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The Optimal Mean-Variance Selling Problem with Finite Horizon

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The optimal mean-variance selling problem seeks to determine a dynamically optimal stopping time in the nonlinear problem

$$\sup_{0 \le \tau \le T} \big[\mathsf{E}(X_\tau) - c\,\mathsf{V}ar(X_\tau) \big]$$

where X is a geometric Brownian motion with strictly positive drift, the supremum is taken over stopping times τ of X, and $c > 0$ is a given and fixed constant. The solution to the problem is known when the horizon T is infinite (cf. [11]), however, the method of proof developed to solve the problem in that case is not applicable in the case when the horizon T is finite. In this paper we develop a new method of proof which solves the problem when the horizon T is finite. In this way we find that the dynamically optimal stopping time is given by

$$\tau_* = \inf \left\{ t \ge 0 \;\middle|\; X_t \ge \tfrac{b(t)}{c(1-2b(t))} \right\}$$

where the function b can be characterised as a unique solution to a nonlinear Volterra integral equation. We also prove that the dynamically optimal stopping time τ_* satisfies the smooth fit principle. To our knowledge this is the first time that such a nonlinear phenomenon of 'dynamic smooth fit' has been derived in the literature.

1. Introduction

Imagine an investor who owns a stock which he wishes to sell so as to *maximise his return* and *minimise his risk* upon selling. In line with the mean-variance analysis of Markowitz [8] we identify the return with the expectation of the stock price and the risk with the variance of the stock price. The quadratic nonlinearity of the variance then moves the resulting optimal stopping problem outside the scope of the standard/linear optimal stopping theory (see e.g. [14]). Consequently the results and methods of the standard/linear optimal stopping theory are no longer applicable in this new/nonlinear setting. The solution to the nonlinear problem when the horizon is infinite has been found in [11], however, the method of proof used to solve the nonlinear problem in that case is not applicable in the case when the horizon is finite. The purpose of the present paper is to develop a new method of proof which solves the nonlinear problem when the horizon is finite.

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Assuming that the stock price X follows a geometric Brownian motion one could consider the constrained problem in which the investor aims to maximise the expectation of X_τ over all stopping times τ of X such that the variance of X_τ is bounded above by a positive constant. Similarly the investor could aim to minimise the variance of X_τ over all stopping times τ of X such that the expectation of X_τ is bounded below by a positive constant. An application of Lagrange multipliers implies that the Lagrange function (Lagrangian) for either/both constrained problems can be expressed as a linear combination of the expectation of X_τ and the variance of X_τ with opposite signs. Optimisation of the Lagrangian over all stopping times τ of X thus yields the central optimal stopping problem under consideration. The constrained problems themselves will not be considered in the present paper as these extensions are somewhat lengthy and more routine.

Due to the quadratic nonlinearity of the variance one can no longer apply standard/linear results of the optimal stopping theory to solve the problem. Moreover, as shown in [11] (see also [10] as well as [1], [2], [3], [4], [7], [9] for subsequent developments), in addition to the *static* formulation of the nonlinear problem where the maximisation takes place relative to the initial time-space point of X that is given and fixed, one is also naturally led to consider a *dynamic* formulation of the nonlinear problem in which each new time-space point of X yields a new optimal stopping problem to be solved upon overruling all the past problems. These overarching aims are formalised in Definitions 1 and 2 recalled below.

The main result of the paper is presented in Theorem 3 below. In its first part we derive the statically optimal stopping time and in its second part we disclose the dynamically optimal stopping time (see also Figure 1 below). The proof of Theorem 3 is divided into nine parts. Each part has a heading describing its contents for ease of reading and overall understanding. In Remark 4 we shed light on the optimality condition derived as part of the main result. In Remarks 5 and 6 we briefly compare the method of proof developed in the present paper to solve the problem when the horizon is finite with the method of proof used in [11] to solve the problem when the horizon is infinite. Both methods, although different, may be seen as methods of *linearisation*, which in turn are intimately related to the method of Lagrange multipliers. Moreover, the fact that nonlinearity of the variance is *quadratic* is *not essential* and both methods are applicable in the cases of *more general* nonlinearities as well. Finally in Theorem 7 we prove the remarkable fact that the static value function associated with the dynamically optimal stopping time in the nonlinear problem satisfies the *smooth fit principle*. To our knowledge this is the first time that such a nonlinear phenomenon of 'dynamic smooth fit' has been derived in the literature.

2. Formulation of the problem

Let X be a geometric Brownian motion solving

$$dX_{t+s} = \mu X_{t+s}\, ds + \sigma X_{t+s}\, dB_s \tag{2.1}$$

with $X_t = x$ for $(t,x) \in [0,T]\times(0,\infty)$, where $\mu \in I\!R$ is the drift, $\sigma > 0$ is the volatility, B is a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$, and $T > 0$ is the time horizon. It is well known that the stochastic differential equation (2.1) has a unique strong solution given by

$$X^x_{t+s} = x \exp\left(\sigma B_s + (\mu - \tfrac{\sigma^2}{2})s\right) \tag{2.2}$$

for $s \in [0, T-t]$. The law of the process (2.2) on the canonical space will be denoted by $\mathsf{P}_{t,x}$. Thus under $\mathsf{P}_{t,x}$ the coordinate process X at time t starts at position x . It is well known that X is a strong Markov process with respect to $\mathsf{P}_{t,x}$ for $(t,x) \in [0,T] \times (0,\infty)$.

Consider the optimal stopping problem

$$V(t,x) = \sup_{0 \le \tau \le T-t} \big[\mathsf{E}_{t,x}(X_{t+\tau}) - c \, \mathsf{V}ar_{t,x}(X_{t+\tau}) \big] \tag{2.3}$$

for $(t,x) \in [0,T] \times (0,\infty)$, where the supremum is taken over stopping times τ of X (i.e. with respect to the natural filtration of X), and $c > 0$ is a given and fixed constant. Clearly, if $\mu \le 0$ then $\mathsf{E}_{t,x}(X_{t+\tau}) \le x$ in (2.3) by the supermartingale property of X so that it is optimal to stop at once. For this reason we will assume that $\mu > 0$ in the sequel. Due to the quadratic nonlinearity of the second term in $\mathsf{V}ar_{t,x}(X_\tau) = \mathsf{E}_{t,x}(X_\tau^2) - (\mathsf{E}_{t,x}(X_\tau))^2$ it is then well known that the problem (2.3) falls outside the scope of the standard/linear optimal stopping theory for Markov processes (see e.g. [14]). In addition to the *static* formulation of the *nonlinear* problem (2.3) where the maximisation takes place relative to the initial time-space point (t,x) that is given and fixed, one is also naturally led to consider a *dynamic* formulation of the nonlinear problem (2.3) in which each new time-space point of the process X yields a new optimal stopping problem to be solved upon overruling all the past problems. These aims will be formalised in two definitions to be recalled shortly below.

The problem (2.3) seeks to maximise the investor's return identified with the expectation of $X_{t+\tau}$ and minimise the investor's risk identified with the variance of $X_{t+\tau}$ upon selling the stock at time $t+\tau$. This identification is made in line with the mean-variance analysis of Markowitz [8]. The linear combination of the expectation and the variance represents the Lagrangian and once the optimal stopping problem has been solved in that form this will also lead to the solution of the constrained problems where either an upper bound is imposed on the size of the variance or a lower bound is imposed on the size of the expectation.

We now recall definitions of the static and dynamic optimality alluded to above. All stopping times throughout refer to stopping times of X .

Definition 1 (Static optimality). A stopping time τ_* is *statically optimal* in (2.3) for $(t,x) \in [0,T] \times (0,\infty)$ given and fixed, if

$$\mathsf{E}_{t,x}(X_{t+\sigma}) - c \, \mathsf{V}ar_{t,x}(X_{t+\sigma}) \le \mathsf{E}_{t,x}(X_{t+\tau_*}) - c \, \mathsf{V}ar_{t,x}(X_{t+\tau_*}) \tag{2.4}$$

for all stopping times $\sigma \le T-t$.

Note that the static optimality refers to the optimality relative to the initial time-space point (t,x) which is given and fixed. Changing the initial time-space point may yield a different optimal stopping time in the nonlinear problem since the statically optimal stopping time may and generally will depend on the initial time-space point in an essential way.

Definition 2 (Dynamic optimality). A stopping time τ_* is *dynamically optimal* in (2.3) if for every $(t,x) \in [0,T] \times (0,\infty)$ given and fixed, we have

$$\mathsf{E}_{t,x}(X_{t+\sigma}) - c \, \mathsf{V}ar_{t,x}(X_{t+\sigma}) \le x \tag{2.5}$$

for all stopping times $\sigma \le T-t$ if and only if $\mathsf{P}_{t,x}(\tau_* = 0) = 1$.

Note that the dynamic optimality corresponds to solving infinitely many optimal stopping problems dynamically in time where each new time-space point of the process X yields a new optimal stopping problem to be solved upon overruling all the past problems. The optimal decision at each time tells us either to stop (if no other stopping time from that time and position could do better) or to continue (if such a stopping time exists). While the static optimality remembers the past (through the initial time-space point) the dynamic optimality completely ignores it and only looks ahead. For more details on the static and dynamic optimality we refer to [11] (see also Section 4 in that paper for a historical account).

3. Solution to the problem

In this section we present solutions to the problems formulated in the previous section. Recall that $\Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^{x} e^{-y^2/2} dy$ denotes the standard normal distribution function for $x \in I\!R$.

Theorem 3. *Consider the optimal stopping problem* (2.3) *where the process* X *solves* (2.1) *with* $X_t = x$ *under* $\mathsf{P}_{t,x}$ *for* $(t,x) \in [0,T] \times (0,\infty)$ *and* $\mu > 0$.

(I) The stopping time

$$\tau_*^s = \inf \big\{ s \in [0, T-t] \;\big|\; X_{t+s} \geq \lambda_c(t,x)\, b(t+s) \big\} \tag{3.1}$$

is statically optimal in (2.3) *for* $(t,x) \in [0,T] \times (0,\infty)$ *given and fixed, where* $\lambda = \lambda_c(t,x)$ *is the unique positive solution to the equation*

$$\lambda = \frac{1}{c} + 2\, \mathsf{E}_{t,x} \big(X_{t+\tau_b(\lambda)} \big) \tag{3.2}$$

with $\tau_b(\lambda) = \inf \{ s \in [0, T-t] \mid X_{t+s} \geq \lambda\, b(t+s) \}$, *and* $t \mapsto b(t)$ *can be characterised as the unique continuous decreasing solution to the nonlinear Volterra integral equation*

$$\begin{aligned} & b(t) \big[e^{(2\mu+\sigma^2)(T-t)} - 1 \big] - \big[e^{\mu(T-t)} - 1 \big] \\ & \quad = (2\mu+\sigma^2)\, b(t) \int_0^{T-t} e^{(2\mu+\sigma^2)s}\, \Phi \left(\frac{\log\big(\frac{b(t)}{b(t+s)}\big) + (\mu + \frac{3\sigma^2}{2})\, s}{\sigma \sqrt{s}} \right) ds \\ & \qquad - \mu \int_0^{T-t} e^{\mu s}\, \Phi \left(\frac{\log\big(\frac{b(t)}{b(t+s)}\big) + (\mu + \frac{\sigma^2}{2})\, s}{\sigma \sqrt{s}} \right) ds \end{aligned} \tag{3.3}$$

satisfying $b(t) > \gamma/(2(1+\gamma))$ *for* $t \in [0,T)$ *where* $\gamma := \mu/(\sigma^2/2)$. *[The solution* b *satisfies* $b(0) < (1/2) \wedge (\gamma/(1+\gamma))$ *and* $b(T) = \gamma/(2(1+\gamma))$ *(see Figure 1 below).]*

(II) The stopping time

$$\tau_*^d = \inf \left\{ s \in [0, T-t] \;\middle|\; X_{t+s} \geq \frac{b(t+s)}{c\big(1-2b(t+s)\big)} \right\} \tag{3.4}$$

is dynamically optimal in (2.3) *for* $t \in [0,T]$ *(see Figure 1 below).*

Proof. We assume throughout that the process X solves (2.1) with $X_t = x$ under $\mathsf{P}_{t,x}$ and can be realised by (2.2) under P for $(t,x) \in [0,T] \times (0,\infty)$ given and fixed. We will divide

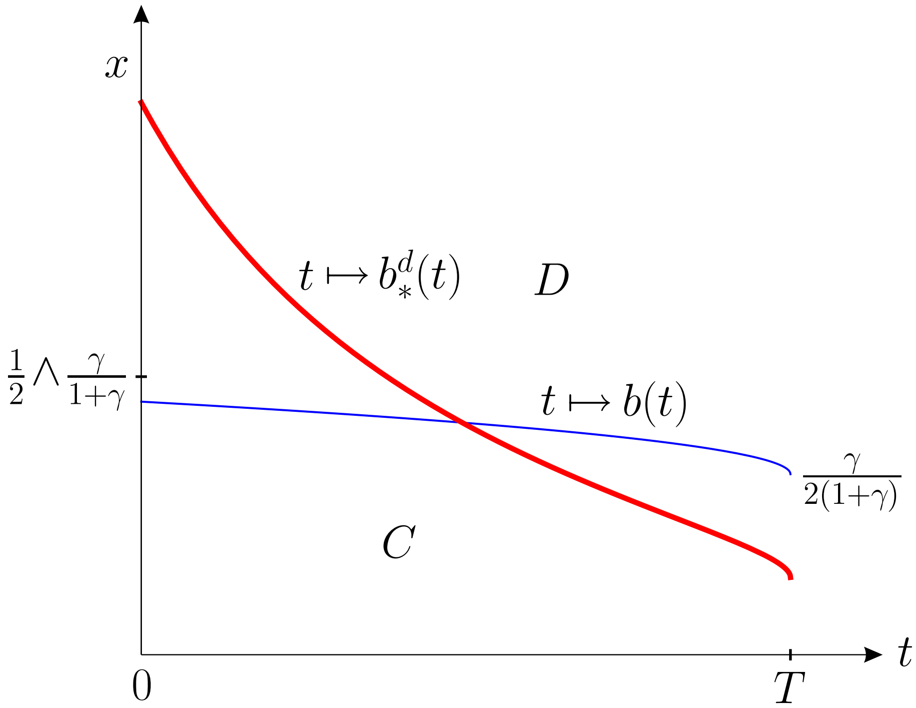

Figure 1. A computer drawing of the dynamically optimal boundary $x \mapsto b_*^d(x)$ $:= b(t)/(c(1-2b(t)))$ in the *nonlinear* problem (2.3) for $c > 0$ and $\mu > \sigma^2/2$, where $t \mapsto b(t)$ is the optimal stopping boundary in the *linear* problem (3.15) uniquely solving the *nonlinear* Volterra integral equation (3.3).

the proof into nine parts. The first eight parts will establish static optimality (Part I) and the final ninth part will establish dynamic optimality (Part II).

(I): In the first part of the proof we will establish that the stopping time (3.1) is statically optimal in (2.3). We will divide this part of the proof into eight parts as follows.

1. *Linearisation.* Note that the optimal stopping problem (2.3) reads

$$V(t,x) = \sup_{0 \le \tau \le T-t} F\big(\mathsf{E}_{t,x}(X_{t+\tau}), \mathsf{E}_{t,x}(X_{t+\tau}^2)\big) \tag{3.5}$$

where the convex *nonlinear* function $F : (0,\infty)\times(0,\infty)$ is given by

$$F(x_1,x_2) = x_1 - c\,(x_2 - x_1^2) \tag{3.6}$$

for $(x_1,x_2) \in (0,\infty)\times(0,\infty)$. Applying a first-order convexity condition we find that

$$\begin{aligned} F(y_1,y_2) &\ge F(x_1,x_2) + F_{x_1}(x_1,x_2)(y_1-x_1) + F_{x_2}(x_1,x_2)(y_2-x_2) \\ &= F(x_1,x_2) + (1+2cx_1)(y_1-x_1) - c(y_2-x_2) \\ &= F(x_1,x_2) - c\big[(y_2-x_2) - (\tfrac{1}{c}+2x_1)(y_1-x_1)\big] \\ &= F(x_1,x_2) - c\big[G_\lambda(y_1,y_2) - G_\lambda(x_1,x_2)\big] \end{aligned} \tag{3.7}$$

where the *linear* function $G_\lambda : (0,\infty)\times(0,\infty)$ is given by

$$G_\lambda(x_1,x_2) = x_2 - \lambda x_1 \tag{3.8}$$

and we set $\lambda = \frac{1}{c}+2x_1$ for (x_1,x_2) and (y_1,y_2) in $(0,\infty)\times(0,\infty)$. This shows that if (x_1,x_2) is given and (y_1,y_2) is chosen so that $G_\lambda(y_1,y_2) \le G_\lambda(x_1,x_2)$ with $\lambda = \frac{1}{c}+2x_1$, then $F(y_1,y_2) \ge F(x_1,x_2)$. In particular, if $(x_1(\sigma),x_2(\sigma)) = \big(\mathsf{E}_{t,x}(X_{t+\sigma}),\mathsf{E}_{t,x}(X_{t+\sigma}^2)\big)$ for a stopping time σ with values in $[0,T{-}t]$ given and fixed, then the minimiser $(y_1(\lambda),y_2(\lambda))$ of $G_\lambda(x_1(\tau),x_2(\tau))$ over all stopping times τ with values in $[0,T{-}t]$ for $\lambda = \frac{1}{c}{+}2x_1(\sigma)$ satisfies

$$F\big(y_1(\lambda),y_2(\lambda)\big) \ge F\big(x_1(\sigma),x_2(\sigma)\big)\,. \tag{3.9}$$

This motives us to consider the optimal stopping problem

$$\inf_{0\le\tau\le T-t} G_\lambda\big(x_1(\tau),x_2(\tau)\big) \tag{3.10}$$

with $\lambda > 0$ given and fixed. After we solve that problem we will apply exactly the same argument using (3.7) above to $(y_1(\lambda),y_2(\lambda))$ in place of $(x_1(\sigma),x_2(\sigma))$ and proceed inductively in exactly the same way further until we reach the maximiser of $F(x_1(\sigma),x_2(\sigma))$ over all σ in the limit. This will be realised in the remaining seven parts of the proof.

2. *Linear problem.* The optimal stopping problem (3.10) reads more explicitly as follows

$$V_\lambda(t,x) = \inf_{0\le\tau\le T-t} \mathsf{E}_{t,x}\big(X_{t+\tau}^2 - \lambda X_{t+\tau}\big) \tag{3.11}$$

for $(t,x) \in [0,T]\times(0,\infty)$ and $\lambda > 0$ where τ is a stopping time of X. This is a linear problem and we can make use of known techniques to solve it (see e.g. [14]).

3. *Measure change.* Looking at (3.11) and recalling (2.2) above note that

$$\begin{aligned}\mathsf{E}_{t,x}\big(X_{t+\tau}^2 - \lambda X_{t+\tau}\big) &= \mathsf{E}_{t,x}\big[X_{t+\tau}(X_{t+\tau}-\lambda)\big] \\ &= x\,\mathsf{E}\big[Z_\tau\, e^{\mu\tau}(X_\tau^x-\lambda)\big] = x\,\tilde{\mathsf{E}}\big[e^{\mu\tau}(X_\tau^x-\lambda)\big] \\ &= \lambda x\,\tilde{\mathsf{E}}_{t,x/\lambda}\big[e^{\mu\tau}(X_{t+\tau}-1)\big]\end{aligned} \tag{3.12}$$

where $d\tilde{\mathsf{P}} = Z_T\, d\mathsf{P}$ with $Z_T = e^{\sigma B_T - (\sigma^2/2)T}$. By the Girsanov theorem we know that $\tilde{B}_t := B_t{-}\sigma t$ is a standard Brownian motion under $\tilde{\mathsf{P}}$ for $t \in [0,T]$ so that X under $\tilde{\mathsf{P}}_{t,x}$ solves

$$dX_{t+s} = (\mu{+}\sigma^2)X_{t+s}\,ds + \sigma X_{t+s}\,d\tilde{B}_s \tag{3.13}$$

with $X_t = x$ for $(t,x) \in [0,T]\times(0,\infty)$. From (3.12) we see that

$$V_\lambda(t,x) = \lambda x \tilde{V}(t,x/\lambda) \tag{3.14}$$

where the value function $\tilde{V}$ is given by

$$\tilde{V}(t,x) = \inf_{0\le\tau\le T-t} \tilde{\mathsf{E}}_{t,x}\big[e^{\mu\tau}(X_{t+\tau}-1)\big] \tag{3.15}$$

for $(t,x) \in [0,T]\times(0,\infty)$. To solve (3.15) we first look at its infinite horizon version.

4. *Infinite horizon.* If the horizon T in (3.15) is infinite then the value function reads

$$\tilde{V}(x) = \inf_{\tau} \tilde{\mathsf{E}}_x \big[e^{\mu\tau}(X_\tau - 1) \big] \tag{3.16}$$

where X under $\tilde{\mathsf{P}}_x$ solves

$$dX_t = (\mu + \sigma^2) X_t \, dt + \sigma X_t \, d\tilde{B}_t \tag{3.17}$$

with $X_0 = x$ for $x > 0$ and hence has the infinitesimal generator given by

$$I\!\!L_X = (\mu + \sigma^2)\, x \frac{d}{dx} + \frac{\sigma^2}{2} x^2 \frac{d^2}{dx^2} \, . \tag{3.18}$$

Solving (3.16) we find that the stopping time

$$\tau_b = \inf \{ \, t \ge 0 \mid X_t \ge b \, \} \tag{3.19}$$

is optimal, where b is given by

$$\begin{aligned} b &= \frac{\gamma}{1+\gamma} \quad \text{if } \gamma \le 1 \text{ i.e. } \mu \le \sigma^2/2 \\ &= \frac{1}{2} \quad \text{if } \gamma > 1 \text{ i.e. } \mu > \sigma^2/2 \end{aligned} \tag{3.20}$$

upon recalling that $\gamma = \mu/(\sigma^2/2)$, and the value function $\tilde{V}$ is given by

$$\begin{aligned} \tilde{V}(x) &= -\frac{1}{\gamma} \Big(\frac{\gamma}{1+\gamma} \Big)^{1+\gamma} x^{-\gamma} \quad \text{if } \gamma \le 1 \text{ i.e. } \mu \le \sigma^2/2 \\ &= \quad -\frac{1}{4} x^{-1} \quad \text{if } \gamma > 1 \text{ i.e. } \mu > \sigma^2/2 \end{aligned} \tag{3.21}$$

for $0 < x \le b$ with $\tilde{V}(x) = x - 1$ for $x \ge b$.

Indeed, from (3.16) we see that $\tilde{V}$ and b should solve the free-boundary problem

$$(I\!\!L_X + \mu)\tilde{V}(x) = 0 \ \text{ for } x \in (0, b) \tag{3.22}$$

$$\tilde{V}(b) = b - 1 \ \text{ (instantaneous stopping)} \tag{3.23}$$

$$\tilde{V}'(b) = 1 \ \text{ (smooth fit)} \, . \tag{3.24}$$

On closer look one also finds from (3.16) that $x \mapsto \tilde{V}(x)$ is increasing and concave on $(0, \infty)$ with $\tilde{V}(x) < x - 1$ for $x \in (0, b)$ and $\tilde{V}(x) = x - 1$ for $x \ge b$ where both $\tilde{V}$ and b are to be found. Solving the Cauchy-Euler equation (3.22) we find that

$$\begin{aligned} \tilde{V}(x) &= Ax^{-\gamma} + Bx^{-1} \ \text{ if } \gamma \ne 1 \\ &= Ax^{-1} + Bx^{-1} \log x \ \text{ if } \gamma = 1 \end{aligned} \tag{3.25}$$

for $x > 0$. By Itô's formula and the optional sampling theorem one finds that the value function (3.16) dominates every solution (3.25) to the free-boundary problem (3.22)-(3.24). For this reason we are naturally led to choose the maximal solution (3.25) which (for small

$x > 0$) is obtained by setting $B = 0$ when $\gamma \le 1$ and $A = 0$ when $\gamma > 1$. This yields (3.20) and (3.21) above as candidate solutions. Verification that these are true solutions can then be carried out by Itô's formula and the optional sampling theorem using standard arguments (see e.g. [14]).

5. *Finite horizon.* Returning to (3.15) when the horizon T is finite, and setting $G(x) := x-1$ for $x > 0$, we see that the value function reads

$$\tilde{V}(t,x) = \inf_{0 \le \tau \le T-t} \tilde{\mathsf{E}}_{t,x}\big[e^{\mu\tau} G(X_{t+\tau})\big] \tag{3.26}$$

where X solves (3.13) with $X_t = x$ under $\tilde{\mathsf{P}}_{t,x}$ for $(t,x) \in [0,T]\times(0,\infty)$. Using (3.18) we find that $H(x) := (G_t + I\!\!L_X G + \mu G)(x) = (2\mu+\sigma^2)x - \mu$ for $x > 0$. Solving (3.26) based on the shape of H we find that the stopping time

$$\tau_b = \inf\{\, s \in [0, T-t] \mid X_{t+s} \ge b(t+s)\,\} \tag{3.27}$$

is optimal, where $t \mapsto b(t)$ can be characterised as the unique continuous decreasing solution to the nonlinear Volterra integral equation

$$G\big(b(t)\big) = e^{\mu(T-t)} \mathsf{E}_{t,b(t)}\big[G(X_T)\big] - \int_0^{T-t} e^{\mu s}\, \mathsf{E}_{t,b(t)}\big[H(X_{t+s})\, I(X_{t+s} \ge b(t+s))\big]\, ds \tag{3.28}$$

satisfying $b(t) > \gamma/(2(1+\gamma))$ for $t \in [0,T)$, and the value function $\tilde{V}$ is given by

$$\begin{aligned} \tilde{V}(t,x) &= e^{\mu(T-t)} \mathsf{E}_{t,x}\big[G(X_T)\big] - \int_0^{T-t} e^{\mu s}\, \mathsf{E}_{t,x}\big[H(X_{t+s})\, I(X_{t+s} \ge b(t+s))\big]\, ds \\ &= e^{\mu(T-t)}\big[x e^{(\mu+\sigma^2)(T-t)} - 1\big] \\ &\quad - (2\mu+\sigma^2)\, x \int_0^{T-t} e^{(2\mu+\sigma^2)s}\, \Phi\left(\frac{\log\big(\frac{x}{b(t+s)}\big) + (\mu + \frac{3\sigma^2}{2})s}{\sigma\sqrt{s}}\right) ds \\ &\quad + \mu \int_0^{T-t} e^{\mu s}\, \Phi\left(\frac{\log\big(\frac{x}{b(t+s)}\big) + (\mu + \frac{\sigma^2}{2})s}{\sigma\sqrt{s}}\right) ds \quad \text{if} \quad 0 < x \le b(t) \\ &= x - 1 \quad \text{if} \quad x \ge b(t) \end{aligned} \tag{3.29}$$

for $(t,x) \in [0,T]\times(0,\infty)$. [The solution b satisfies $b(0) < (1/2) \wedge (\gamma/(1+\gamma))$, where the right-hand side equals (3.20) above, and $b(T) = \gamma/(2(1+\gamma))$ (see Figure 1 above).]

Indeed, noting that $t \mapsto \tilde{V}(t,x)$ and $x \mapsto \tilde{V}(t,x)$ are increasing and $x \mapsto \tilde{V}(t,x)$ is concave, and applying the change-of-variable formula with local time on curves from [12] to $e^{\mu s}\, \tilde{V}(t+s, X_{t+s})$, one obtains (3.29) by the optional sampling theorem upon using that the (vertical) smooth fit holds at b (cf. [5]). Inserting $x = b(t)$ in (3.29) one obtains (3.28) or equivalently (3.3) above. To show that b is a unique solution to (3.3) i.e. (3.28) one can adopt the four-step procedure from the proof of uniqueness given in [6, Theorem 4.1] extending and further refining the original uniqueness arguments from [13, Theorem 3.1]. Given that the present setting creates no additional difficulties we will omit further details of these verifications.

6. *Solution to linear problem.* Returning to the linear problem (3.11) and recalling (3.14) we see that $V_\lambda(t,x) = x^2 - \lambda x$ if and only if $\tilde{V}(t, x/\lambda) = (x/\lambda) - 1$ showing that (t,x) belongs

to the stopping set in (3.11) if and only if $(t, x/\lambda)$ belongs to the stopping set in (3.15). It follows therefore that the stopping time

$$(3.30) \qquad \tau_b(\lambda) = \inf\{\, s \in [0, T{-}t] \mid X_{t+s} \geq \lambda b(t{+}s)\,\}$$

is optimal in (3.11) where $\lambda > 0$ and b is given in (3.27)-(3.28) above.

7. *Linearisation continued.* Having solved the linear problem (3.11) and recalling that (3.11) is equivalent to (3.10), we know that the minimiser in (3.10) equals

$$(3.31) \qquad (y_1(\lambda), y_2(\lambda)) = \big(\mathsf{E}_{t,x}(X_{t+\tau_b(\lambda)}), \mathsf{E}_{t,x}(X^2_{t+\tau_b(\lambda)})\big)$$

and satisfies (3.9) above when $\lambda = \frac{1}{c}{+}2x_1(\sigma)$ for $(x_1(\sigma), x_2(\sigma)) = \big(\mathsf{E}_{t,x}(X_{t+\sigma}), \mathsf{E}_{t,x}(X^2_{t+\sigma})\big)$ with any stopping time σ with values in $[0, T{-}t]$ given and fixed. Applying exactly the same argument using (3.7) above to $(y_1(\lambda), y_2(\lambda))$ in place of $(x_1(\sigma), x_2(\sigma))$ upon setting

$$(3.32) \qquad \lambda_1 = \tfrac{1}{c}{+}2y_1(\lambda)$$

we obtain the first inequality in

$$(3.33) \qquad F\big(y_1(\lambda_1), y_2(\lambda_1)\big) \geq F\big(y_1(\lambda), y_2(\lambda)\big) \geq F\big(x_1(\sigma), x_2(\sigma)\big)\,.$$

where $(y_1(\lambda_1), y_2(\lambda_1)) = \big(\mathsf{E}_{t,x}(X_{t+\tau_b(\lambda_1)}), \mathsf{E}_{t,x}(X^2_{t+\tau_b(\lambda_1)})\big)$ is the minimiser in (3.10) with λ_1 in place of λ while the second inequality in (3.33) follows by (3.9) above. Continuing by induction and applying exactly the argument using (3.7) above to $(y_1(\lambda_n), y_2(\lambda_n))$ in place of $(y_1(\lambda_{n-1}), y_2(\lambda_{n-1}))$ with $\lambda_0 := \lambda$ upon setting

$$(3.34) \qquad \lambda_{n+1} = \frac{1}{c} + 2y_1(\lambda_n)$$

we obtain the first inequality in

$$(3.35) \quad F\big(y_1(\lambda_{n+1}), y_2(\lambda_{n+1})\big) \geq F\big(y_1(\lambda_n), y_2(\lambda_n)\big) \geq \ldots \geq F\big(y_1(\lambda), y_2(\lambda)\big) \geq F\big(x_1(\sigma), x_2(\sigma)\big)$$

where $(y_1(\lambda_{n+1}), y_2(\lambda_{n+1})) = \big(\mathsf{E}_{t,x}(X_{t+\tau_b(\lambda_{n+1})}), \mathsf{E}_{t,x}(X^2_{t+\tau_b(\lambda_{n+1})})\big)$ is the minimiser in (3.10) with λ_{n+1} in place of λ for $n \geq 1$ while the remaining inequalities in (3.35) follow inductively from (3.33) above. It is now tempting to pass to the limit in (3.34) as $n \to \infty$. We will show in the next part below that this is possible. The limit $\lambda_c = \lambda_c(t, x)$ of λ_n as $n \to \infty$ solves

$$(3.36) \qquad \lambda_c = \frac{1}{c} + 2y_1(\lambda_c)$$

which is the same equation as (3.2) above. Moreover, the convergence of λ_n to λ_c is monotone so that not only $y_1(\lambda_n)$ converges to $y_1(\lambda_c)$ by construction (fixed point) but also $y_2(\lambda_n)$ converges to $y_2(\lambda_c)$ by the monotone convergence theorem as $n \to \infty$. Passing to the limit in (3.35) as $n \to \infty$ then yields

$$(3.37) \qquad F\big(y_1(\lambda_c), y_2(\lambda_c)\big) \geq F\big(x_1(\sigma), x_2(\sigma)\big)\,.$$

for any stopping time σ with values in $[0, T{-}t]$. Recalling (3.5) this shows that the stopping time (3.1) is statically optimal in (2.3) as claimed.

8. *Fixed point.* The right-hand side of (3.34) may be viewed as the n-th iterate of the mapping $f:(0,\infty)\rightarrow(0,\infty)$ defined by

$$f(\lambda) = \frac{1}{c} + 2y_1(\lambda) \tag{3.38}$$

for $\lambda>0$ where $y_1(\lambda)=\mathsf{E}_{t,x}(X_{t+\tau_b(\lambda)})$ and $\tau_b(\lambda)$ is given by (3.30) above. The convergence in (3.34) can then be seen as the statement of a fixed-point theorem. To examine the properties of f to this end note that

$$x \mapsto \mathsf{E}_{t,x}(X_{t+\tau_b(\lambda)}) \;\text{ is increasing} \tag{3.39}$$

$$\lambda \mapsto \mathsf{E}_{t,x}(X_{t+\tau_b(\lambda)}) \;\text{ is increasing} \tag{3.40}$$

where (3.39) follows from the fact that b is decreasing and (3.40) follows from the fact that X is a submartingale due to $\mu>0$. From (3.39) we see that $\lambda\mapsto f(\lambda)$ is increasing on $(0,\infty)$. Moreover, noting that $\mathsf{E}_{t,x}(X_{t+\tau_b(\lambda)})=x$ if $x\geq\lambda b(t)$ i.e. $\lambda\leq x/b(t)$, we see that $f'(\lambda)=0$ for $\lambda\in(0,x/b(t)]$. If $\lambda>x/b(t)$ i.e. $x<\lambda b(t)$ then

$$\begin{aligned} f'(\lambda) &= 2y_1'(\lambda) = 2\frac{\partial}{\partial\lambda}\mathsf{E}_{t,x}(X_{t+\tau_b(\lambda)}) = 2\frac{\partial}{\partial\lambda}\Big[\lambda\,\mathsf{E}_{t,x/\lambda}(X_{t+\tau_b})\Big] \\ &= 2\Big[\mathsf{E}_{t,x/\lambda}(X_{t+\tau_b}) - \frac{x}{\lambda}\frac{\partial}{\partial x}\mathsf{E}_{t,x}(X_{t+\tau_b})\big|_{x=x/\lambda}\Big] \\ &\leq 2\,\mathsf{E}_{t,x/\lambda}(X_{t+\tau_b}) = \frac{2}{\lambda}\mathsf{E}_{t,x}(X_{t+\tau_b(\lambda)}) \\ &< \frac{2}{\lambda}\lambda b(t) = 2b(t) \leq 1 \end{aligned} \tag{3.41}$$

where in the third equality we use (2.2) above, in the first inequality we use (3.39) above, in the second (strict) inequality we use the fact that b is (strictly) decreasing, and in the third (final) inequality we use (3.20) above. This shows that f is a contractive mapping and hence by the Banach fixed-point theorem there exists a unique fixed-point of f on $(0,\infty)$, i.e. a unique point $\lambda_c=\lambda_c(t,x)$ in $(0,\infty)$ such that $f(\lambda_c)=\lambda_c$. This establishes (3.36) or equivalently (3.2) as claimed. Moreover, using that $\lambda\mapsto f(\lambda)$ is increasing (and differentiable) one can easily verify that the iterates $\lambda_{n+1}:=f(\lambda_n)$ for $n\geq0$ satisfy $\lambda_n\uparrow\lambda_c$ if $\lambda_0\leq\lambda_c$ and $\lambda_n\downarrow\lambda_c$ if $\lambda_0\geq\lambda_c$ as $n\rightarrow\infty$. This completes the first part of the proof.

(II): In the second part of the proof we will make use of the facts derived in the first part of the proof and establish that the stopping time (3.4) is dynamically optimal in (2.3).

9. *Dynamic optimality.* Recall that for $(t,x)\in[0,T]\times(0,\infty)$ given and fixed there exists a unique $\lambda=\lambda_c(t,x)$ solving (3.2) where $\tau_b(\lambda)$ is given by (3.30) above. From this fact we can derive the following characterisation of the time-space points (t,x) in $[0,T]\times(0,\infty)$ at which it is optimal to stop at once

$$x\geq\lambda b(t) \Longleftrightarrow \lambda = \frac{1}{c}+2x\,. \tag{3.42}$$

Indeed, if $x\geq\lambda b(t)$ then $\tau_b(\lambda)=0$ by (3.30) so that $\lambda=\frac{1}{c}+2x$ by (3.2). Conversely, if $\lambda=\frac{1}{c}+2x$ then λ solves (3.2) with $\tau_b(\lambda)=0$ so that $x\geq\lambda b(t)$ by (3.30) as claimed. From

(3.42) we can then infer further that

$$x \ge \lambda b(t) \Leftrightarrow \Big(x \ge \lambda b(t) \;\;\&\;\; \lambda = \frac{1}{c} + 2x \Big) \Leftrightarrow x \ge \Big(\frac{1}{c} + 2x \Big) b(t) \Leftrightarrow x \ge \frac{b(t)}{c(1-2b(t))} \tag{3.43}$$

where the first equivalence follows by (3.42) and the third equivalence is evident. For the second equivalence note that if $x \ge \lambda b(t)$ & $\lambda = \frac{1}{c}+2x$ then evidently $x \ge (\frac{1}{c}+2x)b(t)$. On the other hand, if $x \ge (\frac{1}{c}+2x)b(t)$ then setting $\lambda = \frac{1}{c}+2x$ we see that (3.2) is satisfied because $x \ge \lambda b(t)$ implies that $\tau_b(\lambda) = 0$. By the uniqueness of the solution to (3.2) we can then concluded that $\lambda = \frac{1}{c}+2x$ indeed and $x \ge \lambda b(t)$ as needed. From the equivalence between the first expression and the final expression in (3.43) we can then conclude that a time-space point (t, x) in $[0, T] \times (0, \infty)$ is dynamically optimal if and only if $x \ge b(t)/(c(1-2b(t)))$ (see Figure 1 above). This establishes that the stopping time (3.4) is dynamically optimal in (2.3) as claimed and the proof is complete. □

Remark 4. To shed light on the optimality condition (3.2), note that (3.9) combined with (3.10) and (3.31) shows that the optimal stopping time in (2.3) i.e. (3.5) is of the form (3.30) for some $\lambda > 0$ to be found. Recalling the notation from (3.31) we thus see that the problem (3.10) can be equivalently rewritten as follows

$$\inf_{\kappa > 0} G_\lambda \big(y_1(\kappa), y_2(\kappa) \big) \tag{3.44}$$

where G_λ is given by (3.8) above, so that a first-order condition at the optimal point $\lambda > 0$ to be found reads as follows

$$y_2'(\lambda) - \lambda y_1'(\lambda) = 0 \,. \tag{3.45}$$

Moreover, the problem (3.5) can then be equivalently rewritten as follows

$$\sup_{\kappa > 0} F \big(y_1(\kappa), y_2(\kappa) \big) \tag{3.46}$$

where F is given by (3.6) above, so that a first-order condition at the optimal point $\lambda > 0$ to be found reads as follows

$$y_1'(\lambda) - c\big(y_2'(\lambda) - 2y_1(\lambda) y_1'(\lambda) \big) = y_1'(\lambda) - c\big(\lambda y_1'(\lambda) - 2y_1(\lambda) y_1'(\lambda) \big) = 0 \tag{3.47}$$

where we use (3.45) in the first equality. From (3.47) we see that a first-order condition in the problem (2.3) i.e. (3.5) reads

$$\lambda = \frac{1}{c} + 2y_1(\lambda) \tag{3.48}$$

and this is exactly the condition (3.2) above. These somewhat informal arguments show that the condition (3.2) is necessary for the (static) optimality in (2.3). The result of Theorem 3 above establishes its sufficiency.

Remark 5. The method of proof used in [11] to solve the problem (2.3) when the horizon T is infinite is not applicable when T is finite. This is because that method of proof makes an essential use of explicit expressions for the value function and the optimal stopping boundary in (3.11) that are no longer available when T is finite. The method of proof developed above to solve the problem when T is finite is applicable when T is infinite as well. Indeed, the proof in the latter case can be carried out in exactly the same way and the conclusions remain the same. For example, from (3.20) we see that $b = (1/2) \wedge (\gamma/(1+\gamma))$ and from (3.43) we see that the dynamically optimal stopping point equals $b/(c(1-2b))$ which thus further equals $\gamma/(c(1-\gamma))$ if $\gamma \in (0,1)$ i.e. $\mu \in (0, \sigma^2/2)$ and formally equals ∞ if $\gamma \geq 1$ i.e. $\mu \geq \sigma^2/2$ as both found in [11] as well.

Remark 6. On closer look one sees that the *linear* problem (3.11) plays a central role in both methods of proof addressed in the previous remark. The method of proof used in [11] relies upon *conditioning* on the size of the expected value of the process stopped at a given stopping time. This conditioning removes the (quadratic) *nonlinearity* of the variance and leads to (3.11) as the Lagrangian of the constrained problem. The method of proof developed above relies upon a *first-order convexity condition* which also removes the (quadratic) *nonlinearity* of the variance and provides (3.11) as a generator of more optimal values. Both methods may thus be seen as methods of *linearisation* and both of them are intimately related to the method of Lagrange multipliers itself. Note that the nonlinearity being *quadratic* is *not essential* and both methods are applicable in the cases of *more general* nonlinearities as well.

4. Dynamic smooth fit principle

In this section we prove the remarkable fact that the static value function associated with the dynamically optimal stopping time (3.4) in the nonlinear problem (2.3) satisfies the smooth fit principle. More specifically, let

$$V_*^d(t,x) := \mathsf{E}_{t,x}(X_{t+\tau_*^d}) - c\,\mathsf{V}ar_{t,x}(X_{t+\tau_*^d}) \tag{4.1}$$

for $(t,x) \in [0,T] \times (0,\infty)$ and $c > 0$, where τ_*^d is the dynamically optimal stopping time in (2.3) given by (3.4) above. Setting

$$b_*^d(t) = \frac{b(t)}{c(1-2b(t))} \tag{4.2}$$

where $t \mapsto b(t)$ is the unique continuous decreasing solution to (3.3), we see that (3.4) reads

$$\tau_*^d = \inf\{\, s \in [0, T-t] \mid X_{t+s} \geq b_*^d(t+s) \,\} \tag{4.3}$$

for $t \in [0,T]$ (see Figure 1 above).

Theorem 7 (Smooth fit). *We have*

$$x \mapsto V_*^d(t,x) \text{ is continuously differentiable at } b_*^d(t) \text{ with} \tag{4.4}$$

$$\frac{\partial V_*^d}{\partial x}(t, b_*^d(t)) = 1 \tag{4.5}$$

for every $t \in [0,T]$.

Proof. Let $t \in [0,T)$ be given and fixed (note that both (4.4) and (4.5) hold trivially when $t=T$ because $V_*^d(T,x) = x$ for $x>0$). Set $x := b_*^d(t)$ and take any $x_n \uparrow x$ as $n \rightarrow \infty$. Passing to a subsequence if needed there is no loss of generality in assuming that

$$\liminf_{n\rightarrow\infty} \frac{\partial V_*^d}{\partial x}(t,x_n) = \lim_{n\rightarrow\infty} \frac{V_*^d(t,x_n+\varepsilon_n)-V_*^d(t,x_n)}{\varepsilon_n} \tag{4.6}$$

for some $\varepsilon_n \downarrow 0$ as $n \rightarrow \infty$ with $x_n+\varepsilon_n < x$ for $n \geq 1$. Let $\tau_n := \tau_b(\lambda_n)$ with $\lambda_n := \lambda_c(t,x_n)$ denote the statically optimal stopping time (3.1) in the problem (2.3) for $n \geq 1$.

1. We claim that

$$\tau_n \rightarrow 0 \tag{4.7}$$

P-almost surely as $n \rightarrow \infty$. For this, first note that

$$x_n < \lambda_n b(t) \leq b_*^d(t) \tag{4.8}$$

for $n \geq 1$. Indeed, for the first (strict) inequality suppose that $x_n \geq \lambda_n b(t)$ for some $n \geq 1$. Then by (3.43) we have $x_n \geq b_*^d(t)$ which is a contradiction because $x_n < x = b_*^d(t)$. Moreover, for the second inequality recall using (3.40) that $\lambda \mapsto f(\lambda)$ defined in (3.38) is increasing so that $x \mapsto \lambda_c(t,x)$ solving (3.36) uniquely is increasing as well on $(0,\infty)$. It thus follows that

$$\lambda_n b(t) = \lambda_c(t,x_n)b(t) \leq \lambda_c(t,x)b(t) = b_*^d(t) \tag{4.9}$$

for $n \geq 1$ as claimed, where the second (final) equality follows upon recalling that $x = b_*^d(t)$ and using (3.43) with equalities throughout. This establishes (4.8) as claimed. Next note that by letting $n \rightarrow \infty$ in (4.8) and using that $x_n \uparrow x = b_*^d(t)$ we find that

$$\lambda_n b(t) \uparrow b_*^d(t) \tag{4.10}$$

as $n \rightarrow \infty$. Hence using that $t \mapsto b(t)$ is decreasing we see by (2.2) above that

$$\begin{aligned} \tau_n = \tau_b(\lambda_n) &= \inf\{\, s \in [0,T-t] \mid X_{t+s}^{x_n} \geq \lambda_n b(t+s)\,\} \\ &\leq \inf\{\, s \in [0,T-t] \mid X_{t+s}^{x_n} \geq \lambda_n b(t)\,\} \\ &= \inf\Big\{\, s \in [0,T-t] \;\Big|\; B_s \geq \tfrac{1}{\sigma}\log\big(\tfrac{\lambda_n b(t)}{x_n}\big) - \big(\tfrac{\mu}{\sigma}-\tfrac{\sigma}{2}\big)s \,\Big\} \rightarrow 0 \end{aligned} \tag{4.11}$$

as $n \rightarrow \infty$ because $a_n := (1/\sigma)\log(\lambda_n b(t)/x_n) \rightarrow 0$ as $n \rightarrow \infty$ by (4.10) above and $s \mapsto bs$ is an upper function for B for every $b>0$ (as well as $b \leq 0$ trivially). From (4.11) we see that (4.7) holds as claimed.

2. Focusing on the right-hand side in (4.6) we find using (2.2) that

$$\begin{aligned} V_*^d(t,x_n+\varepsilon_n)-V_*^d(t,x_n) &\geq \mathsf{E}(X_{t+\tau_n}^{x_n+\varepsilon_n}) - c\,\mathsf{V}ar(X_{t+\tau_n}^{x_n+\varepsilon_n}) - \mathsf{E}(X_{t+\tau_n}^{x_n}) + c\,\mathsf{V}ar(X_{t+\tau_n}^{x_n}) \\ &= \mathsf{E}\big(X_{t+\tau_n}^{x_n+\varepsilon_n} - X_{t+\tau_n}^{x_n}\big) - c\,\mathsf{E}\big[(X_{t+\tau_n}^{x_n+\varepsilon_n})^2 - (X_{t+\tau_n}^{x_n})^2\big] \\ &\quad + c\big[(\mathsf{E}X_{t+\tau_n}^{x_n+\varepsilon_n})^2 - (\mathsf{E}X_{t+\tau_n}^{x_n})^2\big] \\ &= \mathsf{E}(\varepsilon_n X_{t+\tau_n}^1) - c\,\mathsf{E}(2\xi_n(X_{t+\tau_n}^1)^2\varepsilon_n) + c2\eta_n[\mathsf{E}(X_{t+\tau_n}^1)]^2\varepsilon_n \end{aligned} \tag{4.12}$$

for some ξ_n and η_n in $(x_n, x_n+\varepsilon_n)$ for $n \geq 1$ by the mean value theorem. Dividing both sides in (4.12) by ε_n and letting $n \to \infty$ we find from (4.6) that

$$\liminf_{n\to\infty} \frac{\partial V_*^d}{\partial x}(t, x_n) \geq 1 \tag{4.13}$$

upon using (4.7) above.

3. Similarly, there is no loss of generality in assuming that

$$\limsup_{n\to\infty} \frac{\partial V_*^d}{\partial x}(t, x_n) = \lim_{n\to\infty} \frac{V_*^d(t, x_n) - V_*^d(t, x_n - \varepsilon_n)}{\varepsilon_n} \tag{4.14}$$

for some $\varepsilon_n \downarrow 0$ as $n \to \infty$ with $x_n - \varepsilon_n > 0$ for $n \geq 1$. Focusing on the right-hand side in (4.14) we find using (2.2) that

$$\begin{aligned} V_*^d(t, x_n) - V_*^d(t, x_n - \varepsilon_n) &\leq \mathsf{E}(X_{t+\tau_n}^{x_n}) - c\,\mathsf{V}ar(X_{t+\tau_n}^{x_n}) - \mathsf{E}(X_{t+\tau_n}^{x_n-\varepsilon_n}) + c\,\mathsf{V}ar(X_{t+\tau_n}^{x_n-\varepsilon_n}) \\ &= \mathsf{E}\big(X_{t+\tau_n}^{x_n} - X_{t+\tau_n}^{x_n-\varepsilon_n}\big) - c\,\mathsf{E}\big[(X_{t+\tau_n}^{x_n})^2 - (X_{t+\tau_n}^{x_n-\varepsilon_n})^2\big] \\ &\quad + c\big[(\mathsf{E}X_{t+\tau_n}^{x_n})^2 - (\mathsf{E}X_{t+\tau_n}^{x_n-\varepsilon_n})^2\big] \\ &= \mathsf{E}(\varepsilon_n X_{t+\tau_n}^1) - c\,\mathsf{E}(2\xi_n' (X_{t+\tau_n}^1)^2 \varepsilon_n) + c\,2\eta_n' [\mathsf{E}(X_{t+\tau_n}^1)]^2 \varepsilon_n \end{aligned} \tag{4.15}$$

for some ξ_n' and η_n' in $(x_n - \varepsilon_n, x_n)$ for $n \geq 1$ by the mean value theorem. Dividing both sides in (4.15) by ε_n and letting $n \to \infty$ we find from (4.14) that

$$\limsup_{n\to\infty} \frac{\partial V_*^d}{\partial x}(t, x_n) \leq 1 \tag{4.16}$$

upon using (4.7) above. Combining (4.13) and (4.16) we see that (4.4) and (4.5) hold as claimed and the proof is complete. □

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