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Optimal Prediction of Resistance and Support Levels

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Assuming that the asset price X follows a geometric Brownian motion we study the optimal prediction problem

$$\inf_{0 \le \tau \le T} \mathsf{E}|X_\tau - \ell|$$

where the infimum is taken over stopping times τ of X and ℓ is a hidden aspiration level (having a potential of creating a resistance or support level for X). Adopting the 'aspiration level hypothesis' and assuming that ℓ is independent from X we show that a wide class of admissible (non-oscillatory) laws of ℓ lead to unique optimal trading boundaries that can be viewed as the 'conditional median curves' for the resistance and support levels (with respect to X and T). We prove the existence of these boundaries and derive the (nonlinear) integral equations which characterise them uniquely. The results are illustrated through some specific examples of admissible laws and their conditional median curves.

1. Introduction

An important problem in technical analysis of asset prices is to determine/predict their *resistance* and *support* levels (see e.g. [3]). The struggle between buyers (demand) and sellers (supply) moves the price up and down often creating its upward or downward trend as well. Resistance and support levels are the price levels at which a majority of traders are willing to sell the asset (rather than buy it) and vice versa respectively. When these levels are reached the price is being pushed down (in the case of a resistance level) or up (in the case of a support level) for a period of time. Knowledge of the resistance and support levels may therefore be exploited to derive favourable trading strategies and make profits. The key issue in this regard is that the resistance and support levels are not directly observable and hence may be seen as *hidden targets* (in the terminology of [5]). Methods and practices of technical analysis used to determine/predict the resistance and support levels appear to be largely based on heuristic arguments and open to various interpretations. The aim of the present paper is to build on established empirical facts and lay the foundations for a simple and rigourous method that yields the optimal trading strategies and provides a platform for further development.

The approach to be introduced rests on the *aspiration level hypothesis* (dating back to [9]) stating that traders buying/selling an asset already have a target price in mind at which they are willing to sell/buy the asset in the future (see [10] for an agreement with observed data). This

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aspiration level for the asset price can be formed by combining the knowledge of past data of the asset price and volume of its trade (including estimates of the asset's fundamental value), insider information, and the market psychology. For example, the *volume level hypothesis* states that the price levels at which there are higher accumulation of volume (the number of shares/contracts traded) are more likely to become future reference points at which traders are willing to sell or buy assets (see [2] for an agreement with observed data).

It appears to be unrealistic however to expect that all small/individual traders will have the same aspiration level in mind. To reflect this fact we consider a representative trader (as an amalgam of small/individual traders [8, Section 15.1]) and assume that his/her aspiration level (as an amalgam of individual aspiration levels) is a random variable denoted by ℓ. Based on the knowledge referred to above the representative trader then forms a distribution function F of ℓ. In addition, the representative trader chooses a horizon $T>0$ by which the trading decision (sell or buy) should take place, and then considers the optimal prediction problem

$$\inf_{0\le\tau\le T} \mathsf{E}|X_\tau - \ell| \tag{1.1}$$

where X is the observed asset price process and the infimum is taken over stopping times τ of X. A stopping time τ_* at which the infimum in (1.1) is attained then yields the optimal trading strategy for the representative trader.

It is important to understand in this context that the aspiration level ℓ has only a *potential* of creating a resistance or support level for the observed asset price (consistent with the past price/volume data and other possible input referred to above). This does not mean however that a resistance or support level will be created exactly at ℓ. Rather ℓ serves as a reference point to the representative trader to determine the optimal selling or buying strategy which in turn (when accumulated in higher volume) will create a resistance or support level respectively. The key point in this approach (to distinguish it from the classical detection approach described in [7, Chapter VI] and the references therein) is that a resistance or support level does not realise itself independently from the representative trader (observer) but only through the representative trader's action (similarly to the observer's role in quantum mechanics vs classical mechanics for instance). For this reason there is no need for the representative trader to assume in (1.1) that X changes its probabilistic characteristics at ℓ. Given that ℓ is not directly observable and that X naturally develops local maxima and minima (each of which could potentially correspond to a resistance or support level) we see that this approach makes it more challenging to detect ℓ (reflecting the fact that its indirect disclosure in the market may be weak) but also offers better chances to avoid delay in detecting it (as there is no benefit from waiting to spot probabilistic changes in the observed asset price directly).

To illustrate the methodology described above, in the present paper we assume that the asset price X follows a geometric Brownian motion (starting at $x>0$) with drift $\mu \in I\!R$ and volatility $\sigma>0$. Both μ and σ can be formed using the past price/volume data and other available input as described above. This yields a subjective view/estimate of μ (given that more accurate estimates require long horizons) and a more reliable estimate of σ. Setting aside accuracy of these estimates we formally assume that both drift $\mu \in I\!R$ and volatility $\sigma>0$ of X are given and fixed. In the case $\mu>0$ (reflecting a positive trend) the representative trader aims to detect/predict when to sell the asset (or equivalently when X is due to reach its resistance level), and in the case $\mu<0$ (reflecting a negative trend) the representative trader aims to detect/predict when to buy the asset (or equivalently when X is due to reach its support level). We assume that the aspiration level ℓ is independent from the observed price X after its initial value x. Note that this assumption is broadly consistent with the aspiration level hypothesis and the volume level hypothesis recalled above where we *emphasise* that the law of ℓ may still *depend* on the past price/volume data. We further assume that a distribution function F of ℓ and a horizon $T>0$ are given and fixed as described above.

Under these hypotheses we show (in Sections 2 and 3) that a wide class of admissible (non-oscillatory) laws of ℓ lead to unique optimal trading boundaries in the problem (1.1) that can be viewed as the 'conditional median curves' for the resistance and support levels (with respect to X and T). We prove the existence of these boundaries and derive the (nonlinear) integral equations which characterise them uniquely (Theorem 3). The results are illustrated through some specific examples of admissible laws and their conditional median curves (Section 4).

To address what happens subsequently to the optimal trade it is important to realise that the asset price X is naturally assumed to follow a geometric Brownian motion with the given drift μ and volatility σ only up to the optimal stopping time τ_* in (1.1). Indeed, in line with the general arguments exposed following (1.1) above, the optimal trading action triggered at τ_* (when accumulated in higher volume) creates a resistance or support level and changes the probabilistic characteristics of X (often reversing the sign of μ), after which the representative trader is naturally faced with a new problem (1.1), and this procedure can then be sequentially continued as long as needed. In the present paper we do not discuss the subsequent action in detail and leave this open for future development.

The optimal prediction problem (1.1) may be viewed as a finite horizon version of the space domain problem studied in [5] (for a finite horizon solution in the time domain see Remark 3.6 in that paper). We refer to [6] for a quickest detection formulation of the time domain problem in infinite horizon and to [1] for a rigourous optimality argument for the choice of the golden retracement in technical analysis of asset prices. The methodologies used in these papers are different from the methodology developed in the present paper.

2. Formulation of the problem

In this section we formulate the optimal prediction problem (1.1) when the asset price X follows a geometric Brownian motion and introduce a wide class of admissible (non-oscillatory) laws for the aspiration level ℓ as discussed in the previous section. These considerations will be continued in the next section.

1. Let X be a geometric Brownian motion solving

$$dX_t = \mu X_t\, dt + \sigma X_t\, dB_t \tag{2.1}$$

with $X_0 = x$ for $x>0$, where $\mu \in I\!R$ is the drift, $\sigma>0$ is the volatility, and B is a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$. It is well known that the stochastic differential equation (2.1) has a unique strong solution given by

$$X_t^x = x \exp\big(\sigma B_t + (\mu - \tfrac{\sigma^2}{2})t\big) \tag{2.2}$$

for $t \ge 0$. The law of the process X^x on the canonical space will be denoted by P_x. Thus under P_x the coordinate process X starts at x. It is well known that X is a strong Markov process with respect to P_x for $x>0$.

Consider the optimal prediction problem

$$V_*(x) = \inf_{0 \le \tau \le T} \mathsf{E}|X_\tau^x - \ell| \tag{2.3}$$

where the infimum is taken over all stopping times τ of X that are bounded above by a given and fixed horizon $T > 0$, and $\ell > 0$ is a random variable that is independent from X. Recall that X represents the observed asset price and ℓ corresponds to the aspiration level of the representative trader aiming to solve (2.3) and derive the optimal trading strategy. Denoting the distribution function of ℓ by F we begin by disclosing the underlying structure of the problem (2.3) in fuller detail.

Lemma 1. *The following identity holds*

$$\mathsf{E}|x - \ell| = 2\int_0^x \big(F(y) - \tfrac{1}{2}\big)\, dy + \mathsf{E}\ell \tag{2.4}$$

for all $x > 0$.

Proof. Note that

$$\begin{aligned} \mathsf{E}|x - \ell| &= \int_0^\infty |x-y|\, F(dy) = \int_0^x (x-y)\, F(dy) + \int_x^\infty (y-x)\, F(dy) \\ &= 2\int_0^x (x-y)\, F(dy) + \int_0^\infty (y-x)\, F(dy) = 2\int_0^x (x-y)\, F(dy) + \mathsf{E}\ell - x \end{aligned} \tag{2.5}$$

for $x > 0$ given and fixed where we use that $F(0) = 0$ and $F(\infty) = 1$. Setting $u = x - y$ and $dv = F(dy)$ we see that integration by parts yields

$$\int_0^x (x-y)\, F(dy) = (x-y)F(y)\Big|_0^x + \int_0^x F(y)\, dy = \int_0^x F(y)\, dy\, . \tag{2.6}$$

Combining (2.5) and (2.6) we obtain (2.4) as claimed. $\square$

2. Assuming that $\mathsf{E}\ell < \infty$ and setting

$$G(x) = \int_0^x \big(F(y) - \tfrac{1}{2}\big)\, dy \tag{2.7}$$

for $x > 0$ we see from (2.4) using the independence of ℓ from X that the optimal prediction problem (2.3) reduces to the optimal stopping problem

$$V(x) = \inf_{0 \le \tau \le T} \mathsf{E}\, G(X_\tau^x) \tag{2.8}$$

where the infimum is taken over stopping times τ of X. Note that $V_*(x) = 2V(x) + \mathsf{E}\ell$ for $x > 0$. From (2.7) we see that $x \mapsto G'(x) = F(x) - 1/2$ is increasing on $(0, \infty)$ and this shows that $x \mapsto G(x)$ is convex on $(0, \infty)$. Moreover, the form of G' suggests to recall that a number $m \in (0, \infty)$ is called a *median* of ℓ if $F(m-) \le 1/2 \le F(m)$. The set of all medians of ℓ is a bounded and closed interval $[m, M]$ where m is the lowest median of ℓ and M is the highest median of ℓ (they can also be equal in which case ℓ has a unique median). From the facts that $G' = F-1/2 < 0$ on $(0,m)$ and $G' = F-1/2 > 0$ on (M,∞) we see that $x \mapsto G(x)$ is strictly decreasing on $(0,m)$ and strictly increasing on (M,∞) with G being 0 at 0 and constant on $[m,M]$.

3. To tackle the problem (2.8) we will enable the process X to start at arbitrary points at any allowable time and consider the extended optimal stopping problem

$$V(t,x) = \inf_{0 \le \tau \le T-t} \mathsf{E}_{t,x} G(X_{t+\tau}) \tag{2.9}$$

where $X_t = x$ under the probability measure $\mathsf{P}_{t,x}$ for $(t,x) \in [0,T]\times(0,\infty)$. Note that $V(x) = V(0,x)$ for $x > 0$. Note also from (2.2) that $\mathsf{E}_{t,x} G(X_{t+\tau})$ in (2.9) can be replaced by $\mathsf{E}_x G(X_\tau) = \mathsf{E}\, G(X_\tau^x) = \mathsf{E}\, G(xX_\tau^1)$ when analysing the problem and we will often do that with no explicit mention. General optimal stopping theory for Markov processes (see e.g. [7, pp. 46-49]) implies that the continuation set C and the stopping set D are given by

$$C = \{\, (t,x) \in [0,T]\times(0,\infty) \mid V(t,x) < G(x) \,\} \tag{2.10}$$

$$D = \{\, (t,x) \in [0,T]\times(0,\infty) \mid V(t,x) = G(x) \,\} . \tag{2.11}$$

It means that the first entry time of X to D given by

$$\tau_D := \inf \{\, s \in [0,T-t] \mid (t+s, X_{t+s}) \in D \,\} \tag{2.12}$$

is optimal in (2.9) (in the sense that the infimum in (2.9) is attained at this stopping time).

4. Our aim in the sequel is to determine the optimal stopping set D . We begin by making a simple observation that if $\mu = 0$ then it is optimal to stop at once in (2.9). Indeed, since G is convex we find by Jensen's inequality that

$$\mathsf{E}_x G(X_\tau) \ge G(\mathsf{E}_x X_\tau) = G(x\,\tilde{\mathsf{E}} e^{\mu\tau}) = G(x) \tag{2.13}$$

for every stopping time τ of X with values in $[0,T-t]$ where we use (2.2) and the fact that $M_s := \exp(\sigma B_s - \frac{\sigma^2}{2} s)$ is a martingale for $s \in [0,T-t]$ with $t \in [0,T]$ so that $d\tilde{\mathsf{P}} = M_\tau d\mathsf{P}$ defines a new probability measure. From (2.13) we see that the optimal stopping time in (2.9) equals zero as claimed. Since the problems (2.3) and (2.9) are equivalent it follows in particular that when $\mu = 0$ it is optimal to stop at once in (2.3) as well. This fact has a transparent interpretation in terms of the representative trader as the martingale asset price will on average reach any aspiration level with equal return as selling or buying at once.

A small modification of the argument in (2.13) also shows that the set $[0,T]\times[m,\infty)$ is contained in D when $\mu > 0$ and the set $[0,T]\times(0,M]$ is contained in D when $\mu < 0$. Indeed, if $\mu > 0$ then $x\,\tilde{\mathsf{E}}\, e^{\mu\tau} \ge x$ for $x > 0$ and since G is increasing on $[m,\infty)$ we see that $G(x\,\tilde{\mathsf{E}} e^{\mu\tau}) \ge G(x)$ in place of the final equality in (2.13). Similarly, if $\mu < 0$ then $x\,\tilde{\mathsf{E}} e^{\mu\tau} \le x$ for $x > 0$ and since G is decreasing on $(0,M)$ we see again that $G(x\,\tilde{\mathsf{E}} e^{\mu\tau}) \ge G(x)$ in place of the final equality in (2.13). Using then the same arguments as following (2.13) above we see that the two inclusions hold as claimed.

5. The structure of D below m when $\mu > 0$ and above M when $\mu < 0$ can generally be complicated. It turns out that the complexity of this structure depends heavily on certain oscillation properties of the law of ℓ . We will now introduce a wide class of distribution

functions F of ℓ that yield a simple structure of D that can be described by means of a *single* time-dependent boundary. This fact will be established in the next section. Recall that a continuous probability distribution function F on $I\!R$ with $F(0)=0$ is said to be piecewise C^1 if there exists a partition $[x_{i-1},x_i)$ of $I\!R_+$ for $i\ge 1$ such that F restricted on each $[x_{i-1},x_i]$ is C^1. To simplify the notation we will set $F'(x_i):=F'(x_i+)$ as the single value is irrelevant for the arguments to be used when F' is discontinuous at x_i for $i\ge 0$.

Definition 2 (Admissible aspiration level laws). For $\mu>0$ and $\sigma>0$ we let $\mathcal{A}(\mu,\sigma)$ denote the family of piecewise C^1 probability distribution functions F on $I\!R$ with $F(0)=0$ for which there exists $\alpha\in(0,m)$ such that

$$xF'(x)<\tfrac{\mu}{\sigma^2/2}\big(\tfrac{1}{2}-F(x)\big)\ \text{ for } x\in(0,\alpha)\ \ \&\ \ xF'(x)>\tfrac{\mu}{\sigma^2/2}\big(\tfrac{1}{2}-F(x)\big)\ \text{ for } x\in(\alpha,m) \tag{2.14}$$

where m is the lowest median of F. For $\mu<0$ and $\sigma>0$ we let $\mathcal{A}(\mu,\sigma)$ denote the family of piecewise C^1 probability distribution functions F on $I\!R$ with $F(0)=0$ for which there exists $\beta\in(M,\infty)$ such that

$$xF'(x)>\tfrac{\mu}{\sigma^2/2}\big(\tfrac{1}{2}-F(x)\big)\ \text{ for } x\in(M,\beta)\ \ \&\ \ xF'(x)<\tfrac{\mu}{\sigma^2/2}\big(\tfrac{1}{2}-F(x)\big)\ \text{ for } x\in(\beta,\infty) \tag{2.15}$$

where M is the highest median of F.

Setting $\nu=\sigma^2/2$ we see that $x\mapsto(\mu/\nu)\,(1/2-F(x))$ is decreasing when $\mu>0$ and increasing when $\mu<0$ so that (2.14) and (2.15) may be viewed as oscillatory conditions on $x\mapsto xF'(x)$ in relation to μ/ν and $x\mapsto 1/2-F(x)$. A quick sufficient condition for (2.14) is obtained when $x\mapsto xF'(x)$ is strictly increasing on $(0,m)$ and a quick sufficient condition for (2.15) is obtained when $x\mapsto xF'(x)$ is strictly decreasing on (M,∞). These simple sufficient conditions are far from being necessary and there also are other more elaborate ways to establish (2.14) and (2.15). A higher oscillation of $x\mapsto xF'(x)$ on $(0,m)$ or (M,∞) may also cause (2.14) or (2.15) to fail respectively and when this happens the problem needs to be studied on a case-by-case basis (note that the laws of this kind appear to be of secondary interest in the applied context discussed in the present paper). This also includes the setting in which F is not piecewise C^1 and may exhibit jumps for instance. We do not discuss these more singular cases in the present paper.

3. Solution to the problem

In this section we solve the optimal stopping problem (2.9) when the asset price X is given by (2.1)-(2.2) and the loss function G is given by (2.7) where F belongs to the admissible class of probability distribution functions specified in Definition 2 above. Recalling that the problems (2.3) and (2.9) are equivalent we see that this also solves the optimal prediction problem (2.3) and yields the optimal trading strategies for the representative investor having the aspiration level ℓ with the distribution function F. These results will be illustrated through some specific examples in the next section.

Below we make use of the following functions

$$J(t,x)=\mathsf{E}_x\big[G(X_{T-t})\big]=\int_0^\infty G(z)\,f(T-t,x,z)\,dz \tag{3.1}$$

$$H(x) = \mu x\big(F(x)-\tfrac{1}{2}\big) + \tfrac{\sigma^2}{2}x^2F'(x) \tag{3.2}$$

$$K(s,x,y) = \mathsf{E}_x\big[H(X_s)\,I(X_s>y)\big] = \int_y^\infty H(z)\,f(s,x,z)\,dz \tag{3.3}$$

$$L(s,x,y) = \mathsf{E}_x\big[H(X_s)\,I(X_s<y)\big] = \int_0^y H(z)\,f(s,x,z)\,dz \tag{3.4}$$

for $t \in [0,T]$, $x \in (0,\infty)$, $s \in (0,T-t]$ and $y \in (0,\infty)$ where $z \mapsto f(s,x,z)$ is the probability density function of X_s under P_x given by

$$f(s,x,z) = \frac{1}{\sigma\sqrt{s}\,z}\,\varphi\bigg(\frac{1}{\sigma\sqrt{s}}\Big[\log\Big(\frac{z}{x}\Big)+\Big(\frac{\sigma^2}{2}-\mu\Big)s\Big]\bigg) \tag{3.5}$$

for $s>0$, $x>0$ and $z>0$. Recall that φ denotes the standard normal density function given by $\varphi(x) = (1/\sqrt{2\pi})e^{-x^2/2}$ for $x \in I\!R$.

The main result of this section may now be stated as follows.

Theorem 3. *Consider the problems* (2.3) *and* (2.9) *where* X *solves* (2.1) *and* $\ell > 0$ *satisfying* $\mathsf{E}\ell < \infty$ *is independent from* X . *Suppose that the distribution function* F *of* ℓ *belongs to the class* $\mathcal{A}(\mu,\sigma)$ *for* $\mu \neq 0$ *(recall that it is optimal to stop at once in both* (2.3) *and* (2.9) *when* $\mu = 0$ *).*

If $\mu > 0$ *then the stopping set in* (2.9) *is given by* $D = \{\,(t,x) \in [0,T)\times(0,\infty) \mid x \geq b(t)\,\} \cup (\{T\}\times(0,\infty))$ *where the optimal stopping boundary* $b : [0,T) \to I\!R$ *can be characterised as the unique continuous decreasing solution to the nonlinear integral equation*

$$J(t,b(t)) = G(b(t)) + \int_t^T K(s-t,b(t),b(s))\,ds \tag{3.6}$$

satisfying $\alpha \leq b(t) \leq m$ *for* $t \in [0,T)$. *The optimal stopping boundary* b *satisfies* $b(T-) = \alpha$ *and the stopping time* τ_D *from* (2.12) *is optimal in* (2.9). *The stopping time*

$$\tau_b = \inf\{\, t \in [0,T) \mid X_t \geq b(t)\,\} \tag{3.7}$$

is optimal in (2.3). *The value function* V *from* (2.9) *admits the following representation*

$$V(t,x) = J(t,x) - \int_t^T K(s-t,x,b(s))\,ds \tag{3.8}$$

for $(t,x) \in [0,T]\times(0,\infty)$. *The value* $V_*(x)$ *from* (2.3) *equals* $2V(0,x)+\mathsf{E}\ell$ *for* $x>0$.

If $\mu < 0$ *then the stopping set in* (2.9) *is given by* $D = \{\,(t,x) \in [0,T)\times(0,\infty) \mid x \leq b(t)\,\} \cup (\{T\}\times(0,\infty))$ *where the optimal stopping boundary* $b : [0,T) \to I\!R$ *can be characterised as the unique continuous increasing solution to the nonlinear integral equation*

$$J(t,b(t)) = G(b(t)) + \int_t^T L(s-t,b(t),b(s))\,ds \tag{3.9}$$

satisfying $M \leq b(t) \leq \beta$ *for* $t \in [0,T)$. *The optimal stopping boundary* b *satisfies* $b(T-) = \beta$ *and the stopping time* τ_D *from* (2.12) *is optimal in* (2.9). *The stopping time*

$$\tau_b = \inf\{\, t \in [0,T) \mid X_t \leq b(t)\,\} \tag{3.10}$$

is optimal in (2.3). *The value function* V *from* (2.9) *admits the following representation*

$$V(t,x) = J(t,x) - \int_t^T L(s-t,x,b(s))\,ds \tag{3.11}$$

for $(t,x) \in [0,T]\times(0,\infty)$. *The value* $V_*(x)$ *from* (2.3) *equals* $2V(0,x)+\mathsf{E}\ell$ *for* $x>0$.

Proof. The proof is motivated by the facts/ideas exposed in [7] and will be carried out in several steps. We will only treat the case $\mu>0$ in full detail since the case $\mu<0$ can be dealt with analogously and details in this direction will be omitted. Thus we will assume throughout that $\mu>0$ is given and fixed.

1. We show that the value function $(t,x)\mapsto V(t,x)$ is continuous on $[0,T]\times(0,\infty)$. For this, let $t\in[0,T]$ and $x,y\in(0,\infty)$ be given and fixed. Without loss of generality we may assume that $V(t,x)\le V(t,y)$ and let $\tau_x := \tau_*(t,x)$ denote the optimal stopping time for $V(t,x)$. By the mean value theorem we then have

$$\begin{aligned} |V(t,y)-V(t,x)| &= V(t,y)-V(t,x) \le \mathsf{E}G(yX^1_{\tau_x})-\mathsf{E}G(xX^1_{\tau_x}) \\ &= \mathsf{E}\big[G'(\xi)(y-x)X^1_{\tau_x}\big] \le \tfrac{1}{2}|y-x|\,\mathsf{E}X^1_{T-t} = \tfrac{1}{2}e^{\mu(T-t)}|y-x| \end{aligned} \tag{3.12}$$

where we use that $|G'|\le 1/2$ and X^1 is a submartingale. From (3.12) we see that $x\mapsto V(t,x)$ is continuous on $(0,\infty)$ uniformly over $t\in[0,T]$. It is therefore enough to show that $t\mapsto V(t,x)$ is continuous on $[0,T]$ for each $x\in(0,\infty)$ given and fixed. For this, fix arbitrary $0\le t_1<t_2\le T$ and $x\in(0,\infty)$, and let $\tau_1 := \tau_*(t_1,x)$ denote the optimal stopping time for $V(t_1,x)$. Setting $\tau_2 := \tau_1\wedge(T-t_2)$ and noting that $t\mapsto V(t,x)$ is increasing on $[0,T]$ we see that the mean value theorem yields

$$\begin{aligned} 0 &\le V(t_2,x)-V(t_1,x) \le \mathsf{E}G(xX^1_{\tau_2})-\mathsf{E}G(xX^1_{\tau_1}) = \mathsf{E}\big[G'(\eta)\,x\,(X^1_{\tau_2}-X^1_{\tau_1})\big] \\ &= x\,\mathsf{E}\big[G'(\eta)(X^1_{T-t_2}-X^1_{\tau_1})\,I(T-t_2\le\tau_1\le T-t_1)\big] \\ &\le x\,\mathsf{E}\big[|G'(\eta)|\,|X^1_{T-t_2}-X^1_{\tau_1}|\,I(T-t_2\le\tau_1\le T-t_1)\big] \\ &\le \tfrac{1}{2}x\,\mathsf{E}\Big[\sup_{0\le s\le t_2-t_1}|X^1_{T-t_2}-X^1_{T-t_2+s}|\Big] \longrightarrow 0 \end{aligned} \tag{3.13}$$

as $t_2-t_1\to 0$ by the dominated convergence theorem since the sample path $t\mapsto X^1_t$ is (uniformly) continuous on $[0,T]$ and $\mathsf{E}(\sup_{0\le t\le T}X^1_t)<\infty$ as needed. Thus the value function $(t,x)\mapsto V(t,x)$ is continuous on $[0,T]\times(0,\infty)$ as claimed.

2. We show that the stopping set in (2.9) is given by $D=\{\,(t,x)\in[0,T)\times(0,\infty)\mid x\ge b(t)\,\}\cup(\{T\}\times(0,\infty))$ where the function $b:[0,T)\to I\!R$ is decreasing and satisfies $b(t)\in[\alpha,m]$ for $t\in[0,T)$ (recall that $\alpha\in(0,m)$ is defined in (2.14) and m is the lowest median of ℓ). For this, note that Itô-Tanaka's formula yields

$$\begin{aligned} G(X_{t+s}) &= G(x) + \int_0^s (I\!L_X G)(X_{t+u})\,du + \int_0^s \sigma X_{t+u}G'(X_{t+u})\,dB_u \\ &= G(x) + \int_0^s H(X_{t+u})\,du + M_s \end{aligned} \tag{3.14}$$

under $\mathsf{P}_{t,x}$ for $s \in [0, T{-}t]$ with $(t,x) \in [0,T){\times}(0,\infty)$ given and fixed, where $I\!\!L_X = \mu(d/dx)$ $+(\sigma^2 x^2/2)(d^2/dx^2)$ is the infinitesimal generator of X and M_s is a continuous local martingale for $s \in [0, T{-}t]$. Note that $H = I\!\!L_X G$ is given by (3.2) and M_s is a true martingale for $s \in [0, T{-}t]$ since $\mathsf{E}_{t,x}\langle M, M\rangle_{T-t} = \int_0^{T-t} \mathsf{E}_{t,x}(\sigma^2 X_{t+u}^2 (G'(X_{t+u}))^2)\, du \le (\sigma^2/4) \int_0^{T-t} \mathsf{E}_{t,x}(X_{t+u}^2)\, du = (\sigma^2 x^2/4) \int_0^{T-t} e^{(2\mu+\sigma^2)u}\, du < \infty$. Moreover, from (2.14) we see that $H(x) < 0$ for $x \in (0,\alpha)$ so that replacing s in (3.14) by the stopping time $\sigma_\alpha = \inf\{\, u \in [0, T{-}t] \mid X_{t+u} \ge \alpha \,\}$ and applying the optional sampling theorem we see that the set $[0,T){\times}(0,\alpha)$ is contained in the continuation set C. From Section 2 we know that the set $[0,T]\times[m,\infty)$ is contained in the stopping set D. Since G in (2.9) does not depend on time we see that if a point (t,x) from $[0,T){\times}[\alpha,m)$ belongs to D then every other point (s,x) with $s \in (t,T]$ belongs to D. Moreover, if a point (t,x) from $[0,T){\times}[\alpha,m)$ belongs to D then every other point (t,y) with $y > x$ belongs to D. This is evident for $y \ge m$ and for $y \in (x,m)$ it can be seen by replacing s in (3.14) by the optimal stopping time from (2.12) and applying the optional sampling theorem. Recalling from (2.14) that $H(z) > 0$ for $z \in (\alpha,m)$ while both sets $[t,T]{\times}\{m\}$ and $[t,T]{\times}\{x\}$ are contained in D this yields $\mathsf{E}_{t,y} G(X_{t+\tau_D}) > G(y)$ unless τ_D equals zero with $\mathsf{P}_{t,y}$-probability one so that (t,y) must belong to D in this case as well. Combining these conclusions with the fact that V and G are continuous we see that $b(t) := \min\{\, x \in (0,\infty) \mid V(t,x) = G(x)\,\}$ defines a unique number in $[\alpha,m]$ for $t \in [0,T)$ depicting the decreasing boundary between the (open) continuation set C and the (closed) stopping set D as claimed.

3. We show that b is continuous on $[0,T)$ and $b(T-) = \alpha$. For this, let us first show that b is right-continuous on $[0,T)$. Fix an arbitrary point $t \in [0,T)$ and consider any sequence $t_n \downarrow t$ as $n \to \infty$. Since b is decreasing we see that the right-hand limit $b(t+)$ exists. Because $(t_n, b(t_n)) \in D$ for all $n \ge 1$ and D is closed it follows that $(t, b(t+)) \in D$. Hence by the description of D in terms of b established above we see that $b(t+) \ge b(t)$. Since the reverse inequality follows from the fact that b is decreasing this shows that $b(t+) = b(t)$ and thus b is right-continuous as claimed.

Suppose now that b makes a jump at some $t \in (0,T)$ and set $x_1 = b(t)$ and $x_2 = b(t-)$ so that $x_1 < x_2$. For $\varepsilon \in (0, (x_2{-}x_1)/2)$ given and fixed, let $\psi_\varepsilon : (0,\infty) \to [0,1]$ be a C^2-function satisfying (i) $\psi_\varepsilon(x) = 1$ for $x \in [x_1{+}\varepsilon, x_2{-}\varepsilon]$ and (ii) $\psi_\varepsilon(x) = \psi_\varepsilon'(x) = 0$ for $x \in (0, x_1{+}\varepsilon/2] \cup [x_2{-}\varepsilon/2, \infty)$. Letting $I\!\!L_X^* f = -(\mu x f)' + ((\sigma^2 x^2/2) f)''$ denote the adjoint of $I\!\!L_X$ applied to f, recalling that $t \mapsto V(t,x)$ is increasing on $[0,T]$ and that $V_t + I\!\!L_X V = 0$ on C, we find integrating by parts (twice) that

$$
\begin{aligned}
(3.15)\qquad 0 &\le \int_{x_1}^{x_2} V_t(t{-}\delta, x)\, \psi_\varepsilon(x)\, dx = -\int_{x_1}^{x_2} (I\!\!L_X V)(t{-}\delta, x)\, \psi_\varepsilon(x)\, dx \\
&= -\int_{x_1}^{x_2} V(t{-}\delta, x)\, (I\!\!L_X^* \psi_\varepsilon)(x)\, dx
\end{aligned}
$$

for $\delta \in (0, t \wedge (\varepsilon/2))$ so that $\psi_\varepsilon(x_2{-}\delta) = \psi_\varepsilon'(x_2{-}\delta) = 0$ as needed and used. Letting $\delta \downarrow 0$ it follows using the dominated convergence theorem and integrating by parts (twice) that

$$
\begin{aligned}
(3.16)\qquad 0 &\le -\int_{x_1}^{x_2} V(t,x)\, (I\!\!L_X^* \psi_\varepsilon)(x)\, dx = -\int_{x_1}^{x_2} G(x)\, (I\!\!L_X^* \psi_\varepsilon)(x)\, dx \\
&= -\int_{x_1}^{x_2} (I\!\!L_X G)(x)\, \psi_\varepsilon(x)\, dx = -\int_{x_1}^{x_2} H(x)\, \psi_\varepsilon(x)\, dx\,.
\end{aligned}
$$

Letting $\varepsilon \downarrow 0$ we obtain

$$0 < \int_{x_1}^{x_2} H(x)\, dx \le 0 \tag{3.17}$$

since $x \mapsto H(x)$ is strictly positive on $(x_1, x_2]$. We have thus derived a contradiction and hence we can conclude that b is continuous on $[0,T)$ as claimed.

To see that $b(T-) = \alpha$ note first that $b(T-) \ge \alpha$ since $[0,T)\times(0,\alpha)$ is contained in C as established above. Assuming that $b(T-) > \alpha$ and setting $x_1 = \alpha$ and $x_2 = b(T-)$ so that $x_1 < x_2$ we can then proceed in exactly the same way as above to derive a contradiction. This shows that $b(T-) = \alpha$ as claimed.

4. We show that the smooth fit holds at b meaning that $x \mapsto V(t,x)$ is differentiable at $b(t)$ and $V_x(t,b(t)) = G'(b(t))$ for every $t \in [0,T)$. For this, take any $t \in [0,T)$ and set $x = b(t)$. For $\varepsilon > 0$ we have

$$\frac{V(t,x-\varepsilon)-V(t,x)}{-\varepsilon} \ge \frac{G(x-\varepsilon)-G(x)}{-\varepsilon} \tag{3.18}$$

so that letting $\varepsilon \downarrow 0$ we obtain

$$\liminf_{\varepsilon\downarrow 0} \frac{V(t,x-\varepsilon)-V(t,x)}{-\varepsilon} \ge G'(x)\, . \tag{3.19}$$

On the other hand, for $\varepsilon > 0$ let $\tau_\varepsilon := \tau_*(t, x-\varepsilon)$ denote the optimal stopping time for $V(t,x-\varepsilon)$. By the mean value theorem we then find that

$$\frac{V(t,x-\varepsilon)-V(t,x)}{-\varepsilon} \le \frac{\mathsf{E}G((x-\varepsilon)X^1_{\tau_\varepsilon})-\mathsf{E}G(xX^1_{\tau_\varepsilon})}{-\varepsilon} = \mathsf{E}\big[G'(\xi_\varepsilon)X^1_{\tau_\varepsilon}\big] \tag{3.20}$$

for some $\xi_\varepsilon \in [(x-\varepsilon)X^1_{\tau_\varepsilon}, xX^1_{\tau_\varepsilon}]$. Using that $s \mapsto -((\mu-\sigma^2/2)/\sigma)s$ is a lower function of B and that $s \mapsto b(s)$ is decreasing on $[t,T)$ it is easily seen that $\tau_\varepsilon \to 0$ P-a.s. as $\varepsilon \downarrow 0$. Letting $\varepsilon \downarrow 0$ in (3.20) we thus obtain

$$\limsup_{\varepsilon\downarrow 0} \frac{V(t,x-\varepsilon)-V(t,x)}{-\varepsilon} \le G'(x) \tag{3.21}$$

by the dominated convergence theorem. Combining (3.19) and (3.21) we see that the smooth fit holds at b as claimed.

5. Combining the facts derived above with the Markovian structure of X solving (2.1) we see that V and b solve the free-boundary problem

$$V_t(t,x)+\mu x V_x(t,x)+\tfrac{\sigma^2}{2}x^2 V_{xx}(t,x) = 0 \quad \text{for } x \in (0,b(t)) \text{ and } t \in [0,T) \tag{3.22}$$

$$V(t,x) = G(x) \quad \text{for } x \in [b(t),\infty) \text{ and } t \in [0,T) \quad \text{(instantaneous stopping)} \tag{3.23}$$

$$V_x(t,x) = G'(x) \quad \text{for } x = b(t) \text{ and } t \in [0,T) \quad \text{(smooth fit)} \tag{3.24}$$

$$V(t,x) < G(x) \quad \text{for } x \in (0,b(t)) \text{ and } t \in [0,T) \tag{3.25}$$

$$V(T,x) = G(x) \quad \text{for } x \in (0,\infty)\, . \tag{3.26}$$

In line with a solution method for free-boundary problems based on the local time-space calculus (see e.g. [7] and the references therein) we will now express V in terms of b and show that b can be characterised as the unique solution to a nonlinear integral equation.

6. We show that V is given by the formula (3.8) and that b solves the equation (3.6). For this, set $D^o = \{ (t,x) \in [0,T)\times((0,\infty)\setminus\{x_i \,|\, i \geq 1\}) \mid x > b(t) \}$ and note that (i) V is $C^{1,2}$ on $C \cup D^o$ (cf. [7, p. 131]) as well as C^1 at each (t,x_i) with $x_i > b(t)$ and (ii) $V_t + I\!\!L_X V$ is locally bounded on $C \cup D^o$ (in the sense of being bounded on $K \cap (C \cup D^o)$ for every compact set K in $[0,T)\times(0,\infty)$) where we recall that $C = \{ (t,x) \in [0,T)\times(0,\infty) \mid x < b(t) \}$. Moreover, due to the smooth fit condition (3.24) we see that (iii) $t \mapsto V_x(t,b(t)\pm)$ is continuous on $[0,T)$. Finally, we claim that (iv) $V_{xx} = f + g$ on $C \cup D^o$ where $f \leq 0$ on C and $f \geq 0$ on D^o while g is continuous on $\bar{C}$ and D where we set $\bar{C} = \{ (t,x) \in [0,T)\times(0,\infty) \mid x \leq b(t) \}$.

To verify (iv) note from (3.22) that

$$V_{xx}(t,x) = \tfrac{2}{\sigma^2 x^2}\big(-V_t(t,x) - \mu x V_x(t,x)\big) \leq -\tfrac{2\mu}{\sigma^2 x} V_x(t,x) \tag{3.27}$$

for all $(t,x) \in C$ since $t \mapsto V(t,x)$ is increasing on $[0,T]$ for every $x > 0$. To bound V_x from below take any $(t,x) \in C$ and let $\tau_\varepsilon := \tau_*(t,x+\varepsilon)$ denote the optimal stopping time for $V(t,x+\varepsilon)$ where $\varepsilon > 0$ is given and fixed. By the mean value theorem we then have

$$\frac{V(t,x+\varepsilon) - V(t,x)}{\varepsilon} \geq \frac{\mathsf{E}G((x+\varepsilon)X^1_{\tau_\varepsilon}) - \mathsf{E}G(xX^1_{\tau_\varepsilon})}{\varepsilon} = \mathsf{E}\big[G'(\xi_\varepsilon)X^1_{\tau_\varepsilon}\big] \geq -\frac{m}{2x} \tag{3.28}$$

for some $\xi_\varepsilon \in [xX^1_{\tau_\varepsilon}, (x+\varepsilon)X^1_{\tau_\varepsilon}]$ where in the final inequality we use that $G' = F - 1/2 \geq -1/2$ and $(x+\varepsilon)X^1_{\tau_\varepsilon} \leq m$ so that $X^1_{\tau_\varepsilon} \leq m/(x+\varepsilon) \leq m/x$. Letting $\varepsilon \downarrow 0$ we obtain

$$V_x(t,x) \geq -\frac{m}{2x} \tag{3.29}$$

for all $(t,x) \in C$. Combining (3.27) and (3.29) we see that

$$V_{xx}(t,x) \leq \frac{\mu m}{\sigma^2 x^2} \tag{3.30}$$

for all $(t,x) \in C$. This shows that (iv) holds with $f := V_{xx} - g$ on C and $g(t,x) := (\mu m / \sigma^2 x^2)$ for $(t,x) \in \bar{C}$ while (iv) holds with $f := V_{xx} = G'' = F' \geq 0$ on D^o and $g = 0$ on D as claimed. For future reference note also that if we choose $\tau_x := \tau_*(t,x)$ to be the optimal stopping time for $V(t,x)$ instead, then the first inequality in (3.28) is reversed and this gives

$$\frac{V(t,x+\varepsilon) - V(t,x)}{\varepsilon} \leq \mathsf{E}\big[G'(\eta_\varepsilon)X^1_{\tau_x}\big] \leq \Big(F\big((1+\tfrac{\varepsilon}{x})m\big) - \tfrac{1}{2}\Big)\tfrac{m}{x} \tag{3.31}$$

for some $\eta_\varepsilon \in [xX^1_{\tau_x}, (x+\varepsilon)X^1_{\tau_x}]$ where in the final inequality we use that $G' = F - 1/2$ and $xX^1_{\tau_x} \leq m$ so that $\eta_\varepsilon \leq (x+\varepsilon)X^1_{\tau_x} \leq (1+\varepsilon/x)m$. Letting $\varepsilon \downarrow 0$ we obtain

$$V_x(t,x) \leq 0 \tag{3.32}$$

for all $(t,x) \in C$. Note that this inequality also holds for all $(t,x) \in D \cap ([0,T]\times(0,m])$ as is easily seen from (3.23) and (3.26) above.

Since the conditions (i)-(iv) are satisfied we can apply the change-of-variable formula with local time on curves (see [4, Theorem 3.1 & Remark 3.2]) and this gives

$$V(t+s, X_{t+s}) = V(t,x) + \int_0^s \big(V_t + I\!L_X V\big)(t+u, X_{t+u})\, I\big(X_{t+u} \neq b(t+u)\big)\, du \tag{3.33}$$
$$+ \int_0^s \sigma X_{t+u} V_x(t+u, X_{t+u})\, I\big(X_{t+u} \neq b(t+u)\big)\, dB_{t+u}$$
$$+ \frac{1}{2} \int_0^s \big(V_x(t+u, X_{t+u}+) - V_x(t+u, X_{t+u}-)\big)\, I\big(X_{t+u} = b(t+u)\big)\, d\ell^b_{t+u}(X)$$

under $\mathsf{P}_{t,x}$ for $s \in [0, T-t]$ with $(t,x) \in [0,T) \times (0,\infty)$ given and fixed where $\ell^b(X)$ denotes the local time of X at b. Making use of (3.22) in the first integral, upon noting by (3.23) that $V_t + I\!L_X V = G_t + I\!L_X G = H$ strictly above b where in the final equality we also use (2.7) and (3.2) above, and (3.24) in the third integral (which consequently vanishes) we obtain

$$V(t+s, X_{t+s}) = V(t,x) + \int_0^s H(X_{t+u})\, I(X_{t+u} > b(t+u))\, du + M_s \tag{3.34}$$

under $\mathsf{P}_{t,x}$ where $M_s = \int_0^s \sigma X_{t+u} V_x(t+u, X_{t+u}) I(X_{t+u} \neq b(t+u))\, dB_{t+u}$ is a continuous local martingale for $s \in [0, T-t]$. Moreover, making use of (3.29) and (3.32) combined with the fact that $|V_x| = |G'| = |F - 1/2| \leq 1/2$ above b it is easily seen using $\mathsf{E}(\sup_{0 \leq t \leq T} (X_t^1)^2) < \infty$ that $\mathsf{E}_{t,x} \langle M, M \rangle_{T-t} < \infty$ and hence M_s is a true martingale for $s \in [0, T-t]$. Setting $s = T-t$ in (3.34) making use of (3.26) and taking $\mathsf{E}_{t,x}$ on both sides we find that

$$V(t,x) = \mathsf{E}_{t,x}\big[G(X_T)\big] - \int_0^{T-t} \mathsf{E}_{t,x}\big[H(X_{t+u})\, I(X_{t+u} > b(t+u))\big]\, du \tag{3.35}$$

for all $(t,x) \in [0,T) \times (0,\infty)$. Recalling (3.1) and (3.3) we see that this establishes the formula (3.8). Moreover, inserting $x = b(t)$ in (3.8) and using (3.23) we see upon making a simple change of variables that b solves the equation (3.6) as claimed.

7. We show that b is the unique solution to (3.6) in the class of continuous functions $t \mapsto b(t)$ on $[0,T)$ satisfying $\alpha \leq b(t) \leq m$ for $t \in [0,T)$. For this, take any continuous function $c : [0,T) \to I\!R$ which solves (3.6) and satisfies $\alpha \leq c(t) \leq m$ for $t \in [0,T)$. Motivated by (3.35) define the function $V^c : [0,T] \times (0,\infty) \to I\!R$ by setting

$$V^c(t,x) = \mathsf{E}_{t,x}\big[G(X_T)\big] - \int_0^{T-t} \mathsf{E}_{t,x}\big[H(X_{t+u})\, I(X_{t+u} > c(t+u))\big]\, du \tag{3.36}$$

for $(t,x) \in [0,T] \times (0,\infty)$. Observe that c solving (3.6) means exactly that $V^c(t, c(t)) = G(c(t))$ for all $t \in [0,T)$. Note also that $V^c(T,x) = G(x)$ for all $x > 0$.

(i) We show that $V^c(t,x) = G(x)$ for all $(t,x) \in [0,T) \times (0,\infty)$ such that $x \geq c(t)$. For this, take any such (t,x) and note that the Markov property of X implies that

$$V^c(t+s, X_{t+s}) - \int_0^s H(X_{t+u})\, I(X_{t+u} > c(t+u))\, du \tag{3.37}$$

is a continuous martingale under $\mathsf{P}_{t,x}$ for $s \in [0, T-t]$. Consider the stopping time

$$\sigma_c = \inf\{\, s \in [0, T-t] \mid X_{t+s} \leq c(t+s) \,\} \tag{3.38}$$

under $\mathsf{P}_{t,x}$. Since $V^c(t,c(t)) = G(c(t))$ for all $t \in [0,T)$ and $V^c(T,x) = G(x)$ for all $x > 0$ we see that $V^c(t+\sigma_c, X_{t+\sigma_c}) = G(X_{t+\sigma_c})$ under $\mathsf{P}_{t,x}$. Replacing s by σ_c in (3.37), taking $\mathsf{E}_{t,x}$ on both sides and applying the optional sampling theorem, we find that

$$
\begin{aligned}
(3.39) \qquad V^c(t,x) &= \mathsf{E}_{t,x}\big[V^c(t+\sigma_c, X_{t+\sigma_c})\big] - \mathsf{E}_{t,x}\Big[\int_0^{\sigma_c} H(X_{t+u})\, I(X_{t+u} > c(t+u))\, du\Big] \\
&= \mathsf{E}_{t,x}\big[G(X_{t+\sigma_c})\big] - \mathsf{E}_{t,x}\Big[\int_0^{\sigma_c} H(X_{t+u})\, du\Big] = G(x)
\end{aligned}
$$

where in the last equality we use (3.14). This shows that V^c equals G above c as claimed.

(ii) We show that $V^c(t,x) \ge V(t,x)$ for all $(t,x) \in [0,T] \times (0,\infty)$. For this take any such (t,x) and consider the stopping time

$$
(3.40) \qquad \tau_c = \inf\{\, s \in [0,T-t] \mid X_{t+s} \ge c(t+s) \,\}
$$

under $\mathsf{P}_{t,x}$. We claim that $V^c(t+\tau_c, X_{t+\tau_c}) = G(X_{t+\tau_c})$ under $\mathsf{P}_{t,x}$. Indeed, if $x \ge c(t)$ with $t < T$ then $\tau_c = 0$ so that $V^c(t,x) = G(x)$ by (i) above. On the other hand, if $x < c(t)$ with $t < T$ then the claim follows since $V^c(t,c(t)) = G(c(t))$ for all $t \in [0,T)$ and $V^c(T,x) = G(x)$ for all $x > 0$ showing also that the identity holds when $t = T$. Replacing s by τ_c in (3.37), taking $\mathsf{E}_{t,x}$ on both sides and applying the optional sampling theorem, we find that

$$
\begin{aligned}
(3.41) \qquad V^c(t,x) &= \mathsf{E}_{t,x}\big[V^c(t+\tau_c, X_{t+\tau_c})\big] - \mathsf{E}_{t,x}\Big[\int_0^{\tau_c} H(X_{t+u})\, I(X_{t+u} > c(t+u))\, du\Big] \\
&= \mathsf{E}_{t,x}\big[G(X_{t+\tau_c})\big] \ge V(t,x)
\end{aligned}
$$

where in the second equality we use the definition of τ_c . This shows that $V^c \ge V$ as claimed.

(iii) We show that $c(t) \le b(t)$ for all $t \in [0,T)$. For this, suppose that there exists $t \in [0,T)$ such that $c(t) > b(t)$. Take any $x \ge c(t)$ and consider the stopping time

$$
(3.42) \qquad \sigma_b = \inf\{\, s \in [0,T-t] \mid X_{t+s} \le b(t+s) \,\}
$$

under $\mathsf{P}_{t,x}$. Replacing s by σ_b in (3.34) and (3.37), taking $\mathsf{E}_{t,x}$ on both sides of these identities and applying the optional sampling theorem, we find

$$
(3.43) \qquad \mathsf{E}_{t,x}\big[V(t+\sigma_b, X_{t+\sigma_b})\big] = V(t,x) + \mathsf{E}_{t,x}\Big[\int_0^{\sigma_b} H(X_{t+u})\, du\Big]
$$

$$
(3.44) \qquad \mathsf{E}_{t,x}\big[V^c(t+\sigma_b, X_{t+\sigma_b})\big] = V^c(t,x) + \mathsf{E}_{t,x}\Big[\int_0^{\sigma_b} H(X_{t+u})\, I(X_{t+u} > c(t+u))\, du\Big].
$$

Since $x \ge c(t)$ we see by (i) above that $V^c(t,x) = G(x) = V(t,x)$ where the second equality holds since $x \ge b(t)$. Moreover, by (ii) above we know that $V^c(t+\sigma_b, X_{t+\sigma_b}) \ge V(t+\sigma_b, X_{t+\sigma_b})$ so that (3.43) and (3.44) imply that

$$
(3.45) \qquad \mathsf{E}_{t,x}\Big[\int_0^{\sigma_b} H(X_{t+u})\, I(X_{t+u} \le c(t+u))\, du\Big] \le 0\,.
$$

The fact that $c(t) > b(t)$ and the continuity of b and c imply that there exists $\varepsilon > 0$ sufficiently small such that $c(t+u) > b(t+u)$ for all $u \in [0,\varepsilon]$. Consequently the $\mathsf{P}_{t,x}$-probability of X spending a strictly positive amount of time below c (with respect to Lebesgue measure) before hitting b within ε units of time is strictly positive. Combined with the facts that b lies between α and m (note that $b(t) < m$ since $b(t) < c(t) \le m$ by hypothesis) and that H is strictly positive on (α, m) this forces the expectation in (3.45) to be strictly positive and provides a contradiction. Hence $c \le b$ as claimed.

(iv) We show that $b(t) = c(t)$ for all $t \in [0,T)$. For this, suppose that there exists $t \in [0,T)$ such that $c(t) < b(t)$. Take any $x \in (c(t), b(t))$ and consider the stopping time

$$\tau_b = \inf \{ \, s \in [0, T{-}t] \mid X_{t+s} \ge b(t{+}s) \, \} \tag{3.46}$$

under $\mathsf{P}_{t,x}$. Replacing s by τ_b in (3.34) and (3.37), taking $\mathsf{E}_{t,x}$ on both sides of these identities and applying the optional sampling theorem, we find

$$\mathsf{E}_{t,x}\big[V(t{+}\tau_b, X_{t+\tau_b})\big] = V(t,x) \tag{3.47}$$

$$\mathsf{E}_{t,x}\big[V^c(t{+}\tau_b, X_{t+\tau_b})\big] = V^c(t,x) + \mathsf{E}_{t,x}\Big[\int_0^{\tau_b} H(X_{t+u})\, I(X_{t+u} \,{>}\, c(t{+}u))\, du\Big]\,. \tag{3.48}$$

Since $b \ge c$ by (iii) above and V^c equals G above c by (i) above, we see that $V^c(t{+}\tau_b, X_{t+\tau_b}) = G(X_{t+\tau_b}) = V(t{+}\tau_b, X_{t+\tau_b})$ where the second equality holds since V equals G above b (recall also that $V^c(T,x) = G(x) = V(T,x)$ for all $x > 0$). Moreover, by (ii) above we know that $V^c(t,x) \ge V(t,x)$ so that (3.47) and (3.48) imply that

$$\mathsf{E}_{t,x}\Big[\int_0^{\tau_b} H(X_{t+u})\, I(X_{t+u} \,{>}\, c(t{+}u))\, du\Big] \le 0\,. \tag{3.49}$$

But then as in (iii) above the continuity of b and c combined with the facts that c lies between α and m (note that $x < m$ since $x < b(t) \le m$) and that H is strictly positive on (α, m) forces the expectation in (3.49) to be strictly positive and provides a contradiction. Thus $c = b$ as claimed and the proof is complete. $\square$

Remark 4 (Conditional median sets and curves). When the horizon in the optimal prediction problem (2.3) is infinite it is known that the first entry time of X into the median set $[m, M]$ of ℓ is optimal (whenever finite valued with probability one). This follows from the well known characterisation of the median in terms of the L^1 norm (cf. [5]). In view of this fact we see that the optimal stopping set/boundary derived in Theorem 3 may be viewed as the 'conditional median set/curve' of ℓ (or its law) with respect to X (or its law) and T. (This identification also extends to the case when the horizon is infinite and the first entry time above is not finite valued with probability one.) The result of Theorem 3 establishes the existence and uniqueness of the conditional median sets/curves for admissible laws specified in Definition 2. In this case we know that the conditional median curves can be characterised as the unique solutions to the nonlinear integral equations (3.6) and (3.9). These equations cannot be solved in closed form generally but can be used to find the conditional median curves numerically (see e.g. [7, Remark 22.4] for a possible method relying on backward induction). When the law of ℓ is not admissible in the sense of Definition 2 then the structure of the conditional median set of ℓ can generally be more complicated and when this happens the problem of determining this set (and its boundary) needs to be studied on a case-by-case basis. We leave a fuller classification of the conditional median sets and their boundaries open for future studies.

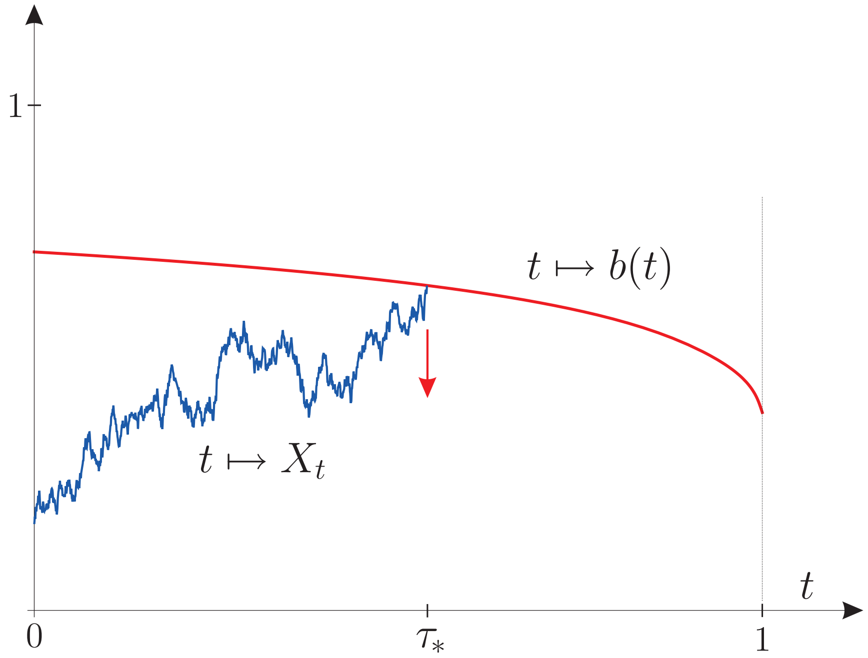

Figure 1. The optimal selling boundary b from Example 1 when the aspiration level ℓ is exponentially distributed.

4. Examples

In this section we illustrate the results derived in the previous section through some specific examples. We assume throughout that the asset price is given by (2.1)-(2.2) and the loss function G is given by (2.7) where F is the distribution function of the aspiration level ℓ.

Example 1 (Resistance level). Consider the case when $\mu > 0$ and the distribution function F of the aspiration level ℓ is exponentially distributed with parameter $\lambda > 0$. This corresponds to the most uncertain realisation of ℓ for the selling level (in terms of entropy) with the expected value $1/\lambda$. Thus $F'(x) = \lambda e^{-\lambda x}$ for $x > 0$ and $F'(x) = 0$ for $x \le 0$. A direct calculation shows that $m = M = (1/\lambda)\log 2$ and (2.14) is satisfied with $\alpha \in (0, m)$ solving uniquely the equation

$$\lambda\alpha = \frac{\mu}{\sigma^2/2}\Big(1 - \frac{1}{2}e^{\lambda\alpha}\Big) \tag{4.1}$$

on $(0, \infty)$ so that F belongs to $\mathcal{A}(\mu, \sigma)$ for all $\sigma > 0$. By the result of Theorem 3 we therefore know that the optimal selling time is given by

$$\tau_* = \inf\{\, t \in [0, T] \mid X_t \ge b(t)\,\} \tag{4.2}$$

where $t \mapsto b(t)$ is the unique solution to (3.6) satisfying $\alpha \le b(t) \le m$ for $t \in [0, T)$ with $b(T-) = \alpha$. Figure 1 shows the optimal selling boundary b when $\mu = 1$, $\lambda = 1/2$, $\sigma = 2$ and $T = 1$. In this case $\alpha = 0.39$, $m = 1.38$ and $b(0) = 0.70$ approximately. The optimal selling action triggered at τ_* creates a resistance level at X_{τ_*} and pushes the price down.

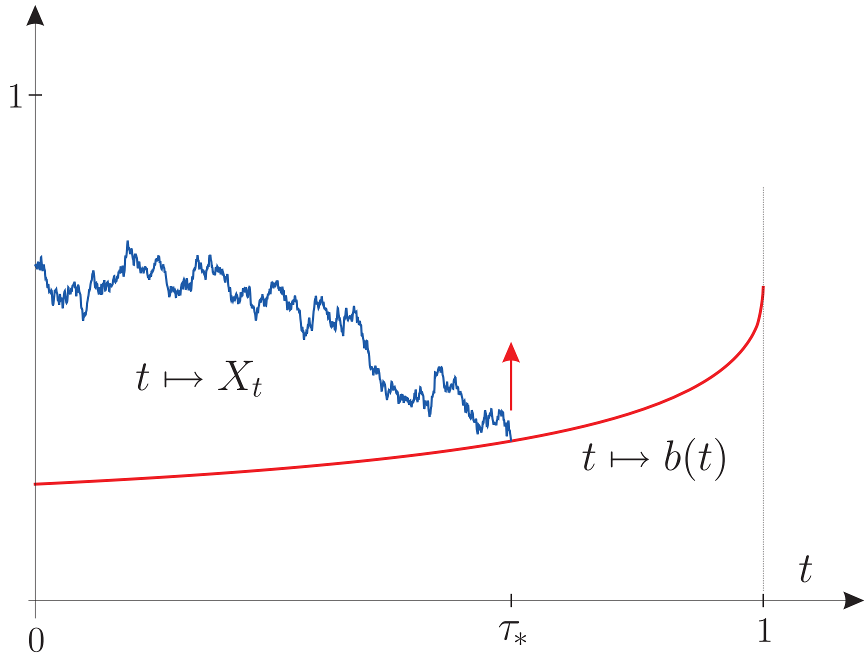

Figure 2. The optimal buying boundary b from Example 2 when the aspiration level ℓ is exponentially distributed.

Example 2 (Support level). Consider the case when $\mu < 0$ and the distribution function F of the aspiration level ℓ is exponentially distributed with parameter $\lambda > 0$. This corresponds to the most uncertain realisation of ℓ for the buying level (in terms of entropy) with the expected value $1/\lambda$. A direct calculation shows that (2.15) is satisfied with $\beta \in (m, \infty)$ solving uniquely the equation

$$\lambda\beta = \frac{\mu}{\sigma^2/2}\Big(1 - \frac{1}{2}e^{\lambda\beta}\Big) \tag{4.3}$$

on $(0, \infty)$ so that F belongs to $\mathcal{A}(\mu, \sigma)$ for all $\sigma > 0$. By the result of Theorem 3 we therefore know that the optimal buying time is given by

$$\tau_* = \inf\{\, t \in [0, T] \mid X_t \leq b(t)\,\} \tag{4.4}$$

where $t \mapsto b(t)$ is the unique solution to (3.9) satisfying $m \leq b(t) \leq \beta$ for $t \in [0, T)$ with $b(T-) = \beta$. Figure 2 shows the optimal buying boundary b when $\mu = -1$, $\lambda = 4$, $\sigma = 2$ and $T = 1$. In this case $m = 0.17$, $\beta = 0.61$ and $b(0) = 0.23$ approximately. The optimal buying action triggered at τ_* creates a support level at X_{τ_*} and pushes the price up.

Example 3 (Aspiration level at the ultimate maximum). Consider the case when $\mu \in (0, \sigma^2/2)$ and the aspiration level is equally distributed as the ultimate maximum of X given by $S = \sup_{t \geq 0}[x_0 \exp(\sigma B_t + (\mu - (\sigma^2/2))t)]$ where x_0 is given and fixed. Recalling the well-known expression for the law of S (see e.g. [7, Remark 25.2]) this means that the distri-

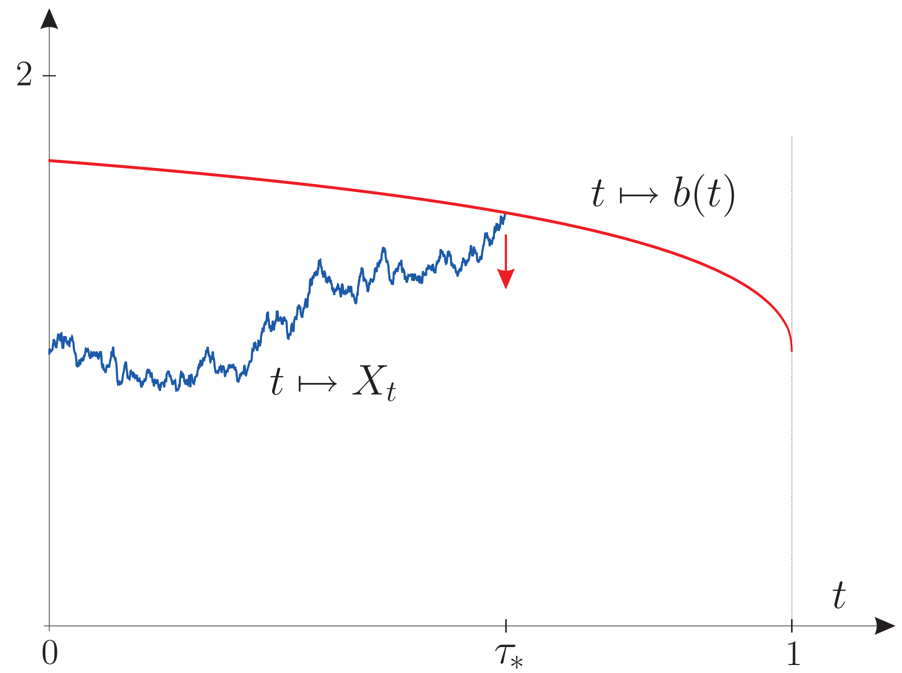

Figure 3. The optimal selling boundary b from Example 3 when the aspiration level ℓ is equally distributed as the ultimate maximum.

bution function F of ℓ is given by

$$F(x) = 1 - \Big(\frac{x_0}{x}\Big)^{1-\mu/(\sigma^2/2)} \tag{4.5}$$

for $x \geq x_0$ and $F(x) = 0$ for $x < x_0$. Note that F' is only piecewise C^1 in this case due to its discontinuity at x_0 . A direct calculation shows that $m = M = 2^{1/(1-\mu/(\sigma^2/2))}x_0$ and (2.14) is satisfied with $\alpha \in (0, m)$ given by

$$\alpha = \Big(2\Big(2 - \frac{\sigma^2/2}{\mu}\Big)\Big)^{1/(1-\mu/(\sigma^2/2))} x_0 \tag{4.6}$$

for $\mu \in [\sigma^2/3, \sigma^2/2)$ and $\alpha = x_0$ for $\mu \in (0, \sigma^2/3)$ so that F belongs to $\mathcal{A}(\mu, \sigma)$. By the result of Theorem 3 we therefore know that the optimal buying time is given by

$$\tau_* = \inf\{\, t \in [0, T] \mid X_t \geq b(t)\,\} \tag{4.7}$$

where $t \mapsto b(t)$ is the unique solution to (3.9) satisfying $\alpha \leq b(t) \leq m$ for $t \in [0, T)$ with $b(T-) = \alpha$. Figure 3 shows the optimal selling boundary b when $\mu = 1$, $x_0 = 1$, $\sigma = 2$ and $T = 1$. In this case $\alpha = 1$, $m = 4$ and $b(0) = 1.68$ approximately. The optimal selling action triggered at τ_* creates a resistance level at X_{τ_*} and pushes the price down.

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