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The Integral Analogue of the Hardy-Littlewood $L \log L$-Inequality for Brownian Motion

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Let $B = (B_t)_{t \geq 0}$ be standard Brownian motion started at zero. Then the following inequality is shown to be satisfied:

$$E\Big(\max_{0 \leq t \leq \tau} |B_t|\Big) \leq cE\bigg(\int_0^\tau \frac{dt}{1+|B_t|}\bigg) + \frac{1}{2c-1}$$

for all stopping times τ for B and all $c > 1/2$. The stopping times at which the equality is attained are of the form:

$$\tau_c = \inf\ \big\{\, t > 0 \ \mid\ S_t - \alpha X_t \geq \beta \,\big\}$$

where $\alpha = 1 + 1/(2c-1)$, $\beta = 1/(2c-1)$, $X_t = |B_t|$ and $S_t = \max_{0 \leq r \leq t} |B_r|$. Taking infimum over all $c > 1/2$ we obtain:

$$E\Big(\max_{0 \leq t \leq \tau} |B_t|\Big) \leq \frac{1}{2}\, E\bigg(\int_0^\tau \frac{dt}{1+|B_t|}\bigg) + \sqrt{2}\bigg(E\int_0^\tau \frac{dt}{1+|B_t|}\bigg)^{1/2}$$

for all stopping times τ for B. This inequality is sharp (the equality is attained at each τ_c for all $c > 1/2$). In view of Itô-Tanaka's formula these inequalities may be thought of as the integral analogues (for reflected Brownian motion) of the classical $L \log L$-inequality of Hardy and Littlewood. The proof is based upon solving the optimal stopping problem:

$$V = \sup_\tau E\big(S_\tau - cI_\tau\big)$$

where $I_\tau = \int_0^\tau (1+|B_t|)^{-1} dt$. The payoff V is shown to be finite if and only if $c > 1/2$, and in this case $V = 1/(2c-1)$. The optimal stopping problem is solved by applying the principle of smooth fit and the maximality principle. All results extend to the case when Brownian motion B starts at any given point.

1. Description of the problem and results

1. Let $B = (B_t)_{t \geq 0}$ be standard Brownian motion started at zero. Then the following integral inequalities are known to be valid (see [8]):

$$A_p E\big(\tau^{1+p/2}\big) \leq E\bigg(\int_0^\tau |B_t|^p dt\bigg) \leq B_p E\big(\tau^{1+p/2}\big) \tag{1.1}$$

for all stopping times τ for B, and all $p > -1$, where A_p and B_p are some universal constants. Recalling Burkholder-Gundy's inequalities (see [1]):

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$$C_q E\big(\tau^{q/2}\big) \le E\Big(\max_{0\le t\le\tau} |B_t|^q\Big) \le D_q E\big(\tau^{q/2}\big) \tag{1.2}$$

for $q>0$ where C_q and D_q are some universal constants, we find that:

$$F_p E\Big(\int_0^\tau |B_t|^p dt\Big) \le E\Big(\max_{0\le t\le\tau} |B_t|^{2+p}\Big) \le G_p E\Big(\int_0^\tau |B_t|^p dt\Big) \tag{1.3}$$

for all stopping times τ for B, and all $p>-1$, where F_p and G_p are some universal constants. The best values for A_p and B_p in (1.1) are known (see [8]). The best values for C_q and D_q in (1.2) are only known in the case $q=2$. In this case (due to $E(\tau)=E|B_\tau|^2$ whenever $E(\tau)<\infty$) the right-hand inequality in (1.2) reduces to Doob's maximal inequality (see [2] and [10]). The same fact extends to all inequalities (1.3). By Itô's formula and the optional sampling theorem we find that:

$$E\Big(\int_0^\tau |B_t|^p dt\Big) = H_p\, E|B_\tau|^{p+2} \tag{1.4}$$

with $H_p = 2/(p+2)(p+1)$, for all $p>-1$, and all stopping times τ for B for which $\{\, |B_{\tau\wedge n}|^{p+2} \colon\, n\ge 1\,\}$ is uniformly integrable. This shows that the right-hand inequality in (1.3) is in fact Doob's maximal inequality [2], so that the best values for F_p and G_p in (1.3) are known (see [10]). The advantage of the integral formulation (1.3) may lie in the fact that these inequalities hold for all stopping times τ for B, and no (uniform) integrability condition has to be imposed (as in the case of Doob's maximal inequality).

2. In this paper we shall address the case $p=-1$ in (1.3) when these inequalities fail to hold. The principal problem lies in the fact that:

$$E\Big(\int_0^\tau \frac{dt}{|B_t|}\Big) = \infty \tag{1.5}$$

for any stopping time τ for B for which $B_\tau \not\equiv 0$ P-a.s. (see (1.11) below). Thus we want to address the problem on how to modify the functional $t\mapsto |B_t|^{-1}$ in an optimal way so that (1.3) remain valid. In this paper we only focus to the right-hand inequality in (1.3), but the left-hand side could be treated similarly. Our main observation is that the functional $t\mapsto |B_t|^{-1}$ can be optimally replaced by the functional $t\mapsto (1+|B_t|)^{-1}$. In Theorem 2.1 below we prove that:

$$E\Big(\max_{0\le t\le\tau} |B_t|\Big) \le cE\Big(\int_0^\tau \frac{dt}{1+|B_t|}\Big) + \frac{1}{2c-1} \tag{1.6}$$

for all stopping times τ for B and all $c>1/2$. This inequality is sharp for any given and fixed $c>1/2$, and the equality is attained at the stopping time:

$$\tau_c = \inf\ \big\{\, t>0 \ \mid\ S_t - \alpha X_t \ge \beta \,\big\} \tag{1.7}$$

where $\alpha = 1+1/(2c-1)$, $\beta = 1/(2c-1)$, $X_t = |B_t|$ and $S_t = \max_{0\le r\le t} |B_r|$. Taking infimum in (1.6) over all $c>1/2$, we obtain the following inequality:

$$E\Big(\max_{0\le t\le\tau} |B_t|\Big) \le \frac{1}{2} E\Big(\int_0^\tau \frac{dt}{1+|B_t|}\Big) + \sqrt{2}\Big(E\int_0^\tau \frac{dt}{1+|B_t|}\Big)^{1/2} \tag{1.8}$$

for all stopping times τ for B. This inequality is again sharp, and moreover the equality in (1.8) is attained at each τ_c from (1.7) for all $c > 1/2$. In addition, we see that this inequality gives a more precise information on the limiting case $c \downarrow 1/2$ in (1.6), as well as a better bound on its left-hand side for small stopping times (when $\tau \equiv 0$ the equality in (1.8) is attained while this fails in (1.6)). A disadvantage of the inequality (1.8) is that it involves two terms on the right-hand side, so that it doesn't appear as elegant as the inequalities (1.3). However, it seems to be a heart of the matter in the case $p = -1$.

3. In view of Itô-Tanaka's formula (applied to $F(X_t)$ with $F(x) = (1{+}x)\log(1{+}x)$ so that $F''(x) = 1/(1{+}x)$) we see that the inequalities (1.6) and (1.8) may be thought of as the integral analogous of the classical $L\log L$-inequality of Hardy and Littlewood (see [7], [2] and [4]). In fact, we shall see in the proof of Theorem 2.1 below that:

$$E\left(\int_0^\tau \frac{dt}{1+|B_t|}\right) = 2E\Big(\big(1{+}|B_\tau|\big)\log\big(1{+}|B_\tau|\big) - |B_\tau|\Big) \tag{1.9}$$

for all stopping times τ for B satisfying $E(\tau^r) < \infty$ for some $r > 1/2$. For comparison, recall that the classical $L\log L$-inequality of Hardy and Littlewood states:

$$E\Big(\max_{0\le t\le\tau} |B_t|\Big) \le K\Big(1 + E\big(|B_\tau|\log^+|B_\tau|\big)\Big) \tag{1.10}$$

for all stopping times τ for B. (This inequality remains valid if the plus sign is removed from $\log^+$. The best values for the constant K are known in both cases (see [4]). For a new probabilistic proof in both cases which exhibit the optimal stopping times we refer to [6].) From (1.9) and (1.10) we see that the bound obtained on the right-hand side in (1.6) is in essence an $L\log L$-bound of Hardy and Littlewood, and thus it is generally known to be best possible (for $|B_\tau|$ large). It is interesting to observe that in the classical $L\log L$-inequality we should have $F(x) = x\log x$ in order that $F''(x) = 1/x$, which would (after applying Itô-Tanaka's formula to $F(X_t)$) correspond to the case $p = -1$ in (1.3). Note, however, that (1.9) extends as follows:

$$E\left(\int_0^\tau \frac{dt}{\varepsilon+|B_t|}\right) = 2E\Big(\big(\varepsilon{+}|B_\tau|\big)\log\big(\varepsilon{+}|B_\tau|\big) - (1{+}\log\varepsilon)|B_\tau|\Big) \tag{1.11}$$

for all $\varepsilon > 0$ and all stopping times τ for B satisfying $E(\tau^r) < \infty$ for some $r > 1/2$. Letting $\varepsilon \downarrow 0$ we see that this expression tends to infinity (whenever $B_\tau \not\equiv 0$ P-a.s.) due to the log term on the right-hand side. The functional $t \mapsto (1{+}|B_t|)^{-1}$ seems particularly interesting since $F'(x) = 1 + \log(1{+}x)$ for $F(x) = (1{+}x)\log(1{+}x)$, so that $F'(0) = 1$, and after applying Itô-Tanaka's formula to $F(X_t)$, the first derivative term equals a Brownian motion plus the local time of X at zero (see proof of Theorem 2.1 below) which leads to the identity (1.9). Finally, note that the inequality (1.8) is a refinement of the inequalities (1.6) and (1.10). Its clear advantage upon (1.6) and (1.10) is its sharpness for small stopping times τ.

4. The proof is based upon solving the optimal stopping problem with the payoff:

$$V = \sup_\tau E\big(S_\tau - cI_\tau\big) \tag{1.12}$$

where $I_\tau = \int_0^\tau (1{+}|B_t|)^{-1}dt$ and the supremum is taken over all stopping times τ for B

satisfying $E(I_\tau) < \infty$. We show that V is finite if and only if $c > 1/2$, and in this case $V = 1/(2c-1)$. The optimal stopping problem is two-dimensional (due to the fact that (X, S) is a Markov process) so that the main difficulty is to choose the optimal stopping boundary out of all possible candidates obtained by the principle of smooth fit of Kolmogorov (see [3]). Motivated by the maximality principle (see [5]) we find a natural solution to this problem (see proof of Theorem 2.1 below). Consequently, this leads to the quantitative expression for V stated above, and that the supremum in (1.12) is attained at the stopping times of the form (1.7). The inequalities (1.6) and (1.8) are then obtained as straightforward consequences. The sharpness of (1.8) is proved by applying a new simple argument (see proof of Corollary 2.2). Finally, all these results extend to the case when Brownian motion B starts at any given point (Corollary 2.3).

2. The results and proofs

In this section we present the main results and proofs. Our principal result is contained in the next theorem. This is reformulated afterwards (Corollary 2.2) into a more precise form. Both formulations extend to the case when the Brownian motion starts at any given point (Corollary 2.3).

Theorem 2.1

Let $B = (B_t)_{t \geq 0}$ be standard Brownian motion started at zero under P . Then the following inequality is shown to be satisfied:

$$E\left(\max_{0 \leq t \leq \tau} |B_t| \right) \leq cE\left(\int_0^\tau \frac{dt}{1 + |B_t|} \right) + \frac{1}{2c-1} \tag{2.1}$$

for all stopping times τ for B and all $c > 1/2$. Moreover, for each $c > 1/2$ given and fixed, the equality in (2.1) is attained at the stopping time:

$$\tau_c = \inf \left\{ t > 0 \mid S_t - \alpha X_t \geq \beta \right\} \tag{2.2}$$

where $\alpha = 1 + 1/(2c-1)$, $\beta = 1/(2c-1)$, $X_t = |B_t|$ and $S_t = \max_{0 \leq r \leq t} |B_r|$.

Proof. 1. Consider the following optimal stopping problem:

$$V(x, s) = \sup_\tau E_{x,s}\big(S_\tau - cI_\tau\big) \tag{2.3}$$

with τ's satisfying $E_{x,s}(I_\tau) < \infty$, where we denote $X_t = |B_t + x|$ and set:

$$S_t = \left(\max_{0 \leq r \leq t} X_r \right) \vee s \tag{2.4}$$

$$I_t = \int_0^t \frac{dr}{1 + X_r} \tag{2.5}$$

for $0 \leq x \leq s$ and $c > 1/2$ given and fixed. Note that the Markov process (X, S) under $P_{x,s} := P$ starts at (x, s) . Supposing that the supremum in (2.3) is attained at τ_* :

$$V(x, s) = E_{x,s}\big(S_{\tau_*} - cI_{\tau_*}\big) \tag{2.6}$$

we see that $x \mapsto V(x,s)$ is to satisfy:

$$\mathbf{L_X} V(x,s) = \frac{c}{1+x} \quad (\, g_*(s) < x < s \,) \tag{2.7}$$

where $\mathbf{L_X} = \partial^2/2\partial x^2$ is the infinitesimal operator of X in $]0,\infty[$ and $s \mapsto g_*(s)$ is the optimal stopping boundary to be found. From (2.7) we readily find that:

$$V(x,s) = A(s)x + B(s) + 2c(x+1)\log(x+1) \tag{2.8}$$

for $g_*(s) < x < s$. To determine the unknown functions $s \mapsto A(s)$, $s \mapsto B(s)$ and $s \mapsto g_*(s)$ we shall make use of the following boundary conditions:

$$V(x,s)\Big|_{x=g_*(s)+} = s \quad \text{(instantaneous stopping)} \tag{2.9}$$

$$\frac{\partial V}{\partial x}(x,s)\Big|_{x=g_*(s)+} = 0 \quad \text{(smooth fit)} \tag{2.10}$$

$$\frac{\partial V}{\partial s}(x,s)\Big|_{x=s-} = 0 \quad \text{(normal reflection)} \; . \tag{2.11}$$

Note that (2.7) with (2.9)-(2.11) is a Stephan's problem with moving (free) boundary $s \mapsto g_*(s)$. From (2.8)+(2.10) we find that:

$$A(s) = -2c\Big(1+\log\big(1+g_*(s)\big)\Big) \; . \tag{2.12}$$

Inserting this into (2.8) and using (2.9) we find that:

$$B(s) = s - 2c\log\big(1+g_*(s)\big) + 2cg_*(s) \; . \tag{2.13}$$

Inserting (2.12) and (2.13) into (2.8) we obtain:

$$V(x,s) = s - 2c\bigg((1+x)\log\Big(\frac{1+g_*(s)}{1+x}\Big) + x - g_*(s)\bigg) \tag{2.14}$$

for $g_*(s) \le x \le s$. Hence by (2.11) we see that $s \mapsto g_*(s)$ is to satisfy:

$$g_*'(s)\left(\frac{s-g_*(s)}{1+g_*(s)}\right) = \frac{1}{2c} \tag{2.15}$$

for $s > 0$. This is the total of information obtained by the principle of smooth fit (2.10). The basic problem now is how to determine the optimal stopping boundary $s \mapsto g_*(s)$ out of all possible solutions to (2.15)

2. Consider the first order (nonlinear) differential equation:

$$\lambda(1+y) + (y-x)\,y' = 0 \tag{2.16}$$

for $x > 0$ where $\lambda = 1/2c$. This equation is not exact. However, multiplying through (2.16) by

$\mu(x,y) = (1+y)^{-\alpha}$ with $\alpha = 1 + 1/\lambda$, the resulting equation becomes exact:

$$\lambda(1+y)^{1-\alpha} + (y-x)(1+y)^{-\alpha}\, y' \,=\, 0 \ . \tag{2.17}$$

Thus the general solution of (2.16) is of the form:

$$\lambda(1+y)^{1-\alpha}x + f(y) \,=\, K \tag{2.18}$$

where K is a numerical constant, and $y \mapsto f(y)$ satisfies:

$$f'(y) \,=\, y(1+y)^{-\alpha} \ . \tag{2.19}$$

Hence we find that:

$$\begin{aligned} f(y) \ &= \ \frac{1}{1+y} + \log(1+y) \ \text{ if } \ \alpha \,=\, 2 \\ &= \ \frac{(1+y)^{2-\alpha}}{2-\alpha} - \frac{(1+y)^{1-\alpha}}{1-\alpha} \ \text{ if } \ \alpha \neq 2 \ . \end{aligned} \tag{2.20}$$

Since we expect that $y(x) \rightarrow \infty$ for $x \rightarrow \infty$, only the case $\alpha > 2$ (or equivalently $c > 1/2$) seems to be of interest. Then by (2.18) the general solution of (2.16) is given by:

$$\frac{(1+x)}{(\alpha-1)(1+y)^{\alpha-1}} - \frac{1}{(\alpha-2)(1+y)^{\alpha-2}} = K \tag{2.21}$$

where K is a numerical constant. If we let $K = 0$, we see that (2.16) admits a linear solution:

$$y_*(x) \,=\, \frac{\alpha-2}{\alpha-1}\, x \ - \ \frac{1}{\alpha-1} \ . \tag{2.22}$$

Moreover, by using this fact and letting $x \mapsto \infty$ in (2.21) we see that $K = 0$ corresponds to the maximal solution of (2.16) which does not hit the diagonal $y = x$. This is in accordance with the maximality principle which holds in a similar context (see [5]).

3. The previous considerations show that a unique candidate for the optimal stopping boundary is given by the expression:

$$g_*(s) \,=\, \frac{2c-1}{2c}\, s \ - \ \frac{1}{2c} \tag{2.23}$$

for $s \geq s_*$, where $g_*(s_*) \,=\, 0$ so that:

$$s_* \,=\, \frac{1}{2c-1} \ . \tag{2.24}$$

To determine the corresponding payoff, note by the strong Markov property that:

$$V(x,s) = V(s_*,s_*) - c\, E_{x,s}\bigg(\int_0^{\tau_{s_*}} \frac{dr}{1+X_r} \bigg) \tag{2.25}$$

for all $0 \,\leq\, x \,\leq\, s \,\leq\, s_*$, where we set:

(2.26) $$\tau_{s_*} = \inf\,\{\, t>0 \mid X_t = s_* \,\}\ .$$

To compute the expectation in (2.25) we shall apply Itô-Tanaka's formula to $F(X_t)$ with $F(x) = (1+x)\log(1+x)$. This yields:

(2.27) $$\begin{aligned}(1+X_t)\log(1+X_t) &= (1+x)\log(1+x) + \int_0^t \big(1+\log(1+X_r)\big)\, dX_r \\ &+ \frac{1}{2}\int_0^t \frac{1}{1+X_r}\, d\big\langle X,X\big\rangle_r = (1+x)\log(1+x) \\ &+ \int_0^t \big(1+\log(1+X_r)\big)\, d(\beta_r+L_r) + \frac{1}{2}\int_0^t \frac{1}{1+X_r}\, dr \\ &= (1+x)\log(1+x) + M_t + L_t + \frac{1}{2}\int_0^t \frac{1}{1+X_r}\, dr\end{aligned}$$

where we used Tanaka's formula $X_t = x+\beta_t+L_t$ with $\beta=(\beta_t)_{t\ge 0}$ being a Brownian motion and $L=(L_t)_{t\ge 0}$ being the local time of X at zero, so that $M_t = \int_0^t (1+\log(1+X_r))\, d\beta_r$ is a local martingale. Clearly $E_{x,s}(L_\tau) = E_{x,s}(X_\tau)-x$ for all stopping times τ for B satisfying $E(\sqrt{\tau})<\infty$. Hence we get (as in (2.43)-(2.45) for (2.41) below):

(2.28) $$E_{x,s}\bigg(\int_0^\tau \frac{dr}{1+X_r}\bigg) = 2E_{x,s}\Big((1+X_\tau)\log\big(1+X_\tau\big)-X_\tau+x\Big) - 2(1+x)\log(1+x)$$

for all stopping times τ for B satisfying $E(\tau^r)<\infty$ for some $r>1/2$. In particular:

(2.29) $$E_{x,s}\bigg(\int_0^{\tau_{s_*}} \frac{dr}{1+X_r}\bigg) = 2\Big((1+s_*)\log\big(1+s_*\big)-s_*+x\Big) - 2(1+x)\log(1+x)\ .$$

Inserting this into (2.25) and using (2.14) we find:

(2.30) $$V(x,s) = s_* + 2c\Big((1+x)\log\big(1+x\big)-x\Big)$$

for all $0\le x\le s\le s_*$.

4. In view of the statement of the theorem, note that (2.30) gives:

(2.31) $$V(0,0) = s_* = \frac{1}{2c-1}\ .$$

Moreover, according to our considerations above, the optimal stopping time (at which the equality in (2.31) is attained) is to be of the form:

(2.32) $$\tau_* = \inf\,\big\{\, t>0 \mid X_t \le g_*(S_t)\,\big\}$$

with $s\mapsto g_*(s)$ given in (2.23). Noting that (2.31)+(2.32) is exactly (2.1)+(2.2), we see that the proof will be completed if we show that the candidate for the payoff given by (2.14)+(2.23)+(2.30) is indeed the payoff (2.3). This verification is the content of the next final step. For convenience, this candidate is denoted by $V_*(x,s)$. It should be noted that $V_*(x,s)=s$ for $0\le x\le g_*(s)$.

5. By Itô-Tanaka's formula (being applied two-dimensionally) we get:

$$V_*(X_t,S_t) = V_*(X_0,S_0) + \int_0^t \frac{\partial V_*}{\partial x}(X_r,S_r)\; dX_r + \int_0^t \frac{\partial V_*}{\partial s}(X_r,S_r)\; dS_r \tag{2.33}$$

$$+ \frac{1}{2}\int_0^t \frac{\partial^2 V_*}{\partial x^2}(X_r,S_r)\; d\langle X,X\rangle_r = V_*(x,s) + \int_0^t \frac{\partial V_*}{\partial x}(X_r,S_r)\; d(\beta_r + L_r)$$

$$+ \int_0^t \frac{\partial V_*}{\partial s}(X_r,S_r)\; dS_r + \frac{1}{2}\int_0^t \frac{\partial^2 V_*}{\partial x^2}(X_r,S_r)\; dr\;.$$

Since the increment dS_r equals zero outside the diagonal $x = s$, and $V_*(x,s)$ at the diagonal satisfies (2.11), the integral over dS_r in (2.33) is identically zero. Moreover, note that:

$$\begin{aligned} \frac{\partial V_*}{\partial x}(x,s) &= 2c \log\left(\frac{1+x}{1+(g_*(s)\vee 0)}\right) \quad \text{if } \; x > g_*(s) \\ &= 0 \quad \text{if } \; x \le g_*(s) \end{aligned} \tag{2.34}$$

for all $0 \le x \le s$. Since the increment dL_r equals zero outside $\{\, r \mid X_r = 0\,\}$, we see from (2.34) that the integral over dL_r in (2.33) is identically zero. Finally, note that:

$$\begin{aligned} \frac{\partial^2 V_*}{\partial x^2}(x,s) \;&=\; \frac{2c}{1+x} \quad \text{if } \; x > g_*(s) \\ &=\; 0 \quad \text{if } \; x < g_*(s) \end{aligned} \tag{2.35}$$

(the value of $\partial^2 V_*/\partial x^2$ at $(g_*(s),s)$ is irrelevant, since the set of all r for which $X_r = g_*(S_r)$ is of Lebesgue measure zero). Hence by (2.33) we obtain:

$$\begin{aligned} V_*(X_t,S_t) &= V_*(x,s) + \int_0^t \frac{\partial V_*}{\partial x}(X_r,S_r)\; d\beta_r + \frac{1}{2}\int_0^t \frac{\partial^2 V_*}{\partial x^2}(X_r,S_r)\; dr \\ &\le V_*(x,s) + M_t + cI_t \end{aligned} \tag{2.36}$$

where $M = (M_t)_{t\ge 0}$ is a continuous local martingale given by:

$$M_t \;=\; \int_0^t \frac{\partial V_*}{\partial x}(X_r,S_r)\; d\beta_r \tag{2.37}$$

and $I = (I_t)_{t\ge 0}$ is defined in (2.5).

Given any stopping time τ for B satisfying $E_{x,s}(I_\tau) < \infty$, from (2.36) we get:

$$V_*(X_\tau,S_\tau) \le V_*(x,s) + M_\tau + cI_\tau\;. \tag{2.38}$$

Moreover, it is easily seen by (2.35) that:

$$V_*(X_\tau,S_\tau) \;=\; V_*(x,s) + M_\tau + cI_\tau \tag{2.39}$$

for all stopping times τ for B satisfying $\tau \le \tau_*$. Since $V_*(x,s) \ge s$ for all $0 \le x \le s$, from (2.38) and (2.39) we get:

$$S_\tau - cI_\tau \le V_*(x,s) + M_\tau \tag{2.40}$$

for all τ satisfying $E_{x,s}(I_\tau) < \infty$, with the equality if $\tau = \tau_*$. So if we show that:

$$E_{x,s}(M_\tau) = 0 \tag{2.41}$$

then by (2.40) (with the equality for $\tau = \tau_*$) we get:

$$V(x,s) = \sup_\tau E_{x,s}\big(S_\tau - cI_\tau\big) = V_*(x,s) \tag{2.42}$$

for all $0 \le x \le s$. Thus the proof will be completed as soon as we show that (2.41) holds for all bounded stopping times τ for B , and for $\tau = \tau_*$. For this we shall need to know when $E(\tau_*)^r < \infty$ for $r > 0$. The answer to this question is known and can be found in [10]. It follows from there that $E(\tau_*)^r < \infty$ if and only if $r < c$. Since $c > 1/2$ this indicates that it is enough to prove (2.41) for all stopping times τ for B satisfying $E(\tau^r) < \infty$ for some $r > 1/2$.

Let τ be a stopping time for B satisfying $E(\tau^r) < \infty$ for some $r > 1/2$ given and fixed. To prove (2.41), by Burkholder-Gundy's inequality for continuous local martingales and Doob's optional sampling theorem (see [9]), it is enough to show that:

$$E_{x,s}\bigg(\int_0^\tau \Big(\frac{\partial V_*}{\partial x}(X_r,S_r)\Big)^2 dr\bigg)^{1/2} < \infty\ . \tag{2.43}$$

For this, note that from (2.34) by Hölder's inequality we have:

$$\begin{aligned} E_{x,s}\bigg(\int_0^\tau \Big(\frac{\partial V_*}{\partial x}(X_r,S_r)\Big)^2 dr\bigg)^{1/2} &\le 2c\, E_{x,s}\bigg(\int_0^\tau \log^2\big(1+X_r\big)\, dr\bigg)^{1/2} \\ \le 2c\, E_{x,s}\Big(\sqrt{\tau}\log\big(1+S_\tau\big)\Big) &\le 2c\, \Big(E_{x,s}\big(\tau^{p/2}\big)\Big)^{1/p}\Big(E_{x,s}\big(\log^q\big(1+S_\tau\big)\big)\Big)^{1/q} \end{aligned} \tag{2.44}$$

where $p\, ,\, q > 1$ and $1/p + 1/q = 1$. Since $\log(1+x) \le \lambda_\delta x^\delta$ for all $x \ge 0$ and all $\delta > 0$ (with $\lambda_\delta = (1/\delta) \vee 1$) , setting $\delta = p/q$ by Burkholder-Gundy's inequality (1.2) we get:

$$\begin{aligned} E_{x,s}\big(\log^q\big(1+S_\tau\big)\big) &\le \lambda_\delta^q E_{x,s}\big(S_\tau^p\big) \le \lambda_\delta^q 2^{p-1}\Big(s^p + E_{x,s}\Big(\max_{0\le t\le\tau}|B_t|^p\Big)\Big) \\ &\le \lambda_\delta^q 2^{p-1}\Big(s^p + D_p E_{x,s}\big(\tau^{p/2}\big)\Big)\ . \end{aligned} \tag{2.45}$$

From (2.44) and (2.45) we see that (2.43) follows if we take $p = 2r$. The proof is complete. □

In the next step we refine the inequality (2.1) by taking the infimum over all $c > 1/2$ on the right-hand side. To prove the sharpness of the resulting inequality we present a new simple argument which can be used in similar contexts (compare with [3]).

Corollary 2.2

Let $B=(B_t)_{t\ge 0}$ *be standard Brownian motion started at zero under* P *. Then the following inequality is shown to be satisfied:*

$$E\Big(\max_{0\le t\le\tau}|B_t|\Big)\le\frac{1}{2}E\Big(\int_0^\tau\frac{dt}{1+|B_t|}\Big)+\sqrt{2}\Big(E\int_0^\tau\frac{dt}{1+|B_t|}\Big)^{1/2} \tag{2.46}$$

for all stopping times τ *for* B *. Moreover, this inequality is sharp, and the equality in (2.46) is attained at each stopping time* τ_c *from (2.2) whenever* $c>1/2$ *. In particular, we have:*

$$E\Big(\int_0^{\tau_c}\frac{dt}{1+|B_t|}\Big)=\frac{2}{(2c-1)^2} \tag{2.47}$$

for all stopping times τ_c *from (2.2) with* $c>1/2$ *.*

Proof. In view of (2.1) consider the following function:

$$F(c;E(I_\tau))=cE(I_\tau)+\frac{1}{2c-1} \tag{2.48}$$

for $c>1/2$ with $I_\tau=\int_0^\tau(1+|B_t|)^{-1}dt$, where τ is a stopping time for B satisfying $E(I_\tau)<\infty$. Then $c\mapsto F(c;E(I_\tau))$ attains its minimum at $c_*=c_*(E(I_\tau))=(1/2)+1/\sqrt{2E(I_\tau)}$ with:

$$\inf_{c>1/2}F(c;E(I_\tau))=F(c_*(E(I_\tau));E(I_\tau))=\frac{1}{2}E(I_\tau)+\sqrt{2}\sqrt{E(I_\tau)}\ . \tag{2.49}$$

This expression combined with (2.1) gives (2.46). To prove the sharpness of (2.46), note that by (2.1) with $S_\tau=\max_{0\le t\le\tau}|B_t|$ we have:

$$E\big(S_\tau\big)\le F(c_*(E(I_\tau));E(I_\tau))\le F(c;E(I_\tau)) \tag{2.50}$$

for all $c>1/2$ for all stopping times τ satisfying $E(I_\tau)<\infty$. It was proved in Theorem 2.1 that for each $c>1/2$ we have $E\big(S_{\tau_c}\big)=F(c;E(I_{\tau_c}))$, so that both inequalities in (2.50) are equalities when $\tau=\tau_c$. This shows the sharpness of (2.46) for each given and fixed τ_c with $c>1/2$. Moreover, the second equality in (2.50) for $\tau=\tau_c$ is equivalently written as follows:

$$\frac{1}{2}E(I_{\tau_c})+\sqrt{2}\sqrt{E(I_{\tau_c})}=cE(I_{\tau_c})+\frac{1}{2c-1} \tag{2.51}$$

which can be solved in $E(I_{\tau_c})$, and this gives (2.47). The proof is complete.

□

The results of Theorem 2.1 and Corollary 2.2 extend to the case when the Brownian motion starts at any given point. This extension is indicated in the next corollary. For simplicity, some of the explicit expressions are omitted.

Corollary 2.3

Let $B=(B_t)_{t\ge 0}$ *be standard Brownian motion started at zero under* P *. Then the following inequality is shown to be satisfied:*

$$E\Big(\max_{0\le t\le\tau}|B_t+x|\Big)\le cE\Big(\int_0^\tau\frac{dt}{1+|B_t+x|}\Big)+V_c(x) \tag{2.52}$$

for all stopping times τ *for* B *all* $c>1/2$ *and all* $x\ge 0$ *, where* $V_c(x)$ *is given by:*

$$\begin{aligned} V_c(x) &= \frac{1}{2c-1}+2c\Big((1+x)\log(1+x)-x\Big) \;\; \text{if } x\le 1/(2c-1) \\ &= 2c(1+x)\log\Big(1+\frac{1}{2c-1}\Big)-1 \;\; \text{if } x>1/(2c-1)\,. \end{aligned} \tag{2.53}$$

Moreover, for each $c>1/2$ *and* $x\ge 0$ *given and fixed, the equality in (2.52) is attained at the stopping time* τ_c *from (2.2). Finally, taking the infimum over all* $c>1/2$ *on the right-hand side in (2.52) gives a sharp inequality (the equality is attained at each stopping time* τ_c *from (2.2) whenever* $c>1/2$ *and* $x\ge 0$ *).*

Proof. These facts follow from the proofs of Theorem 2.1 and Corollary 2.2 with $V_c(x)$ being equal to $V(x,x)$ from (2.3) and explicitly given by (2.14)+(2.23)+(2.30).

□

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