J. Theoret. Probab. Vol. 11 No. 1 1998, (259-277)
Research Report No. 335, 1995, *Dept. Theoret. Statist.* Aarhus (18 pp)

Optimal Stopping and Maximal Inequalities for Linear Diffusions

S. E. Graversen[1] **and G. Peskir**[1, 2]

Received March 25, 1996; revised September 19, 1996

Given a linear diffusion the solution is found to the optimal stopping problem where the gain is given by the maximum of the process and the cost is proportional to the duration of time. The optimal stopping boundary is shown to be the maximal solution of a nonlinear differential equation expressed in terms of the scale function and the speed measure. Applications to maximal inequalities are indicated.

KEY WORDS: Optimal stopping; linear diffusion; speed measure.

1. INTRODUCTION

1.1. Formulation of the Problem

Let $((X_t), \mathbf{P}_x)$ denote the canonical diffusion on an interval $I \subseteq \mathbb{R}$ (see Ref. 4, p. 271) with local characteristic operator on $\mathring{I}$ (the interior of I) given by

$$\tfrac{1}{2}\sigma^2(x)\, \partial^2/\partial x^2 + \mu(x)\, \partial/\partial x$$

where σ^2 and μ are assumed to be continuous and $\sigma^2 > 0$ on $\mathring{I}$. Denoting by (S_t) the maximal process $(\max_{0 \leq s \leq t} X_s)$ the object of this note is to study the following kind of optimal stopping problems:

For fixed $c > 0$ determine the value function

$$V_c(x) := \sup_{\tau} V_c(x, \tau) \tag{1.1}$$

[1] Institute of Mathematics, University of Aarhus, Ny Munkegade, 8000 Aarhus, Denmark. E-mail: matseg{goran}@mi.aau.dk.
[2] Department of Mathematics, University of Zagreb, Bijenička 30, 41000 Zagreb, Croatia.

where for stopping times τ

$$V_c(x,\tau) := \begin{cases} \mathbf{E}_x[S_\tau - c\tau] & \text{if } \mathbf{E}_x[\tau] < \infty \\ -\infty & \text{if } \mathbf{E}_x[\tau] = \infty \end{cases}$$

Problems of this type are important in many applied contexts (see Refs. 2, 3, and 5). But they are also interesting from a theoretical point of view since they provide maximal inequalities like

$$\mathbf{E}_x[\max_{0 \leqslant s \leqslant \tau} X_s] \leqslant \inf_{c>0}(V_c(x) + c \cdot \mathbf{E}_x[\tau]) \tag{1.2}$$

The solutions to (1.1) will generally be of the form $V_c(x) = V_c(x, \tau_{g_c})$ where

$$\tau_{g_c} := \inf\{t > 0 \mid X_t \leqslant g_c(S_t)\}$$

for some real valued continuous functions g_c defined on I satisfying $g_c(x) < x$ for $x \in \mathring{\mathrm{I}}$. The existence of such optimal strategies indicates that the maximal inequalities (1.2) are rather sharp.

2. THE VALUE FUNCTION

In view of the applications we have in mind we shall assume that $\mathrm{I} = [0, \infty)$, but it will be obvious that the considerations are generally valid. In accordance with standard notation we denote by (θ_t) the usual shift operators and let the term stopping time refer to $(\mathfrak{F}^\circ_{t+})$-stopping times where $(\mathfrak{F}^\circ_t)$ is the natural filtration. Likewise we use the notation.

$$\tau_x := \inf\{t > 0 \mid X_t = x\} \qquad \text{for} \quad x \geqslant 0$$

$$\tau_{a,b} := \inf\{t > 0 \mid X_t \notin (a,b)\} \qquad \text{for} \quad 0 \leqslant a < b$$

Less standard we define for any $m \geqslant 0$ and any continuous function g on I satisfying $g(x) < x$ for $x \in \mathring{\mathrm{I}}$

$$\tau_g := \inf\{t > 0 \mid X_t \leqslant g(S_t)\}$$

$$\tau_g^m := \inf\{t > 0 \mid X_t \leqslant g(m \vee S_t)\}$$

Observe that τ_g and τ_g^m are stopping times and that $(m, \omega) \to \tau_g^m(\omega)$ is measurable as a function of two variables. Furthermore if $\tau_g > \tau_{a,b}$ $\mathbf{P}_x$ a.s. for $x \in (a, b)$ where $0 < a < b$ then $\mathbf{P}_x$ a.s.

$$\tau_g = \tau_{a,b} + \tau_g^m \cdot \theta_{\tau_{a,b}} \qquad \text{where} \quad m = S_{\tau_{a,b}} \tag{2.1}$$

Equation (2.1) is a special case of the following property which we shall use several times:

If τ_2 is a stopping time of the form τ_x, $\tau_{a,b}$ or τ_g as defined earlier then for any stopping time τ_1 there exists a function $\tau'(\cdot, \cdot)$ such that

$$\tau_1 = \tau_2 + \tau'(\cdot, \theta_{\tau_2}) \qquad \text{on} \quad \{\tau_1 \geqslant \tau_2\} \quad \mathbf{P}_x \text{ a.s.} \qquad \text{for all} \quad x \in \mathring{\mathrm{I}} \tag{2.2}$$

and $\tau'(\cdot, \cdot)$ is measurable w.r.t. $\mathfrak{F}^\circ_{\tau_2+} \times \mathfrak{F}^\circ_\infty$ being a stopping time as a function of the second variable for every fixed first argument. (see Courrège and Priouret,[1] for a more general result). Defining for $x \in \mathring{\mathrm{I}}$ the scale

$$S(x) := \int_1^x \varphi(y)\, dy$$

where

$$\varphi(x) = \exp\left(-\int_1^x 2\mu(t)/\sigma^2(t)\, dt\right)$$

the following identities for $0 < a < x < b$ are well known

$$\mathbf{P}_x(X_{\tau_{a,b}} = a) = 1 - \mathbf{P}_x(X_{\tau_{a,b}} = b) = \frac{\mathrm{S}(b) - \mathrm{S}(x)}{\mathrm{S}(b) - \mathrm{S}(a)} \tag{2.3}$$

$$\begin{aligned} \mathbf{E}_x[\tau_{a,b}] &= 2\int_a^x \frac{\mathrm{S}(b) - \mathrm{S}(x)}{\mathrm{S}(b) - \mathrm{S}(a)} \cdot \frac{\mathrm{S}(y) - \mathrm{S}(a)}{\sigma^2(y)\,\varphi(y)}\, dy \\ &\quad + 2\int_x^b \frac{\mathrm{S}(x) - \mathrm{S}(a)}{\mathrm{S}(b) - \mathrm{S}(a)} \cdot \frac{\mathrm{S}(b) - \mathrm{S}(y)}{\sigma^2(y)\,\varphi(y)}\, dy \end{aligned} \tag{2.4}$$

$$\begin{aligned} \mathbf{P}_x(S_{\tau_{a,b}} \geqslant t, X_{\tau_{a,b}} = a) &= \mathbf{P}_x(X_{\tau_{a,t}} = t, X_{\tau_{a,b}} = a) \\ &= P_x(X_{\tau_{a,t}} = t) \cdot \mathbf{P}_t(X_{\tau_{a,b}} = a) \\ &= \frac{\mathrm{S}(x) - \mathrm{S}(a)}{\mathrm{S}(b) - \mathrm{S}(a)} \cdot \frac{\mathrm{S}(b) - \mathrm{S}(t)}{\mathrm{S}(t) - \mathrm{S}(a)} \qquad \text{for} \quad x \leqslant t < b \end{aligned} \tag{2.5}$$

Now let us fix a $c > 0$. For sake of convenience we shall drop c in the notation. The main result of this section concerns the value function V.

Proposition 1. If V is finite then it is continuous, increasing and $V(x) > x$ for all $x > 0$.

Proof. Assume V is finite. We shall divide up the proof into four parts.

Step 1. $V(x) > x$ for all $x > 0$.

Taking $\tau \equiv 0$ we see that $V(x) \geqslant x$ for all $x > 0$. Let $x > 0$ be given. The assumptions on σ^2 and μ make it possible to choose points $r_n \downarrow x$ and $l_n \uparrow x$ such that

$$\mathbf{P}_x(X_{\tau_{r_n, l_n}} = r_n) = 1/2$$

and

$$A < (r_n - x)/(r_n - l_n) < 1$$

for all n for some positive constant $0 < A < 1$. Furthermore using formula (2.4) we see that for some constant $B > 0$ independent of n $\mathbf{E}_x[\tau_{l_n, r_n}] \leqslant B \cdot (r_n - l_n)^2$ which together with the inequality

$$V(x) \geqslant V(x, \tau_{l_n, r_n}) \geqslant x + 1/2(r_n - x) - c \cdot \mathbf{E}_x[\tau_{l_n, r_n}]$$

shows that $V(x) > x$.

Step 2. V is increasing.

Let $0 < h < x$ be given. For all bounded stopping times τ we have by strong Markov property

$$\begin{aligned} V(x-h, \tau) &= \mathbf{E}_{x-h}[S_\tau - c\tau, \tau \leqslant \tau_x] + \mathbf{E}_{x-h}[S_\tau - c\tau, \tau_x < \tau] \\ &\leqslant x \cdot \mathbf{P}_{x-h}[\tau \leqslant \tau_x] + \mathbf{E}_{x-h}[V(x), \tau_x < \tau] \\ &\quad - c \cdot \mathbf{E}_{x-h}[\tau_x \wedge \tau] \leqslant V(x) \end{aligned}$$

thus V is increasing.

Step 3. V is right continuous.

Let $x > 0$ be given. Choose points $r_n \downarrow x$ and $l_n \uparrow x$ such that $\mathbf{P}_x(X_{\tau_{l_n, r_n}} = r_n) \to 1$ for $n \to \infty$. For $n \geqslant 1$ denote by $\tau(r_n)$ and $\tau(l_n)$ $1/n$-optimal stopping times for r_n and l_n respectively. Define

$$\tau(n) := \tau_{l_n, r_n} + \tau(r_n) \circ \theta_{\tau_{l_n, r_n}} \mathbf{1}_{\{X_{\tau_{l_n, r_n}} = r_n\}} + \tau(l_n) \circ \theta_{\tau_{l_n, r_n}} \mathbf{1}_{\{X_{\tau_{l_n, r_n}} = l_n\}}$$

Applying strong Markov property we get

$$\begin{aligned} V(x, \tau(n)) &= \mathbf{E}_x[S_{\tau(n)} - c\tau(n), X_{\tau_{l_n, r_n}} = r_n] + \mathbf{E}_x[S_{\tau(n)} - c\tau(n), X_{\tau_{l_n, r_n}} = l_n] \\ &= \mathbf{E}_x[\mathbf{E}_{r_n}[S_{\tau(r_n)} - c\tau(r_n)], X_{\tau_{l_n, r_n}} = r_n] \\ &\quad + \mathbf{E}_x[\mathbf{E}_{l_n}[m \vee S_{\tau(l_n)} - c\tau(l_n)]_{m = S_{\tau_{l_n, r_n}}}, X_{\tau_{l_n, r_n}} = l_n] - c \cdot \mathbf{E}_x[\tau_{l_n, r_n}] \\ &\geqslant (V(r_n) - 1/n) \cdot \mathbf{P}_x(X_{\tau_{l_n, r_n}} = r_n) + (V(l_n) - 1/n) \cdot \mathbf{P}_x(X_{\tau_{l_n, r_n}} = l_n) \\ &\quad - c \cdot \mathbf{E}_x[\tau_{l_n, r_n}] \xrightarrow[n \to \infty]{} V(x+) \end{aligned}$$

i.e., $V(x) \geqslant V(x+)$ and therefore V is right continuous at x.

Step 4. V is left continuous.
Let $x > 0$ be given. Choose points $l_n < x_n < x$ such that $l_n \uparrow x$ and

$$\mathbf{P}_{x_n}(X_{\tau_{l_n, x}} = x) \to 1 \qquad \text{for} \quad n \to \infty.$$

$V(x-) \geqslant V(x_n) \geqslant V(x_n, \tau(n))$ where

$$\tau(n) := \tau_{l_n, x} + \tau(x, n) \circ \theta_{\tau_{l_n, x}} \mathbf{1}_{\{X_{\tau_{l_n, x}} = x\}} + \tau(l_n) \circ \theta_{\tau_{l_n, x}} \mathbf{1}_{\{X_{\tau_{l_n, x}} = l_n\}}$$

and $\tau(x, n)$ and $\tau(l_n)$ are $1/n$-optimal stopping times for x and l_n. Arguing as before we get

$$\begin{aligned} V(x_n, \tau(n)) &= \mathbf{E}_{x_n}[S_{\tau(n)} - c\tau(n), X_{\tau_{l_n, x}} = x] + \mathbf{E}_{x_n}[S_{\tau(n)} - c\tau(n), X_{\tau_{l_n, x}} = l_n] \\ &= \mathbf{E}_{x_n}[\mathbf{E}_x[S_{\tau(x, n)} - c\tau(x, n)], X_{\tau_{l_n, x}} = x] \\ &\quad + \mathbf{E}_{x_n}[\mathbf{E}_{l_n}[m \vee S_{\tau(l_n)} - c\tau(l_n)]_{m = S_{\tau_{l_n, x}}}, X_{\tau_{l_n, x}} = l_n] - c \cdot \mathbf{E}_{x_n}[\tau_{l_n, x}] \\ &\geqslant (V(x) - 1/n) \cdot \mathbf{P}_{x_n}(X_{\tau_{l_n, x}} = x) + (V(l_n) - 1/n) \cdot \mathbf{P}_{x_n}(X_{\tau_{l_n, x}} = l_n) \\ &\quad - c \cdot \mathbf{E}_{x_n}[\tau_{l_n, x}] \xrightarrow[n \to \infty]{} V(x) \end{aligned}$$

i.e., $V(x-) \geqslant V(x)$ and thus V is left continuous. □

As an important corollary of Proposition 1 we have the following result on existence of uniform $1/n$-optimal stopping rules. V is supposed to be finite.

Corollary 1. For every $0 < a < b$ and every $n \geqslant 1$ there exists a stopping time $\tau(n)$ such that

$$V(x, \tau(n)) > V(x) - 1/n \qquad \text{for all} \quad x \in [a, b]$$

Proof. Let a, b and n be given. Using the continuity of V and formula (2.4) we can find $m \geqslant 1$ such that

$$\sup_{\substack{x,\, y \in [a, b] \\ |x-y| \leqslant 2^{-m}}} |V(x) - V(y)| + \sup_{\substack{x \in [a, b],\, x_1 < x < x_2 \\ |x_1 - x_2| \leqslant 2^{-m}}} c \cdot \mathbf{E}_x[\tau_{x_1, x_2}] \leqslant 1/2n$$

Now choose for each $j \geqslant 1$ an $1/2n$-optimal stopping rule $\tau^{j,m}$ for $j/2^m$ and define a stopping time $\tau(n)$ as follows

$$\tau(n) := \sum_j (\tau_{j-1/2^m,\, j/2^m} + \tau^{j,m} \circ \theta_{\tau_{(j-1)/2^m,\, j/2^m}} \mathbf{1}_{\{X_{\tau_{(j-1)/2^m,\, j/2^m}} = j/2^m\}}$$
$$+ \tau^{j-1,m} \circ \theta_{\tau_{(j-1)/2^m,\, j/2^m}} \mathbf{1}_{\{X_{\tau_{(j-1)/2^m,\, j/2^m}} = (j-1)/2^m\}}) \cdot \mathbf{1}_{](j-1)/2^m,\, j/2^m]}(X_0)$$

It is now immediately seen that $\tau(n)$ so defined has the right property. □

Assume from now on that V is finite all over. Let g: $[0, \infty) \to [0, \infty)$ be continuous such that $g(x) < x$ for all $x > 0$ and such that $\{g = 0\}$ is of the form $[0, s_g]$ for some $s_g \geqslant 0$. Assume furthermore that $\mathbf{E}_x[\tau_g]$ is finite for all $x > s_g$. Notice by (2.1) that in this case the interval (s_g, ∞) can be covered by intervals (a, b) such that $g(t) < a$ for all $t \in (a, b)$ and $\sup_{a<x<b} \mathbf{E}_x[\tau_g] < \infty$. Writing $V_g(x)$ for $V(x, \tau_g)$ the following results will be useful.

Lemma 1. V_g is absolutely continuous on $\{g > 0\}$.

Lemma 2. If g is a.e. differentiable then V_g is $\mathscr{C}^1$ on $\{g > 0\}$ and satisfies the equation

$$V_g'(x) = \frac{\varphi(x)}{S(x) - S(g(x))} \cdot \left((V_g(x) - x) + 2c \int_{g(x)}^{x} \frac{S(y) - S(g(x))}{\sigma^2(y)\, \varphi(y)}\, dy \right)$$

Proof of Lemma 1. It is sufficient to study V_g on intervals (a, b) of the type mentioned above. For $x \in (a, b)$ $\tau_g > \tau_{a,b}$ $\mathbf{P}_x$ a.s. therefore by (2.1) and strong Markov property we have

$$V_g(x) = \mathbf{E}_x[S_{\tau_g} - c\tau_g, \tau_g > \tau_{a,b}]$$
$$= \mathbf{E}_x[\mathbf{X}_{X_{\tau_{a,b}}}[m \vee S_{\tau_g^m} - c\tau_g^m]_{m = S_{\tau_{a,b}}} - c\tau_{a,b}]$$
$$= V_g(b) \cdot \mathbf{P}_x(X_{\tau_{a,b}} = b) + \mathbf{E}_x[H(S_{\tau_{a,b}}), X_{\tau_{a,b}} = a] - c \cdot \mathbf{E}_x[\tau_{a,b}]$$

where $H(m) = \mathbf{E}_a[m \vee S_{\tau_g^m}] - c \cdot \mathbf{E}_a[\tau_g^m]$ for $x \leqslant m \leqslant b$. The assumptions imply that H is bounded. Thus for any $x \in (a, b)$ we have

$$V_g(x) = \frac{\mathrm{S}(x) - \mathrm{S}(a)}{\mathrm{S}(b) - \mathrm{S}(a)} \cdot V_g(b) + (\mathrm{S}(x) - \mathrm{S}(a))$$

$$\times \int_x^b H(t) \cdot \frac{\varphi(t)}{(\mathrm{S}(t) - \mathrm{S}(a))^2} \, dt - c \cdot \mathbf{E}_x[\tau_{a,b}]$$

which proves Lemma 1. □

Proof of Lemma 2. Fix $x > 0$ such that $g(x) > 0$ and V_g and g are both differentiable at x. For $\varepsilon > 0$ denote by τ^ε the exit time $\tau_{g(x)+\varepsilon, r^\varepsilon + x}$ where $r^\varepsilon = \varepsilon/2(|g'(x)| + 1)$. Then for ε small enough we have $\tau^\varepsilon < \tau_g$ $\mathbf{P}_x$ a.s. As before this implies

$$V_g(x) = V_g(x + r^\varepsilon) \cdot \mathbf{P}_x(X_{\tau^\varepsilon} = x + r^\varepsilon)$$

$$+ \mathbf{E}_x[\tilde{H}(X_{\tau^\varepsilon}), X_{\tau^\varepsilon} = g(x) + \varepsilon] - c \cdot \mathbf{E}_x[\tau^\varepsilon]$$

where $\tilde{H}(m) = \mathbf{E}_{g(x)+\varepsilon}[m \vee S_{\tau_g^m}] - c \cdot \mathbf{E}_{g(x)+\varepsilon}[\tau_g^m]$ for $x \leqslant m \leqslant x + r^\varepsilon$, and therefore

$$V_g(x + r^\varepsilon) - V_g(x) = (V_g(x + r^\varepsilon) - x) \cdot \mathbf{P}_x(X_{\tau^\varepsilon} = g(x) + \varepsilon)$$

$$+ c \cdot \mathbf{E}_x[\tau^\varepsilon] - \mathbf{E}_x[(\tilde{H}(S_{\tau^\varepsilon}) - x), X_{\tau^\varepsilon} = g(x) + \varepsilon]$$

Observe that $\lim_{\varepsilon \to 0} \mathbf{E}_x[(\tilde{H}(S_{\tau^\varepsilon}) - x), X_{\tau^\varepsilon} = g(x) + \varepsilon]/\varepsilon = 0$. Dividing by r^ε and letting ε tend to 0 we derive using (2.3) and (2.4) the identity

$$V_g'(x) = \frac{\varphi(x)}{\mathrm{S}(x) - \mathrm{S}(g(x))} \cdot \left((V_g(x) - x) + 2c \int_{g(x)}^x \frac{\mathrm{S}(y) - \mathrm{S}(g(x))}{\sigma^2(y)\, \varphi(y)} \, dy \right) \qquad (2.6)$$

Since V_g is absolutely continuous and the right-hand side is continuous (2.6) holds for all $x > 0$ proving Lemma 2. □

3. A RELATED OPTIMAL STOPPING PROBLEM

In order to proceed we shall consider yet another optimal stopping problem. For given $s > 0$ and $a_s > s$ determine

$$\tilde{V}(s, x) := \sup_\tau \mathbf{E}_x[h(X_\tau) - c\tau] \qquad \text{for} \quad x \leqslant s \qquad (3.1)$$

where

$$h(x)=\begin{cases} s & x<s \\ a_s & x \geqslant s \end{cases}$$

Taking $\tau \equiv 0$ we see that

$$s \leqslant \tilde{V}(s,x) \leqslant \tilde{V}(s,s)=a_s \qquad \text{for all} \quad x \leqslant s$$

Lemma 3. $\tilde{V}(s,\cdot)$ is increasing for all s.

Proof. Fix $s>0$ and let $x<y<s$ be given. Subtracting s out we may assume $h=a_s \mathbf{1}_{[s,\infty)}$. For all bounded stopping times τ we have using strong Markov property

$$\begin{aligned} \mathbf{E}_x[h(X_\tau)-c\tau] &= a_s \cdot \mathbf{P}_x(X_\tau \geqslant s)-c \cdot \mathbf{E}_x[\tau] \\ &= a_s \cdot \mathbf{P}_x(X_\tau \geqslant s, \tau>\tau_y)-c \cdot \mathbf{E}_x[\tau, \tau>\tau_y]-c \cdot \mathbf{E}_x[\tau, \tau \leqslant \tau_y] \\ &\leqslant \tilde{V}(s,y) \cdot \mathbf{P}_x(\tau>\tau_y)-c \cdot \mathbf{E}_x[\tau \wedge \tau_y] \leqslant \tilde{V}(s,y) \end{aligned}$$

which proves that $\tilde{V}(s,\cdot)$ is increasing. □

Define

$$\mathrm{C}(s):=\{x \in (0,s) \mid \exists \varepsilon>0 \ \mathbf{E}_x[h(X_{\tau_{y,s}})-c\tau_{y,s}]>s \text{ for } y \in (x-\varepsilon, x)\}$$

that is $x \in \mathrm{C}(s)$ if for $y<x$ and sufficiently close to x

$$(a_s-s) \cdot \mathbf{P}_x(X_{\tau_{y,s}}=s)-c \cdot \mathbf{E}_x[\tau_{y,s}]=(a_s-s) \cdot \frac{\mathrm{S}(x)-\mathrm{S}(y)}{\mathrm{S}(s)-\mathrm{S}(y)}-c \cdot \mathbf{E}_x[\tau_{y,s}]>0$$

or equivalently

$$\frac{\mathbf{E}_x[\tau_{y,s}]}{\mathrm{S}(x)-\mathrm{S}(y)} \cdot (\mathrm{S}(s)-\mathrm{S}(y))<(a_s-s)/c$$

Letting $y \uparrow x$ and using (2.4) the left-hand side tends to

$$2\int_x^s \frac{\mathrm{S}(s)-\mathrm{S}(t)}{\sigma^2(t)\,\varphi(t)}\,dt$$

and since this expression is continuous and strictly decreasing in x with limit 0 for $x \uparrow s$ we see that

$$(g(s), s) \subseteq \mathrm{C}(s) \subseteq [g(s), s)$$

Here $g(s)$ denotes the strictly positive solution to

$$2c \int_{g(s)}^{s} \frac{\mathrm{S}(s)-\mathrm{S}(t)}{\sigma^2(t)\,\varphi(t)}\,dt = a_s - s \tag{3.2}$$

if such a solution exists and $g(s) := 0$ otherwise. Notice that $g(s) < s$ and that g is continuous if $s \to a_s$ is continuous.

Remark. Before stating the solution to the optimal stopping problem (3.1) we shall for later use make a few comments concerning the Eq. (3.2). Using standard notation for classification of boundary points we see that for all $s > 0$

$$\lim_{x \to 0} 2c \int_{x}^{s} \frac{\mathrm{S}(s)-\mathrm{S}(t)}{\sigma^2(t)\,\varphi(t)}\,dt = \infty$$

if 0 is either a natural or an exit boundary point, i.e., in these cases $g(s) > 0$ for all s. Among the remaining cases we shall only consider that of 0 being either a regular instantaneously reflecting boundary point or an entrance boundary point. Important here is that

$$\mathbf{E}_0[\tau_s] = 2 \int_{0}^{s} \frac{\mathrm{S}(s)-\mathrm{S}(t)}{\sigma^2(t)\,\varphi(t)}\,dt \qquad \text{for all} \quad s > 0$$

in the reflecting regular case and that a similar identity holds in the entrance case with the left-hand side replaced by $\mathbf{E}_{0+}[\tau_s]$.

The solution to the optimal stopping problem (3.1) can now be stated.

Proposition 3. If $g(s) > 0$ or $\tilde{V}(s, 0+) = s$ then $\tau_{g(s),s}$ is a (unique) optimal stopping rule for every $x > 0$, if $g(s) = 0$ and $\tilde{V}(s, 0+) > s$ the (unique) optimal stopping time is τ_s.

Proof. We shall only consider the case $g(s) > 0$ and leave the rest to the reader. From the definition of $\mathrm{C}(s)$ it is clear that for any x only stopping times which satisfy the condition $\mathbf{P}_x(X_\tau \in (g(s), s)) = 0$ have a chance of being optimal. Now let $x \in (g(s), s)$ be given and consider a stopping time τ such that $\tau \geqslant \tau_{(g(s),s}$ $\mathbf{P}_x$ a.s. Using strong Markov property we have

$$\begin{aligned}\mathbf{E}_x[h(X_\tau)-c\tau] &= \mathbf{E}_x[h(X_\tau), \tau \geqslant \tau_{g(s),s}] - c\cdot\mathbf{E}_x[\tau, \tau\geqslant \tau_{g(s),s}]\\ &\leqslant \tilde{V}(s,s)\cdot \mathbf{P}_x(X_{\tau_{g(s),s}}=s)\\ &+\tilde{V}(s,g(s))\cdot\mathbf{P}_x(X_{\tau_{g(s),s}}=g(s))-c\cdot\mathbf{E}_x[\tau_{g(s),s}]\\ &\leqslant \tilde{V}_*(s,x)+\tilde{V}(s,g(s))\cdot\mathbf{P}_x(X_{\tau_{g(s),s}}=g(s))-s\cdot\frac{\mathrm{S}(s)-\mathrm{S}(x)}{\mathrm{S}(s)-\mathrm{S}(g(s))}\end{aligned}$$

where

$$\begin{aligned}\tilde{V}_*(s,x) &= \mathbf{E}_x[h(X_{\tau_{g(s),s}})-c\tau_{g(s),s}]\\ &= a_s\cdot\frac{\mathrm{S}(x)-\mathrm{S}(g(s))}{\mathrm{S}(s)-\mathrm{S}(g(s))}+s\cdot\frac{\mathrm{S}(s)-\mathrm{S}(x)}{\mathrm{S}(s)-\mathrm{S}(g(s))}-c\cdot\mathbf{E}_x[\tau_{g(s),s}]\end{aligned}$$

Since τ was arbitrary we may conclude

$$0\leqslant \tilde{V}(s,x)-\tilde{V}(s,g(s))\leqslant(\tilde{V}_*(s,x)-s)-(\tilde{V}(s,g(s))-s)\cdot\frac{\mathrm{S}(x)-\mathrm{S}(g(s))}{\mathrm{S}(s)-\mathrm{S}(g(s))}$$

Dividing through with $x-g(s)$ and letting $x\downarrow g(s)$ we see that $\tilde{V}(s,g(s))$ equals s if

$$\frac{\tilde{V}_*(s,x)-s}{x-g(s)}\to 0 \qquad \text{for} \quad x\downarrow g(s) \tag{3.3}$$

Straightforward computations based on the definition of $g(s)$ and formula (2.4) show that (3.3) is satisfied. Thus $\tilde{V}(s,g(s))=s$ and therefore $\tilde{V}(s,\cdot)=\tilde{V}_*(s,\cdot)$ that is $\tau_{g(s),s}$ is an optimal stopping rule for every $x>0$. (Uniqueness is clear.) □

4. THE OPTIMAL STOPPING RULE

The connection to our original optimal stopping problem (1.1) takes place through the particular problem of (3.1)-type specified by $a_s=V(s)$. Let for simplicity $\tilde{V}_*(s,\cdot)$ denote the corresponding value function and g_* the related function determined by (3.2). Recall that $0\leqslant g_*(s)<s$ and that

$$2c\int_{g_*(s)}^{s}\frac{\mathrm{S}(s)-\mathrm{S}(t)}{\sigma^2(t)\,\varphi(t)}\,dt=V(s)-s \tag{4.1}$$

whenever $0<g_*(s)$. To see the connection between the two problems observe that for given $0<x<s$ and stopping time τ with $\mathbf{E}_x[\tau]<\infty$ we have

$$\begin{aligned}\mathbf{E}_x[s \vee S_\tau - c\tau] &= \mathbf{E}_x[s \vee S_\tau - c\tau, \tau \leqslant \tau_s] + \mathbf{E}_x[s \vee S_\tau - c\tau, \tau > \tau_s] \\ &= \mathbf{E}_x[s - c\tau, \tau \leqslant \tau_s] + \mathbf{E}_x[\mathbf{E}_s[S_{\tau'(\cdot,\,)} - c\tau'(\cdot,\,)] - c\tau_s, \tau > \tau_s] \\ &\leqslant \mathbf{E}_x[s - c\tau, \tau \leqslant \tau_s] + \mathbf{E}_x[V(s) - c\tau_s, \tau > \tau_s] \\ &\leqslant \mathbf{E}_x[h(X_{\tau \wedge \tau_s}) - c\tau \wedge \tau_s] \leqslant \tilde{V}_*(s, x)\end{aligned}$$

where $h(x)=s$ if $x<s$ and $h(x)=V(s)$ otherwise. Motivated by this we are lead to think that τ_{g_*} is an optimal stopping rule for (1.1). But before proving this we shall use this remark to derive some more information about g_*. In the case of 0 being a natural or an exit boundary point $g_*(s)>0$ for all $s>0$. In the remaining two cases under consideration we see that

$$2\int_0^{s+\varepsilon} \frac{\mathrm{S}(s+\varepsilon)-\mathrm{S}(t)}{\sigma^2(t)\,\varphi(t)}\,dt - 2\int_0^{s} \frac{\mathrm{S}(s)-\mathrm{S}(t)}{\sigma^2(t)\,\varphi(t)}\,dt = \mathbf{E}_s[\tau_{s+\varepsilon}]$$

for all s and ε strictly positive and since by strong Markov property

$$(V(s+\varepsilon)-(s+\varepsilon))-(V(s)-s) \leqslant c\cdot \mathbf{E}_s[\tau_{s+\varepsilon}]-\varepsilon$$

we may conclude that if 0 is either a regular instantaneously reflecting boundary point or an entrance boundary point then the following description is true:

There exist $s_* \geqslant 0$ such that

$$2c\int_0^{s} \frac{\mathrm{S}(s)-\mathrm{S}(t)}{\sigma^2(t)\,\varphi(t)}\,dt < (=)(>)\, V(s)-s$$

according to if $s<s_*$, $s=s_*$ or $s>s_*$ respectively. That is $g_*(s)=0$ for all $s \leqslant s_*$ and $g_*(s)>0$ for $s>s_*$. Setting $s_*=0$ in the natural or exit boundary point case we see that this statement is generally true.

A first step in the direction of proving that τ_{g_*} is an optimal stopping rule for (1.1) is contained in the following result.

Lemma 4. $V(x,\tau) \leqslant V(x, \tau \wedge \tau_{g_*})$ for every $x>s_*$ and every stopping time τ.

Proof. Let x and τ be given. Without loss of generality we can assume τ to be finite and having finite expectation. Then

$$\begin{aligned}
V(x,\tau) &= \mathbf{E}_x[S_\tau - c\tau, \tau \leqslant \tau_{g_*}] + \mathbf{E}_x[S_\tau - c\tau, \tau > \tau_{g_*}] \\
&= \mathbf{E}_x[S_\tau - c\tau, \tau \leqslant \tau_{g_*}] + \mathbf{E}_x[\mathbf{E}_{X_{\tau_{g_*}}}[s \vee S_{\tau'} - c\tau']_{s=S_{\tau_{g_*}}} - c\tau_{g_*}, \tau > \tau_{g_*}] \\
&\leqslant \mathbf{E}_x[S_\tau - c\tau, \tau \leqslant \tau_{g_*}] + \mathbf{E}_x[\tilde{V}_*(S_{\tau_{g_*}}, X_{\tau_{g_*}}) - c\tau_{g_*}, \tau > \tau_{g_*}] \\
&\leqslant \mathbf{E}_x[S_\tau - c\tau, \tau \leqslant \tau_{g_*}] + \mathbf{E}_x[S_{\tau_{g_*}} - c\tau_{g_*}, \tau > \tau_{g_*}] \\
&\leqslant V(x, \tau \wedge \tau_{g_*}) \quad \square
\end{aligned}$$

To prove the optimality of τ_{g_*} we need still another technical lemma.

Lemma 5. Let $x > s_*$ be given such that $\mathbf{E}_x[\tau_{g_*}]$ is finite. Then to any stopping time $\tau \leqslant \tau_{g_*}$ with $\mathbf{P}_x(\tau < \tau_{g_*}) > 0$ there exists a stopping time τ_1 such that: (i) $\tau \leqslant \tau_1 \leqslant \tau_{g_*}$, (ii) $\mathbf{P}_x(\tau_1 < \tau_{g_*}) < \mathbf{P}_x(\tau < \tau_{g_*})$ and (iii) $V(x,\tau) \leqslant V(x,\tau_1)$.

Proof. Since $\mathbf{P}_x(\tau < \tau_{g_*}) > 0$ there exist $a < x < b$ and $n \geqslant 1$ such that the $\mathfrak{F}^\circ_{\tau+}$-measurable set $B := \{X_\tau > g_*(S_\tau),\ a < X_\tau \leqslant S_\tau < b,\ \tilde{V}_*(S_\tau, X_\tau) > S_\tau + 1/n,\ \tau < \tau_{g_*}\}$ has strictly positive probability under $\mathbf{P}_x$. Define

$$\tau^* := \tau + \tau_{g_*(S_\tau), S_\tau}(\theta_\tau \cdot) \qquad \text{and} \qquad \bar{\tau} = \tau^* + \tau(n) \circ \theta_{\tau^*} \cdot \mathbf{1}_{\{X_{\tau^*} = S_\tau\}}$$

where $\tau(n)$ is defined like in Corollary 1 corresponding to the given constants a, b and n. Now let

$$\tilde{\tau} := \tau \cdot \mathbf{1}_{B^c} + \bar{\tau} \cdot \mathbf{1}_B$$

$\tilde{\tau}$ is clearly a stopping time bigger than τ having finite expectation and by construction we see that $\mathbf{P}_x(\tilde{\tau} = \tau_{g_*}) > \mathbf{P}_x(\tau = \tau_{g_*})$ since the diffusion is regular and $\tilde{\tau} = \tau_{g_*}$ on $B \cap \{X_{\tau^*} = g_*(S_\tau)\}$. Furthermore

$$V(x, \tilde{\tau}) = \mathbf{E}_x[S_\tau - c\tau, B^c] + \mathbf{E}_x[S_{\bar{\tau}} - c\bar{\tau}, B]$$

and since by strong Markov property the last term equals

$$\begin{aligned}
&\mathbf{E}_x[(\mathbf{E}_y[s \vee S_{\tau_{g_*(s),s} + \tau(n)\circ\theta_{\tau_{g_*(s),s}}} - c\tau(n) \circ \theta_{\tau_{g_*(s),s}} - c\tau_{g_*(s),s}, X_{\tau_{g_*(s),s}} = s] \\
&\quad + \mathbf{E}_y[s \vee S_{\tau_{g_*(s),s}} - c\tau_{g_*(s),s}, X_{\tau_{g_*(s),s}} = g_*(s)])_{s=S_\tau}^{y=X_\tau} - c\tau, B] \\
&= \mathbf{E}_x[(\mathbf{E}_y[\mathbf{E}_s[S_{\tau(n)} - c\tau(n)], X_{\tau_{g_*(s),s}} = s] + s \cdot \mathbf{P}_y(X_{\tau_{g_*(s),s}} = g_*(s)) \\
&\quad - c \cdot \mathbf{E}_y[\tau_{g_*(s),s}])_{s=S_\tau}^{y=X_\tau} - c\tau, B] \geqslant \mathbf{E}_x[\tilde{V}_*(S_\tau, X_\tau) - 1/n - c\tau, B]
\end{aligned}$$

we have that

$$V(x, \tilde{\tau}) \geqslant \mathbf{E}_x[S_\tau - c\tau, B^c] + \mathbf{E}_x[S_\tau - c\tau, B] = V(x, \tau)$$

thus using Lemma 4 $\tau_1 := \tilde{\tau} \wedge \tau_{g_*}$ is seen to have the right properties. □

We are now in position to state and prove the main theorem.

Theorem 1. Let $x > s_*$ be given such that $\mathbf{E}_x[\tau_{g_*}]$ is finite. Then τ_{g_*} is a (unique) optimal stopping rule, i.e. $V(x, \tau) \leqslant V(x, \tau_{g_*})$ for every stopping time τ.

Proof. From Lemma 4 we know that it suffices to consider stopping times $\tau \leqslant \tau_{g_*}$ such that $\mathbf{P}_x(\tau < \tau_{g_*}) > 0$. Making repeated use of Lemma 5 we can construct a sequence $(\tau_n)_{n \geqslant 1}$ of stopping times such that for all n:

(i) $\tau_n \leqslant \tau_{n+1} \leqslant \tau_{g_*}$,

(ii) $\mathbf{P}_x(\tau_{n+1} < \tau_{g_*}) < \mathbf{P}_x(\tau_n < \tau_{g_*})$

(iii) $V(x, \tau_n) \leqslant V(x, \tau_{n+1})$.

Defining $\tau_\infty := \sup_n \tau_n$ we see that τ_∞ is again a stopping time dominated by τ_{g_*} such that $\mathbf{P}_x(\tau_\infty < \tau_{g_*}) < \mathbf{P}_x(\tau_n < \tau_{g_*})$ for all n. By the monotone convergence theorem we also have $V(x, \tau_n) \leqslant V(x, \tau_\infty)$ for all n from which a straightforward argument based upon induction along the ordinals proves the statement. The uniqueness of τ_{g_*} follows from Proposition 3 and Lemma 5. □

Theorem 1 treats only starting points bigger than s_* but using Proposition 3 and the continuity of V similar arguments can be applied to handle the general case. Recalling that hitting times to any level bigger than the starting point have finite expectation under our boundary assumption we have the following corollary to Theorem 1.

Corollary. Assume that $\mathbf{E}_x[\tau_{g_*}]$ is finite for all $x \geqslant s_*$. Then $\tau_{s_*} + \tau_{g_*} \cdot \theta_{\tau_{s_*}}$ is an optimal stopping rule for all $x < s_*$, i.e. $V(x, \tau) \leqslant V(x, \tau_{s_*} + \tau_{g_*} \cdot \theta_{\tau_{s_*}}) = V(s_*, \tau_{g_*}) - c \cdot \mathbf{E}_x[\tau_{s_*}]$ for every stopping time τ.

Assume from now on that $\mathbf{E}_x[\tau_{g_*}]$ is finite for all $x > s_*$ and let us derive a differential equation for g_*. From (4.1) we know that

$$V(s) = s + 2c \int_{g_*(s)}^{s} \frac{S(s) - S(t)}{\sigma^2(t)\, \varphi(t)}\, dt \qquad \text{for} \quad s > s_* \tag{4.2}$$

from which we deduce by means of Proposition 1 and the implicit function theorem that g_* is a.e. differentiable. Thus using Theorem 1 and Lemma 2 we get for $s > s_*$

$$V'(s) = \frac{\varphi(s)}{S(s) - S(g_*(s))} \cdot \left((V(s) - s) + 2c \int_{g_*(s)}^{s} \frac{S(y) - S(g_*(s))}{\sigma^2(y)\,\varphi(y)}\,dy \right) \qquad (4.3)$$

Inserting (4.2) into (4.3) gives

$$V'(s) = 2c\varphi(s) \int_{g_*(s)}^{s} \frac{1}{\sigma^2(y)\,\varphi(y)}\,dy \qquad \text{for} \quad s > s_* \qquad (4.4)$$

Repeated use of the implicit function theorem shows that g_* is continuously differentiable on (s_*, ∞) and by differentiating (4.2) and combining the result with (4.4) we deduce that g_* on (s_*, ∞) solves the nonlinear differential equation

$$g'(s) = \frac{1}{2c} \cdot \frac{\sigma^2(g(s)) \cdot \varphi(g(s))}{S(s) - S(g(s))} \qquad (4.5)$$

Observe that the right-hand is strictly positive, i.e., all solutions to (4.5) are strictly increasing functions.

5. THE MAXIMALITY PRINCIPLE

Our goal for this section is to give sufficient conditions for g_* to be the maximal solution to the differential equation (4.5) subject to the condition $g(s) < s$. We believe the statement is generally true. We shall assume that there exists a standard Wiener process (B_t) such that for every $x > 0$

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dB_t \quad \mathbf{P}_x \qquad \text{a.s.}$$

and that $\mathcal{M}$ is nonempty where $\mathcal{M}$ denotes the set of functions g satisfying

$$0 \leqslant g(s) < s,\ g \neq 0 \qquad \text{and} \qquad g \text{ solves (4.5) on } (s_g, \infty) = \{g > 0\}$$

Simple arguments shows that $\mathcal{M}$ contains a maximal element g^*. Associate to g^* the following nonnegative functions

$$D_{g^*}(x, s) := \begin{cases} 2c \cdot \varphi(x) \displaystyle\int_{g^*(s)}^{x} \frac{1}{\sigma^2(t)\,\varphi(t)}\,dt & \text{for} \quad g^*(s) < x < s \\ 0 & \text{for} \quad 0 \leqslant x \leqslant g^*(s) \end{cases} \qquad (5.1)$$

and

$$W_{g^*}(s) := s + 2c\int_{g^*(s)}^{s} \frac{S(s)-S(t)}{\sigma^2(t)\,\varphi(t)}\,dt \qquad \text{for} \quad s>0 \tag{5.2}$$

The main result of this section can now be stated as follows.

Theorem 2. $g_* = g^*$ if $E_x[\tau_{g^*}] + E_x[(\int_0^{\tau_{g^*}} D_{g^*}^2(X_t, S_t)\cdot\sigma^2(X_t)\,dt)^{1/2}] < \infty$ for all $x > s_{g^*}$.

Proof. For simplicity we shall consider only the case where g^* is strictly positive. Let $\tilde{W}_{g^*}$ denote the real valued function defined for $0 < x \leqslant s$ by the formulas:

$$\tilde{W}_{g^*}(x,s) = W_{g^*}(s) \qquad \text{for} \quad 0<x=s,$$

$$\tilde{W}_{g^*}(x,s) = s \qquad \text{for} \quad 0<x\leqslant g^*(s), \qquad \text{and}$$

$$\begin{aligned}\tilde{W}_{g^*}(x,s) = s + (W_{g^*}(s)-s)\cdot\frac{S(x)-S(g^*(s))}{S(s)-S(g^*(s))} \\ -2c\int_{g^*(s)}^{x}\frac{S(s)-S(x)}{S(s)-S(g^*(s))}\cdot\frac{S(y)-S(g^*(s))}{\sigma^2(y)\,\varphi(y)}\,dy \\ -2c\int_{x}^{s}\frac{S(x)-S(g^*(s))}{S(s)-S(g^*(s))}\cdot\frac{S(s)-S(y)}{\sigma^2(y)\,\varphi(y)}\,dy\end{aligned}$$

for $g^*(s) < x < s$.

Straightforward calculations show that

$$\partial/\partial s\ \tilde{W}_{g^*}(x,s)_{|x=s-} \equiv 0$$

$$\partial/\partial x\ \tilde{W}_{g^*}(x,s) = D_{g^*}(x,s)$$

and

$$L_x \tilde{W}_{g^*}(x,s) = c \qquad \text{for} \quad g^*(s)<x<s$$

where $L_x \tilde{W}_{g^*}(x,s) := \frac{1}{2}\sigma^2(x)\,\partial^2/\partial x^2\ \tilde{W}_{g^*}(x,s) + \mu(x)\,\partial/\partial x\ \tilde{W}_{g^*}(x,s)$. Note that

$$\partial/\partial x\ \tilde{W}_{g^*}(x,s)_{|x=g^*(s)+} \equiv 0 \qquad \text{and} \qquad s \leqslant \tilde{W}_{g^*}(x,s)$$

Applying now Itô's formula we get $\mathbf{P}_x$ a.s. for every $x, t > 0$

$$\tilde{W}_{g^*}(X_t, S_t) = \tilde{W}_{g^*}(x, x) + \int_0^t \partial/\partial x\ \tilde{W}_{g^*}(X_s, S_s) \cdot \sigma(X_s)\, dB_s$$

$$+ \int_0^t \mathbf{L}_x \tilde{W}_{g^*}(X_s, S_s)\, ds + \int_0^t \partial/\partial s\ \tilde{W}_{g^*}(X_s, S_s)\, dS_s$$

$$= \tilde{W}_{g^*}(x, x) + \int_0^t D_{g^*}(X_s, S_s) \cdot \sigma(X_s)\, dB_s + \int_0^t \mathbf{L}_x \tilde{W}_{g^*}(X_s, S_s)\, ds$$

and therefore using the before-mentioned properties

$$\tilde{W}_{g^*}(X_\tau, S_\tau) \leqslant \tilde{W}_{g^*}(x, x) + \int_0^\tau D_{g^*}(X_s, S_s) \cdot \sigma(X_s)\, dB_s + c \cdot \tau$$

for all finite stopping times τ with equality if $\tau \leqslant \tau_{g^*}$. In particular

$$S_{\tau_{g_*}} = \tilde{W}_{g^*}(X_{\tau_{g_*}}, S_{\tau_{g_*}})$$

$$= \tilde{W}_{g^*}(x, x) + \int_0^{\tau_{g_*}} D_{g^*}(X_s, S_s) \cdot \sigma(X_s)\, dB_s + c \cdot \tau_{g_*}$$

which by the assumptions imply that $S_{\tau_{g^*}}$ is $\mathbf{P}_x$-integrable and that we have

$$\tilde{W}_{g^*}(x, x) = \mathbf{E}_x[S_{\tau_{g_*}} - c\tau_{g_*}]$$

Now choose a localization $(\tau_n)_{n \geqslant 1}$ such that

$$\left(\int_0^{t \wedge \tau_n} D_{g^*}(X_s, S_s) \cdot \sigma(X_s)\, dB_s \right)$$

is a martingale for every n. For every bounded stopping time τ and every n we then have

$$\mathbf{E}_n[S_{\tau \wedge \tau_n}] \leqslant \tilde{W}_{g^*}(x, x) + c \cdot \mathbf{E}_x[\tau \wedge \tau_n]$$

Letting n tend to infinity and taking supremum over all such τ's we obtain

$$V(x) = \tilde{W}_{g^*}(x, x) = W_{g^*}(x)$$

Comparing formulas (4.1) and (5.2) we conclude that $g_* = g^*$. □

Observe that g^* is the only $g \in \mathcal{M}$ for which the integral condition in Theorem 2 may be satisfied.

6. AN EXAMPLE

Let $((X_t), \mathbf{P}_x)$ denote the Bessel process of dimension α, where $\alpha > 1$ but $\alpha \neq 2$ (the remaining cases can be treated similarly). That is a diffusion on $[0, \infty)$ with local characteristic operator given by

$$\frac{1}{2}\partial^2/\partial x^2 + \frac{\alpha-1}{2x}\partial/\partial x$$

and 0 being a regular instantaneously reflecting boundary point if $\alpha < 2$ and an entrance boundary point if $\alpha > 2$. Straightforward calculations show that for any $g \in \mathcal{M}$

$$D_g(x, s) = \frac{2c}{\alpha}\cdot(x - x^{1-\alpha}g^{\alpha}(x)) \qquad \text{for} \quad g(s) < x < s \tag{6.1}$$

Using this formula we get for every x by applying Hölder's and Burkholder–Davis–Gundy's inequalities for Bessel processes that

$$\begin{aligned}
&\mathbf{E}_x\left[\sqrt{\int_0^{\tau} D^2_{g^*}(X_t, S_t)\cdot\sigma^2(X_t)\,dt}\right] \\
&\quad \leqslant \frac{2c}{\alpha}\cdot\mathbf{E}_x\left[\sqrt{\int_0^{\tau} X_t^2\,dt}\right] \\
&\quad \leqslant \frac{2c}{\alpha}\cdot\mathbf{E}_x[S_\tau\cdot\sqrt{\tau}] \\
&\quad \leqslant \frac{2c}{\alpha}\cdot\sqrt{\mathbf{E}_x[S_\tau^2]}\cdot\sqrt{\mathbf{E}_x[\tau]} \\
&\quad \leqslant \frac{2c}{\alpha}\cdot\sqrt{K_\alpha(x^2 + \mathbf{E}_x[\tau])}\cdot\sqrt{\mathbf{E}_x[\tau]} \qquad \text{for some constant } K_\alpha
\end{aligned}$$

for all finite stopping times τ. Thus to verify the condition in Theorem 2 it is sufficient to show that $\mathbf{E}_x[\tau_{g_*}]$ is finite for all x. As already remarked Burkholder-Davis-Gundy's inequalities for Bessel processes show that the value function V grows at most linearly. Combining this with (4.2) gives that

$$\lim_{s\to\infty} g_*(s)/s = 1 \tag{6.2}$$

Recalling that hitting times to any level bigger than the starting point have finite expectation, we see using (6.2) that $E_x[\tau_{g_*}]$ will be finite for all x if $E_x[\tau_\lambda]$ is finite for some $\lambda < 1$ where

$$\tau_\lambda := \inf\{t > 0 \mid X_t \leqslant \lambda S_t\}$$

To do this we shall for $(2/\alpha)^{1/(\alpha-2)} < \lambda < 1$ consider the function m_λ defined by

$$m_\lambda(x, s) = -\frac{x^2}{\alpha} + \left(\frac{\lambda^\alpha}{\alpha} - \frac{\lambda^{2\alpha-2}}{\alpha\lambda^{\alpha-2} - 2}\right) \cdot \frac{s^\alpha}{x^{\alpha-2}} + \frac{\lambda^\alpha}{\alpha\lambda^{\alpha-2} - 2} \cdot s^2 \qquad (6.3)$$

for $\lambda s \leqslant x \leqslant s$. Now it is easily verified that m_λ is nonnegative and satisfies

$$\partial/\partial s \, m_\lambda(x, s)_{|x=s-} \equiv 0$$

$$m_\lambda(x, s)_{|x=\lambda s+} \equiv 0$$

and

$$\mathbf{L}_x m_\lambda(x, s) = -1 \qquad \text{for} \quad \lambda s < x < s$$

From this we get for all $x, t > 0$ by applying Itô's formula and arguing as before that

$$m_\lambda(X_t, S_t) = m_\lambda(x, x) + \int_0^t \partial/\partial x \, m_\lambda(X_s, S_s)\, dB_s$$

$$+ \int_0^t \mathbf{L}_x m_\lambda(X_s, S_s)\, ds + \int_0^t \partial/\partial x \, m_\lambda(X_s, S_s)\, dS_s$$

$$\leqslant m_\lambda(x, x) + \int_0^t \partial/\partial x \, m_\lambda(X_s, S_s)\, dB + \int_0^t \mathbf{L}_x m_\lambda(X_s, S_s)\, ds$$

Fixing $x > 0$ and denoting by τ_D the exit time from any relatively compact open subset D of $\{(u, v) \mid \lambda v < u \leqslant v\}$ containing (x, x), we deduce from this identity using the properties of m_λ that $E_x[\tau_D] \leqslant m_\lambda(x, x)$. Letting D increase to $\{(u, v) \mid \lambda v < u \leqslant v\}$ we conclude using monotone convergence that $E_x[\tau_\lambda]$ is smaller than $m_\lambda(x, x)$ and therefore finite. Thus Theorem 2 is applicable and therefore $g_* = g^*$ i.e., an optimal stopping boundary is the maximal solution to the differential equation (4.5) subject to the condition that the solution shall stay below the diagonal. Observe that this example is also studied in Ref. 2.

REFERENCES

1. Courrège, Ph. and Priouret, P. (1965). Temps d'arrêt d'une fonction aléatoire, *Publ. Inst. Stat. Paris* 14, 245–274.
2. Dubins, L. E., Shepp, L. and Shiryaev, A. N. (1993). Optimal stopping rules and maximal inequalities for Bessel processes, *Theory Prob. Appl.* **38**, 226–261.
3. Graversen, S. E. and Peskir, G. (1995). Optimal stopping and maximal inequalities for geometric Brownian motion, Research Report No. 330, Dept. Theoretical Statistics Aarhus, p. 18.
4. Rogers, L. C. G. and Williams, D. (1987). *Diffusions, Markov Processes, and Martingales, Vol. 2, Itô Calculus*, John Wiley and Sons.
5. Shepp, L. and Shiryaev, A. N. (1993). The Russian option: reduced regret, *Ann. Appl. Prob.* **3**, 631–640.