)^{1/4} \Big] \tag{7}$$

being valid for all $a_1, \dots, a_n \in \mathbf{R}$ and all $n \geq 1$. Our next aim is to show that the exponent $1/4$ in inequality (7) may be replaced by the exponent $1/3$ in an optimal way, see section 4 in [14]. We proceed these considerations under the same hypotheses and notation as above. Suppose that $A_n \leq 1$, let $1 < p < \infty$ be given, and let q be the conjugate exponent of p , that is $1/p + 1/q = 1$. Then $2\,(A_n)^{1/q}/C^2 < 1$ and therefore by (5) in the proof of theorem 3.1 we have:

$$\begin{aligned} \int_\Omega \exp \Big(\frac{1}{C^2 (A_n)^{1/p}} \cdot (M_n)^2 \Big)\, dP &= \int_\Omega \exp \Big(\frac{(A_n)^{1/q}}{C^2 A_n} \cdot (M_n)^2 \Big)\, dP \\ &\leq 2 \cdot \Big(1 - \frac{2\,(A_n)^{1/q}}{C^2} \Big)^{-1/2} - 1 \ . \end{aligned} \tag{8}$$

Let us define:

$$q^* = \sup \Big\{ q \geq 2 \ \Big| \ \Big(1 - \frac{2}{C^2}\, x^{1/q} \Big)^{-1/2} \leq 1 + \frac{C}{2} \cdot x^{1/2 - 1/2q} \ , \forall x \in [0,1] \Big\}$$

and let p^* be the conjugate exponent of q^* , that is $1/p^* + 1/q^* = 1$. Then by (8), the definition of q^* , and the definition of the Orlicz norm $\| \cdot \|_{T_\psi}$ we may easily conclude:

$$\| M_n \|_{T_\psi} \leq C \cdot (A_n)^{1/2p^*} \ . \tag{9}$$

Furthermore it is easily verified that the inequality in the definition of q^* for $q = 4$ is equivalent to the following inequality:

$$x^6 - \frac{C^2}{2} \cdot x^4 + \frac{4}{C} \cdot x^3 - 2C \cdot x + \frac{4}{C^2} \leq 0$$

being valid for all $0 \leq x \leq 1$, which is obviously not satisfied. Thus $q^* < 4$. Moreover it is easily verified that the inequality in the definition of q^* for $q = 3$ is equivalent to the following easily checking inequality:

$$x^2 + \Big(\frac{4}{C} - \frac{C^2}{2} \Big) \cdot x + \frac{4}{C^2} - 2C \leq 0$$

being valid for all $0 \leq x \leq 1$. Therefore $3 \leq q^* < 4$. Finally it is easily verified that the inequality in the definition of q^* for $q = 3 + \varepsilon$ with $0 < \varepsilon < 1$ is equivalent to the following inequality:

$$x^{2+\varepsilon} - \frac{C^2}{2} \cdot x^{1+\varepsilon} + \frac{4}{C} \cdot x^{1+\varepsilon/2} - 2C \cdot x^{\varepsilon/2} + \frac{4}{C^2} \leq 0$$

for all $0 < x \leq 1$. However the left side of this inequality takes the value $4/C^2 > 0$ at $x = 0$ for every $0 < \varepsilon < 1$. Therefore $q^* = 3$, and by (9) we get:

$$\| M_n \|_{T_\psi} \leq C \cdot (A_n)^{1/3} \ . \tag{10}$$

In this way we have proved the following theorem.

Theorem 4.7.

Let $\{ \varepsilon_i \mid i \geq 1 \}$ *be a Bernoulli sequence defined on a probability space* $(\Omega, \mathcal{F}, P)$ *, and let* $\| \cdot \|_{T_\psi}$ *denote the Orlicz norm on* $(\Omega, \mathcal{F}, P)$ *as defined in section 2. Then the following maximal inequality is satisfied:*

$$\text{(1)} \qquad \| \max_{1 \leq j \leq n} | \sum_{i=1}^{j} a_i \varepsilon_i \, | \, \|_{T_\psi} \leq C \cdot \Big[\Big(\sum_{i=1}^{n} |a_i|^2 \Big)^{1/2} \vee \Big(\sum_{i=1}^{n} |a_i|^2 \Big)^{1/3} \Big]$$

for all $a_1, \ldots, a_n \in \mathbf{R}$ *and all* $n \geq 1$ *, where* C *is the numerical constant appearing in (1) in theorem 4.1.*

Proof. Straight forward by (1) and (10) above. □

Problem 4.8.

What is the best possible exponent that can take the place of $1/3$ *in inequality (1) in theorem 4.7?* Note that according to results deduced above we may conclude that this number belongs to the interval $[1/3, 1/2[$. For more details in this direction see problem 4.8 in [14].

Using the classical symmetrization technique we shall extend the result of theorem 4.7 from the Bernoulli case to the case of more general real valued random variables. Again this procedure has several steps and the final result may be stated as follows.

Theorem 4.9.

Let $\{ X_i \mid i \geq 1 \}$ *be a sequence of independent* $a.s.$ *bounded real valued random variables defined on a probability space* $(\Omega, \mathcal{F}, P)$ *, let* $\| \cdot \|_{T_\psi}$ *denote the Orlicz norm on* $(\Omega, \mathcal{F}, P)$ *as defined in section 2, let* $\| \cdot \|_\infty$ *denote the usual sup-norm on* $(\Omega, \mathcal{F}, P)$ *, and let* C *be the numerical constant appearing in (1) in theorem 4.1. Then for every* $\alpha > 0$ *and all* $n \geq 1$ *we have:*

$$\text{(1)} \qquad \| \max_{1 \leq j \leq n} | \sum_{i=1}^{j} (X_i - EX_i) \, | \, \|_{T_\psi} \leq C_n(\alpha) \cdot \Big[\Big(\sum_{i=1}^{n} \| X_i - EX_i \|_\infty^\alpha \Big)^{1/\alpha} \vee \Big(\sum_{i=1}^{n} \| X_i - EX_i \|_\infty^\alpha \Big)^{2/3\alpha} \Big]$$

where $C_n(\alpha)$ *is given by:*

$$C_n(\alpha) = \begin{cases} 2C & , \ \textit{if} \ 0 < \alpha \leq 2 \\ 2C \cdot n^{\frac{1}{2} - \frac{1}{\alpha}} & , \ \textit{if} \ 2 < \alpha < \infty \, . \end{cases}$$

Moreover, if $X_1, X_2, \ldots$ *are symmetric, then for every* $\alpha > 0$ *and all* $n \geq 1$ *we have:*

$$\text{(2)} \qquad \| \max_{1 \leq j \leq n} | \sum_{i=1}^{j} X_i \, | \, \|_{T_\psi} \leq D_n(\alpha) \cdot \Big[\Big(\sum_{i=1}^{n} \| X_i \|_\infty^\alpha \Big)^{1/\alpha} \vee \Big(\sum_{i=1}^{n} \| X_i \|_\infty^\alpha \Big)^{2/3\alpha} \Big]$$

where $D_n(\alpha)$ *is given by:*

$$D_n(\alpha) \;=\; \begin{cases} \quad C \quad , & \textit{if } \; 0 < \alpha \le 2 \\[2ex] C \cdot n^{\,\frac{1}{2} - \frac{1}{\alpha}} \; , & \textit{if } \; 2 < \alpha < \infty \; . \end{cases}$$

Proof. It might be proved in exactly the same way as inequalities (1) and (2) in theorem 3.6 by using theorem 4.7 and inequalities (2.20) and (2.21) in [14]. For this purpose the following inequality is turned out to be valid:

$$\| \, M_n \, \|_{T_\psi} \le \| \, \hat{M}_n \, \|_{T_\psi}$$

with M_n and $\hat{M}_n$ as in the proof of theorem 3.6. We shall leave the details to the reader.

□

5. Maximal inequalities in the Orlicz space $L^{\Upsilon_\psi}(P)$

This section consists of maximal inequalities involving the Orlicz norm $\| \cdot \|_{\Upsilon_\psi}$ as defined in section 2. The method relies upon the facts obtained in the previous two sections. The deduced estimates are sharp and near to be as optimal as possible.

Theorem 5.1.

Let $\{ \, \varepsilon_i \mid i \ge 1 \, \}$ *be a Bernoulli sequence defined on a probability space* $(\Omega, \mathcal{F}, P)$ *, and let* $\| \cdot \|_{\Upsilon_\psi}$ *denote the Orlicz norm on* $(\Omega, \mathcal{F}, P)$ *as defined in section 2. Then for every* $C > \sqrt{2}$ *the following maximal inequality is satisfied:*

(1) $$\Bigg\| \, \frac{1}{C \cdot \Big(\sum_{i=1}^{n} |a_i|^2 \Big)^{1/2}} \cdot \max_{1 \le j \le n} \Big| \sum_{i=1}^{j} a_i \varepsilon_i \Big| \, \Bigg\|_{\Upsilon_\psi} \le \; \frac{2\,C}{\sqrt{C^2 - 2}} \; - \; 2$$

for all $a_1, \dots, a_n \in \mathbf{R}$ *and all* $n \ge 1$ *.*

Proof. Given $a_1, \dots, a_n \in \mathbf{R}$ for some $n \ge 1$, we denote $A_n = \sum_{i=1}^{n} |a_i|^2$ and $M_n = \max_{1 \le j \le n} | \, S_j \, |$ with $S_j = \sum_{i=1}^{j} a_i \varepsilon_i$ for $1 \le j \le n$. Then by (5) in the proof of theorem 3.1 we have:

$$\begin{aligned} \Bigg\| \, \frac{1}{C \cdot \Big(\sum_{i=1}^{n} |a_i|^2 \Big)^{1/2}} \cdot \max_{1 \le j \le n} \Big| \sum_{i=1}^{n} a_i \varepsilon_i \Big| \, \Bigg\|_{\Upsilon_\psi} &= \int_\Omega \exp \Big(\, \frac{1}{C^2 A_n} \cdot (M_n)^2 \, \Big) \, dP - 1 \\ &\le 2 \cdot \Big(\, 1 - \frac{2}{C^2} \, \Big)^{-1/2} - \; 2 \; = \; \frac{2\,C}{\sqrt{C^2 - 2}} \; - \; 2 \end{aligned}$$

for all $C > \sqrt{2}$. This fact completes the proof. □

Theorem 5.2.

Let $\{ \, X_i \mid i \ge 1 \, \}$ *be a sequence of independent symmetric real valued random variables* *defined on a probability space* $(\Omega, \mathcal{F}, P)$ *, and let* $\| \cdot \|_{\Upsilon_\psi}$ *denote the Orlicz norm on* $(\Omega, \mathcal{F}, P)$ *as defined in section 2. Then for every* $C > \sqrt{2}$ *and all* $n \geq 1$ *the following maximal inequality is satisfied:*

(1)
$$\Big\| \frac{1}{C \cdot \Big(\sum_{i=1}^{n} |X_i|^2 \Big)^{1/2}} \cdot \max_{1 \leq j \leq n} \Big| \sum_{i=1}^{j} X_i \Big| \Big\|_{\Upsilon_\psi} \leq \frac{2C}{\sqrt{C^2 - 2}} - 2$$

Proof. It might be proved in exactly the same way as inequality (1) in theorem 5.2 in [14] by using theorem 5.1 and working with the function g defined by:

$$g(x, C) = E\Big\{ exp \Big[\frac{1}{C^2 \cdot \sum_{i=1}^{n} |x_i|^2} \cdot \max_{1 \leq j \leq n} \Big| \sum_{i=1}^{j} x_i \varepsilon_i \Big|^2 \Big] \Big\}$$

for $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ and $C > \sqrt{2}$, where $\varepsilon_1, \varepsilon_2 \ \ldots$ is a Bernoulli sequence. We shall leave the details to the reader. $\square$

Remark 5.3.

By (4.8) we may easily deduce the following "dual" estimate which extends the result of theorem 5.1:

(1)
$$\Big\| \frac{1}{C \cdot \Big(\sum_{i=1}^{n} |a_i|^2 \Big)^{1/2p}} \cdot \max_{1 \leq j \leq n} \Big| \sum_{i=1}^{j} a_i \varepsilon_i \Big| \Big\|_{\Upsilon_\psi} \leq 2 \Big(1 - \frac{2}{C^2} \cdot \Big(\sum_{i=1}^{n} |a_i|^2 \Big)^{1/q} \Big)^{-1/2} - 2$$

being valid for all $a_1, \ldots a_n \in \mathbf{R}$, all $n \geq 1$, and all $C > 0$ for which $\big(\sum_{i=1}^{n} |a_i|^2 \big)^{1/2} < \big(C/\sqrt{2} \big)^{p/p-1}$ with $p > 1$ and $1/p + 1/q = 1$. And as in the proof of theorem 5.2 one might be able to conclude that the following inequality extends inequality (1) in theorem 5.2: If $X_1, X_2, \ldots$ are independent symmetric $a.s.$ bounded real valued random variables, then we have:

(2)
$$\Big\| \frac{1}{C \cdot \Big(\sum_{i=1}^{n} |X_i|^2 \Big)^{1/2p}} \cdot \max_{1 \leq j \leq n} \Big| \sum_{i=1}^{j} X_i \Big| \Big\|_{\Upsilon_\psi}$$
$$\leq 2 \Big(1 - \frac{2}{C^2} \cdot \Big(\sum_{i=1}^{n} \| X_i \|_\infty^2 \Big)^{1/q} \Big)^{-1/2} - 2$$

for all $C > 0$ and all $n \geq 1$ for which $\big(\sum_{i=1}^{n} \| X_i \|_\infty^2 \big)^{1/2} < \big(C/\sqrt{2} \big)^{p/p-1}$ with $p > 1$ and $1/p + 1/q = 1$. The given estimates are sharp and near to be as optimal as possible.

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REFERENCES

[1] BILLINGSLEY, P. (1968). *Convergence of probability measures*. John Wiley & Sons, Inc., New York.

[2] EDGAR, G. A. *and* SUCHESTON, L. (1989). On maximal inequalities in Orlicz spaces. *Measure and Measurable Dynamics*, Proc. Conf. Rochester, *Contemp. Math*. 94 (113-129).

[3] GLUSKIN, E. D. PIETSCH, A. *and* PUHL, J. (1980). A generalization of Khintchine's inequality and its application in the theory of operator ideals. *Studia Math*. 67 (149-155).

[4] The continuity principle in exponential type Orlicz spaces.

[5] GRAVERSEN, S. E. PESKIR, G. *and* WEBER, M. (1992). The continuity principle in exponential type Orlicz spaces. *Math. Inst. Aarhus, Preprint Ser.* No. 32, (19 pp). *Nagoya Math. J.* 137, 1995 (55–75).

[6] HAAGERUP, U. (1978-1982). The best constants in the Khintchine inequality. *Operator algebras, ideals, and their applications in theoretical physics*, Proc. Int. Conf. Leipzig (69-79). *Studia Math.* 70 (231-283).

[7] HOFFMANN-JØRGENSEN, J. (1991). Function norms. *Math. Inst. Aarhus, Preprint Ser.* No. 40, (19 pp).

[8] HOFFMANN-JØRGENSEN, J. (1991). Inequalities for sums of random elements. *Math. Inst. Aarhus, Preprint Ser.* No. 41, (27 pp).

[9] HOFFMANN-JØRGENSEN, J. (1994). *Probability with a view toward statistics*. Chapman and Hall.

[10] JOHNSON, W. B. SCHECHTMAN, G. *and* ZINN, J. (1985). Best constants in moment inequalities for linear combinations of independent and exchangeable random variables. *Ann. Probab*. 13 (234-253).

[11] KAHANE, J. P. (1968-1985). *Some random series of functions*. D. C. Heath & Co. (first edition). Cambridge University Press (second edition).

[12] KOMOROWSKI, R. (1988). On the best possible constants in the Khintchine inequality for $p \geq 3$. *Bull. London Math. Soc*. 20 (73-75).

[13] KRASNOSEL'SKII, M. A. *and* RUTICKII, Ya. B. (1961). *Covex functions and Orlicz spaces*. P. Noordhoff, Ltd. Groningen.

[14] MARCUS, M. B. *and* PISIER, G. (1985). Stochastic processes with sample paths in exponential Orlicz spaces. *Proc. Probab. Banach Spaces* V, *Lecture Notes in Math*. 1153 (328-358).

[15] PESKIR, G. (1992). Best constants in Kahane-Khintchine inequalities in Orlicz spaces. *Math. Inst. Aarhus, Preprint Ser.* No. 10, (42 pp). *J. Multivariate Anal.* 45, 1993 (183-216).

[16] PICK, L. (1992). Two-weight weak type maximal inequalities in Orlicz classes. *Studia Math*. 100 (207-218).

[17] RAO, M. M. *and* REN, Z. D. (1991). *Theory of Orlicz spaces*. Marcel Dekker Inc., New York.

[18] RODIN, V. A. *and* SEMYONOV, E. M. (1975). Rademacher series in symmetric spaces. *Analysis Mathematica* 1 (207-222).

[19] SAWA, J. (1985). The best constant in the Khintchine inequality for complex Steinhaus variables, the case $p = 1$. *Studia Math*. 81 (107-126).

[20] STECHKIN, S. B. (1961). On the best lacunary systems of functions. *Izv. Akad. Nauk SSSR Ser. Mat*. 25 (in Russian) (357-366).

[21] SZAREK, S. J. (1978). On the best constant in the Khintchine inequality. *Studia Math*. 58 (197-208).

[22] TIGNOL, J. P. (1988). *Galois' theory of algebraic equations*. Institut de Mathématique Pure et Appliquée, UCL, Louvain-la-Neuve, Belgium.

[23] WEBER, M. (1983). Analyse infinitesimale de fonctions aleatoires. *Ecole d'Eté Probabilités de Saint-Flour* XI, *Lecture Notes in Math*. 976 (383-465).

[24] YOUNG, R. M. G. (1976). On the best possible constants in the Khintchine inequality. *J. London Math. Soc*. 14 (496-504).

Goran Peskir
Department of Mathematical Sciences
University of Aarhus, Denmark
Ny Munkegade, DK-8000 Aarhus
home.imf.au.dk/goran
goran@imf.au.dk