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Best Constants in Kahane-Khintchine Inequalities in Orlicz Spaces

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Several inequalities of Kahane-Khintchine's type in certain Orlicz spaces are proved. For this the classical symmetrization technique is used and four basically different methods have been presented. The first two are based on the well-known estimates for subnormal random variables, see [9], the third one is a consequence of a certain Gaussian-Jensen's majorization technique, see [6], and the fourth one is obtained by Haagerup-Young-Stechkin's best possible constants in the classical Khintchine inequalities, see [4]. Moreover, by using the central limit theorem it is shown that this fourth approach gives the best possible numerical constant in the inequality under consideration: *If* $\{ \varepsilon_i \mid i \geq 1 \}$ *is a Bernoulli sequence, and* $\| \cdot \|_\psi$ *denotes the Orlicz norm induced by the function* $\psi(x) = e^{x^2} - 1$ *for* $x \in \mathbf{R}$ *, then the best possible numerical constant* C *satisfying the following inequality:*

$$\| \sum_{i=1}^n a_i \varepsilon_i \|_\psi \leq C \cdot \Big(\sum_{i=1}^n | a_i |^2 \Big)^{1/2}$$

for all $a_1, \ldots, a_n \in \mathbf{R}$ *and all* $n \geq 1$ *, is equal to* $\sqrt{8/3}$. Similarly, the best possible estimates of that type are also deduced for some other inequalities in Orlicz spaces, discovered in this paper.

1. Introduction

Let $\{ \varepsilon_i \mid i \geq 1 \}$ be a Bernoulli sequence defined on a probability space $(\Omega, \mathcal{F}, P)$, and let $\| \cdot \|_\psi$ denote the gauge norm on $(\Omega, \mathcal{F}, P)$, that is:

$$\| X \|_\psi = \inf \{ a > 0 \mid E[\psi(X/a)] \leq 1 \}$$

for all real valued random variables X on $(\Omega, \mathcal{F}, P)$, where $\psi(x) = e^{x^2} - 1$ for $x \in \mathbf{R}$, and with $\inf \emptyset = \infty$. Then the following inequality is satisfied:

$$\| \sum_{i=1}^n a_i \varepsilon_i \|_\psi \leq C \cdot \Big(\sum_{i=1}^n | a_i |^2 \Big)^{1/2} \tag{1}$$

for all $a_1, \ldots, a_n \in \mathbf{R}$ and all $n \geq 1$, where C is a numerical constant, see for instance [9], [13], [14], [19], [22], [30]. In this paper we shall show that the best possible numerical constant C

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in inequality (1) is equal to $\sqrt{8/3}$, and the present work is devoted to the study of various ways for proving (1), as well as to the study of the analogous question for some other Orlicz norms, see section 2 below. Moreover, by using the classical symmetrization technique, inequality (1), as well as some other inequalities of that type which will be deduced later, will be extended in an appropriate way to more general cases. Let us say that the inequality given in (1) has a number of applications. In particular, using that result together with certain modulus of continuity results of Preston's type, see [15], [20] and [30], one can obtain a connection between the central limit theorem in a Banach space and the uniform law of large numbers on the unit ball of its dual space, see [17] and [18]. For applications to the law of iterated logarithm in a Banach space, see [30].

2. Preliminary facts

Orlicz functionals, norms and spaces. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $(B, \|\cdot\|)$ be a Banach space. Let φ be an increasing left continuous function from $[0,\infty[$ into $[0,\infty[$ such that $\varphi(0) = \lim_{t\downarrow 0} \varphi(t) = 0$. Then *the Orlicz functionals* τ_φ , T_φ and Υ_φ generated by φ are defined as follows:

$$\begin{aligned}
\tau_\varphi(R_f) &= \inf \{ \; a > 0 \;\; | \;\; \int_0^\infty R_f(at) \; \varphi(dt) \leq 1 \; \} \\
\mathrm{T}_\varphi(R_f) &= \inf \{ \; a > 0 \;\; | \;\; \int_0^\infty R_f(at) \; \varphi(dt) \leq a \; \} \\
\Upsilon_\varphi(R_f) &= \int_0^\infty R_f(t) \; \varphi(dt)
\end{aligned}$$

where $R_f(t) = \mu^*\{ \; \| \; f \; \| > t \; \}$ for $f \in B^X$. Recall that $\int_0^\infty f(t) \; \varphi(dt)$ denotes the integral of a function f with respect to the Lebesgue-Stieltjes measure λ_φ defined by $\lambda_\varphi([0,x[) = \varphi(x)$ for all $x \geq 0$, that μ^* denotes the outer μ-measure, and that B^X denotes the set of all functions from X into B . Also the following formula is valid, see [5]:

$$\int^* \varphi \circ \xi \; d\mu \; = \; \int_0^\infty \mu^*\{ \; \xi > t \; \} \; \varphi(dt)$$

for every function ξ from $[0,\infty]$ into $[0,\infty]$, where we put $\varphi(\infty) = \lim_{t\to\infty} \varphi(t)$. Then *the function norm* and *the function space*, see [5], induced by functionals τ_φ , T_φ and Υ_φ are given respectively by:

$$\begin{aligned}
\| \; f \; \|_{\tau_\varphi} &= \; \inf \{ \; a > 0 \;\; | \;\; \int^* \varphi(\; \frac{1}{a} \; \| \; f \; \| \;) \; d\mu \;\; \leq 1 \; \} \\
L^{\tau_\varphi}(\mu^*, B) &= \{ \; f \in B^X \;\; | \;\; \lim_{\varepsilon\downarrow 0} \| \; \varepsilon f \; \|_{\tau_\varphi} = 0 \; \} \\
\| \; f \; \|_{\mathrm{T}_\varphi} &= \; \inf \{ \; a > 0 \;\; | \;\; \int^* \varphi(\; \frac{1}{a} \; \| \; f \; \| \;) \; d\mu \;\; \leq a \; \} \\
L^{\mathrm{T}_\varphi}(\mu^*, B) &= \{ \; f \in B^X \;\; | \;\; \lim_{\varepsilon\downarrow 0} \| \; \varepsilon f \; \|_{\mathrm{T}_\varphi} = 0 \; \}
\end{aligned}$$

$$\| f \|_{\Upsilon_\varphi} = \int^* \varphi(\| f \|) \, d\mu$$

$$L^{\Upsilon_\varphi}(\mu^*, B) \; = \; \{ \; f \in B^X \;\; | \;\; \lim_{\varepsilon \downarrow 0} \| \varepsilon f \|_{\Upsilon_\varphi} = 0 \; \}$$

for all $f \in B^X$. Recall that $\int^* f \, d\mu$ denotes the upper μ-integral of a function f . It is shown in [5] that the following statements are satisfied:

(1) $(\; B^X, \; \| \cdot \|_{\mathrm{T}_\varphi} \;)$ *is a pseudo-F^*-space, and consequently* $(\; L^{\mathrm{T}_\varphi}(\mu^*, B) \; , \; \| \cdot \|_{\mathrm{T}_\varphi} \;)$ *is a pseudo-Frechet space.*

(2) *If* φ *is convex, then* $(\; B^X, \; \| \cdot \|_{\tau_\varphi} \;)$ *is a pseudo-B^*-space, and consequently* $(\; L^{\tau_\varphi}(\mu^*, B) \; , \; \| \cdot \|_{\tau_\varphi} \;)$ *is a pseudo-Banach space.*

(3) *If* φ *is subadditive (for instance, concave), then* $(\; B^X, \; \| \cdot \|_{\Upsilon_\varphi} \;)$ *is a pseudo-F^*-space, and thus* $(\; L^{\Upsilon_\varphi}(\mu^*, B) \; , \; \| \cdot \|_{\Upsilon_\varphi} \;)$ *is a pseudo-Frechet space. Moreover, then we have:*

$$\sqrt{\frac{1}{2} \| f \|_{\Upsilon_\varphi}} \; \wedge \; \| f \|_{\Upsilon_\varphi} \; \leq \; \| f \|_{\mathrm{T}_\varphi} \; \leq \; \sqrt{2 \| f \|_{\Upsilon_\varphi}} \; \vee \; \| f \|_{\Upsilon_\varphi}$$

for all $f \in B^X$.

(4) *If* φ *is* <u>*convex*</u> *at* 0 *, i.e.* $\varphi(\lambda t) \leq \lambda \varphi(t)$ *for all* $\lambda \in [0,1]$ *and all* $t \geq 0$ *, then* $(\; B^X, \; \sqrt{\| \cdot \|_{\tau_\varphi}} \;)$ *is a pseudo-F^*-space, and we have:*

$$\sqrt{\| f \|_{\tau_\varphi}} \; \wedge \; \| f \|_{\tau_\varphi} \; \leq \; \| f \|_{\mathrm{T}_\varphi} \; \leq \; \sqrt{\| f \|_{\tau_\varphi}} \; \vee \; \| f \|_{\tau_\varphi}$$

for all $f \in B^X$.

(5) *If* φ *is* <u>*concave*</u> *at* 0 *, i.e.* $\varphi(\lambda t) \geq \lambda \varphi(t)$ *for all* $\lambda \in [0,1]$ *and all* $t \geq 0$ *, then* $(\; B^X, \; \sqrt{\| \cdot \|_{\Upsilon_\varphi}} \;)$ *is a pseudo-F^*-space, and we have:*

$$\sqrt{\| f \|_{\Upsilon_\varphi}} \; \wedge \; \| f \|_{\Upsilon_\varphi} \; \leq \; \| f \|_{\mathrm{T}_\varphi} \; \leq \; \sqrt{\| f \|_{\Upsilon_\varphi}} \; \vee \; \| f \|_{\Upsilon_\varphi}$$

for all $f \in B^X$.

In the next considerations we shall mainly work with the function ψ defined by $\psi(x) = e^{x^2} - 1$ for $x \in \mathbf{R}$. The Banach space B will be equal to $\mathbf{R}$, and the measure μ will be a probability measure, that is $\mu(X) = 1$. We shall write $L^\psi(\mu)$, $L^{T_\psi}(\mu)$ and $L^{\Upsilon_\psi}(\mu)$ to denote the spaces of all μ-measurable functions in $L^{\tau_\psi}(\mu^*, \mathbf{R})$, $L^{T_\psi}(\mu^*, \mathbf{R})$ and $L^{\Upsilon_\psi}(\mu^*, \mathbf{R})$, respectively. One can verify that the given spaces are closed and the natural injections are continuous. The function norm $\| \cdot \|_{\tau_\psi}$ will be shortly denoted by $\| \cdot \|_\psi$. The Orlicz space $(L^\psi(\mu), \| \cdot \|_\psi)$ will be called *the gauge space*, and the Orlicz norm $\| \cdot \|_\psi$ will be called *the gauge norm*. For more informations in this direction we shall refer the reader to [21].

Subnormal random variables. Let X be a real valued random variable defined on a probability space $(\Omega, \mathcal{F}, P)$. Then *the Laplace transform* of X is given by:

$$L_X(z) = E(e^{zX})$$

for all $z \in \mathcal{D}(L_X)$, where $\mathcal{D}(L_X) = \{\ z \in \mathbf{C} \ \mid\ E\,|e^{zX}\,| < \infty\ \}$ is *the complex domain* of L_X . *The real domain* of L_X is given by $\mathcal{R}(L_X) = \mathcal{D}(L_X) \cap \mathbf{R}$. Let us recall that a *normal* distributed random variable $N \sim N(\mu, \sigma^2)$ with mean μ and variance $\sigma^2 \geq 0$ has the Laplace transform given by:

$$L_N(z) = e^{\ \mu z\ +\ \frac{1}{2}\sigma^2 z^2}$$

for all $z \in \mathcal{D}(L_N) = \mathbf{C}$. And a real random variable X is called *subnormal*, if its Laplace transform is dominated on the real line by the Laplace transform of some normally distributed random variable. In other words, X is subnormal, if there exists $\mu \in \mathbf{R}$ and $\sigma^2 \geq 0$ such that:

(6) $$L_X(t) \leq e^{\ \mu t\ +\ \frac{1}{2}\sigma^2 t^2}$$

for all $t \in \mathbf{R}$. It is well-known that for a subnormal random variable X we have:

(7) $$EX = \mu \quad \text{and} \quad VarX \leq \sigma^2 \ .$$

And if (6) is satisfied with $\mu = 0$ and $\sigma^2 = 1$, then X is said to be a *standard subnormal* random variable. Then by (7) we have $EX = 0$ and $VarX \leq 1$. A sequence of (standard) subnormal random variables will be called a (standard) subnormal sequence. For more informations in this direction we shall refer the reader to [9] (p.62).

Gaussian-Jensen's majorization technique. Recall that a finite or infinite sequence of independent and identically distributed random variables $\varepsilon_1,\ \varepsilon_2,\ \dots$ taking values ± 1 with the same probability $1/2$ is called a Bernoulli sequence. Certain inequalities involving Bernoulli sequences can be easily obtained by using a technique which will be called *the Gaussian-Jensen's majorization technique* and which may be described as follows, see [6] (p.11-12): Suppose that we have a convex and sign-symmetric function $g : \mathbf{R}^n \to \,]-\infty\,,\infty]$, i.e. g is convex and $g\ (\pm x_1,\ \dots\ ,\pm x_n) = g\ (x_1,\ \dots\ ,x_n)$ for all choices of signs $\pm$ and all $(x_1,\ \dots\ ,x_n) \in \mathbf{R}^n$, and let $X_1,\ \dots\ ,X_n \in L^1(P)$ be real valued random variables. Then we evidently have:

$$g\,(x_1,\ \dots\ ,x_n)\ =\ g\,(|x_1|\,,\ \dots\ ,|x_n|)$$

for all $(x_1, \dots, x_n) \in \mathbf{R}^n$, and thus by Jensen's inequality we get:

(8) $$g\,(x_1,\ \dots\ ,x_n)\ \leq\ Eg\,(X_1,\ \dots\ ,X_n)$$

where $E|X_i| = |x_i|$, for $i = 1, \dots, n$. Moreover, suppose that we have:

(9) $$g\,(x_1,\ \dots\ ,x_n)\ =\ E\,h\Big(\sum_{i=1}^n x_i\varepsilon_i\Big)$$

where $\varepsilon_1, \dots, \varepsilon_n$ is a Bernoulli sequence and $h : \mathbf{R} \to \bar{\mathbf{R}}$ is a given function such that the right side in (9) is well-defined and belongs to $]-\infty\,,\infty]$ for all $(x_1, \dots, x_n) \in \mathbf{R}^n$. Then taking $X_{i,x_i} \sim N(0, |x_i|^2\pi/2)$ to be mutually independent for $i = 1, \dots, n$, as well as independent of

$\varepsilon_1, \dots, \varepsilon_n$, by (8), Fubini's theorem and the fact that $E\,|\,X_{i,x_i}| = |\,x_i\,|$ for $i = 1, \dots, n$, we get:

$$g\,(x_1,\ \dots\ ,x_n) \ \le\ E\,h\Big(\sum_{i=1}^{n} X_{i,x_i} \cdot \varepsilon_i\Big) \tag{10}$$

for $(x_1, \dots, x_n) \in \mathbf{R}^n$. A usefulness of above inequality lies in the fact that we obviously have $\sum_{i=1}^{n} X_{i,x_i} \cdot \varepsilon_i \sim N(0\ , \sum_{i=1}^{n} |x_i|^2 \pi/2)$, so it might be a "good" chance to compute exactly the right side in (10), or at least to make a "good" estimate for it, and in this way to obtain an estimate for the left side in (10), that is for the function g defined by (9). For an application of this technique see proof 3 of theorem 3.1 below.

Classical Khintchine inequalities. Let $\{\ \varepsilon_i\ \mid\ i \ge 1\ \}$ be a Bernoulli sequence, and let $0 < p, q < \infty$ and $n \ge 1$. Then *the Khintchine constant* $K_n(p,q)$ is defined to be the smallest number satisfying the following inequality:

$$E\ |\sum_{i=1}^{n} \varepsilon_i x_i\ |^p \le K_n(p,q)\ \Big(\sum_{i=1}^{n} |x_i|^q\Big)^{p/q} \tag{11}$$

for all $x_1, \dots, x_n \in \mathbf{R}$. Let us put $\kappa(p,q) = p \cdot (\ \frac{1}{2} - \frac{1}{q}\ )^+$, and define:

$$K(p,q)\ =\ \sup_{n \ge 1}\ \Big(\ \frac{K_n(p,q)}{n^{\kappa(p,q)}}\ \Big) \tag{12}$$

for all $0 < p, q < \infty$. Then we have, see [6] (p.10):

$$1 \le K_n(p,q) \le n^{p(1-\frac{1}{q})^+},\ \textit{ for all }\ p,q > 0\ \textit{ and all }\ n \ge 1 \tag{13}$$

$$K_n(p,q)\ \textit{ and }\ [K_n(p,q)]^{1/p}\ \textit{ are increasing in }\ (n,p,q) \tag{14}$$

$$K_n(p,q)\ \textit{ and }\ \log K_n(p,q)\ \textit{ are convex in }\ p\ \textit{, for all }\ q > 0\ \textit{ and all }\ n \ge 1 \tag{15}$$

$$K_n(p,q) \le n^{p(\frac{1}{r}-\frac{1}{q})^+} K_n(p,r)\ \textit{, for all }\ p,q,r > 0\ \textit{ and all }\ n \ge 1 \tag{16}$$

$$K(p,q) = \begin{cases} \dfrac{2^{\frac{p}{2}}\Gamma(\frac{p+1}{2})}{\sqrt{\pi}} & \textit{if}\ \ 2 \le p,q < \infty \\[2ex] 1 & \textit{if}\ \ 0 < p \le 2\ \ \textit{or}\ \ 0 < q \le 1 \end{cases} \tag{17}$$

$$1\ \le\ K(p,q)\ \le\ \frac{2^{\frac{p}{2}}\Gamma(\frac{p+1}{2})}{\sqrt{\pi}}\ ,\ \textit{ for all }\ 1\ <\ q\ <\ 2\ <\ p \tag{18}$$

Note that $K(p,q) = E|N|^p$ for $p,q \ge 2$, where $N \sim N(0,1)$ is a standard normal random variable. U. Haagerup in [4], R.M.G. Young in [31] and S.B. Stechkin in [24] have shown that the constants $K(p,q)$ in (17) are indeed the best possible in (1) for $p,q \ge 2$. Let us in addition remind that *Stirling's formula* states:

$$n!\ =\ \sqrt{2\pi n} \cdot n^n \cdot e^{-n} \cdot e^{r_n}\ ,\quad \tfrac{1}{12n+1}\ <\ r_n\ <\ \tfrac{1}{12n} \tag{19}$$

for $n \ge 1$. Recall that a concave function $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ is subadditive, i.e. we have $\varphi(x+y) \le \varphi(x) + \varphi(y)$ for all $x, y \ge 0$. Using this fact and Jensen's inequality we may easily obtain the following two inequalities:

$$\Big(\sum_{i=1}^{n}|x_i|^2\Big)^{1/2} \le \Big(\sum_{i=1}^{n}|x_i|^\alpha\Big)^{1/\alpha}, \quad \text{for all} \quad 0 < \alpha \le 2 \tag{20}$$

$$\Big(\sum_{i=1}^{n}|x_i|^2\Big)^{1/2} \le n^{(\frac{1}{2}-\frac{1}{\alpha})} \Big(\sum_{i=1}^{n}|x_i|^\alpha\Big)^{1/\alpha}, \quad \text{for all} \quad 2 < \alpha < \infty \tag{21}$$

being valid for all $x_1, \ldots, x_n \in \mathbf{R}$ and all $n \ge 1$.

3. Basic results on the gauge space $L^\psi(P)$

This section is devoted to the study of an inequality of Kahane-Khintchine's type in the gauge space $(L^\psi(P)\ ,\ \|\cdot\|_\psi)$, where we recall that $\psi(x) = e^{x^2} - 1$ for $x \in \mathbf{R}$. The first step in that direction is done in the next theorem, see inequality (1) in theorem 1 below. The result is well-known and our attention is mainly directed to its proof in order to find the best possible numerical constant under consideration, as well as to obtain appropriate tools for closely related questions on some other Orlicz spaces in the next two sections. Using these results we reach this first ambition in corollary 4 below. Then, using the classical symmetrization technique, we extend the given results to more general cases, see remark 7 and corollary 9 below.

Theorem 1. *(**A Kahane-Khintchine inequality in the gauge space**). Let $\{\ \varepsilon_i \mid i \ge 1\ \}$ be a Bernoulli sequence defined on a probability space $(\Omega, \mathcal{F}, P)$, and let $\|\cdot\|_\psi$ denote the gauge norm on $(\Omega, \mathcal{F}, P)$. Then there exists a numerical constant $C > 0$ such that the following inequality is satisfied:*

$$\Big\| \sum_{i=1}^{n} a_i \varepsilon_i \Big\|_\psi \le C \cdot \Big(\sum_{i=1}^{n} |a_i|^2 \Big)^{1/2} \tag{1}$$

for all $a_1, \ldots, a_n \in \mathbf{R}$ and all $n \ge 1$.

Proof. Given $a_1, \ldots, a_n \in \mathbf{R}$ for some $n \ge 1$, we denote $S_n = \sum_{i=1}^{n} a_i \varepsilon_i$, $T_n = (S_n)^2$ and $A_n = \sum_{i=1}^{n} |a_i|^2$. Then by the definition of the gauge norm $\|\cdot\|_\psi$, it is enough to show the following inequality:

$$\int_\Omega exp\Big(\frac{1}{C^2 A_n} \cdot T_n \Big)\, dP \le 2 \tag{2}$$

In order to illustrate various ways for proving (1), as well as to make possible their comparisons, we shall present four basically different proofs of (2):

Proof 1. The first proof is based on a classical Kahane-Khintchine inequality for subnormal sequences of random variables and Stirling's formula, and we shall show that (2), and thus (1) also, holds with $C = \sqrt{2 + 2\sqrt{2} \cdot e^{1/156}} = 2.20\ldots$. For this put $C_0 = 2 + 2\sqrt{2} \cdot e^{1/156}$, and let us consider the Laplace transform L_{S_n} of S_n . Since $\varepsilon_1, \ldots, \varepsilon_n$ are independent and $\cosh(x) \le e^{x^2/2}$ for $x \in \mathbf{R}$, then we have:

$$(3) \qquad L_{S_n}(t) \;=\; E\;[\;exp\;(t \cdot S_n)\;] \;=\; \prod_{i=1}^{n} E\;[\;exp\;(ta_i \cdot \varepsilon_i)\;] \;=$$

$$= \prod_{i=1}^{n} \cosh\;(ta_i) \;\le\; \prod_{i=1}^{n} exp\;\Big(\;\frac{t^2|a_i|^2}{2}\;\Big) \;=\; exp\;\Big(\;\frac{t^2 A_n}{2}\;\Big)$$

for all $t \in \mathbf{R}$. Thus if we put $D_n = \dfrac{S_n}{\sqrt{A_n}}$, then for its Laplace transform L_{D_n} we get: $L_{D_n}(t) \le e^{t^2/2}$, for all $t \in \mathbf{R}$, or in other words:

$$(4) \qquad \Big\{\;\frac{S_n}{\sqrt{A_n}}\;\mid\; n \ge 1\;\Big\} \text{ is a standard subnormal sequence}$$

Now we apply an idea presented in Kahane's book [9], see p. 43. Since S_n is symmetric, then so is $(S_n)^{2k+1}$ and thus $E\;(S_n)^{2k+1} = 0$ for all $k = 0, 1, \ldots$, and we have:

$$L_{S_n}(t) = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!}\; E(S_n)^{2k}$$

for all $t \in \mathbf{R}$. Hence by (3) we have:

$$E(S_n)^{2k} \;\le\; \frac{(2k)!}{t^{2k}} \cdot exp\;\Big(\;\frac{t^2 A_n}{2}\;\Big)$$

for all $k = 1, 2, \ldots$, and all $t > 0$. In that way we may obtain the following estimate for the Laplace transform L_{T_n} of T_n :

$$L_{T_n}(u) = E\;[\;exp\;(u \cdot T_n)\;] = E\;[\;exp\;(u \cdot (S_n)^2)\;] \;=$$

$$= \sum_{k=0}^{\infty} \frac{u^k}{k!}\; E\;(S_n)^{2k} \;\le\; 1 \;+\; \sum_{k=1}^{\infty} \frac{u^k}{k!}\;\frac{(2k)!}{(t_k)^{2k}} \cdot exp\;\Big(\;\frac{(t_k)^2 A_n}{2}\;\Big)$$

where $t_k > 0$ are arbitrary numbers for $k = 1, 2, \ldots$. In order to find a suitable choice for t_k's let us put $(t_k)^2 = \dfrac{2}{A_n}\,\tau_k$ for $k = 1, 2, \ldots$. Then we get:

$$L_{T_n}(u) \;\le\; 1 \;+\; \sum_{k=1}^{\infty} \frac{u^k}{k!}\;\frac{(2k)!}{(t_k)^{2k}} \cdot exp\;(\tau_k) =$$

$$= \;1 \;+\; \sum_{k=1}^{\infty} \frac{(2k)!}{2^k \cdot k!} \cdot \frac{1}{(\tau_k)^k}\;(uA_n)^k \cdot exp\;(\tau_k)$$

Using Stirling's formula (2.19), one can easily obtain:

$$\frac{(2k)!}{2^k \cdot k!} \;\le\; \sqrt{2} \cdot (2k)^k \cdot exp\;\Big(\;\frac{1}{12k} - \frac{1}{12k+1} - k\;\Big)$$

for all $k = 1, 2, \ldots$. Thus putting $\tau_k = k$ for $k = 1, 2, \ldots$, we get:

$$L_{T_n}(u) \;\leq\; 1 \;+\; \sum_{k=1}^{\infty} \;\sqrt{2}\cdot(2uA_n)^k \cdot exp\;\Big(\;\frac{1}{12k} - \frac{1}{12k+1}\;\Big) \;\leq$$

$$\leq\; 1 \;+\; \sqrt{2}\cdot e^{1/156}\cdot \;\sum_{k=1}^{\infty}\,(2uA_n)^k \;=$$

$$=\; 1 \;+\; \sqrt{2}\cdot e^{1/156}\cdot \;\Big(\;\frac{1}{1-2uA_n}\;-\;1\;\Big)$$

provided that $\mid 2uA_n \mid < 1$. Putting $u = 1/C_0A_n$ we may conclude:

$$\int_\Omega exp\;\Big(\;\frac{1}{C_0A_n}\;\cdot\;T_n\;\Big)\;dP \;=\; L_{T_n}\big(\;\frac{1}{C_0A_n}\;\big)\;\leq$$

$$\leq\; 1 \;+\; \sqrt{2}\cdot e^{1/156}\cdot\;\Big(\;\frac{1}{1-\frac{2}{C_0}}\;-\;1\;\Big)\;=\;2$$

Thus (2) is satisfied and proof 1 is complete.

Proof 2. The second proof is based on a classical Kahane-Khintchine inequality for tail probabilities of symmetric subnormal random variables and on the real representation of the P-integral, and we shall show that (2), and thus (1) also, holds with $C = \sqrt{6} = 2.44\ldots$. Put for this $C_0 = 6$ and $E_n = T_n/A_n$, and let us consider the Laplace transform L_{E_n} of E_n . By the real representation of the P-integral we have:

(5) $$L_{E_n}(u) \;=\; E\;[\;exp\;(u\cdot E_n)\;]\;=\;\int_\Omega exp\;(u\cdot E_n)\;dP\;=$$

$$=\;\int_0^\infty P\{\;exp\;(u\cdot E_n) > t\;\}\;dt\;=\;1\;+\;\int_1^\infty P\{\;exp\;\Big(u\cdot\frac{T_n}{A_n}\;\Big) > t\;\}\;dt\;=$$

$$=\;1\;+\;\int_0^\infty e^t\cdot P\{\;\frac{T_n}{A_n} > \frac{t}{u}\;\}\;dt\;=\;1\;+\;\int_0^\infty e^t\cdot P\{\;|D_n| > \sqrt{\tfrac{t}{u}}\;\}\;dt$$

for all $u > 0$, where D_n is given as above by $D_n = \dfrac{S_n}{\sqrt{A_n}}$. By (4) we have:

$$E\;[\;exp\;(v\cdot D_n)\;]\;\leq e^{v^2/2}$$

for all $v \in \mathbf{R}$, and since S_n is symmetric, hence by Markov's inequality we get:

$$P\{\;|D_n| \geq t\;\}\;=\;2\cdot P\{\;D_n\;\geq\;t\;\}\;=\;2\cdot P\{\;exp\;(v\cdot D_n) \geq exp\;(vt)\;\}\;\leq$$

$$\leq\;2\cdot exp\;(-vt)\cdot E\;[\;exp\;(v\cdot D_n)\;]\;\leq\;2\cdot exp\;\Big(\;\frac{v^2}{2} - vt\;\Big)$$

for all $t, v \geq 0$. Now it is easy to check that the function $S(v) = \frac{v^2}{2} - vt$ attains its minimal value on $\mathbf{R}_+$ at the point $v_{min} = t$, and $S(v_{min}) = -t^2/2$. In this way we may conclude:

$$P\{\;|D_n| \geq t\;\}\;\leq\;2\cdot e^{-t^2/2}$$

for all $t > 0$. Inserting this last inequality into (5) we find that:

$$L_{E_n}(u) \ \leq \ 1 \ + \ 2 \cdot \int_0^\infty e^{\ (1-\frac{1}{2u})\ t} \ dt \ = $$
$$= \ 1 \ + \ \frac{2}{\frac{1}{2u} - 1} \ = \ 1 \ + \ \frac{4u}{1-2u}$$

for all $0 < u < \dfrac{1}{2}$. Since $0 < \dfrac{1}{C_0} < \dfrac{1}{2}$, thus we may conclude:

$$\int_\Omega exp \ \Big(\ \frac{1}{C_0 A_n} \ \cdot \ T_n \ \Big) \ dP \ = \ L_{E_n}\big(\ \frac{1}{C_0} \ \big) \ \leq \ 1 \ + \ \frac{\frac{4}{C_0}}{1 - \frac{2}{C_0}} \ = \ 2$$

Thus (2) is satisfied and proof 2 is complete.

Proof 3. The third proof is based on an application of the Gaussian-Jensen's majorization technique which is described in section 2, and we show that (2), and thus (1) also, holds with $C = \sqrt{4\pi/3} = 2.04\ldots$. Put $C_0 = 4\pi/3$ and define:

$$g(x_1, \ldots, x_n) \ = \ E \ h\Big(\sum_{i=1}^n x_i \varepsilon_i \ \Big)$$

for $(x_1, \ldots, x_n) \in \mathbf{R}^n$, where $h(x) = exp \ \big(\frac{1}{C_0 A_n} \cdot x^2 \big)$ for $x \in \mathbf{R}$. Then evidently g is a convex function from $\mathbf{R}^n$ into $\mathbf{R}_+$, and one may easily observe that g is sign-symmetric. Thus by (2.10) we get:

$$E \ [\ exp \ \Big(\ \frac{1}{C_0 A_n} \cdot T_n \ \Big) \ ] \ = \ g(a_1, \ldots, a_n) \ \leq \ E \ h\Big(\sum_{i=1}^n X_i \varepsilon_i \ \Big) \ = $$
$$= \ E \ [\ exp \ \Big(\ \frac{1}{C_0 A_n} \cdot \Big(\ \sum_{i=1}^n X_i \varepsilon_i \ \Big)^2 \ \Big) \ ]$$

where $X_i \sim N(0, |a_i|^2 \pi/2)$ for $i = 1, \ldots, n$ are independent, as well as independent of $\varepsilon_1, \ldots, \varepsilon_n$. Since $\varepsilon_i X_i \sim X_i$ for $i = 1, \ldots, n$, then by independence of $X_1, \ldots, X_n, \varepsilon_1, \ldots, \varepsilon_n$ we have $\sum_{i=1}^n X_i \varepsilon_i \sim N(0, A_n \pi/2)$ and thus:

$$\frac{1}{C_0 A_n} \cdot \Big(\ \sum_{i=1}^n X_i \varepsilon_i \ \Big)^2 \ = \ \Big(\ \frac{1}{C\sqrt{A_n}} \cdot \sum_{i=1}^n X_i \varepsilon_i \ \Big)^2 \sim Y^2$$

where $Y \sim N(0, \dfrac{\pi}{2C_0})$. Hence we get:

$$E \ [\ exp \ \Big(\ \frac{1}{C_0 A_n} \cdot T_n \ \Big) \ ] \ \leq \ E \ [\ exp \ (Y^2) \ ] \ = $$
$$= \ \frac{1}{\sigma\sqrt{2\pi}} \ \int_{-\infty}^\infty exp \ \Big(\ x^2 - \frac{x^2}{2\sigma^2} \ \Big) \ dx \ = $$
$$= \ \frac{1}{\sigma\sqrt{2\pi}} \ \int_{-\infty}^\infty exp \ \Big[\ \big(\ 1 - \frac{1}{2\sigma^2} \ \big) \cdot x^2 \ \Big] \ dx$$

where $\sigma^2 = \dfrac{\pi}{2C_0}$. Since $\int_{-\infty}^{\infty} e^{-ax^2}\, dx = \sqrt{\frac{\pi}{a}}$ for $a > 0$, then we may conclude:

$$\int_\Omega exp\ \Big(\frac{1}{C_0 A_n} \cdot T_n \Big)\ dP \ \le\ \frac{1}{\sigma\sqrt{2\pi}} \cdot \sqrt{\frac{\pi}{\frac{1}{2\sigma^2} - 1}} = \frac{1}{\sqrt{1 - \frac{\pi}{C_0}}} = 2$$

provided that $\big(1 - \dfrac{1}{2\sigma^2} \big) < 0$, or in other words that $C^2 = C_0 > \pi$. Thus (2) is satisfied and proof 3 is complete.

Proof 4. The idea of the fourth proof is very simple. Namely, we shall expand the integrand in (2) into Taylor's series and then we shall apply the classical Khintchine inequalities (2.11) with (2.17). Thus there is a good reason to believe that the given result will be very deep, see [4]. Indeed, we shall see in corollary 4 below that the soon given constant $C = \sqrt{8/3} = 1.63\ldots$ is really the best possible. Let us say that during author's computations on the subject, this fact was conjectured by J. Hoffmann-Jørgensen. So, we shall show that (2), and thus (1) also, holds with $C = \sqrt{8/3}$. Let us consider the left side in (2). Then we have:

$$\int_\Omega exp\ \Big(\frac{1}{C^2 A_n} \cdot T_n \Big)\ dP \ =\ E\ [\ exp\ \Big(\frac{1}{C^2 A_n} \cdot (S_n)^2 \Big)\ ]\ =$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{1}{(C^2 A_n)^k} \cdot E\ (S_n)^{2k}$$

By the classical Kahane-Khintchine inequalities (2.11) with (2.17) we find:

$$E\ |\sum_{i=1}^{n} a_i \varepsilon_i\ |^{2k} \ \le\ K(2k,2) \cdot \Big(\sum_{i=1}^{n} |a_i|^2 \Big)^k$$

where $K(2k,2) = \dfrac{2^k \cdot \Gamma(k+\frac{1}{2})}{\sqrt{\pi}}$ for $k = 1, 2, \ldots$. Since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, then we have:

$$(6)\qquad \int_\Omega exp\ \Big(\frac{1}{C^2 A_n} \cdot T_n \Big)\ dP \ \le\ \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{1}{(C^2 A_n)^k} \cdot \frac{2^k \cdot \Gamma(k+\frac{1}{2})}{\sqrt{\pi}} \cdot (A_n)^k \ =$$

$$= \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \Big(\frac{2}{C^2} \Big)^k \cdot \Gamma(k+\frac{1}{2})$$

Now one can easily check that $\Gamma(k+\dfrac{1}{2}) = \dfrac{(2k-1)!! \cdot \sqrt{\pi}}{2^k}$ for $k \ge 1$, where $(2k-1)!! = (2k-1)\cdot(2k-3)\cdot\ldots\cdot 3 \cdot 1$. Since evidently $|\,2/C^2| < 1$, thus we may conclude:

$$(7)\qquad \int_\Omega exp\ \Big(\frac{1}{C^2 A_n} \cdot T_n \Big)\ dP \ \le\ \sum_{k=0}^{\infty} \frac{(2k-1)!!}{2^k \cdot k!} \cdot \Big(\frac{2}{C^2} \Big)^k \ =$$

$$= \Big(1 - \frac{2}{C^2} \Big)^{-1/2} \ =\ 2$$

Thus (2) is satisfied and proof 4 is complete.

$\square$

In order to prove that $C = \sqrt{8/3}$ is the best possible constant in inequality (1) in theorem 1, we shall first turn out the next two auxiliary results which are also of interest in themselves.

Lemma 2. *Let* $\{ X_i \mid i \geq 1 \}$ *be a sequence of independent identically distributed random variables such that* $E\,(X_1)^2 < \infty$ *, and let* $Z_n = \frac{1}{\sigma\sqrt{n}}\,(S_n - ES_n)$ *, where* $S_n = \sum_{i=1}^n X_i$ *and* $\sigma^2 = Var X_1$ *, for* $n \geq 1$ *. Suppose that* $\{ Z_n \mid n \geq 1 \}$ *is a symmetric standard subnormal sequence, that is,* Z_n *is symmetric and we have* $L_{Z_n}(t) \leq e^{t^2/2}$ *for all* $t \in \mathbf{R}$ *and all* $n \geq 1$ *. Then for every* $C > \sqrt{2}$ *, the sequence*

$$\{\ exp\ [\ \Big(\ \frac{Z_n}{C}\ \Big)^2\ ]\ \mid\ n \geq 1\ \}$$

is uniformly integrable.

Proof. Let $C > \sqrt{2}$ be given, then it is enough to show that for some $p > 1$ we have:

$$\sup_{n \geq 1}\ E\ [\ exp\ \Big(\ \frac{Z_n}{C}\ \Big)^2\ ]^p\ <\ \infty$$

Thus let us denote $I(p) = \sup_{n\geq 1}\ E\ [\,exp\ \big(Z_n/C\big)^2\ ]^p$. Then by the real representation of the P-integral we have:

$$\begin{aligned} I(p)\ &=\ \sup_{n\geq 1}\ E\ \{\ exp\ [\ p \cdot \Big(\frac{Z_n}{C}\ \Big)^2\ ]\ \}\ = \\ &=\ \sup_{n\geq 1}\ \int_0^\infty P\{\ exp\ [\ p \cdot \Big(\frac{Z_n}{C}\ \Big)^2\ ] > t\ \}\ dt\ = \\ &=\ 1\ +\ \sup_{n\geq 1}\ \int_1^\infty P\{\ \mid Z_n \mid\ > \frac{C}{\sqrt{p}} \cdot \sqrt{\log t}\ \}\ dt \end{aligned}$$

Since by our assumption $\{ Z_n \mid n \geq 1 \}$ is a symmetric standard subnormal sequence, then by the estimate established in proof 2 of theorem 1 we have:

$$\begin{aligned} I(p)\ &\leq\ 1\ +\ 2\ \int_1^\infty\ exp\ \Big(\ -\frac{1}{2} \cdot \frac{C^2}{p} \cdot \log t\ \Big)\ dt\ = \\ &=\ 1\ +\ 2\ \int_1^\infty\ \frac{1}{t^{C^2/2p}}\ dt \end{aligned}$$

Thus $I(p) < \infty$, if $C^2/2p > 1$. Since by our assumption $C^2/2 > 1$, then we see that there exists $p \in\ ]\ 1\ ,\ C^2/2\ [$ for which $I(p) < \infty$, and this fact completes the proof.

$\square$

Proposition 3. *Let* $\{ X_i \mid i \geq 1 \}$ *be a sequence of independent identically distributed* *random variables defined on a probability space* $(\Omega, \mathcal{F}, P)$ *such that* $E(X_1)^2 < \infty$ *, let* $Z_n = \frac{1}{\sigma\sqrt{n}}(S_n - ES_n)$ *, where* $S_n = \sum_{i=1}^n X_i$ *and* $\sigma^2 = Var X_1$ *for* $n \geq 1$ *, let* $N \sim N(0,1)$ *be a standard normal random variable, and let* $\|\cdot\|_\psi$ *denote the gauge norm on* $(\Omega, \mathcal{F}, P)$ *. If* $\{ Z_n \mid n \geq 1 \}$ *is a symmetric standard subnormal sequence, then*

$$\| Z_n \|_\psi \longrightarrow \| N \|_\psi \tag{1}$$

as $n \to \infty$ *, where* $\| N \|_\psi = \sqrt{8/3}$ *.*

Proof. Let us put $\sigma_n = \| Z_n \|_\psi$ for $n \geq 1$, and $C = \| N \|_\psi$. Then one can easily compute that $C = \sqrt{8/3}$. In order to prove (1) we shall first prove:

$$C \geq \limsup_{n\to\infty} \sigma_n \tag{2}$$

Indeed suppose $C < \limsup_{n\to\infty} \sigma_n$. Thus $C + \varepsilon < \sigma_{n_k}$ for some $n_1 < n_2 < \dots$ and some $\varepsilon > 0$. Since $C > \sqrt{2}$, then by lemma 2 the sequence $\{ exp\,[\,(\frac{Z_n}{C+\varepsilon})^2\,] \mid n \geq 1 \}$ is uniformly integrable, and therefore by the central limit theorem and the definition of the gauge norm $\|\cdot\|_\psi$ we get:

$$2 = \int_\Omega exp\,\Big[\Big(\frac{N}{C}\Big)^2\Big]\,dP > \int_\Omega exp\,\Big[\Big(\frac{N}{C+\varepsilon}\Big)^2\Big]\,dP =$$

$$= \lim_{n\to\infty} \int_\Omega exp\,\Big[\Big(\frac{Z_n}{C+\varepsilon}\Big)^2\Big]\,dP =$$

$$= \lim_{k\to\infty} \int_\Omega exp\,\Big[\Big(\frac{Z_{n_k}}{C+\varepsilon}\Big)^2\Big]\,dP \geq 2$$

Thus $C < \limsup_{n\to\infty} \sigma_n$ leads to a contradiction, and hence (2) is proved. Second we show:

$$C \leq \liminf_{n\to\infty} \sigma_n \tag{3}$$

Again suppose $C > \liminf_{n\to\infty} \sigma_n$. Thus $C - \varepsilon > \sigma_{n_k}$ for some $n_1 < n_2 < \dots$ and some $\varepsilon > 0$ with $C - \varepsilon > \sqrt{2}$. Hence by lemma 2, the central limit theorem, and the definition of the gauge norm $\|\cdot\|_\psi$ we get:

$$2 \geq \liminf_{k\to\infty} \int_\Omega exp\,\Big[\Big(\frac{Z_{n_k}}{\sigma_{n_k}}\Big)^2\Big]\,dP \geq$$

$$\geq \liminf_{k\to\infty} \int_\Omega exp\,\Big[\Big(\frac{Z_{n_k}}{C-\varepsilon}\Big)^2\Big]\,dP =$$

$$= \lim_{n\to\infty} \int_\Omega exp\,\Big[\Big(\frac{Z_n}{C-\varepsilon}\Big)^2\Big]\,dP =$$

$$= \int_\Omega exp\,\Big[\Big(\frac{N}{C-\varepsilon}\Big)^2\Big]\,dP > 2$$

Thus $C > \liminf_{n\to\infty} \sigma_n$ leads to a contradiction, and consequently (3) is proved. But then (2) and (3) evidently complete the proof.

$\square$

Corollary 4. *The best possible numerical constant C in inequality (1) in theorem 1 is equal to $\sqrt{8/3}$.*

Proof. We have shown in proof 4 of theorem 1 that inequality (1) in theorem 1 is satisfied with $\sqrt{8/3}$. Thus the best possible constant in inequality (1) in theorem 1, say D , is less than $\sqrt{8/3}$. To prove $D = \sqrt{8/3}$, let us take $a_1 = \ldots = a_n = 1/\sqrt{n}$ for $n \geq 1$. Then (1) in theorem 1 gets the following form:

$$\| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \varepsilon_i \|_\psi \leq D \tag{1}$$

being valid for all $n \geq 1$. And in order to apply proposition 3 we should know that $\{ (1/\sqrt{n}) \sum_{i=1}^{n} \varepsilon_i \mid n \geq 1 \}$ is a symmetric standard subnormal sequence. But let us note that this fact is established in proof 1 of theorem 1, see its relation (4). Thus letting $n \to \infty$ in (1), by proposition 3 we get $\sqrt{8/3} \leq D$, and hence we may conclude that the best possible constant D is indeed equal to $\sqrt{8/3}$. This fact completes the proof.

$\square$

Problem 5. In order to set up a natural question related to the result in proposition 3, let us recall some basic facts from [5] which are in the background of our section on Orlicz functionals, norms and spaces. Let $\mathcal{L}$ denote the set of all increasing left continuous functions φ from $[0,\infty[$ into $[0,\infty[$ with $\varphi(0) = \lim_{t\to 0+} \varphi(t) = 0$, and let $\mathcal{R}$ denote the set of all decreasing right continuous functions R from $[0,\infty[$ into $[0,\infty]$. Let $\mathcal{Q}$ denote the set of all functions q from $\mathcal{R}$ into $[0,\infty]$ such that $q(0) = 0$ and $q(R) \leq \sup_{n\geq 1} q(R_n)$ whenever $R \leq \liminf_{n\to\infty} R_n$ for $R, R_1, R_2 \ldots \in \mathcal{R}$. Recall that $q \in \mathcal{Q}$ is said to be:

(i) *subadditive*, if $q(R) \leq q(R_1) + q(R_2)$ whenever $R, R_1, R_2 \in \mathcal{R}$ and $R \preceq R_1 \oplus R_2$, i.e. $\forall \varphi, \psi, \xi \in \mathcal{L}$ with $\varphi \ltimes \psi \oplus \xi$, which means $\varphi(x+y) \leq \psi(x) + \xi(y)$ $\forall x, y \geq 0$, we have:

$$\int_0^\infty R(x)\ \varphi(dx) \ \leq \ \int_0^\infty R_1(x)\ \psi(dx) + \int_0^\infty R_2(x)\ \xi(dx)$$

(ii) *strongly subadditive*, if $q(R) \leq q(R_1) + q(R_2)$ whenever $R, R_1, R_2 \in \mathcal{R}$ and $R \ltimes R_1 \oplus R_2$, i.e. $R(x+y) \leq R_1(x) + R_2(y)$ $\forall x, y \geq 0$

(iii) *weakly subadditive*, if $q(R) \leq q(R_1) + q(R_2)$ whenever $R, R_1, R_2 \in \mathcal{R}$ and $R \leq R_1 + R_2$, i.e. $R(x) \leq R_1(x) + R_2(x)$ $\forall x \geq 0$

(iv) *homogeneous*, if $q(R^{(\lambda)}) = \frac{1}{\lambda} \cdot q(R)$ for all $\lambda > 0$, whenever $R \in \mathcal{R}$

(v) *non-degenerate*, if $q(R) = 0$ implies $R = 0$

(vi) *moderated*, if $q(R^{(\alpha)}) = \frac{1}{\alpha} \cdot q(R)$ for all $0 < \alpha \leq 1$, whenever $R \in \mathcal{R}$

Recall that $R^{(\lambda)}(t) = R(\lambda t)$ for $t \geq 0$, whenever $R \in \mathcal{R}$. And a natural question related to the result in proposition 3, see also proposition 4.3 and (2) in theorem 5.1 below, may be stated as follows: *What are (necessary and) sufficient conditions for* $R_n(t) \to R(t)$ $(\forall t \in S)$ *to imply* $q(R_n) \to q(R)$ *, where* $R, R_1, R_2 \ldots \in \mathcal{R}$ *,* $q \in \mathcal{Q}$ *and* $S \subset [0,\infty[$ *is a given subset ?* Since condition (i) plays an important role in cases when function norms and function spaces are induced by a measure space, see [5] (p.8), from that point of view, it is not a big restriction to assume that q in our question satisfies that condition also. Let us say that the answer to that generally stated problem will have interesting consequences related to Orlicz spaces, as well as Laurent spaces, and for more informations in that direction we refer the reader to [5].

$\square$

Using the classical symmetrization technique we shall now extend the result from theorem 1 in an appropriate way to a more general case. First we shall look at the sign-symmetric case in the next theorem, and then we shall pass to the general case in theorem 8 below.

Theorem 6. *Let* $\{ X_i \mid i \geq 1 \}$ *be a sequence of independent* $a.s.$ *bounded symmetric real valued random variables defined on a probability space* $(\Omega, \mathcal{F}, P)$ *, let* $\| \cdot \|_\psi$ *denote the gauge norm, and let* $\| \cdot \|_\infty$ *denote the usual sup-norm on* $(\Omega, \mathcal{F}, P)$ *. Then for every* $n \geq 1$ *the following inequality is satisfied:*

$$\| \sum_{i=1}^n X_i \|_\psi \leq \sqrt{8/3} \cdot \Big(\sum_{i=1}^n \| X_i \|_\infty^2 \Big)^{1/2} \tag{1}$$

Moreover, the numerical constant $\sqrt{8/3}$ *is the best possible in (1).*

Proof. Given $n \geq 1$, we denote $X = (X_1, \ldots, X_n)$, and let $\varepsilon_1, \ldots, \varepsilon_n$ be a Bernoulli sequence such that $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ is independent of $X = (X_1, \ldots, X_n)$. It is no restriction to assume that X and ε are defined on the same probability space $(\Omega, \mathcal{F}, P)$. Put $C_0 = 8/3$ and let us define a function f from $\mathbf{R}^n \times \mathbf{R}^n$ into $\mathbf{R}$ as follows:

$$f(x_1, \ldots, x_n, \delta_1, \ldots, \delta_n) \;=\; exp \left[\frac{1}{C_0 \cdot \sum_{i=1}^n |x_i|^2} \cdot \Big(\sum_{i=1}^n x_i \delta_i \Big)^2 \right]$$

for $(x_1, \ldots, x_n, \delta_1, \ldots, \delta_n) \in \mathbf{R}^n \times \mathbf{R}^n$. Since $X = (X_1, \ldots, X_n)$ and $(\varepsilon_1, \ldots, \varepsilon_n)$ are independent, then by Fubini's theorem we may conclude:

$$E \left\{ exp \left[\frac{1}{C_0 \cdot \sum_{i=1}^n |X_i|^2} \cdot \Big(\sum_{i=1}^n X_i \varepsilon_i \Big)^2 \right] \right\} \;=\; E\ g(X)$$

where

$$g(x) = E\ f(x, \varepsilon) = E \left\{ exp \left[\frac{1}{C_0 \cdot \sum_{i=1}^n |x_i|^2} \cdot \Big(\sum_{i=1}^n x_i \varepsilon_i \Big)^2 \right] \right\}$$

for $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$. By proof 4 of theorem 1 one directly finds that $g(x) \leq 2$ for all

$x \in \mathbf{R}^n$, and thus we may obtain $Eg(X) \le 2$. Since by our assumptions $X = (X_1, \dots, X_n)$ is sign-symmetric, then by the inequality just established we may conclude:

$$(2) \qquad E \left\{ exp \left[\frac{1}{C_0 \cdot \sum_{i=1}^n |X_i|^2} \cdot \Big(\sum_{i=1}^n X_i \Big)^2 \right] \right\} =$$

$$= E \left\{ exp \left[\frac{1}{C_0 \cdot \sum_{i=1}^n |X_i|^2} \cdot \Big(\sum_{i=1}^n X_i \varepsilon_i \Big)^2 \right] \right\} \le 2$$

But then we have:

$$E \left\{ exp \left[\frac{1}{C_0 \cdot \sum_{i=1}^n \| X_i \|_\infty^2} \cdot \Big(\sum_{i=1}^n X_i \Big)^2 \right] \right\} \le$$

$$\le E \left\{ exp \left[\frac{1}{C_0 \cdot \sum_{i=1}^n |X_i|^2} \cdot \Big(\sum_{i=1}^n X_i \Big)^2 \right] \right\} \le 2$$

Thus we may conclude:

$$\| \sum_{i=1}^n X_i \|_\psi \le \sqrt{C_0} \cdot \Big(\sum_{i=1}^n \| X_i \|_\infty^2 \Big)^{1/2}$$

and the proof of (1) is complete. The last statement follows directly by corollary 4. These facts complete the proof.

□

Remark 7. Note that (2) in the proof of theorem 6 states: *If* $X_1, \dots, X_n$ *are independent symmetric real valued random variables, then the following inequality is satisfied:*

$$\Big\| \frac{1}{\Big(\sum_{i=1}^n |X_i|^2 \Big)^{1/2}} \cdot \sum_{i=1}^n X_i \Big\|_\psi \le \sqrt{8/3}$$

Moreover, the numerical constant $\sqrt{8/3}$ *is the best possible (the smallest) with that property.*

Theorem 8. *Let* $\{ X_i \mid i \ge 1 \}$ *be a sequence of independent* $a.s.$ *bounded real valued random variables defined on a probability space* $(\Omega, \mathcal{F}, P)$ *, let* $\| \cdot \|_\psi$ *denote the gauge norm, and let* $\| \cdot \|_\infty$ *denote the usual sup-norm on* $(\Omega, \mathcal{F}, P)$ *. Then for every* $n \ge 1$ *the following inequality is satisfied:*

$$(1) \qquad \| \sum_{i=1}^n (X_i - EX_i) \|_\psi \le \sqrt{32/3} \cdot \Big(\sum_{i=1}^n \| X_i - EX_i \|_\infty^2 \Big)^{1/2}$$

Proof. Given $n \geq 1$, we denote $X = (X_1, \dots, X_n)$, and let $Y = (Y_1, \dots, Y_n)$ be a random vector such that X and Y are independent and identically distributed. It is no restriction to assume that X and Y are defined on the same probability space $(\Omega, \mathcal{F}, P)$, as well as that $EX_i = 0$ for $i = 1, \dots, n$. Put $S_n = \sum_{i=1}^n X_i$ and $T_n = \sum_{i=1}^n Y_i$, then we have:

(2) $$\| S_n \|_\psi \leq \| S_n - T_n \|_\psi$$

For this it is enough to show that

$$E \left\{ exp \left[\left(\frac{S_n}{C(n)} \right)^2 \right] \right\} \leq 2$$

where $C(n) = \| S_n - T_n \|_\psi$. Thus define:

$$f(s,t) = exp \left[\left(\frac{s-t}{C(n)} \right)^2 \right]$$

for $s, t \in \mathbf{R}$. Then evidently $t \mapsto f(s,t)$ is a convex function on $\mathbf{R}$, for all $s \in \mathbf{R}$, and moreover by our assumptions we have $T_n \in L^1(P)$ with $ET_n = 0$. Therefore by Fubini's theorem and Jensen's inequality we may easily obtain:

$$Ef(S_n, ET_n) \leq Ef(S_n, T_n)$$

Since $ET_n = 0$, then by the definition of the Orlicz norm $\| \cdot \|_\psi$ we get:

$$E \left\{ exp \left[\left(\frac{S_n}{C(n)} \right)^2 \right] \right\} = Ef(S_n, ET_n) \leq Ef(S_n, T_n) =$$

$$= E \left\{ exp \left[\left(\frac{S_n - T_n}{C(n)} \right)^2 \right] \right\} \leq 2$$

Thus (2) is proved. Now since $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_n)$ are independent and identically distributed, and $X_1, \dots, X_n$ are independent, then obviously $X - Y = (X_1 - Y_1, \dots, X_n - Y_n)$ is sign-symmetric, and thus by (2) and theorem 6 we may conclude:

$$\| S_n \|_\psi \leq \| S_n - T_n \|_\psi = \left\| \sum_{i=1}^n (X_i - Y_i) \right\|_\psi \leq$$

$$\leq \sqrt{8/3} \cdot \left(\sum_{i=1}^n \| X_i - Y_i \|_\infty^2 \right)^{1/2} \leq$$

$$\leq \sqrt{8/3} \cdot \left[\sum_{i=1}^n 2 \cdot \left(\| X_i \|_\infty^2 + \| Y_i \|_\infty^2 \right) \right]^{1/2} =$$

$$= \sqrt{32/3} \cdot \left(\sum_{i=1}^n \| X_i \|_\infty^2 \right)^{1/2}$$

Thus (1) is showed and the proof is complete.

$\square$

Corollary 9. *Let* $\{ X_i \mid i \geq 1 \}$ *be a sequence of independent* $a.s.$ *bounded real valued random variables defined on a probability space* $(\Omega, \mathcal{F}, P)$ *, let* $\| \cdot \|_\psi$ *denote the gauge norm, and let* $\| \cdot \|_\infty$ *denote the usual sup-norm on* $(\Omega, \mathcal{F}, P)$ *. Then for every* $\alpha > 0$ *and all* $n \geq 1$ *we have:*

(1) $$\| \sum_{i=1}^{n} (X_i - EX_i) \|_\psi \leq C_n(\alpha) \cdot \Big(\sum_{i=1}^{n} \| X_i - EX_i \|_\infty^\alpha \Big)^{1/\alpha}$$

where $C_n(\alpha)$ *is given by:*

$$C_n(\alpha) = \begin{cases} \sqrt{32/3} & , \text{ if } 0 < \alpha \leq 2 \\ \sqrt{32/3} \cdot n^{\frac{1}{2} - \frac{1}{\alpha}} & , \text{ if } 2 < \alpha < \infty . \end{cases}$$

Moreover, if $X_1, X_2, \dots$ *are symmetric, then for every* $\alpha > 0$ *and all* $n \geq 1$ *we have:*

(2) $$\| \sum_{i=1}^{n} X_i \|_\psi \leq D_n(\alpha) \cdot \Big(\sum_{i=1}^{n} \| X_i \|_\infty^\alpha \Big)^{1/\alpha}$$

where $D_n(\alpha)$ *is given by:*

$$D_n(\alpha) = \begin{cases} \sqrt{8/3} & , \text{ if } 0 < \alpha \leq 2 \\ \sqrt{8/3} \cdot n^{\frac{1}{2} - \frac{1}{\alpha}} & , \text{ if } 2 < \alpha < \infty . \end{cases}$$

Proof. Inequality (1) follows by theorem 8, (2.20) and (2.21), and inequality (2) follows by theorem 6, (2.20) and (2.21).

□

4. Basic results on the Orlicz space $L^{T_\psi}(P)$

Recall that the gauge norm $\| \cdot \|_\psi$ from the previous section is the norm $\| \cdot \|_{\tau_\psi}$ induced by the Orlicz functional τ_ψ , as defined in section 2, where $\psi(x) = e^{x^2} - 1$ for $x \in \mathbf{R}$. Consequently, one can be interested to find out inequalities involving the norms $\| \cdot \|_{T_\psi}$ and $\| \cdot \|_{\Upsilon_\psi}$ which correspond to the inequalities presented in corollary 3.9 and remark 3.7. Note that the inequality in (2.4) can be used for this purpose as well as the inequalities in (2.3) and (2.5) for related ones, but we shall try to prove it directly using the facts obtained in previous approaches. The starting point should obviously be the inequality presented in theorem 3.1 for which we have turned out four basically different proofs, or in other words four different techniques. However, note that the Orlicz norm $\| \cdot \|_{T_\psi}$ is not necessarily homogeneous, but one can easily check that we have:

(1) $$\| c \cdot X \|_{T_\psi} \leq | c | \cdot \| X \|_{T_\psi} \text{ , for all } | c | \geq 1$$

(2) $$\| c \cdot X \|_{T_\psi} \geq | c | \cdot \| X \|_{T_\psi} \text{ , for all } | c | \leq 1$$

where X is a given random variable, see also (2.1). And according to the result in corollary 3.4 we may conclude that the fourth approach in the proof of theorem 1 gives the best estimate for the left side under consideration, so it is reasonable to apply that technique in order to get as optimal result as possible. And this is really done in the proof of the next theorem.

Theorem 1. *Let* $\{ \varepsilon_i \mid i \geq 1 \}$ *be a Bernoulli sequence defined on a probability space* $(\Omega, \mathcal{F}, P)$ *, and let* $\| \cdot \|_{T_\psi}$ *denote the Orlicz norm on* $(\Omega, \mathcal{F}, P)$ *as defined in section 2. Then the following inequality is satisfied:*

$$\Big\| \frac{1}{\Big(\sum_{i=1}^{n} |a_i|^2 \Big)^{1/2}} \cdot \sum_{i=1}^{n} a_i \varepsilon_i \Big\|_{T_\psi} \leq C \tag{1}$$

for all $a_1, \dots a_n \in \mathbf{R}$ *and all* $n \geq 1$ *, where the numerical constant* C *is given by:*

$$C = \frac{1}{2} \cdot \Big(\sqrt{ \frac{2}{\sqrt{\gamma}} - \gamma + 7 } + \sqrt{\gamma} - 1 \Big) \; , \; \textit{with} \tag{2}$$

$$\gamma = \frac{7}{3} - 2 \cdot \Big(\sqrt[3]{ \frac{29}{54} - \sqrt{\frac{31}{108}} } + \sqrt[3]{ \frac{29}{54} + \sqrt{\frac{31}{108}} } \Big)$$

Moreover, that numerical constant is the best possible in (1).

Proof. Given $a_1, \dots, a_n \in \mathbf{R}$ for some $n \geq 1$, we denote as before $S_n = \sum_{i=1}^n a_i \varepsilon_i$, $T_n = (S_n)^2$ and $A_n = \sum_{i=1}^n |a_i|^2$. Then by the definition of the Orlicz norm $\| \cdot \|_{T_\psi}$, it is enough to show the following inequality:

$$\int_\Omega exp \Big(\frac{1}{C^2 A_n} \cdot T_n \Big) \, dP \leq 1 + C \tag{3}$$

For this note that by (7) in proof 4 of theorem 3.1 we have:

$$\int_\Omega exp \Big(\frac{1}{x^2 A_n} \cdot T_n \Big) \, dP \leq \Big(1 - \frac{2}{x^2} \Big)^{-1/2} \tag{4}$$

for all $x > \sqrt{2}$. Put $\alpha(x) = (1 - 2/x^2)^{-1/2}$ and $\beta(x) = 1 + x$ for $x > \sqrt{2}$. Then one can easily check that $\alpha > \beta$ on $] \sqrt{2} , C [$, $\alpha < \beta$ on $] C , \infty [$ and $\alpha(C) = \beta(C)$. Moreover, given C satisfies the following equation: $C^4 + 2C^3 - 2C^2 - 4C - 2 = 0$. Thus by using Ferrari's formulas, see [26], it is a matter of routine to check that C is given by (2) above. Hence (3) follows by (4) and the proof of (1) is complete. To prove that the given numerical constant C is the best possible in (1), we shall follow the idea presented in the proof of corollary 3.4. So, let D denote the best possible numerical constant in (1). If we take $a_1 = \dots = a_n = 1/\sqrt{n}$ for $n \geq 1$, then (1) gets the following form:

$$\| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \varepsilon_i \|_{T_\psi} \leq D \tag{5}$$

Now one needs a result similar to that presented in proposition 3.3 but with $\| \cdot \|_{T_\psi}$ instead of $\| \cdot \|_\psi$. And this fact will be established in proposition 3 below, so letting $n \to \infty$ in (5), by proposition 3 below we may conclude that $C \leq D$. Since the inequality $D \leq C$ follows by (1), this fact completes the proof.

$\square$

Remark 2. We have seen in the proof of theorem 1 that the numerical constant C given by (2) in theorem 1 is a unique solution of the equation: $x^4 + 2x^3 - 2x^2 - 4x - 2 = 0$ for $x > \sqrt{2}$. By the well-known criterion for rational solutions for algebraic equations with rational coefficients, see [26], each rational solution of the above equation belongs to the set $\{ \pm 1, \pm 2 \}$. And one can easily check that ± 1 , as well as ± 2 , does not satisfy the above equation, and thus we may conclude that the above equation has no rational solutions at all. Therefore *the numerical constant* C given by (2) in theorem 1 *is not a rational number*. But one can easily check that we have: $C = 1.538615763\ldots$, as well as that the following rational approximations are valid:

$$\tag{1} \tfrac{60}{39} < C < \tfrac{77}{50} \quad \text{and} \quad \sqrt{\tfrac{71}{30}} < C < \sqrt{\tfrac{19}{8}}$$

where $77/50 - C = 0.0013\ldots$, and $\sqrt{19/8} - C = 0.0024\ldots$. Thus inequality (1) in theorem 1 is satisfied with $C = 77/50$, as well as with $C = \sqrt{19/8}$. Note that $\sqrt{19/8} < \sqrt{8/3}$, see corollary 3.4.

Proposition 3. *Let* $\{ X_i \mid i \geq 1 \}$ *be a sequence of independent identically distributed random variables defined on a probability space* $(\Omega, \mathcal{F}, P)$ *such that* $E\,(X_1)^2 < \infty$ *, let* $Z_n = \frac{1}{\sigma\sqrt{n}}\,(S_n - ES_n)$ *, where* $S_n = \sum_{i=1}^n X_i$ *and* $\sigma^2 = Var X_1$ *for* $n \geq 1$ *, let* $N \sim N(0,1)$ *be a standard normal random variable, and let* $\| \cdot \|_{T_\psi}$ *denote the Orlicz norm on* $(\Omega, \mathcal{F}, P)$ *as defined in section 2. If* $\{ Z_n \mid n \geq 1 \}$ *is a symmetric standard subnormal sequence, then*

$$\tag{1} \| Z_n \|_{T_\psi} \longrightarrow \| N \|_{T_\psi}$$

as $n \to \infty$ *, where* $\| N \|_{T_\psi}$ *is equal to the numerical constant* C *given by (2) in theorem 1.*

Proof. Let us put $\sigma_n = \| Z_n \|_{T_\psi}$ for $n \geq 1$, and $C = \| N \|_{T_\psi}$. Then by the definition of the Orlicz norm $\| \cdot \|_{T_\psi}$ one can easily check that $C > \sqrt{2}$ and $C^4 + 2C^3 - 2C^2 - 4C - 2 = 0$. Thus by the proof of theorem 1 we see that C is given by (2) in theorem 1. And in order to prove that $\sigma_n \to C$ as $n \to \infty$, one can repeat the proof of (2) and (3) presented in the proof of proposition 3.3, where the numerical constant 2 should be replaced by the numerical constant $1 + C$ on the right places. These facts easily complete the proof.

$\square$

Theorem 4. *Let* $\{ X_i \mid i \geq 1 \}$ *be a sequence of independent symmetric real valued random variables defined on a probability space* $(\Omega, \mathcal{F}, P)$ *, and let* $\| \cdot \|_{T_\psi}$ *denote the Orlicz norm on* $(\Omega, \mathcal{F}, P)$ *as defined in section 2. Then for every* $n \geq 1$ *the following inequality is satisfied:*

$$\tag{1} \Big\| \frac{1}{\Big(\sum_{i=1}^n |X_i|^2 \Big)^{1/2}} \cdot \sum_{i=1}^n X_i \Big\|_{T_\psi} \leq C$$

where C is the numerical constant given by (2) in theorem 1. Moreover, that numerical constant is the best possible in (1).

Proof. The proof of (1) is completely the same as the proof of (2) in theorem 3.6, where the numerical constant 2 should be replaced by the numerical constant $1+C$ on the right places, with $C_0 = C^2$, and one should use theorem 1 instead of proof 4 of theorem 3.1 to obtain the desired inequality. Also note that the last statement follows directly by the last statement in theorem 1. These facts easily complete the proof.

$\square$

We shall continue our considerations by searching for an analogous inequality to that presented in theorem 3.1 where the gauge norm $\| \cdot \|_\psi$ should be replaced by the Orlicz norm $\| \cdot \|_{T_\psi}$ as defined in section 2. For this, let us consider a Bernoulli sequence $\{ \varepsilon_i \mid i \geq 1 \}$ which is defined on a probability space $(\Omega, \mathcal{F}, P)$. Despite the fact that the Orlicz norm $\| \cdot \|_{T_\psi}$ on $(\Omega, \mathcal{F}, P)$ is not homogeneous, according to (4.2) and inequality (1) in theorem 1, we may conclude that the following inequality is satisfied:

$$(3) \qquad \Big\| \sum_{i=1}^n a_i \varepsilon_i \Big\|_{T_\psi} \leq C \cdot \Big(\sum_{i=1}^n |a_i|^2 \Big)^{1/2}$$

for all $a_1, \dots, a_n \in \mathbf{R}$ and all $n \geq 1$ for which $\sum_{i=1}^n |a_i|^2 \geq 1$, where C is the numerical constant given by (2) in theorem 1. And one can ask is this inequality true in general, or in other words, does (3) hold for all $a_1, \dots, a_n \in \mathbf{R}$ and all $n \geq 1$? We shall now show that the answer to this question is negative. For this, suppose that (3) is satisfied for all $a_1, \dots, a_n \in \mathbf{R}$ and all $n \geq 1$. Then taking $a_1 = \dots = a_n = 1/n$ for $n \geq 1$, we get:

$$\Big\| \frac{1}{n} \sum_{i=1}^n \varepsilon_i \Big\|_{T_\psi} \leq C \cdot \frac{1}{\sqrt{n}}$$

for all $n \geq 1$. Hence by the definition of the Orlicz norm $\| \cdot \|_{T_\psi}$ we may conclude:

$$\int_\Omega exp \Big(\frac{1}{C\sqrt{n}} \sum_{i=1}^n \varepsilon_i \Big)^2 \, dP \; \leq \; 1 + C \cdot \frac{1}{\sqrt{n}}$$

for all $n \geq 1$. Letting $n \to \infty$, by lemma 3.2 and the central limit theorem we easily obtain:

$$\int_\Omega exp \Big(\frac{1}{C} N \Big)^2 \, dP \; = \; \lim_{n \to \infty} \int_\Omega exp \Big(\frac{1}{C\sqrt{n}} \sum_{i=1}^n \varepsilon_i \Big)^2 \, dP \; = \; 1$$

where $N \sim N(0,1)$ is a standard normal random variable. Since this inequality obviously does not hold for any real number C , thus (3) does not hold in that case. By the way, let us note that using the fact established in lemma 3.2 together with the classical Hartman-Wintner law of iterated logarithm one can easily conclude:

$$\lim_{n \to \infty} \int_\Omega exp \Big(\frac{1}{C \cdot n^p} \sum_{i=1}^n \varepsilon_i \Big)^2 \, dP \; = \; 1$$

for any $p > 1/2$ and any $C > 0$. However, note that by the general inequality given in (2.4) and theorem 3.1 we get:

$$\| \sum_{i=1}^{n} a_i \varepsilon_i \|_{T_\psi} \leq \Big[\sqrt{8/3} \cdot \Big(\sum_{i=1}^{n} |a_i|^2 \Big)^{1/2} \Big] \vee \Big[\sqrt{8/3} \cdot \Big(\sum_{i=1}^{n} |a_i|^2 \Big)^{1/2} \Big]^{1/2}$$

for all $a_1, \dots, a_n \in \mathbf{R}$ and all $n \geq 1$. Hence we can deduce the following inequality with a unique constant:

$$\| \sum_{i=1}^{n} a_i \varepsilon_i \|_{T_\psi} \leq \sqrt{8/3} \cdot \Big[\Big(\sum_{i=1}^{n} |a_i|^2 \Big)^{1/2} \vee \Big(\sum_{i=1}^{n} |a_i|^2 \Big)^{1/4} \Big] \tag{4}$$

for all $a_1, \dots, a_n \in \mathbf{R}$ and all $n \geq 1$. In particular, we have:

$$\| \sum_{i=1}^{n} a_i \varepsilon_i \|_{T_\psi} \leq \sqrt{8/3} \cdot \Big(\sum_{i=1}^{n} |a_i|^2 \Big)^{1/4} \tag{5}$$

for all $a_1, \dots, a_n \in \mathbf{R}$ and $n \geq 1$ for which $\sum_{i=1}^{n} |a_i|^2 \leq 1$. Now one can ask, first of all, is the numerical constant $\sqrt{8/3}$ the best possible in (4) ? In other words we may ask, is it the best possible in (5), since by (3) we know that this is not true for $\sum_{i=1}^{n} |a_i|^2 \geq 1$? In order to get some preliminary informations in this direction, let D denote the best possible numerical constant in (5), that is, let (5) be satisfied for all $a_1, \dots, a_n \in \mathbf{R}$ and all $n \geq 1$ for which $\sum_{i=1}^{n} |a_i|^2 \leq 1$, if we replace $\sqrt{8/3}$ by D , and let D be the smallest number with that property. Putting $a_1 = \ldots = a_n = 1/\sqrt{n}$ for all $n \geq 1$, by proposition 3 and (5) we may easily conclude that $C \leq D$, where C denotes the numerical constant given by (2) in theorem 1. Thus $C \leq D \leq \sqrt{8/3}$, and in the next lemma we show that the first inequality is actually equality. These facts answer our first question.

Lemma 5. *The best possible numerical constant in inequality (4) above is equal to* C *given by (2) in theorem 1.*

Proof. Given $a_1, \dots, a_n \in \mathbf{R}$ for some $n \geq 1$, we denote $S_n = \sum_{i=1}^{n} a_i \varepsilon_i$, $T_n = (S_n)^2$ and $A_n = \sum_{i=1}^{n} |a_i|^2$. According to (3) and conclusions which precede lemma 5, the only fact which remains to prove is inequality (5) with the numerical constant C given by (2) in theorem 1, instead of $\sqrt{8/3}$. For this we follow proof 4 of theorem 3.1, and using the same arguments as for relations (6) and (7) there, we may obtain:

$$\int_\Omega exp \Big(\frac{1}{C^2 \sqrt{A_n}} \cdot T_n \Big) \, dP \leq \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{1}{(\, C^2 \sqrt{A_n} \,)^k} \cdot \frac{2^k \cdot \Gamma(k+\frac{1}{2})}{\sqrt{\pi}} \cdot (A_n)^k = \tag{1}$$

$$= \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \Big(\frac{2 \cdot \sqrt{A_n}}{C^2} \Big)^k \cdot \Gamma(k+\frac{1}{2}) =$$

$$= \sum_{k=0}^{\infty} \frac{(2k-1)!!}{2^k \cdot k!} \cdot \Big(\frac{2 \cdot \sqrt{A_n}}{C^2} \Big)^k = \Big(1 - \frac{2 \cdot \sqrt{A_n}}{C^2} \Big)^{-1/2}$$

since evidently by our assumptions $\mid 2\sqrt{A_n} \ / \ C^2 \mid < 1$, where C is the numerical constant given by (2) in theorem 1. Now one can easily check that:

$$2 \cdot x^3 \ + \ \frac{4}{C} \cdot x^2 \ + \ \Big(\frac{2}{C^2} - C^2 \Big) \cdot x \ - \ 2C \ \leq \ 0$$

for all $0 \leq x \leq 1$, and thus the following inequality is satisfied:

$$(2) \qquad \Big(\ 1 - \frac{2}{C^2} \ x \ \Big)^{-1/2} \leq \ 1 \ + \ C \cdot \sqrt{x}$$

for all $0 \leq x \leq 1$. And by (1) and (2) we get:

$$\int_\Omega exp \ \Big(\ \frac{1}{C^2 \sqrt{A_n}} \ \cdot \ T_n \ \Big) \ dP \ \leq \ 1 \ + \ C \cdot \sqrt[4]{A_n}$$

Hence by the definition of the Orlicz norm $\| \cdot \|_{T_\psi}$ we may conclude that $\| \ S_n \ \|_{T_\psi} \leq \ C \cdot \sqrt[4]{A_n}$, and this fact completes the proof.

□

Note that a main reason for exponent $1/4$ appears in inequalities (4) and (5) above is coming from the general inequality given in (2.4). And one can ask is this exponent indeed the best possible in that case? In order to answer this question, let $\{ \ \varepsilon_i \ \mid \ i \geq 1 \ \}$ be a Bernoulli sequence defined on a probability space $(\Omega, \mathcal{F}, P)$, and let $\| \cdot \|_{T_\psi}$ denote the Orlicz norm on $(\Omega, \mathcal{F}, P)$ as defined in section 2. Given $a_1, \dots, a_n \in \mathbf{R}$ for some $n \geq 1$, we denote $S_n = \sum_{i=1}^n a_i \varepsilon_i$, $T_n = (S_n)^2$ and $A_n = \sum_{i=1}^n |a_i|^2$. Suppose that $A_n \leq 1$, let $p > 1$ be a given number, and let q be the conjugate exponent of p , that is: $1 = 1/p + 1/q$. Then the same argument as in (1) in the proof of lemma 5, yields the following estimate:

$$\begin{aligned} (6) \qquad & \int_\Omega exp \ \Big(\ \frac{1}{C^2 \ (A_n)^{1/p}} \ \cdot \ T_n \ \Big) \ dP \ \leq \\ & \leq \ \sum_{k=0}^\infty \ \frac{1}{k!} \cdot \frac{1}{(C^2 \ (A_n)^{1/p})^k} \cdot \frac{2^k \cdot \Gamma(k+\frac{1}{2})}{\sqrt{\pi}} \cdot (A_n)^k \ = \\ & = \ \frac{1}{\sqrt{\pi}} \ \sum_{k=0}^\infty \ \frac{1}{k!} \cdot \Big(\ \frac{2 \ (A_n)^{1/q}}{C^2} \ \Big)^k \cdot \ \Gamma(k + \frac{1}{2}) \ = \\ & = \ \sum_{k=0}^\infty \ \frac{(2k-1)!!}{2^k \cdot k!} \cdot \Big(\ \frac{2 \ (A_n)^{1/q}}{C^2} \ \Big)^k \ = \ \Big(\ 1 - \frac{2 \ (A_n)^{1/q}}{C^2} \ \Big)^{-1/2} \end{aligned}$$

since evidently $\mid 2 \ (A_n)^{1/q} \ / \ C^2 \mid < 1$, where C is the numerical constant given by (2) in theorem 1. Let us now define:

$$q^* \ = \sup \Big\{ \ \ q \geq 2 \ \ \mid \ \ \Big(\ 1 - \frac{2}{C^2} \ x^{1/q} \ \Big)^{-1/2} \leq \ 1 \ + \ C \cdot x^{1/2 - 1/2q} \ , \ \forall x \in [0,1] \ \ \Big\}$$

Recall that by the definition of C , see the proof of theorem 1, we have:

$$\left(1 - \frac{2}{C^2} \right)^{-1/2} = 1 + C$$

Thus the inequality in definition of q^* is always satisfied for $x = 1$, as well as for $x = 0$. Let p^* be the conjugate exponent of q^*, that is: $1/p^* + 1/q^* = 1$. Then we evidently have:

$$\left(1 - \frac{2}{C^2} x^{1/q^*} \right)^{-1/2} \leq 1 + C \cdot x^{1/2 - 1/2q^*} = 1 + C \cdot x^{1/2p^*}$$

for all $0 \leq x \leq 1$, and thus by (6) we may obtain:

$$\int_\Omega exp \left(\frac{1}{C^2 (A_n)^{1/p^*}} \cdot T_n \right) dP \leq 1 + C \cdot (A_n)^{1/2p^*}$$

By the definition of the Orlicz norm $\| \cdot \|_{T_\psi}$ hence we may conclude:

(7) $$\| S_n \|_{T_\psi} \leq C \cdot (A_n)^{1/2p^*}$$

Note that by lemma 5 we have $p^* \leq 2$, or in other words it is no restriction to assume that $q^* \geq 2$. Moreover, it is easy to check that the inequality in definition of q^*, for $q = 4$, is equivalent to the following inequality:

$$2 \cdot x^6 - C^2 \cdot x^4 + \frac{4}{C} \cdot x^3 - 2C \cdot x + \frac{2}{C^2} \leq 0$$

for all $0 \leq x \leq 1$, which is obviously not satisfied, since the left side takes the value $2/C^2 > 0$ at $x = 0$. Thus $q^* < 4$, or in other words $p^* > 4/3$. Consequently, we may conclude:

(8) $$\frac{2}{8} \leq \frac{1}{2p^*} < \frac{3}{8}$$

And the aim of the next lemma is to establish that $q^* = 3$, or in other words that $1/2p^* = 1/3$.

Lemma 6. *The largest number q satisfying the following inequality:*

(1) $$\left(1 - \frac{2}{C^2} x^{1/q} \right)^{-1/2} \leq 1 + C \cdot x^{1/2 - 1/2q}$$

for all $0 \leq x \leq 1$, where C is the numerical constant given by (2) in theorem 1, is equal to 3.

Proof. By (8) above we know that the largest number q satisfying (1), say q^*, is strictly less than 4. Furthermore, one can easily verify that for $q = 3$, inequality (1) is equivalent to the following easy to check inequality:

$$2 \cdot x^2 + \left(\frac{4}{C} - C^2 \right) \cdot x - 2C + \frac{2}{C^2} \leq 0$$

for all $0 \leq x \leq 1$. Thus we may deduce that $3 \leq q^* < 4$. Let us therefore take $0 < \varepsilon < 1$, and let us consider inequality (1) for $q = 3 + \varepsilon$. Then one can easily verify that inequality (1) is equivalent to the following inequality:

$$2 \cdot x^{2+\varepsilon} \; - \; C^2 \cdot x^{1+\varepsilon} \; + \; \frac{4}{C} \cdot x^{1+\varepsilon/2} \; - \; 2C \cdot x^{\varepsilon/2} \; + \; \frac{2}{C^2} \; \leq \; 0$$

for all $0 < x \leq 1$. But note that for every $\varepsilon > 0$, the left side above takes the value $2/C^2 > 0$ at $x = 0$, and therefore we may conclude that (1) is not satisfied for any $q = 3 + \varepsilon$ with some $\varepsilon > 0$. This fact shows that the largest q satisfying inequality (1) is equal exactly to 3 , and the proof is complete.

□

Theorem 7. *Let* $\{ \varepsilon_i \mid i \geq 1 \}$ *be a Bernoulli sequence defined on a probability space* $(\Omega, \mathcal{F}, P)$ *, and let* $\| \cdot \|_{T_\psi}$ *denote the Orlicz norm on* $(\Omega, \mathcal{F}, P)$ *as defined in section 2. Then the following inequality is satisfied:*

$$\| \sum_{i=1}^{n} a_i \varepsilon_i \|_{T_\psi} \leq C \cdot \Big[\Big(\sum_{i=1}^{n} |a_i|^2 \Big)^{1/2} \vee \Big(\sum_{i=1}^{n} |a_i|^2 \Big)^{1/3} \Big] \tag{1}$$

for all $a_1, \ldots, a_n \in \mathbf{R}$ *and all* $n \geq 1$ *, where* C *is the numerical constant given by (2) in theorem 1. Moreover, that numerical constant is the best possible in (1).*

Proof. Inequality (1) follows directly by (3), (7) and lemma 6, and the last statement follows straight forward by lemma 5.

□

Problem 8. Note that the exponent $1/3$ in inequality (1) in theorem 7 is optimal in the framework of our best estimate established in proof 4 of theorem 3.1 which in turn lies on the best possible constants in classical Khintchine inequalities, see (6) and (7) in proof 4 of theorem 3.1. However, note that we did not prove that it is indeed the best possible, mainly because of the fact that something "wrong" may happen for "small" A_n's for which our basic inequality employed above, see (6), possibly does not work in the best way. Hence, we can set up a natural question: *What is the best possible exponent which can take the place of* $1/3$ *in inequality (1) in theorem 7 ?* Note that by results established above this number is greater or equal to $1/3$, and strictly less than $1/2$.

As usual, by using the classical symmetrization technique, we shall extend the result of theorem 7 to more general cases in the next two theorems.

Theorem 9. *Let* $\{ X_i \mid i \geq 1 \}$ *be a sequence of independent* $a.s.$ *bounded symmetric real valued random variables defined on a probability space* $(\Omega, \mathcal{F}, P)$ *, let* $\| \cdot \|_{T_\psi}$ *denote the Orlicz norm as defined in section 2, and let* $\| \cdot \|_\infty$ *denote the usual sup-norm on* $(\Omega, \mathcal{F}, P)$ *. Then for every* $n \geq 1$ *the following inequality is satisfied:*

$$\| \sum_{i=1}^{n} X_i \|_{T_\psi} \leq C \cdot \Big[\Big(\sum_{i=1}^{n} \| X_i \|_\infty^2 \Big)^{1/2} \vee \Big(\sum_{i=1}^{n} \| X_i \|_\infty^2 \Big)^{1/3} \Big] \tag{1}$$

where C *is the numerical constant given by (2) in theorem 1. Moreover, that numerical constant* *is the best possible in (1).*

Proof. The proof of (1) is completely the same as the proof of (1) in theorem 3.6, where the numerical constant 2 should be replaced by $1 + C \cdot [(\sum_{i=1}^n |x_i|^2)^{1/2} \vee (\sum_{i=1}^n |x_i|^2)^{1/3}]$, and then by $1 + C \cdot [(\sum_{i=1}^n \| X_i \|_\infty^2)^{1/2} \vee (\sum_{i=1}^n \| X_i \|_\infty^2)^{1/3}]$ on the right places, as well as expressions $\sum_{i=1}^n |x_i|^2$ and $\sum_{i=1}^n |X_i|^2$ by $(\sum_{i=1}^n |x_i|^2) \vee (\sum_{i=1}^n |x_i|^2)^{2/3}$ and $(\sum_{i=1}^n |X_i|^2) \vee (\sum_{i=1}^n |X_i|^2)^{2/3}$, and where theorem 7 should be used instead of theorem 3.1 on the right places. Also note that the last statement follows directly by the last statement given in theorem 7. These facts easily complete the proof.

$\square$

Theorem 10. *Let $\{ X_i \mid i \geq 1 \}$ be a sequence of independent $a.s.$ bounded real valued random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, let $\| \cdot \|_{T_\psi}$ denote the Orlicz norm as defined in section 2, and let $\| \cdot \|_\infty$ denote the usual sup-norm on $(\Omega, \mathcal{F}, P)$. Then for every $n \geq 1$ the following inequality is satisfied:*

$$
(1) \qquad \| \sum_{i=1}^n (X_i - EX_i) \|_{T_\psi} \leq 2\, C \cdot \Big[\Big(\sum_{i=1}^n \| X_i - EX_i \|_\infty^2 \Big)^{1/2} \vee \vee \Big(\sum_{i=1}^n \| X_i - EX_i \|_\infty^2 \Big)^{1/3} \Big]
$$

where C is the numerical constant given by (2) in theorem 1.

Proof. The proof of (1) is completely the same to the proof of (1) in theorem 3.8, where the numerical constant 2 should be replaced by the numerical constant $1 + C(n) = 1 + \| S_n - T_n \|_{T_\psi}$ on the right places. In this way we may conclude:

$$
\| S_n \|_{T_\psi} \leq \| S_n - T_n \|_{T_\psi}
$$

for all $n \geq 1$, and in the rest one should use theorem 9 instead of theorem 3.6 to deduce the final conclusion. These facts easily complete the proof.

$\square$

Corollary 11. *Let $\{ X_i \mid i \geq 1 \}$ be a sequence of independent $a.s.$ bounded real valued random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, let $\| \cdot \|_{T_\psi}$ denote the Orlicz norm on $(\Omega, \mathcal{F}, P)$ as defined in section 2, let $\| \cdot \|_\infty$ denote the usual sup-norm on $(\Omega, \mathcal{F}, P)$, and let C be the numerical constant given by (2) in theorem 1. Then for every $\alpha > 0$ and all $n \geq 1$ we have:*

$$
(1) \qquad \| \sum_{i=1}^n (X_i - EX_i) \|_{T_\psi} \leq C_n(\alpha) \cdot \Big[\Big(\sum_{i=1}^n \| X_i - EX_i \|_\infty^\alpha \Big)^{1/\alpha} \vee \vee \Big(\sum_{i=1}^n \| X_i - EX_i \|_\infty^\alpha \Big)^{2/3\alpha} \Big]
$$

where $C_n(\alpha)$ is given by:

$$C_n(\alpha) \; = \; \begin{cases} \quad 2C \quad , & \textit{if} \;\; 0 < \alpha \le 2 \\ \\ 2C \cdot n^{\,\frac{1}{2} - \frac{1}{\alpha}} \; , & \textit{if} \;\; 2 < \alpha < \infty \; . \end{cases}$$

Moreover, if $X_1, X_2, \dots$ *are symmetric, then for every* $\alpha > 0$ *and all* $n \ge 1$ *we have:*

(2) $$\| \sum_{i=1}^n X_i \|_{T_\psi} \le \; D_n(\alpha) \cdot \Big[\Big(\sum_{i=1}^n \| X_i \|_\infty^\alpha \Big)^{1/\alpha} \vee \Big(\sum_{i=1}^n \| X_i \|_\infty^\alpha \Big)^{2/3\alpha} \Big]$$

where $D_n(\alpha)$ *is given by:*

$$D_n(\alpha) \; = \; \begin{cases} \quad C \quad , & \textit{if} \;\; 0 < \alpha \le 2 \\ \\ C \cdot n^{\,\frac{1}{2} - \frac{1}{\alpha}} \; , & \textit{if} \;\; 2 < \alpha < \infty \; . \end{cases} \; .$$

Proof. Inequality (1) follows by theorem 10, (2.20) and (2.21), and inequality (2) follows by theorem 9, (2.20) and (2.21).

□

5. Basic results on the Orlicz space $L^{\Upsilon_\psi}(P)$

The considerations in this section are devoted to the study of the questions presented in the last two sections, but now for the Orlicz norm $\| \cdot \|_{\Upsilon_\psi}$ as defined in section 2, where $\psi(x) = e^{x^2} - 1$ for $x \in \mathbf{R}$. Similarly to the previous approach we shall essentially use the estimate established in proof 4 of theorem 3.1. The first result in this direction may be stated as follows:

Theorem 1. *Let* $\{ \, \varepsilon_i \; | \; i \ge 1 \, \}$ *be a Bernoulli sequence defined on a probability space* $(\Omega, \mathcal{F}, P)$ *, and let* $\| \cdot \|_{\Upsilon_\psi}$ *denote the Orlicz norm on* $(\Omega, \mathcal{F}, P)$ *as defined in section 2. Then for every* $C > \sqrt{2}$ *the following inequality is satisfied:*

(1) $$\Big\| \; \frac{1}{C \cdot \Big(\sum_{i=1}^n |a_i|^2 \Big)^{1/2}} \cdot \sum_{i=1}^n a_i \varepsilon_i \; \Big\|_{\Upsilon_\psi} \; \le \; \frac{C}{\sqrt{C^2 - 2}} \; - \; 1$$

for all $a_1, \dots, a_n \in \mathbf{R}$ *and all* $n \ge 1$ *. Moreover, the estimate given by (1) is the best possible in the sense described in the proof below.*

Proof. Given $a_1, \dots, a_n \in \mathbf{R}$ for some $n \ge 1$, we denote as before $S_n = \sum_{i=1}^n a_i \varepsilon_i$, $T_n = (S_n)^2$ and $A_n = \sum_{i=1}^n |a_i|^2$. According to (7) in proof 4 of theorem 3.1 we may conclude:

$$\Big\| \; \frac{1}{C \cdot \Big(\sum_{i=1}^n |a_i|^2 \Big)^{1/2}} \cdot \sum_{i=1}^n a_i \varepsilon_i \; \Big\|_{T_\Upsilon} \; =$$

$$= \; \int_\Omega exp \, \Big(\, \frac{1}{C^2 A_n} \cdot T_n \, \Big) \, dP \; - 1 \; \le \; \Big(\, 1 - \frac{2}{C^2} \, \Big)^{-1/2} - \; 1 \; =$$

$$= \frac{C}{\sqrt{C^2-2}} - 1$$

for all $n \geq 1$ and all $C > \sqrt{2}$. Thus (1) is satisfied and the first part of the proof is complete. And for the last statement, let us take $a_1 = \ldots = a_n = 1$ for $n \geq 1$, then (1) gets the following form:

$$\| \frac{1}{C\sqrt{n}} \cdot \sum_{i=1}^{n} \varepsilon_i \|_{\Upsilon_\psi} \leq \frac{C}{\sqrt{C^2-2}} - 1$$

for all $n \geq 1$ and all $C > \sqrt{2}$. And we shall now show that:

$$\text{(2)} \qquad \| \frac{1}{C\sqrt{n}} \cdot \sum_{i=1}^{n} \varepsilon_i \|_{\Upsilon_\psi} \longrightarrow \frac{C}{\sqrt{C^2-2}} - 1$$

as $n \to \infty$, for all $C > \sqrt{2}$. For this, it is enough to show that for given $C > \sqrt{2}$, with $S_n = \sum_{i=1}^{n} \varepsilon_i$, we have:

$$\text{(3)} \qquad \int_\Omega exp \Big[\Big(\frac{1}{C\sqrt{n}} S_n \Big)^2 \Big] \, dP \longrightarrow \frac{C}{\sqrt{C^2-2}}$$

as $n \to \infty$. In order to prove (3), by the central limit theorem it is enough to verify that the sequence $\{ \, exp \, [\, (Z_n/C)^2 \,] \, \mid \, n \geq 1 \, \}$ is uniformly integrable, where $Z_n = S_n/\sqrt{n}$ for $n \geq 1$. And by lemma 3.2 we may notice that this fact is satisfied, if $\{ \, Z_n \, \mid \, n \geq 1 \, \}$ is a symmetric standard subnormal sequence. But this last fact is established in proof 1 of theorem 3.1, see its relation (4). Thus (3) follows, and therefore (2) is satisfied also. Hence we may conclude that the estimate given by (1) is indeed the best possible, in the sense that given $C > \sqrt{2}$ and $\varepsilon > 0$ with $C - \varepsilon > \sqrt{2}$, we can find $n_\varepsilon \geq 1$ and $a_1, \ldots, a_{n_\varepsilon} \in \mathbf{R}$ such that the following two inequalities are satisfied:

$$\Big\| \frac{1}{(C-\varepsilon) \cdot \Big(\sum_{i=1}^{n_\varepsilon} |a_i|^2 \Big)^{1/2}} \cdot \sum_{i=1}^{n_\varepsilon} a_i \varepsilon_i \Big\|_{\Upsilon_\psi} > \frac{C}{\sqrt{C^2-2}} - 1$$

$$\Big\| \frac{1}{C \cdot \Big(\sum_{i=1}^{n_\varepsilon} |a_i|^2 \Big)^{1/2}} \cdot \sum_{i=1}^{n_\varepsilon} a_i \varepsilon_i \Big\|_{\Upsilon_\psi} > \frac{C+\varepsilon}{\sqrt{(C+\varepsilon)^2-2}} - 1$$

Note that the function $C \mapsto C/\sqrt{C^2-2}$ is decreasing on $]\sqrt{2}, \infty[$. These facts complete the proof. □

Theorem 2. *Let* $\{ \, X_i \, \mid \, i \geq 1 \, \}$ *be a sequence of independent symmetric real valued random variables defined on a probability space* $(\Omega, \mathcal{F}, P)$ *, and let* $\| \cdot \|_{\Upsilon_\psi}$ *denote the Orlicz norm on* $(\Omega, \mathcal{F}, P)$ *as defined in section 2. Then for every* $C > \sqrt{2}$ *and all* $n \geq 1$ *the following inequality is satisfied:*

$$\text{(1)} \qquad \Big\| \frac{1}{C \cdot \Big(\sum_{i=1}^{n} |X_i|^2 \Big)^{1/2}} \cdot \sum_{i=1}^{n} X_i \Big\|_{\Upsilon_\psi} \leq \frac{C}{\sqrt{C^2-2}} - 1$$

Moreover, the estimate given by (1) is the best possible in the sense described in the proof of theorem 1.

Proof. Given $n \geq 1$, we denote $X = (X_1, \ldots, X_n)$, and let $\varepsilon_1, \ldots, \varepsilon_n$ be a Bernoulli sequence such that $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ is independent of $X = (X_1, \ldots, X_n)$. It is no restriction to assume that X and ε are defined on the same probability space $(\Omega, \mathcal{F}, P)$. Exactly as in the proof of theorem 3.6 we may conclude:

$$E\Big\{ \, exp \, \Big[\, \frac{1}{C^2 \cdot \sum_{i=1}^n |X_i|^2} \cdot \Big(\sum_{i=1}^n X_i \varepsilon_i \Big)^2 \, \Big] \, \Big\} \; = \; Eg(X, C)$$

for all $C > \sqrt{2}$, where the function g is given by:

$$g(x, C) = E\Big\{ \, exp \, \Big[\, \frac{1}{C^2 \cdot \sum_{i=1}^n |x_i|^2} \cdot \Big(\sum_{i=1}^n x_i \varepsilon_i \Big)^2 \, \Big] \, \Big\}$$

for $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ and $C > \sqrt{2}$. By (1) in theorem 1 we have:

$$g(x, C) \; \leq \; \frac{C}{\sqrt{C^2 - 2}}$$

for all $x \in \mathbf{R}^n$ and all $C > \sqrt{2}$. Since by our assumptions $X = (X_1, \ldots, X_n)$ is sign-symmetric, then by the inequality above we may conclude:

$$E\Big\{ \, exp \, \Big[\, \frac{1}{C^2 \cdot \sum_{i=1}^n |X_i|^2} \cdot \Big(\sum_{i=1}^n X_i \Big)^2 \, \Big] \, \Big\} \; =$$

$$E\Big\{ \, exp \, \Big[\, \frac{1}{C^2 \cdot \sum_{i=1}^n |X_i|^2} \cdot \Big(\sum_{i=1}^n X_i \varepsilon_i \Big)^2 \, \Big] \, \Big\} \; \leq \; \frac{C}{\sqrt{C^2 - 2}}$$

for all $C > \sqrt{2}$. Hence (1) follows directly, and the first part of the proof is complete. The last statement follows straight forward by the last statement in theorem 1. These facts complete the proof.

$\square$

Remark 3. By (4.6) we may easily deduce the following "dual" estimate which extends the result of theorem 1:

$$(1) \qquad \Big\| \, \frac{1}{C \cdot \Big(\sum_{i=1}^n |a_i|^2 \Big)^{1/2p}} \cdot \sum_{i=1}^n a_i \varepsilon_i \, \Big\|_{\Upsilon_\psi} \leq \; \Big(1 \; - \; \frac{2}{C^2} \cdot \Big(\sum_{i=1}^n |a_i|^2 \Big)^{1/q} \Big)^{-1/2} \; - \; 1$$

for all $a_1, \ldots a_n \in \mathbf{R}$, all $n \geq 1$, and all $C > 0$ for which $\Big(\sum_{i=1}^n |a_i|^2 \Big)^{1/2} < \big(C/\sqrt{2} \big)^{p/p-1}$, where $p > 1$ and $1/p + 1/q = 1$. Now it is a matter of routine, see the proof of theorem 3.6, to conclude that the following inequality extends inequality (1) given in theorem 2: If $X_1, X_2, \dots$ are independent symmetric $a.s.$ bounded real valued random varibles, then we have:

$$(2) \qquad \Big\| \frac{1}{C \cdot \Big(\sum_{i=1}^{n} |X_i|^2 \Big)^{1/2p}} \cdot \sum_{i=1}^{n} X_i \Big\|_{\Upsilon_\psi} \leq \Big(1 - \frac{2}{C^2} \cdot \Big(\sum_{i=1}^{n} \| X_i \|_\infty^2 \Big)^{1/q} \Big)^{-1/2} - 1$$

for all $C > 0$ and all $n \geq 1$ for which $\big(\sum_{i=1}^{n} \| X_i \|_\infty^2 \big)^{1/2} < \big(C/\sqrt{2}\big)^{p/p-1}$, where $p > 1$ and $1/p + 1/q = 1$. Moreover, putting $a_i = 1/\sqrt{n}$ for $i = 1, \dots, n$ in (1), one can easily establish that the estimates (1) and (2) are the best possible, for $C > \sqrt{2}$, in the sense described in the proof of theorem 1. We shall leave the details in this direction to the reader.

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REFERENCES

[1] ARAUJO, A. P. (1978). On the central limit theorem in Banach spaces. *J. Multivariate Anal.* 8 (598-613).

[2] BOLLOBAS, B. (1980). Martingale inequalities. *Math. Proc. Camb. Philos. Soc.* 87 (377-382).

[3] GLUSKIN, E. D. PIETSCH, A. *and* PUHL, J. (1980). A generalization of Khintchine's inequality and its application in the theory of operator ideals. *Studia Math.* 67 (149-155).

[4] HAAGERUP, U. (1978-1982). The best constants in the Khintchine inequality. *Operator algebras, ideals, and their applications in theoretical physics*, Proc. Int. Conf. Leipzig (69-79). *Studia Math.* 70 (231-283).

[5] HOFFMANN-JØRGENSEN, J. (1991). Function norms. *Math. Inst. Aarhus, Preprint Ser.* No. 40, (19 pp).

[6] HOFFMANN-JØRGENSEN, J. (1991). Inequalities for sums of random elements. *Math. Inst. Aarhus, Preprint Ser.* No. 41, (27 pp).

[7] HOFFMANN-JØRGENSEN, J. (1994). *Probability with a view toward statistics*. Chapman and Hall.

[8] JOHNSON, W. B. SCHECHTMAN, G. *and* ZINN, J. (1985). Best constants in moment inequalities for linear combinations of independent and exchangeable random variables. *Ann. Probab.* 13 (234-253).

[9] KAHANE, J. P. (1968-1985). *Some random series of functions*. D. C. Heath & Co. (first edition). Cambridge University Press (second edition).

[10] KOMOROWSKI, R. (1988). On the best possible constants in the Khintchine inequality for $p \geq 3$. *Bull. London Math. Soc.* 20 (73-75).

[11] KRASNOSEL'SKII, M. A. *and* RUTICKII, Ya. B. (1961). *Covex functions and Orlicz spaces*. P. Noordhoff, Ltd. Groningen.

[12] LEDOUX, M. (1985). Sur une inégalité de H. P. Rosenthal et le théorème limite central dans les espaces de Banach. *Israel J. Math*. 50 (290–318).

[13] MARCUS, M. B. *and* PISIER, G. (1984). Characterizations of almost surely continuous p-stable random Fourier series and strongly stationary processes. *Acta Mathematica* 152 (245-301).

[14] MARCUS, M. B. *and* PISIER, G. (1985). Stochastic processes with sample paths in exponential Orlicz spaces. *Proc. Probab. Banach Spaces* V, *Lecture Notes in Math*. 1153 (328-358).

[15] NANOPOULOS, C. *and* NOBELIS, P. (1978). Régularité et propriétés limites des fonctions aléatoires. *Sém. Probab*. XII, *Lecture Notes in Math*. 649 (567-690).

[16] NEWMAN, C. M. (1975). An extension of Khintchine's inequality. *Bull. Amer. Math. Soc*. 81 (913-915).

[17] PESKIR, G. (1992). Note on the connection between the central limit theorem and the uniform law of large numbers in Banach spaces. *Not appeared in written form*.

[18] PESKIR, G. *and* WEBER, M. (1992). Necessary and sufficient conditions for the uniform law of large numbers in the stationary case. *Math. Inst. Aarhus, Preprint Ser*. No. 27, (26 pp). *Proc. Funct. Anal*. IV (Dubrovnik 1993), *Various Publ. Ser*. Vol. 43, 1994 (165-190).

[19] PISIER, G. (1981). De nouvelles caractérisations des ensembles de Sidon. *Adv. in Math. Suppl. Stud. 7b*, Academic Press, New York (686-725).

[20] PRESTON, C. (1971). Banach spaces arising from some integral inequalities. *Indiana Univ. Math. Journal* 20 (997-1015).

[21] RAO, M. M. *and* REN, Z. D. (1991). *Theory of Orlicz spaces*. Marcel Dekker Inc., New York.

[22] RODIN, V. A. *and* SEMYONOV, E. M. (1975). Rademacher series in symmetric spaces. *Analysis Mathematica* 1 (207-222).

[23] SAWA, J. (1985). The best constant in the Khintchine inequality for complex Steinhaus variables, the case $p = 1$. *Studia Math*. 81 (107-126).

[24] STECHKIN, S. B. (1961). On the best lacunary systems of functions. *Izv. Akad. Nauk SSSR Ser. Mat*. 25 (in Russian) (357-366).

[25] SZAREK, S. J. (1978). On the best constant in the Khintchine inequality. *Studia Math*. 58 (197-208).

[26] TIGNOL, J. P. (1988). *Galois' theory of algebraic equations*. Institut de Mathématique Pure et Appliquée, UCL, Louvain-la-Neuve, Belgium.

[27] TOMASZEWSKI, B. (1982). Two remarks on the Khintchine-Kahane inequality. *Colloq. Math*. 46 (283-288).

[28] WANG G. (1991). Sharp square-function inequalities for conditionally symmetric martingales. *Trans. Amer. Math. Soc*. 328 (393-419).

[29] WEBER, M. (1983). Analyse infinitesimale de fonctions aleatoires. *Ecole d'Eté Probabilités de Saint-Flour* XI, *Lecture Notes in Math*. 976 (383-465).

[30] WEBER, M. (1991). New sufficient conditions for the law of the iterated logarithm in Banach spaces. *Sém. Probab.* XXV, *Lecture Notes in Math.* 1495 (310-315).

[31] YOUNG, R. M. G. (1976). On the best possible constants in the Khintchine inequality. *J. London Math. Soc.* 14 (496-504).

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