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Optimal Mean-Variance Selling Strategies

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Assuming that the stock price X follows a geometric Brownian motion with drift $\mu \in I\!R$ and volatility $\sigma > 0$, and letting P_x denote a probability measure under which X starts at $x > 0$, we study the dynamic version of the nonlinear mean-variance optimal stopping problem

$$\sup_{\tau} \Big[\mathsf{E}_{X_t}(X_\tau) - c\,\mathsf{V}ar_{X_t}(X_\tau) \Big]$$

where t runs from 0 onwards, the supremum is taken over stopping times τ of X, and $c > 0$ is a given and fixed constant. Using direct martingale arguments we first show that when $\mu \le 0$ it is optimal to stop at once and when $\mu \ge \sigma^2/2$ it is optimal not to stop at all. By employing the method of Lagrange multipliers we then show that the nonlinear problem for $0 < \mu < \sigma^2/2$ can be reduced to a family of linear problems. Solving the latter using a free-boundary approach we find that the optimal stopping time is given by

$$\tau_* = \inf\,\big\{\, t \ge 0 \mid X_t \ge \tfrac{\gamma}{c(1-\gamma)} \,\big\}$$

where $\gamma = \mu/(\sigma^2/2)$. The dynamic formulation of the problem and the method of solution are applied to the constrained problems of maximising/minimising the mean/variance subject to the upper/lower bound on the variance/mean from which the nonlinear problem above is obtained by optimising the Lagrangian itself.

1. Introduction

Imagine an investor who owns a stock which he wishes to sell so as to *maximise his return*. Optimal stopping problems of this kind have been considered in the literature (see e.g. [5] and the references therein). In the present paper we account for the additional feature of the investor wishing to *minimise his risk* upon selling. In line with the mean-variance analysis of Markowitz [10] we will identify the return with the expectation of the stock price and the risk with the variance of the stock price. The quadratic nonlinearity of the variance then moves the resulting optimal stopping problem outside the scope of the standard optimal stopping theory (see e.g. [13]) which may be viewed as linear programming in the sense of optimising linear functions subject to linear constraints (see [13, Remark 2.12]). Consequently the results and methods of the standard/linear optimal stopping theory are not directly applicable in this new/nonlinear setting. The purpose of the present paper is to develop a new methodology for solving nonlinear optimal stopping problems of this kind and demonstrate its use in the optimal mean-variance selling problem stated above.

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Assuming that the stock price X follows a geometric Brownian motion we first consider the constrained problem in which the investor aims to maximise the expectation of X_τ over all stopping times τ of X such that the variance of X_τ is bounded above by a positive constant. Similarly the investor could aim to minimise the variance of X_τ over all stopping times τ of X such that the expectation of X_τ is bounded below by a positive constant. A first application of Lagrange multipliers implies that the Lagrange function (Lagrangian) for either/both constrained problems can be expressed as a linear combination of the expectation of X_τ and the variance of X_τ with opposite signs. Optimisation of the Lagrangian over all stopping times τ of X thus yields the central optimal stopping problem under consideration. Due to the quadratic nonlinearity of the variance we can no longer apply standard/linear results of the optimal stopping theory to solve the problem.

Conditioning on the size of the expectation we show that a second application of Lagrange multipliers reduces the nonlinear optimal stopping problem to a family of linear optimal stopping problems. Solving the latter using a free-boundary approach we find that the optimal stopping boundary depends on the initial point of X in an essential way. This raises the question whether the optimality obtained is adequate for practical purposes. We refer to this optimality as the *static optimality* (Definition 1) to distinguish it from the *dynamic optimality* (Definition 2) in which each new position of the process X yields a new optimal stopping problem to be solved upon overruling all the past problems. This in effect corresponds to solving infinitely many optimal stopping problems dynamically in time with the aim of determining when it is optimal to stop (in the sense that no other stopping time could produce a more favourable value in the future). While the static optimality has been used in [11] as far as we know the dynamic optimality has not been considered in the nonlinear setting of optimal stopping before (see Section 4 below for a more detailed historical account).

We show that the dynamic formulation of the nonlinear optimal stopping problem admits a simple closed-form solution (Theorem 3) in which the optimal stopping boundary no longer depends on the initial point of the process X. Remarkably we also verify that this solution satisfies the principle of smooth fit (Remark 4) which is known to be a key variational principle for linear optimal stopping problems under natural conditions (see e.g. [13]). To our knowledge this is the first time that such a nonlinear phenomenon of 'dynamic smooth fit' has been observed in the literature. Solutions to the constrained problems are then derived using the solution to the unconstrained problem (Corollaries 5 and 7). These results are of both theoretical and practical interest. We also note that a dynamically optimal stopping strategy in the constrained problem can have a 'bang-bang' character (Remark 6).

The novel problems and methodology of the present paper suggest a number of avenues for further research. In addition to finite horizon formulations of the nonlinear optimal stopping problems we also aim to study optimal prediction problems and optimal stopping games in the nonlinear context. This also includes more general diffusion and Markov processes such as Lévy processes. It will be interesting to examine to what extent the results and methods laid down in the present paper remain valid in this greater generality.

2. Formulation of the problem

Let X be a geometric Brownian motion solving

$$dX_t = \mu X_t \, dt + \sigma X_t \, dB_t \tag{2.1}$$

with $X_0 = x$ for $x > 0$, where $\mu \in I\!R$ is the drift, $\sigma > 0$ is the volatility, and B is a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$. It is well known that the stochastic differential equation (2.1) has a unique strong solution given by

$$X_t^x = x \exp\big(\sigma B_t + (\mu - \tfrac{\sigma^2}{2})t\big) \tag{2.2}$$

for $t \geq 0$. The law of the process X^x on the canonical space will be denoted by P_x. Thus under P_x the coordinate process X starts at x. It is well known that X is a strong Markov process with respect to P_x for $x > 0$.

Consider the optimal stopping problem

$$V(x) = \sup_\tau \big[\mathsf{E}_x(X_\tau) - c\,\mathsf{V}ar_x(X_\tau)\big] \tag{2.3}$$

where the supremum is taken over all stopping times of X such that $\mathsf{E}_x(X_\tau^2) < \infty$ for $x > 0$ and $c > 0$ is a given and fixed constant. Stopping times of X refer to stopping times with respect to the natural filtration of X. All stopping times throughout are always assumed to be stopping times of X and finite valued unless stated differently.

Due to the quadratic nonlinearity of the second term in $\mathsf{V}ar_x(X_\tau) = \mathsf{E}_x(X_\tau^2) - (\mathsf{E}_x(X_\tau))^2$ it is evident that the problem (2.3) falls outside the scope of the standard/linear optimal stopping theory for Markov processes (see e.g. [13]). Moreover, we will see below that in addition to the static formulation of the nonlinear problem (2.3) where the maximisation takes place relative to the initial point x which is given and fixed, one is also naturally led to consider a dynamic formulation of the nonlinear problem (2.3) in which each new position of the process X yields a new optimal stopping problem to be solved upon overruling all the past problems. We believe that this dynamic optimality is of general interest in the nonlinear problems of optimal stopping as well as nonlinear problems of optimal stochastic control that we do not discuss here (see Section 4 below for a more detailed historical account).

The problem (2.3) seeks to maximise the investor's return identified with the expectation of X_τ and minimise the investor's risk identified with the variance of X_τ upon selling the stock at time τ. This identification is done in line with the mean-variance analysis of Markowitz [10]. Moreover, we will see in the proof below that the problem (2.3) is obtained by optimising the Lagrangian of the constrained problems

$$V_1(x) = \sup_{\tau\,:\,\mathsf{V}ar_x(X_\tau) \leq \alpha} \mathsf{E}_x(X_\tau) \tag{2.4}$$

$$V_2(x) = \inf_{\tau\,:\,\mathsf{E}_x(X_\tau) \geq \beta} \mathsf{V}ar_x(X_\tau) \tag{2.5}$$

respectively, where τ is a stopping time of X, and $\alpha > 0$ and $\beta > 0$ are given and fixed constants. Solving (2.3) we will therefore be able to solve (2.4) and (2.5) as well. Note that the constrained problems have transparent interpretations in terms of the investor's return and the investor's risk as discussed above.

We now formalise definitions of the optimalities alluded to above. Recall that all stopping times throughout refer to stopping times of X.

Definition 1 (Static optimality). A stopping time τ_* is *statically optimal* in (2.3) for $x > 0$ given and fixed, if there is no other stopping time σ such that

$$\mathsf{E}_x(X_\sigma) - c\,\mathsf{V}ar_x(X_\sigma) > \mathsf{E}_x(X_{\tau_*}) - c\,\mathsf{V}ar_x(X_{\tau_*})\,. \tag{2.6}$$

A stopping time τ_* is *statically optimal* in (2.4) for $x > 0$ given and fixed, if $\mathsf{V}ar_x(X_{\tau_*}) \le \alpha$ and there is no other stopping time σ satisfying $\mathsf{V}ar_x(X_\sigma) \le \alpha$ such that

$$\mathsf{E}_x(X_\sigma) > \mathsf{E}_x(X_{\tau_*})\,. \tag{2.7}$$

A stopping time τ_* is *statically optimal* in (2.5) for $x > 0$ given and fixed, if $\mathsf{E}_x(X_{\tau_*}) \ge \beta$ and there is no other stopping time σ satisfying $\mathsf{E}_x(X_\sigma) \ge \beta$ such that

$$\mathsf{V}ar_x(X_\sigma) < \mathsf{V}ar_x(X_{\tau_*})\,. \tag{2.8}$$

Clearly it is of interest to look for pointwise P_x-a.s. minimal stopping times that are statically optimal and we will tacitly take this requirement as part of the definition.

Note that the static optimality refers to the optimality relative to the initial point x which is given and fixed. Changing the initial point may yield a different optimal stopping time in the nonlinear problems since the statically optimal stopping time may and generally will depend on the initial point in an essential way (cf. [11]). This stands in sharp contrast with standard/linear problems of optimal stopping where the optimal stopping time equals the first entry time into the set where the value function equals the gain or loss function (under mild regularity conditions) and as such does not depend on the initial point explicitly (on the canonical probability space). This is a key difference between the static optimality of nonlinear problems of optimal stopping and the standard optimality of linear problems of optimal stopping (cf. [13]).

Definition 2 (Dynamic optimality). A stopping time τ_* is *dynamically optimal* in (2.3) if there is no other stopping time σ such that

$$\mathsf{P}_x\big(\mathsf{E}_{X_{\tau_*}}(X_\sigma) - c\,\mathsf{V}ar_{X_{\tau_*}}(X_\sigma) > X_{\tau_*}\big) > 0 \tag{2.9}$$

for some $x > 0$. A stopping time τ_* is *dynamically optimal* in (2.4) if there is no other stopping time σ such that

$$\mathsf{P}_x\big(\mathsf{V}ar_{X_{\tau_*}}(X_\sigma) \le \alpha\big) = 1 \quad \& \quad \mathsf{P}_x\big(\mathsf{E}_{X_{\tau_*}}(X_\sigma) > X_{\tau_*}\big) > 0 \tag{2.10}$$

for some $x > 0$. Clearly it is of interest to look for pointwise P_x-a.s. minimal stopping times that are dynamically optimal and we will tacitly take this requirement as part of the definition. For comments on dynamic optimality in (2.5) we refer to Remark 8 below.

Note that the dynamic optimality corresponds to solving infinitely many optimal stopping problems dynamically in time where each new position of the process X yields a new optimal stopping problem to be solved upon overruling all the past problems. The optimal decision at each time tells us either to stop (if no other stopping time from that time and position could do better) or to continue (if such a stopping time exists). While the static optimality remembers the past (through the initial point) the dynamic optimality completely ignores it and only looks ahead. Nonetheless it is clear that there is a strong link between the static and dynamic optimality (the latter being formed through the beginnings of the former) and this will be exploited in the proof below when searching for the dynamically optimal stopping times. In the case of standard/linear optimal stopping problems for Markov processes it is evident that the static and dynamic optimality coincide under mild regularity conditions due to the fact that the first entry time of the process into the stopping set is optimal (see e.g. [13]). This is not the case for the nonlinear problems of optimal stopping considered in the present paper as it will be seen below.

3. Solution to the problem

In this section we present solutions to the problems formulated in the previous section. We first focus on the unconstrained problem.

Theorem 3. *Consider the optimal stopping problem* (2.3) *where the process* X *solves* (2.1) *with* $X_0 = x$ *under* P_x *for* $x > 0$ *.*

(A) *If* $\mu \le 0$ *then it is both statically and dynamically optimal to stop at once. The value function in* (2.3) *is given by* $V(x) = x$ *for* $x > 0$ *.*

(B) *If* $\mu \ge \sigma^2/2$ *then it is both statically and dynamically optimal not to stop at all. The value function in* (2.3) *is given by* $V(x) = \infty$ *for* $x > 0$ *.*

(C) *If* $\mu \in (0, \sigma^2/2)$ *then the stopping time*

$$\tau_* = \inf \{ \, t \ge 0 \mid X_t \ge b(x) \, \} \tag{3.1}$$

is statically optimal for $x > 0$ *where* $b(x)$ *is the unique solution to*

$$\tfrac{1+\gamma}{\gamma} \, b(x) - 2x^{1-\gamma} \, b^\gamma(x) = \tfrac{1}{c} \tag{3.2}$$

in $(0, \infty)$ *with* $\gamma = \mu/(\sigma^2/2)$ *. The function* $x \mapsto b(x)$ *is strictly increasing on* $(0, \infty)$ *with* $b(0+) = \gamma/(c(1+\gamma))$ *(see Figure 1 below). The value function in* (2.3) *is given by*

$$V(x) = x^{1-\gamma} \, b^\gamma(x) \big(1 + c\big[x^{1-\gamma} \, b^\gamma(x) - b(x)\big]\big) \tag{3.3}$$

for $x \in (0, b(x)]$ *and* $V(x) = x$ *for* $x \ge b(x)$ *which holds if and only if* $x \ge \gamma/(c(1-\gamma))$ *.*

(D) *If* $\mu \in (0, \sigma^2/2)$ *then the stopping time*

$$\tau_* = \inf \Big\{ \, t \ge 0 \mid X_t \ge \tfrac{\gamma}{c(1-\gamma)} \, \Big\} \tag{3.4}$$

is dynamically optimal.

Proof. We assume throughout that the process X solves (2.1) with $X_0 = x$ under P_x and can be realised by (2.2) under P for $x > 0$.

(A): If $\mu \le 0$ then X is a positive supermartingale and hence by Fatou's lemma and the optional sampling theorem we see that $\mathsf{E}_x(X_\tau) = \mathsf{E}_x(\lim_{N\to\infty} X_{\tau\wedge N}) \le \liminf_{N\to\infty} \mathsf{E}_x(X_{\tau\wedge N}) \le x$ for all stopping times τ and $x > 0$. This shows that it is optimal to stop at once both in the static and dynamic sense with $V(x) = x$ for $x > 0$ as claimed.

(B): If $\mu \ge \sigma^2/2$ then $\limsup_{t\to\infty} X_t = \infty$ P_x-a.s. and hence the stopping time $\tau_N = \inf \{ \, t \ge 0 \mid X_t \ge N \, \}$ is finite valued P_x-a.s. for $N \ge x > 0$. It follows that $\mathsf{E}_x(X_{\tau_N}) = N$ and since likewise $\mathsf{V}ar_x(X_{\tau_N}) = 0$ we see that $V(x) \ge N$ for all $N \ge x > 0$. Letting $N \to \infty$ we get $V(x) = \infty$ for $x > 0$ implying also that it is optimal not to stop at all in both static and dynamic sense as claimed.

(C): Assume that $\mu \in (0, \sigma^2/2)$. Then $X_t \to 0$ P_x-a.s. but $\mathsf{E}_x(X_t) = x \, e^{\mu t} \to \infty$ as $t \to \infty$ for $x > 0$. Note that the objective function in (2.3) reads

$$\mathsf{E}_x(X_\tau) - c \, \mathsf{V}ar_x(X_\tau) = \mathsf{E}_x(X_\tau) + c\big(\mathsf{E}_x(X_\tau)\big)^2 - c \, \mathsf{E}_x(X_\tau^2) \tag{3.5}$$

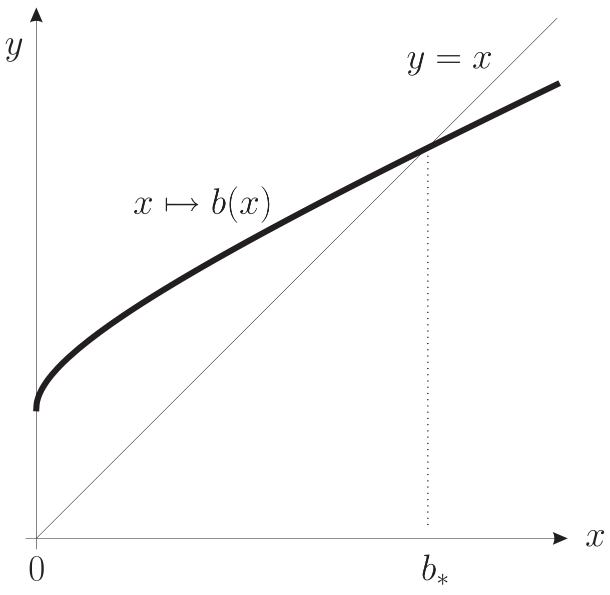

Figure 1. A computer drawing of the statically optimal boundary $x \mapsto b(x)$ from Theorem 3 for $\gamma = 1/2$ and $c = 1$. The dynamically optimal boundary $b_* = \gamma/(c(1-\gamma))$ equals 1 in this case.

where the key difficulty is the quadratic nonlinearity of the middle term on the right-hand side. To overcome this difficulty we will condition on the size of $\mathsf{E}_x(X_\tau)$. This yields

$$
\begin{aligned}
V(x) &= \sup_{M \geq 0} \sup_{\tau \,:\, \mathsf{E}_x(X_\tau) = M} \Big[\mathsf{E}_x(X_\tau) - c\,\mathsf{V}ar_x(X_\tau) \Big] \\
&= \sup_{M \geq 0} \sup_{\tau \,:\, \mathsf{E}_x(X_\tau) = M} \Big[\mathsf{E}_x(X_\tau) + c\big(\mathsf{E}_x(X_\tau)\big)^2 - c\,\mathsf{E}_x(X_\tau^2) \Big] \\
&= \sup_{M \geq 0} \Big[M + cM^2 - c \inf_{\tau \,:\, \mathsf{E}_x(X_\tau) = M} \mathsf{E}_x(X_\tau^2) \Big]
\end{aligned} \tag{3.6}
$$

for $x > 0$. Hence to solve (3.6) and thus (2.3) we need to solve the constrained problem

$$
V_M(x) = \inf_{\tau \,:\, \mathsf{E}_x(X_\tau) = M} \mathsf{E}_x(X_\tau^2) \tag{3.7}
$$

for $x > 0$ and $M > 0$ given and fixed where τ is a stopping time of X.

1. To tackle the problem (3.7) we will apply the method of Lagrange multipliers. For this, define the Lagrangian as follows

$$
L_x(\tau, \lambda) = \mathsf{E}_x(X_\tau^2) - \lambda \big[\mathsf{E}_x(X_\tau) - M \big] \tag{3.8}
$$

for $\lambda > 0$ and let τ_*^λ denote the optimal stopping time in the unconstrained problem

$$
L_x(\tau_*^\lambda, \lambda) := \inf_\tau L_x(\tau, \lambda) \tag{3.9}
$$

upon assuming that it exists. Suppose moreover that there exists $\lambda = \lambda(M,x) > 0$ such that

$$\mathsf{E}_x\big(X_{\tau_*^\lambda}\big) = M\,. \tag{3.10}$$

It then follows from (3.8)-(3.10) that

$$\mathsf{E}_x(X^2_{\tau_*^\lambda}) = L_x(\tau_*^\lambda,\lambda) \le \mathsf{E}_x(X^2_\tau) \tag{3.11}$$

for all stopping times τ such that $\mathsf{E}_x(X_\tau) = M$. This shows that τ_*^λ satisfying (3.9) and (3.10) with $\lambda = \lambda(M,x)$ is optimal in (3.7).

2. To tackle the problem (3.9) with (3.10) we consider the optimal stopping problem

$$V_\lambda(x) = \inf_\tau \mathsf{E}_x\big(X^2_\tau - \lambda X_\tau\big) \tag{3.12}$$

for $x > 0$. Note that the gain function $x \mapsto G_\lambda(x) = x^2 - \lambda x$ in (3.12) is convex on $(0,\infty)$, satisfies $G_\lambda(0) = 0$ and $G_\lambda(\infty) = \infty$, and attains its unique minimum $-\lambda^2/4$ at $\lambda/2$. Since $X_t \to 0$ P_x-a.s. as $t \to \infty$ it is evident that $\tau_*^\lambda := \inf\{\, t \ge 0 \mid X_t \le \lambda/2 \,\}$ is an optimal stopping time in (3.12) for $x \ge \lambda/2$ and $V_\lambda(x) = -\lambda^2/4$ for $x \ge \lambda/2$. Due to the specific structure of G_λ and X we then conjecture that it is optimal to stop at once in (3.12) for $x \in [b, \lambda/2]$ where $b \in (0, \lambda/2)$ is the optimal stopping point for X starting at $x \in (0,b)$ to be determined. This leads to the following free-boundary problem:

$$\mathbb{L}_X V_\lambda = 0 \quad \text{in } (0,b) \tag{3.13}$$

$$V_\lambda(0+) = 0 \tag{3.14}$$

$$V_\lambda(b) = G_\lambda(b) = b^2 - \lambda b \quad \text{(instantaneous stopping)} \tag{3.15}$$

$$V_\lambda'(b) = G_\lambda'(b) = 2b - \lambda \quad \text{(smooth fit)} \tag{3.16}$$

where $\mathbb{L}_X = \mu\, x\, d/dx + (\sigma^2/2)\, x^2\, d^2/dx^2$ is the infinitesimal generator of X.

From (3.13) we find that $V_\lambda(x) = c_1 + c_2 x^{1-\gamma}$ where $\gamma = \mu/(\sigma^2/2) \in (0,1)$. From (3.14) we see that $c_1 = 0$ and from (3.15) we find that $c_2 = b^{1+\gamma} - \lambda b^\gamma$ so that

$$V_\lambda(x) = \big(b^{1+\gamma} - \lambda b^\gamma\big)\, x^{1-\gamma} \tag{3.17}$$

for $x \in (0,b]$. Finally from (3.16) we find that

$$b = \frac{\lambda\gamma}{1+\gamma}\,. \tag{3.18}$$

It is then possible to verify that the stopping time

$$\tau_*^\lambda := \tau_b = \inf\{\, t \ge 0 \mid X_t \ge b \,\} \tag{3.19}$$

is optimal in (3.12) for $x \in (0,b]$ with (3.17) and (3.18) being satisfied as conjectured. This also yields that $[b, \lambda/2]$ is the optimal stopping set in (3.12) with $V_\lambda(x) = G_\lambda(x) = x^2 - \lambda x$ for $x \in [b, \lambda/2]$ as claimed.

Indeed, using that V_λ defined in this way is C^2 everywhere but at b and $\lambda/2$ where it is C^1, we see that the Itô-Tanaka formula reduces to the Itô formula which yields

$$V_\lambda(X_t) = V_\lambda(x) + \int_0^t I\!\!L_X V_\lambda(X_s)\, ds + M_t \tag{3.20}$$

for $x > 0$ where $M_t = \sigma \int_0^t X_s V_\lambda'(X_s)\, dB_s$ is a continuous local martingale for $t \geq 0$ (it is also easily seen that M is a true martingale). Note that $I\!\!L_X G_\lambda(x) = \mu x G_\lambda'(x) + (\sigma^2/2) x^2 G_\lambda''(x) = \mu x\,(2x-\lambda) + \sigma^2 x^2 = (\sigma^2+2\mu)x^2 - \lambda \mu x \geq 0$ if and only if $x \geq \lambda\mu/(\sigma^2+2\mu) = \lambda\gamma/(2(1+\gamma))$. Since $b = \lambda\gamma/(1+\gamma) \geq \lambda\gamma/(2(1+\gamma))$ it follows that $I\!\!L_X V_\lambda(x) = I\!\!L_X G_\lambda(x) \geq 0$ for $x \in (b, \lambda/2)$. Combining this with $I\!\!L_x V_\lambda = 0$ for $x \in (0,b) \cup (\lambda/2, \infty)$ we see that $I\!\!L_X V_\lambda \geq 0$ on $(0,\infty) \setminus \{b, \lambda/2\}$. Since the time that X spends at b or $\lambda/2$ has Lebesgue measure zero, and $G_\lambda \geq V_\lambda$ on $(0,\infty)$, we see from (3.20) using the optional sampling theorem that

$$\mathsf{E}_x[G_\lambda(X_{\tau\wedge\sigma_n})] \geq V_\lambda(x) \tag{3.21}$$

for $x > 0$ where τ is any stopping time of X such that $\mathsf{E}_x(X_\tau^2) < \infty$ and σ_n is a localisation sequence of stopping times for M with $n \geq 1$. Letting $n \to \infty$ in (3.21) and using Fatou's lemma we see upon recalling (3.12) that

$$\inf_\tau \mathsf{E}_x[G_\lambda(X_\tau)] \geq V_\lambda(x) \tag{3.22}$$

for $x > 0$. Taking $\tau_*^\lambda := \inf\{\, t \geq 0 \mid X_t \leq \lambda/2 \,\}$ for $x \geq \lambda/2$ and $\tau_*^\lambda = \inf\{\, t \geq 0 \mid X_t \geq b \,\}$ for $x \in (0, \lambda/2)$ we see from the arguments above that equality in (3.22) is attained. This establishes the optimality claims made following (3.19) above.

3. Having solved the problem (3.9) we still need to meet the condition (3.10). For this, recalling Doob's identity $\mathsf{P}\big(\sup_{t\geq 0}(B_t - \alpha t) \geq \beta\big) = \mathrm{e}^{-2\alpha\beta}$ for $\alpha, \beta > 0$ (cf. [4]) we find that

$$\mathsf{P}_x(\tau_b < \infty) = \mathsf{P}_x\Big(\sup_{t\geq 0} X_t \geq b\Big) = \Big(\frac{x}{b}\Big)^{1-\gamma} \tag{3.23}$$

for $x \in (0, b]$. Recalling that $X_t \to 0$ P_x-a.s. as $t \to \infty$ hence it follows that

$$\mathsf{E}_x(X_{\tau_b}) = b\, \mathsf{P}_x(\tau_b < \infty) + 0\, \mathsf{P}_x(\tau_b = \infty) = b\Big(\frac{x}{b}\Big)^{1-\gamma} = b^\gamma x^{1-\gamma} \tag{3.24}$$

$$\mathsf{E}_x(X_{\tau_b}^2) = b^2\, \mathsf{P}_x(\tau_b < \infty) + 0\, \mathsf{P}_x(\tau_b = \infty) = b^2\Big(\frac{x}{b}\Big)^{1-\gamma} = b^{1+\gamma} x^{1-\gamma} \tag{3.25}$$

for $x \in (0, b]$. To realise (3.10) we need to identify (3.24) with M. This yields

$$M = b^\gamma x^{1-\gamma} \tag{3.26}$$

and hence we find the closed-form expression

$$b = M^{1/\gamma} x^{1-1/\gamma}\,. \tag{3.27}$$

for $x \in (0, b]$. It follows from (3.18) and (3.27) that we need to set

$$\lambda = \tfrac{1+\gamma}{\gamma}\, b = \tfrac{1+\gamma}{\gamma}\, M^{1/\gamma} x^{1-1/\gamma} \tag{3.28}$$

for $x \in (0, b]$. Inserting (3.27) into (3.25) and recalling (3.11) we see that (3.7) is given by

$$V_M(x) = b^{1+\gamma} x^{1-\gamma} = \big(M^{1/\gamma} x^{1-1/\gamma}\big)^{1+\gamma} x^{1-\gamma} = M^{1+1/\gamma} x^{1-1/\gamma} \tag{3.29}$$

for $x \in (0, b]$. Note from (3.26) that $x \le b$ if and only if $M \ge x$ in this case of λ . If $M < x$ then setting $\lambda = 2M$ and recalling that $\tau_*^\lambda := \inf\{\, t \ge 0 \mid X_t \le \lambda/2\,\}$ is optimal in (3.12) for $x > \lambda/2 = M$, we see by (3.10) and (3.11) that (3.7) is given by $V_M(x) = M^2$. Combining this with (3.29) we see that $V_M(x) = M^{1+1/\gamma} x^{1-1/\gamma}$ for $M \ge x$ and $V_M(x) = M^2$ for $M < x$. Inserting this into (3.6) it follows that

$$\begin{aligned} V(x) &= \sup_{M \ge x} \Big[M + cM^2 - c\, x^{1-1/\gamma} M^{1+1/\gamma} \Big] \vee \sup_{M<x} M \\ &= \sup_{M \ge x} \Big[M + cM^2 - c\, x^{1-1/\gamma} M^{1+1/\gamma} \Big] \end{aligned} \tag{3.30}$$

for all $x > 0$ where the second equality holds since $\sup_{M<x} M = x$ and we can take $M = x$ in the first supremum above.

Setting $F(M) := M + cM^2 - cx^{1-1/\gamma} M^{1+1/\gamma}$ for $M \ge 0$ with $x > 0$ given and fixed, and recalling that $\gamma \in (0,1)$, we see that the power $1+1/\gamma$ is strictly larger than 2 . Thus after starting at 0 and exhibiting a strict increase initially we see that $F(M)$ converges to $-\infty$ as M moves from 0 to ∞ . It follows therefore that the function $M \mapsto F(M)$ attains its maximum on $[0, \infty)$ at some $M > 0$. To find this M we can solve $F'(M) = 0$ which reads

$$1 + 2cM - c\big(1 + \tfrac{1}{\gamma}\big) x^{1-1/\gamma} M^{1/\gamma} = 0\,. \tag{3.31}$$

Differentiating further in (3.31) with respect to M we moreover find that the maximum point M of F is unique. From (3.31) we also see that $F'(x) \ge 0$ if and only if $x \le \gamma/(c(1-\gamma))$ which in turn is satisfied if and only if $M \ge x$ as needed. Recalling (3.26) we see that (3.31) can be rewritten as follows

$$1 + 2cx^{1-\gamma} b^\gamma - c\big(1 + \tfrac{1}{\gamma}\big) b = 0 \tag{3.32}$$

and this equation is equivalent to (3.2) above. Combined with the optimality of (3.19) in (3.7) these facts show that the stopping time (3.1) is statically optimal in (2.3) for $x \le b(x)$ or equivalently $x \le \gamma/(c(1-\gamma))$ where $b(x)$ solves uniquely (3.2) as claimed. If $x > b(x)$ or equivalently $x > \gamma/(c(1-\gamma))$ then $F'(x) < 0$ so that the final supremum in (3.30) is attained at $M = x$ with $V(x) = x$. This shows that it is optimal to stop at once in (2.3) so that the stopping time (3.1) is statically optimal in (2.3) for all $x > 0$ as claimed.

Multiplying both sides of (3.2) by $b^{-\gamma}(x)$ and differentiating with respect to x we find that $b'(x) > 0$ for $x > 0$ and hence we see that $x \mapsto b(x)$ is strictly increasing on $(0, \infty)$ with $b(0+) = \gamma/(c(1+\gamma))$ as claimed. Inserting (3.26) into F on the right-hand side of (3.30) it is easily verified that we get (3.3) for $x \in (0, b(x)]$ while we also have $V(x) = x$ for $x \ge b(x)$ since in this case it is optimal to stop at once as shown above. The final equivalence claim following (3.3) has been established above and will also be verified below.

(D): We know that $x \mapsto b(x)$ is strictly increasing on $(0, \infty)$ with $b(0+) = \gamma/(c(1+\gamma)) > 0$. Solving the equation $b(x) = x$ we find from (3.2) that $x_* = \gamma/(c(1-\gamma))$ is the unique solution

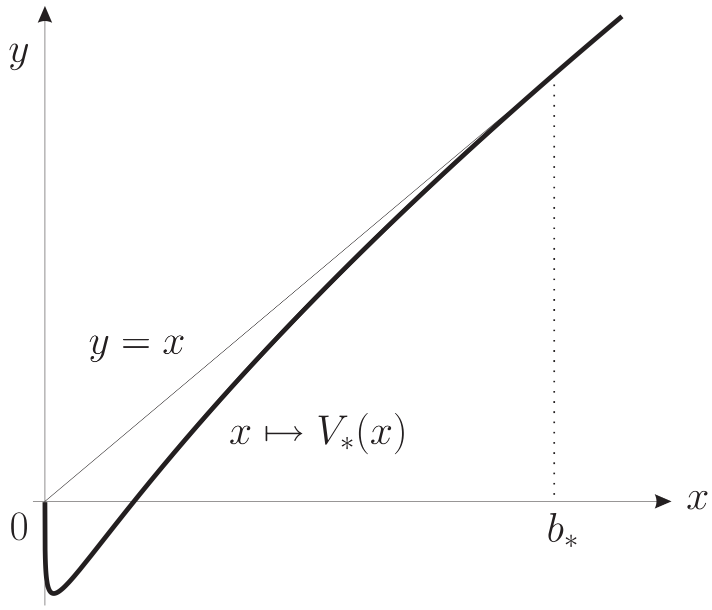

Figure 2. A computer drawing of the value function $x \mapsto V_*(x)$ from (3.34) when $\gamma = 7/10$ and $c = 1$. A smooth fit holds at the dynamically optimal boundary $b_* = \gamma/(c(1-\gamma))$ which equals $7/3$ in this case.

in $(0, \infty)$. Multiplying both sides of (3.2) by $b^{-\gamma}(x)$ and differentiating with respect to x, it is easily verified that

$$b'(x_*) = \frac{2\gamma}{1+2\gamma} < 1 \,. \tag{3.33}$$

This shows that $b(x) > x$ for $x \in (0, x_*)$ and $b(x) < x$ for $x > x_*$ with $b(x_*) = x_*$. It follows therefore that it is statically optimal to stop at once for a given and fixed $x > 0$ if and only if $x \geq x_* = \gamma/(c(1+\gamma))$. This shows that the stopping time (3.4) is dynamically optimal in (2.3) and the proof is complete. □

Remark 4 (Dynamic smooth-fit principle). From (3.3) we read that the static value associated with the dynamically optimal stopping time τ_* from (3.4) is given by

$$V_*(x) = \mathsf{E}_x(X_{\tau_*}) - c\,\mathsf{V}ar_x(X_{\tau_*}) = x^{1-\gamma}\, b_*^{\gamma}\big(1 + c\big[x^{1-\gamma}\, b_*^{\gamma} - b_*\big]\big) \tag{3.34}$$

for $x \in (0, b_*]$ and $V_*(x) = x$ for $x \geq b_*$ where we set $b_* = \gamma/(c(1-\gamma))$. It is interesting to observe that (i) $x \mapsto V_*(x)$ is increasing on $(0, b_*)$ with $V_*(0+) = 0$ and $V_*(x) > x$ for $x \in (0, b_*)$ when $\gamma \in (0, 1/2)$; (ii) $V_*(x) = x$ for $x \in (0, b_*)$ when $\gamma = 1/2$; and (iii) $x \mapsto V_*(x)$ is initially decreasing and negative and then increasing and positive on $(0, b_*)$ with $V_*(0+) = 0$ and $V_*(x) < x$ for $x \in (0, b_*)$ when $\gamma \in (1/2, 1)$ (see Figure 2 above).

Note that the fact that $V_*(x) < x$ for $x \in (0, b_*)$ when $\gamma \in (1/2, 1)$ does not contradict the dynamic optimality of τ_* from (3.4) since stopping at any such x at once would produce a larger value in the static sense relative to P_x, however, it would also violate the fact that continuing at x could produce an even larger value if applying a stopping time that is statically optimal relative to P_x (note that the latter stopping time exists since $x < b(x)$ when $x < b_*$).

From (3.34) we easily deduce the remarkable fact that

$$V_*'(b_*) = 1 \quad \text{(smooth fit).} \tag{3.35}$$

It means that the static value function associated with the dynamically optimal stopping time satisfies the smooth fit principle in the nonlinear problem (2.3). To our knowledge this is the first time that such a nonlinear phenomenon of 'dynamic smooth fit' has been observed in the literature.

We now turn to the constrained problems. Note in the proofs below that the unconstrained problem above is obtained by optimising the Lagrangian of the constrained problems.

Corollary 5. *Consider the optimal stopping problem* (2.4) *where the process* X *solves* (2.1) *with* $X_0 = x$ *under* P_x *for* $x > 0$ *.*

(A) *If* $\mu \le 0$ *then it is both statically and dynamically optimal to stop at once. The value function in* (2.4) *is given by* $V_1(x) = x$ *for* $x > 0$ *.*

(B) *If* $\mu \ge \sigma^2/2$ *then it is both statically and dynamically optimal not to stop at all. The value function in* (2.4) *is given by* $V_1(x) = \infty$ *for* $x > 0$ *.*

(C) *If* $\mu \in (0, \sigma^2/2)$ *then the stopping time*

$$\tau_* = \inf \{ \, t \ge 0 \mid X_t \ge b_\alpha(x) \, \} \tag{3.36}$$

is statically optimal for $x > 0$ *where* $b_\alpha(x)$ *is the unique solution to*

$$x^{1-\gamma} \, b_\alpha^{1+\gamma}(x) - x^{2(1-\gamma)} \, b_\alpha^{2\gamma}(x) = \alpha \tag{3.37}$$

in $(0, \infty)$ *with* $\gamma = \mu/(\sigma^2/2)$ *. The function* $x \mapsto b_\alpha(x)$ *satisfies* $b_\alpha(0+) = \infty$ *and* $b_\alpha(x) > x$ *for* $x > 0$ *. The value function in* (2.4) *is given by*

$$V_1(x) = x^{1-\gamma} \, b_\alpha^\gamma(x) \tag{3.38}$$

for $x \in (0, b_\alpha(x)]$ *and* $V(x) = x$ *for* $x \ge b_\alpha(x)$ *.*

(D) *If* $\mu \in (0, \sigma^2/2)$ *then it is dynamically optimal not to stop at all.*

Proof. We assume throughout that the process X solves (2.1) with $X_0 = x$ under P_x and can be realised by (2.2) under P for $x > 0$.

(A): If $\mu \le 0$ then by part (A) in the proof of Theorem 3 we know that it is optimal to stop at once in the unconstrained problem. It follows therefore that the same conclusion holds for the constrained problem.

(B): If $\mu \ge \sigma^2/2$ then by part (B) in the proof of Theorem 3 we know that the stopping times $\tau_N = \inf \{ \, t \ge 0 \mid X_t \ge N \, \}$ yield the infinite value as $N \to \infty$ implying also that it is optimal not to stop at all in the unconstrained problem. Since these stopping times yield zero variance it follows therefore that the same conclusions hold for the constrained problem.

(C): Assume that $\mu \in (0, \sigma^2/2)$. Note that we can think of (3.5) as (the essential part of) the Lagrangian for the constrained problem (2.4) defined by

$$L_x(\tau, c) = \mathsf{E}_x(X_\tau) - c\big[\mathsf{V}ar_x(X_\tau) - \alpha\big] \tag{3.39}$$

for $x > 0$ and $c > 0$. By the result of Theorem 3 we know that the stopping time τ_*^c given in (3.1) is optimal in the unconstrained problem

$$L_x(\tau_*^c, c) := \sup_\tau L_x(\tau, c) \tag{3.40}$$

for $x > 0$ and $c > 0$. Suppose moreover that there exists $c = c(\alpha, x) > 0$ such that

$$\mathsf{V}ar_x\big(X_{\tau_*^c}\big) = \alpha \tag{3.41}$$

for $x > 0$. It then follows that

$$\mathsf{E}_x(X_{\tau_*^c}) = L_x(\tau_*^c, c) \geq \mathsf{E}_x\big(X_\tau\big) - c\big[\mathsf{V}ar_x(X_\tau) - \alpha\big] \geq \mathsf{E}_x(X_\tau) \tag{3.42}$$

for all stopping times τ such that $\mathsf{V}ar_x(X_\tau) \leq \alpha$ with $x > 0$. This shows that the stopping time τ_*^c from (3.1) satisfying (3.41) with $c = c(\alpha, x)$ is statically optimal in (2.4) for $x > 0$.

To realise (3.41) recall from (3.24) and (3.25) that

$$\mathsf{V}ar_x(X_{\tau_b}) = \mathsf{E}_x(X_{\tau_b}^2) - \big(\mathsf{E}_x(X_{\tau_b})\big)^2 = b^{1+\gamma} x^{1-\gamma} - b^{2\gamma} x^{2(1-\gamma)} \tag{3.43}$$

for $x > 0$ given and fixed. Setting this expression equal to α yields

$$b^{1+\gamma} x^{1-\gamma} - b^{2\gamma} x^{2(1-\gamma)} = \alpha\,. \tag{3.44}$$

This can be rewritten as $(b^{2\gamma})^p x^{1-\gamma} - b^{2\gamma} x^{2(1-\gamma)} = \alpha$ where $p = (1+\gamma)/2\gamma > 1$. Setting $F(b) = (b^{2\gamma})^p x^{1-\gamma} - b^{2\gamma} x^{2(1-\gamma)}$ for $b \geq 0$ with $x > 0$ given and fixed, we see that F is convex on $[0, \infty)$ with $F(0) = 0$, and after exhibiting a strict decrease initially and reaching its unique minimum point, F exhibits a strict increase afterwards with $F(x) = 0$ and $F(\infty) = \infty$. It follows therefore that there exists a unique point $b = b_\alpha(x)$ such that $F(b) = \alpha$. This shows that (3.44) is satisfied and hence (3.41) holds too with τ_*^c from (3.1) where $b(x) = b_\alpha(x)$. Note that the arguments above show that $b_\alpha(x) > x$ for all $x > 0$. Recalling further that $b(x)$ needs to solve (3.2) for the optimality established in Theorem 3 to be applicable, we see that $c = c(\alpha, x) > 0$ forming the right-hand side of (3.2) is uniquely determined. Noting that (3.37) is the same as (3.44) and invoking the optimality established in Theorem 3, we can conclude that the stopping (3.36) is statically optimal in (2.4) for $x > 0$ as claimed. Finally, it is evident from (3.37) that $b_\alpha(0+) = \infty$ while (3.38) follows from (3.24) above.

(D): We have established above that $b_\alpha(x) > x$ for all $x > 0$. This implies that it is dynamically optimal not to stop at all. Indeed, for every $x > 0$ given and fixed the stopping time (3.36) produces strictly more than stopping at once and this shows that x cannot be a dynamically optimal stopping point. This completes the proof. □

Remark 6 (Dynamic bang-bang strategy). Note that the dynamically optimal stopping strategy in the constrained problem (2.4) of Corollary 5 exhibits a 'bang-bang' character at

$\mu = 0$ just as the optimal stopping strategy in the same unconstrained problem (cf. [5, Remark 1]). Note that this is not the case for the statically optimal stopping strategy in the constrained problem (2.4) of Corollary 5.

Corollary 7. *Consider the optimal stopping problem* (2.5) *where the process* X *solves* (2.1) *with* $X_0 = x$ *under* P_x *for* $x > 0$ *.*

(A) *If* $\mu \le 0$ *then it is statically optimal to stop at once for* $x \ge \beta$ *. The value function in* (2.5) *is given by* $V_2(x) = 0$ *for* $x \ge \beta$ *. If* $x \in (0, \beta)$ *then the problem has no solution in the static sense.*

(B) *If* $\mu \ge \sigma^2/2$ *then the stopping time*

$$\tau_* = \inf \{ \, t \ge 0 \mid X_t \ge \beta \, \} \tag{3.45}$$

is statically optimal. The value function in (2.5) *is given by* $V_2(x) = 0$ *for* $x > 0$ *.*

(C) *If* $\mu \in (0, \sigma^2/2)$ *then it is statically optimal to stop at once for* $x \ge \beta$ *. If* $x \in (0, \beta)$ *then the stopping time*

$$\tau_* = \inf \{ \, t \ge 0 \mid X_t \ge \beta^{1/\gamma} x^{1-1/\gamma} \, \} \tag{3.46}$$

is statically optimal. The value function in (2.5) *is given by*

$$V_2(x) = \beta^{1+1/\gamma} x^{1-1/\gamma} - \beta^2 \tag{3.47}$$

for $x \in (0, \beta]$ and $V_2(x) = 0$ for $x \ge \beta$. Note that $V_2(0+) = \infty$.

Proof. We assume throughout that the process X solves (2.1) with $X_0 = x$ under P_x and can be realised by (2.2) under P for $x > 0$.

(A): If $\mu \le 0$ and $x \ge \beta$ then stopping at once we meet the constraint and obtain zero variance so that this stopping time is statically optimal and $V_2(x) = 0$ as claimed. If $x \in (0, \beta)$ then by the supermartingale argument of part (B) in the proof of Theorem 3 we know that $\mathsf{E}_x(X_\tau) \le x < \beta$ for every stopping time τ . This shows that the constraint in the problem (2.5) cannot be met and hence the problem has no solution in the static sense as claimed.

(B): If $\mu \ge \sigma^2/2$ then by part (B) in the proof of Theorem 3 we know that the stopping time τ_* from (3.45) satisfies $\mathsf{E}_x(X_{\tau_*}) = \beta$ for $x \in (0, \beta)$ and $\mathsf{E}_x(X_{\tau_*}) = x$ for $x \ge \beta$ with $\mathsf{V}ar_x(X_{\tau_*}) = 0$ in both cases. This shows that τ_* is statically optimal in (2.5) and $V_2(x) = 0$ for $x > 0$ as claimed.

(C): Assume that $\mu \in (0, \sigma^2/2)$. Note that the Lagrangian for the constrained problem (2.5) is defined by the following relation

$$L_x(\tau, c) = \mathsf{V}ar_x(X_\tau) - c\big[\mathsf{E}_x(X_\tau) - \beta\big] \tag{3.48}$$

for $x > 0$ and $c > 0$. To connect to the results of Theorem 3 observe that

$$\inf_\tau \big(\mathsf{V}ar_x(X_\tau) - c\big[\mathsf{E}_x(X_\tau) - \beta\big]\big) = -c \sup_\tau \big(\mathsf{E}_x(X_\tau) - \tfrac{1}{c}\mathsf{V}ar_x(X_\tau)\big) + c\beta \tag{3.49}$$

which shows that the stopping time $\tau_*^{1/c}$ given in (3.1) is optimal in the unconstrained problem

$$L_x(\tau_*^{1/c}, c) := \inf_\tau L_x(\tau, c) \tag{3.50}$$

for $x > 0$ and $c > 0$. Suppose moreover that there exists $c = c(\beta, x) > 0$ such that

$$\mathsf{E}_x\big(X_{\tau_*^{1/c}}\big) = \beta \tag{3.51}$$

for $x > 0$. It then follows that

$$\mathsf{V}ar_x\big(X_{\tau_*^{1/c}}\big) = L_x(\tau_*^{1/c}, c) \le \mathsf{V}ar_x(X_\tau) - c\big[\mathsf{E}_x(X_\tau) - \beta\big] \le \mathsf{V}ar_x(X_\tau) \tag{3.52}$$

for all stopping times τ such that $\mathsf{E}_x(X_\tau) \ge \beta$ with $x > 0$. This shows that the stopping time $\tau_*^{1/c}$ from (3.1) satisfying (3.51) with $c = c(\beta, x)$ is statically optimal for $x > 0$.

To realise (3.51) recall from (3.24) that

$$\mathsf{E}_x(X_{\tau_b}) = b^\gamma x^{1-\gamma} \tag{3.53}$$

for $x > 0$ given and fixed. Setting this expression equal to β yields

$$b = b_\beta(x) := \beta^{1/\gamma} x^{1-1/\gamma}. \tag{3.54}$$

It follows that (3.51) holds with $\tau_*^{1/c}$ from (3.1) with $b(x) = b_\beta(x)$. Recalling further that $b(x)$ needs to solve (3.2) with $1/c$ in place of c for the optimality established in Theorem 3 to be applicable, we easily find using (3.54) that $c = c(\beta, x) > 0$ forming the reciprocal of the right-hand side in (3.2) is given by

$$c = \tfrac{1+\gamma}{\gamma} \beta^{1/\gamma} x^{1-1/\gamma} - 2\beta. \tag{3.55}$$

Recalling that $\gamma \in (0,1)$ it is easily verified that $c > 0$ for $x \le \beta$. Invoking the optimality established in Theorem 3 we can conclude that the stopping time (3.46) is statically optimal in (2.5) for $x \in (0, \beta]$. If $x > \beta$ then the static optimality of stopping at once is evident directly from (2.5). Note that the latter conclusion agrees with (3.46) for $x = \beta$.

To calculate the value function from (2.5) we can make use of (3.24) and (3.25). Using (3.43) combined with (3.54) this yields

$$\begin{aligned} V_2(x) = \mathsf{V}ar_x(X_{\tau_b}) &= \big(\beta^{1/\gamma} x^{1-1/\gamma}\big)^{1+\gamma} x^{1-\gamma} - \big(\beta^{1/\gamma} x^{1-1/\gamma}\big)^{2\gamma} x^{2(1-\gamma)} \\ &= \beta^{1+1/\gamma} x^{1-1/\gamma} - \beta^2 \end{aligned} \tag{3.56}$$

for $x \in (0, \beta]$ as claimed in (3.47) above. This completes the proof. □

Remark 8. Note that dynamic optimality in the problem (2.5) is trivial since the set of admissible stopping points equals $[\beta, \infty)$ and when the process X is stopped at once after starting at any of these points we see that the resulting variance of its position is zero and hence cannot be made any smaller by continuing beyond the initial point. Thus in (2.5) it is dynamically optimal to wait until the process X enters $[\beta, \infty)$ and stop it then (or not at all). This is consistent with the results on static optimality derived in Corollary 7 above.

4. Static vs dynamic optimality

In this section we address the rationale for introducing the static and dynamic optimality in the nonlinear optimal stopping problems under consideration and explain their relevance for applications of both theoretical and practical interest. We also discuss relation of these results with the existing approaches to similar problems in the literature.

1. To simplify the exposition we focus on the unconstrained problem (2.3) and similar arguments apply to the constrained problems (2.4) and (2.5) as well. Recall that (2.3) represents the optimal selling problem for an investor who owns a stock which he wishes to sell so as to *maximise his return* (identified with the expectation of the stock price) and *minimise his risk* (identified with the variance of the stock price) upon selling. Due to the quadratic *non-linearity* of the variance (as a function of the expectation) the optimal selling strategy (3.1) depends on the initial stock price x through its boundary $b(x)$ in an essential way (we only focus on the most interesting case when $\mu \in (0, \sigma^2/2)$ and exclude other cases). This *spatial inconsistency* (not present in the standard/linear optimal stopping problems) introduces the *time inconsistency* in the problem because the stock price moves from the initial value x in t units of time to a new value y (different from x with probability one) which in turn yields a new optimal selling boundary $b(y)$ that is different from the initial boundary $b(x)$. This time inconsistency repeats itself between any two points in time and the investor may be in doubt which optimal selling boundary to use unless already made up his mind. To tackle these inconsistencies we are naturally led to consider two types of investors and consequently introduce the two notions of optimality as stated in Definitions 1 and 2 respectively. The first investor is a *static investor* who stays 'pre-committed' to the optimal selling strategy evaluated initially and does not re-evaluate the optimality criterion (2.3) at later times. This investor will determine the optimal selling strategy at time zero and follow it blindly to the prescribed time of sale. The second investor is a *dynamic investor* who remains 'non-committed' to the optimal selling strategy evaluated initially as well as subsequently and continuously re-evaluates the optimality criterion (2.3) at each new time. This investor will determine the optimal selling strategy at time zero and continue doing so at each new time until the optimality criterion tells him to stop and sell. Clearly both the static investor and the dynamic investor embody realistic economic behaviour (see below for a more detailed account coming from economics) and Theorem 3 discloses their optimal selling strategies in the unconstrained problem (2.3). Similarly Corollary 5 and Corollary 7 disclose their optimal selling strategies in the constrained problems (2.4) and (2.5). Given that the financial interpretations of these results are easy to draw directly and somewhat lengthy to state explicitly we will omit further details. It needs to be noted that although closely related the three problems (2.3)-(2.5) are still different and hence it is to be expected that their solutions are also different for some values of the parameters. Difference between the static and dynamic optimality is best understood by analysing each problem on its own first as in this case the complexity of the overall comparison is greatly reduced.

2. Apart from the paper [11] where the static optimality was used in a nonlinear problem, we are not aware of any other paper on *optimal stopping* where nonlinear problems of the same kind were studied. In particular the dynamic optimality (Definition 2) and the dynamic smooth-fit principle (Remark 4) appear to be original to the present paper. There are two streams of papers on *optimal control* however, one in the economics literature dating back to the paper

by Strotz[1] [17], and one in the finance literature dating back to the paper by Richardson [15], where closely related problems were considered. Although these problems belong to the field of optimal control (where the observed process is being affected/defined by the observer through the optimal control) so that a direct comparison with the problems of optimal stopping (where the observer does not affect the observed process) is not possible, we present a brief review of these papers to highlight similarities/differences and indicate the applicability of the present methodology in these settings.

3. The stream of papers in the *economics* literature starts with the paper by Strotz [17] who points out a time inconsistency arising from the presence of the initial point in the time domain when the exponential discounting in the utility model of Samuelson [16] is replaced by a non-exponential discounting. For an illuminating exposition of the problem of *intertemporal choices* (decisions involving tradeoffs among costs and benefits occurring at different times) lasting over hundred years and leading to the Samuelson's simplifying model containing a single parameter (discount rate) see [7] and the references therein. To tackle the issue of the time inconsistency Strotz proposed two strategies in his paper: (i) the strategy of '*pre-commitment*' (where the individual commits to the optimal strategy derived initially) and (ii) the strategy of '*consistent planning*' (where the individual rejects any strategy which he will not follow through and aims to find the optimal strategy among those that he will actually follow). Note in particular that Strotz coins the term 'pre-committed' strategy in his paper and this term has since been used in the literature including most recent papers too. Although his setting is deterministic and his time is discrete on closer look one sees that our financial analysis of the static investor above is fully consistent with his economic reasoning and moreover the statically optimal selling strategy derived in the present paper may be viewed as the strategy of 'pre-committment' in Strotz's sense as already indicated above. The dynamically optimal selling strategy derived in the present paper is different however from the strategy of 'consistent planning' in Strotz's sense. The difference is subtle still substantial and it will become clearer through the exposition of the subsequent development that continues to the present time. The next to point out is the paper by Pollak [14] who showed that the derivation of the strategy of 'consistent planning' in the Strotz paper [17] was incorrect (one cannot replace the individual's non-exponential discount function by the exponential discount function having the same slope as the non-exponential discount function at zero). Peleg and Yaari [12] then attempted to find the strategy of 'consistent planning' by backward recursion and concluded that the strategy could exist only under too restrictive hypotheses to be useful. They suggested to look at what we now refer to as a *subgame-perfect* Nash equilibrium (the optimality concept refining Nash equilibrium proposed by Selten in 1965). Goldman [8] then pointed out that the failure of backward recursion does not disprove the existence as suggested in [12] and showed that the strategy of 'consistent planning' does exist under quite general conditions. All these papers deal with problems in discrete time. A continuous-time extension of these results appear more recently in the paper by Ekeland and Pirvu [6] and the paper by Björk and Murgoci [2] (see also the references therein for other unpublished work). The Strotz's strategy of 'consistent planning' is being understood as a subgame-perfect Nash equilibrium in this context (satisfying the natural consumption constraint at present time).

[1] We are indebted to Sven Rady for pointing out possible connections with the economics literature after seeing the results on the static and dynamic optimality exposed in Theorem 3 above.

4. The stream of papers in the *finance* literature starting with the paper by Richardson [15] deals with optimal portfolio selection problems under mean-variance criteria (similar to (2.3)-(2.5) above). Richardson's paper [15] deals with problems in continuous time and the paper by Li and Ng [9] deals with problems in discrete time. There also exist other papers of this kind in the literature but we do not provide a complete list. All of them however derive/deal with 'pre-committed' strategies in the sense of Strotz. This was pointed out by Basak and Chabakauri in their paper [1] where they return to the Strotz's approach of 'consistent planning' and study the subgame-perfect Nash equilibrium in continuous time. The paper by Björk and Murgoci [2] merges this with the stream of papers from the economics literature (as already stated above) and studies general formulations of time inconsistent problems based on the Strotz's approach of 'pre-committment' vs 'consistent planning' in the sense of the subgame-perfect Nash equilibrium. A recent paper by Czichowsky [3] studies analogous formulations and further refinements in a general semimartingale setting.

5. We now return to the question of comparison between the Strotz's definition of 'consistent planning' which is interpreted as the subgame-perfect Nash equilibrium in the literature and the 'dynamic optimality' as defined in the present paper. The key conceptual difference is that the Strotz's definition of 'consistent planning' is *relative* (constrained) in the sense that the 'optimal' control at time t is best among all 'available' controls (the ones which will be actually followed) while the present definition of the 'dynamic optimality' is *absolute* (unconstrained) in the sense that the optimal control at time t is best among all 'possible' controls afterwards. To illustrate this distinction recall that the subgame-perfect Nash equilibrium formulation of the Strotz 'consistent planning' optimality can be informally described as follows. Given the present time t and all future times $s > t$ one identifies the control c_r applied at time $r \geq t$ with an action of the r-th player. The Strotz 'consistent planning' optimality is then obtained through the subgame-perfect Nash equilibrium at a given control $(c_r)_{r \geq 0}$ if the action c_t is best when the actions c_s for $s > t$ are given and fixed, i.e. no other action $\tilde{c}_t$ in place c_t would do better when the actions c_s for $s > t$ are given and fixed (the requirement is clear in discrete time and requires some right-hand limiting argument in continuous time). This game-theoretic interpretation of controls makes also sense in problems of optimal stopping if we let $c_r(x)$ take value 0 or 1 depending on whether the observation of the process at time r and position x should continue or stop respectively. The action c_r may be formally identified with the indicator function of the r-th section of the optimal stopping set (in the time-space domain). For example, the dynamically optimal selling time from (3.4) can be identified with the control $(c_r)_{r \geq 0}$ where $c_r(x) = 0$ if $x < b_*$ and $c_r(x) = 1$ for $x \geq b_*$ where $b_* = \gamma/(c(1-\gamma))$. Invoking the meaning of Definition 2 it is then easy to check that the dynamically optimal selling time from (3.4) is subgame-perfect Nash optimal in the sense of Strotz's 'consistent planning' when $\gamma \in (0, 1/2]$ upon recalling that in this case (Remark 4) the static value associated with the dynamically optimal selling time is greater than or equal to the gain function (obtained by stopping at once). If $\gamma \in (1/2, 1)$ then the latter fact is no longer true however and changing the strategy by stopping at once the investor receives a higher static value and hence the dynamically optimal selling time is no longer subgame-perfect Nash optimal in the sense of Strotz's 'consistent planning'. This shows that the two notions of optimality are different. Moreover, the following example shows that even in the linear problems of optimal stopping there is no complete agreement between the standard optimality and the subgame-perfect Nash (SPN) optimality in the sense of Strotz's 'consistent planning'

(SCP) since the latter notion admits multiple optimal stopping times yielding multiple value functions too. Note that this is not the case with the 'dynamically optimality' where there is a full agreement with the standard optimality as pointed out following Definition 2 above.

Example 9 (Multiple SPN/SCP equilibria). Consider the optimal stopping problem

$$V(x) = \sup_{\tau} \mathsf{E}_x[G(X_\tau)] \tag{4.1}$$

where X is a standard Brownian motion in $(0,1)$ starting at x under P_x for $x \in [0,1]$ and getting absorbed upon reaching either 0 or 1, while the gain function is defined by $G(x) = 0$ for $x \in [0,1/2)$ and $G(x) = x - 1/2$ for $x \in [1/2,1]$. Note that G is continuous on $[0,1]$ and strictly increasing on $[1/2,1]$. Consider the stopping time

$$\tau_b = \inf\{\, t \geq 0 \mid X_t \geq b \,\} \tag{4.2}$$

for $b \in [1/2,1]$. Recalling that V coincides with the smallest concave function which dominates G on $[0,1]$ it is evident that V is linear on $[0,1]$ with $V(0) = G(0)$ and $V(1) = G(1)$ so that $V(x) = x/2$ for $x \in [0,1]$. This also shows that the stopping time τ_1 is optimal in (4.1) and it is clear from the structure of the problem that no other stopping time can be optimal in the standard sense. On the other hand, we claim that each τ_b for $b \in [1/2,1]$ forms a subgame-perfect Nash equilibrium strategy in the sense of Strotz's 'consistent planning'. Indeed, recall that the action of the r-th player corresponding to τ_b can be identified with the control $c_r(x) = 1_{[b,1]}(x)$ for each $r \geq 0$ and all $x \in [0,1]$. To examine whether any other action $\tilde{c}_t$ in place of c_t could do better when the actions c_s for $s > t$ are given and fixed, note that there is no restriction in assuming that the given time t equals 0 because the problem (4.1) and the candidate stopping time τ_b are time-homogeneous. This amounts to setting $\tilde{c}_0(x) = 1_A(x)$ for any measurable subset A of $[0,1]$ in place of the interval $[b,1]$ above and keeping $c_s(x) = 1_{[b,1]}(x)$ for each $s > t$ and all $x \in [0,1]$. If we denote the resulting stopping time by $\tilde{\tau}_b$ then under P_x we evidently have $\tilde{\tau}_b = \tau_b$ if $x \notin A \cap [0,b)$ and $\tilde{\tau}_b = 0$ if $x \in A \cap [0,b)$. The deviation $\tilde{\tau}_b$ from τ_b is clearly suboptimal since the value function V_b associated with τ_b is linear on $[0,b]$ with $V_b(0) = 0$ and $V_b(b) = G(b)$ so that $V_b(x)$ dominates $G(x)$ obtained by stopping at once whenever $x \in A \cap [0,b) \neq \emptyset$. This shows that each τ_b forms a subgame-perfect Nash equilibrium strategy in the sense of Strotz's 'consistent planning' when $b \in [1/2,1]$ as claimed. In particular, each τ_b yields a different value function V_b which is linear on $[0,b]$, takes value 0 at 0, and coincides with G on $[b,1]$ for $b \in [1/2,1]$ given and fixed. Clearly the standard value function $V = V_1$ associated with τ_1 is the largest and hence the 'best' among all of them but this is not visible from the SPN/SCP notion of optimality.

A direct comparison between optimal control problems and optimal stopping problems is not possible and we will present a fuller comparison of the two notions of optimality among problems of optimal control to which the 'dynamic optimality' also applies elsewhere. Setting problems of optimal control aside in this paper it should be noted that the 'dynamic optimality' resolves the issue of space and time inconsistency in the nonlinear optimal stopping problems under consideration and already this fact by itself makes it interesting/useful from the standpoint of optimal stopping and its applications.

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