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A Change-of-Variable Formula with Local Time on Curves

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Let $X = (X_t)_{t \ge 0}$ be a continuous semimartingale and let $b : I\!\!R_+ \to I\!\!R$ be a continuous function of bounded variation. Setting $C = \{ (t,x) \in I\!\!R_+ \times I\!\!R \mid x < b(t) \}$ and $D = \{ (t,x) \in I\!\!R_+ \times I\!\!R \mid x > b(t) \}$ suppose that a continuous function $F : I\!\!R_+ \times I\!\!R \to I\!\!R$ is given such that F is $C^{1,2}$ on $\bar{C}$ and F is $C^{1,2}$ on $\bar{D}$. Then the following change-of-variable formula holds:

$$\begin{aligned} F(t,X_t) = F(0,X_0) &+ \int_0^t \frac{1}{2}\Big(F_t(s,X_s+)+F_t(s,X_s-)\Big)\,ds \\ &+ \int_0^t \frac{1}{2}\Big(F_x(s,X_s+)+F_x(s,X_s-)\Big)\,dX_s \\ &+ \frac{1}{2}\int_0^t F_{xx}(s,X_s)\,I(X_s \ne b(s))\;d\langle X,X\rangle_s \\ &+ \frac{1}{2}\int_0^t \Big(F_x(s,X_s+)-F_x(s,X_s-)\Big)I(X_s = b(s))\;d\ell_s^b(X) \end{aligned}$$

where $\ell_s^b(X)$ is the local time of X at the curve b given by:

$$\ell_s^b(X) = I\!\!P - \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon}\int_0^s I(b(r)-\varepsilon < X_r < b(r)+\varepsilon)\;d\langle X,X\rangle_r$$

and $d\ell_s^b(X)$ refers to the integration with respect to $s \mapsto \ell_s^b(X)$. A version of the same formula derived for an Itô diffusion X under weaker conditions on F has found applications in free-boundary problems of optimal stopping.

1. Introduction

1. Let $X = (X_t)_{t \ge 0}$ be a continuous semimartingale (see e.g. [6]) and let $b : I\!\!R_+ \to I\!\!R$ be a continuous function of bounded variation. Setting:

$$C = \{ (t,x) \in I\!\!R_+ \times I\!\!R \mid x < b(t) \} \tag{1.1}$$

$$D = \{ (t,x) \in I\!\!R_+ \times I\!\!R \mid x > b(t) \} \tag{1.2}$$

suppose that a continuous function $F : I\!\!R_+ \times I\!\!R \to I\!\!R$ is given such that:

$$F \text{ is } C^{1,2} \text{ on } \bar{C} \tag{1.3}$$

$$F \text{ is } C^{1,2} \text{ on } \bar{D}\,. \tag{1.4}$$

More explicitly, it means that F restricted to C coincides with a function F_1 which is $C^{1,2}$ on $I\!\!R_+ \times I\!\!R$, and F restricted to D coincides with a function F_2 which is $C^{1,2}$ on $I\!\!R_+ \times I\!\!R$.

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[We recall that a continuous function $G : I\!R_+ \times I\!R \to I\!R$ is $C^{1,2}$ on $I\!R_+ \times I\!R$ if the partial derivatives G_t , G_x and G_{xx} exist and are continuous as functions from $I\!R_+ \times I\!R$ to $I\!R$.] Moreover, since F is continuous, the two functions F_1 and F_2 must coincide on the curve $\{(t, b(t)) \mid t \in I\!R_+\}$, or in other words:

$$F_1(t, b(t)) = F(t, b(t)) = F_2(t, b(t)) \tag{1.5}$$

for all $t \in I\!R_+$.

Then the natural desire arising in some free-boundary problems of optimal stopping (see [4]-[5]) is to apply a change-of-variable formula to the process $F(t, X_t)$ so to account for possible jumps of $F_x(s, x)$ at $x = b(s)$ being measured by:

$$\ell_s^b(X) = I\!P\!-\!\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^s I(b(r)\!-\!\varepsilon < X_r < b(r)\!+\!\varepsilon)\, d\langle X, X\rangle_r \tag{1.6}$$

which represents the local time of X at the curve b for $s \in [0, t]$. The limit in (1.6) denotes a limit in probability (stronger convergence relations may also hold).

The best known example of such a formula is the Tanaka formula (see e.g. [6] p. 222) where $F(t, x) = x^+$ and $b(t) = 0$ for all $t \in I\!R_+$. Further special cases are derived in [2] (Section 5) when X is a Brownian motion, $F(t, x) = (x\!-\!b(t))^+$, and $b : I\!R_+ \to I\!R$ is a continuous function. We refer to the recent paper [1] for more definite results in the case of Brownian motion and for further references on this topic. Motivated by applications in free-boundary problems mentioned above, the main purpose of the present paper is to consider a more general case of the function F and the process X using also a somewhat different method of proof.

2. Before we state the change-of-variable formula in Section 2 below, let us for further reference list the following consequences of the conditions (1.1)-(1.4) stated above:

$$x \mapsto F(t, x) \text{ is continuous at } b(t) \tag{1.7}$$

$$\text{limits } F_x(t, b(t)\pm) \text{ exist in } I\!R \tag{1.8}$$

$$t \mapsto F_x(t, b(t)\pm) \text{ are continuous} \tag{1.9}$$

$$\text{limits } F_{xx}(t, b(t)\pm) \text{ exist in } I\!R \tag{1.10}$$

$$t \mapsto F_{xx}(t, b(t)\pm) \text{ are continuous.} \tag{1.11}$$

It may appear evident that some of these conditions (and thus (1.3) and (1.4) above as well) may be relaxed. Some of these further extensions will be discussed in Section 3 below.

Yet another consequence of (1.3) and (1.4) that may be useful to note is the following:

$$t \mapsto F(t, b(t)) \text{ is } C^1 \text{ when } b \text{ is } C^1 . \tag{1.12}$$

It follows by recalling (1.5) above from where we also see that:

$$\frac{d}{dt}\Big(F(t, b(t))\Big) = F_t(t, b(t)\pm) + F_x(t, b(t)\pm) b'(t) \tag{1.13}$$

where the two signs $\pm$ are simultaneously equal to either $+$ or $-$ respectively.

2. The first result and proof

The first result of the present paper may now be stated as follows.

Theorem 2.1

Let $X=(X_t)_{t\ge 0}$ be a continuous semimartingale, let $b:I\!R_+\rightarrow I\!R$ be a continuous function of bounded variation, and let $F:I\!R_+\times I\!R\rightarrow I\!R$ be a continuous function satisfying (1.3) and (1.4) above. Then the following change-of-variable formula holds:

$$
\begin{aligned}
F(t,X_t) &= F(0,X_0)+\int_0^t \frac{1}{2}\Big(F_t(s,X_s+)+F_t(s,X_s-)\Big)\,ds \\
&+\int_0^t \frac{1}{2}\Big(F_x(s,X_s+)+F_x(s,X_s-)\Big)\,dX_s \\
&+\frac{1}{2}\int_0^t F_{xx}(s,X_s)\; I(X_s\neq b(s))\; d\langle X,X\rangle_s \\
&+\frac{1}{2}\int_0^t \Big(F_x(s,X_s+)-F_x(s,X_s-)\Big)\, I(X_s=b(s))\; d\ell_s^b(X)
\end{aligned}
\tag{2.1}
$$

where $\ell_s^b(X)$ is the local time of X at the curve b given by (1.6) above, and $d\ell_s^b(X)$ refers to the integration with respect to the continuous increasing function $s\mapsto \ell_s^b(X)$.

Proof. To prove the theorem we shall combine two approximation methods each of which will improve upon weak points of the other. Consequently, the proof can be shortened but we present the longer version for comparison of the methods. Another proof will be given in Section 3 below.

Part I: 1. The first approximation method may be termed simple (linear or quadratic). With $t>0$ given and fixed, we shall "smooth out" a possible discontinuity of the map $x\mapsto F_x(t,x)$ at $b(t)$ by a linear approximation as follows:

$$
\begin{aligned}
F_x^n(t,x) &= F_x(t,x) \quad \text{if } x\notin\,]b(t)-\varepsilon_n, b(t)+\varepsilon_n[\\
&= \text{linear} \quad \text{if } x\in [b(t)-\varepsilon_n, b(t)+\varepsilon_n]
\end{aligned}
\tag{2.2}
$$

for $(t,x)\in I\!R_+\times I\!R$ and $n\ge 1$ where $\varepsilon_n\downarrow 0$ as $n\rightarrow\infty$. We thus have:

$$
F_x^n(t,x)=\frac{F_x(t,b(t)+\varepsilon_n)-F_x(t,b(t)-\varepsilon_n)}{2\varepsilon_n}\Big(x-\big(b(t)-\varepsilon_n\big)\Big)+F_x(t,b(t)-\varepsilon_n)
\tag{2.3}
$$

for $x\in[b(t)-\varepsilon_n, b(t)+\varepsilon_n]$ and $t\in I\!R_+$.

Define the map $F^n:I\!R_+\times I\!R\rightarrow I\!R$ using (2.2) and (2.3) as follows (other definitions are also possible and will lead to the same quantitative result):

$$
\begin{aligned}
F^n(t,x) &= F(t,b(t)-\varepsilon_n)+\int_{b(t)-\varepsilon_n}^x F_x^n(t,y)\; dy \\
&= F(t,b(t)-\varepsilon_n)+\frac{F_x(t,b(t)+\varepsilon_n)-F_x(t,b(t)-\varepsilon_n)}{2\varepsilon_n}\,\frac{(x-(b(t)-\varepsilon_n))^2}{2} \\
&\quad + F_x(t,b(t)-\varepsilon_n)\,(x-(b(t)-\varepsilon_n))
\end{aligned}
\tag{2.4}
$$

for $x \in [b(t)-\varepsilon_n, b(t)+\varepsilon_n]$ and set:

$$
\begin{aligned}
(2.5) \qquad F^n(t,x) &= F(t,x) && \text{if } x < b(t)-\varepsilon_n \\
&= F(t,x)-D_n(t) && \text{if } x > b(t)+\varepsilon_n
\end{aligned}
$$

where $D_n(t) = F(t,b(t)+\varepsilon_n)-F^n(t,b(t)+\varepsilon_n)$ so that using (2.4) we find:

$$
(2.6) \qquad D_n(t) = F(t,b(t)+\varepsilon_n)-F(t,b(t)-\varepsilon_n) - \varepsilon_n\Big(F_x(t,b(t)+\varepsilon_n)+F_x(t,b(t)-\varepsilon_n)\Big)
$$

for $t \in I\!\!R_+$. [It may be noted that the approximating map $x \mapsto F^n(t,x)$ is quadratic on $[b(t)-\varepsilon_n, b(t)+\varepsilon_n]$, and thus as such has three degrees of freedom, while to match the map $x \mapsto F(t,x)$ together with its first derivative with respect to x at $b(t)\pm\varepsilon_n$ introduces four conditions. All of them thus cannot be met in general, and this is the reason that $D_n(t)$ in (2.5) is not necessarily zero.]

2. Using (1.3) and (1.4) it follows that $x \mapsto F^n(t,x)$ is C^2 but possibly at $b(t)\pm\varepsilon_n$ where it is at least C^1 . Since moreover $x \mapsto F^n_x(t,x)$ is clearly of bounded variation on each bounded interval containing $b(t)\pm\varepsilon_n$ (due to the fact that the four limits $F^n_{xx}(t,(b(t)\pm\varepsilon_n)\pm)$ exist in $I\!\!R$ and define continuous functions of t being therefore bounded on bounded intervals) it follows that the Itô formula can be applied to $F^n(t,X_t)$ in the standard form (see e.g. [6] p. 147) as soon as we have that $t \mapsto F^n(t,x)$ is C^1 on $I\!\!R_+$. [If the reader is not familiar with this fact, it may be noted that a proof also follows from the convolution arguments given in Part II below (see (2.53) and Remark 2.3).] From (2.4)-(2.6) we however see that $t \mapsto F^n(t,x)$ will be C^1 if additionally to $x \mapsto F_x(t,x)$ being C^1 at $b(t)\pm\varepsilon_n$, which we have since $x \mapsto F(t,x)$ is C^2 on the open set $C \cup D$, we should also have that $s \mapsto F_x(s,x)$ is C^1 at t for $x = b(t)\pm\varepsilon_n$ and that b is C^1 on $I\!\!R_+$. For these reasons we shall first prove the theorem i.e. establish (2.1) when b is C^1 on $I\!\!R_+$ and the continuous function $F : I\!\!R_+ \times I\!\!R \rightarrow I\!\!R$ satisfying (1.3) and (1.4) is of the form $F(t,x) = G(t)\,H(x)$.

3. Let b be C^1 and let F satisfying (1.3)-(1.5) be of the form $F(t,x) = G(t)\,H(x)$. Then clearly $s \mapsto F_x(s,x) = G(s)H'(x)$ is C^1 at t for $x = b(t)\pm\varepsilon_n$ so that by the arguments exposed above we can apply the Itô formula in its standard form to $F^n(t,X_t)$ giving:

$$
(2.7) \quad F^n(t,X_t) = F^n(0,X_0) + \int_0^t F^n_t(s,X_s)\,ds + \int_0^t F^n_x(s,X_s)\,dX_s + \frac{1}{2}\int_0^t F^n_{xx}(s,X_s)\,d\langle X,X\rangle_s
$$

for all $t \in I\!\!R_+$ and all $n \geq 1$. A natural step is then to pass to the limit in (2.7) for $n \rightarrow \infty$, and for this we need a few preliminary remarks.

A standard localization argument based on using the exit times $\tau_m = \inf\{\, s>0 \,:\, |X_s| \geq m \,\}$ and establishing (2.1) via (2.7) first for $t \wedge \tau_m$ in place of t and then letting $m \rightarrow \infty$ in the resulting formula shows that there is no restriction to assume in the sequel that $|F|$, $|F_t|$, $|F_x|$ and $|F_{xx}|$ are all uniformly bounded by a constant on $C \cup D$ (and thus on $\bar{C} \cup \bar{D}$ as well). This fact then transfers further and enables us to make similar conclusions about the maps $|F^n|$, $|F^n_t|$, $|F^n_x|$ and $|F^n_{xx}|$ for $n \geq 1$. In other words, from definitions (2.2)-(2.6) we see that there is no restriction to assume that these maps remain uniformly bounded on $I\!\!R_+ \times I\!\!R$ by a constant not depending on $n \geq 1$. This will be freely used in what follows.

4. From (2.6) we see that $D_n(t) \to 0$ so that by (2.4) and (2.5) we find that:

$$F^n(t,x) \;\to\; F(t,x) \tag{2.8}$$

as $n \to \infty$ for all $(t,x) \in I\!R_+ \times I\!R$.

Moreover, it is easily seen directly from (2.4) and (2.5) that:

$$\begin{aligned} F_x^n(t,x) &\to F_x(t,x) \quad \text{if} \;\; x \neq b(t) \\ &\to \frac{1}{2}\Big(F_x(t,b(t)+)+F_x(t,b(t)-)\Big) \quad \text{if} \;\; x = b(t) \end{aligned} \tag{2.9}$$

as $n \to \infty$ for all $(t,x) \in I\!R_+ \times I\!R$.

Finally, note firstly that from (2.6) using (1.13) we get:

$$\begin{aligned} D_n'(t) &= F_t(t,b(t)+\varepsilon_n) + F_x(t,b(t)+\varepsilon_n)\,b'(t) - F_t(t,b(t)-\varepsilon_n) - F_x(t,b(t)-\varepsilon_n)\,b'(t) \\ &\quad -\varepsilon_n \frac{d}{dt}\Big(F_x(t,b(t)+\varepsilon_n)+F_x(t,b(t)-\varepsilon_n)\Big) \\ &\to F_t(t,b(t)+) + F_x(t,b(t)+)\,b'(t) - F_t(t,b(t)-) - F_x(t,b(t)-)\,b'(t) = 0 \end{aligned} \tag{2.10}$$

as $n \to \infty$ for all $t \in I\!R_+$. Secondly, let us compute $F_t^n(t,x)$ for $x = b(t)$ using (2.4). For this, note that by differentiating in (2.4) we find:

$$\begin{aligned} F_t^n(t,x) &= F_t(t,b(t)-\varepsilon_n) + F_x(t,b(t)-\varepsilon_n)\,b'(t) \\ &\quad + \frac{d}{dt}\Big(F_x(t,b(t)+\varepsilon_n)-F_x(t,b(t)-\varepsilon_n)\Big)\,\frac{(x-(b(t)-\varepsilon_n))^2}{4\varepsilon_n} \\ &\quad + \Big(F_x(t,b(t)+\varepsilon_n)-F_x(t,b(t)-\varepsilon_n)\Big)\,\frac{(x-(b(t)-\varepsilon_n))}{2\varepsilon_n}\,(-b'(t)) \\ &\quad + \frac{d}{dt}\Big(F_x(t,b(t)-\varepsilon_n)\Big)\,\big(x-(b(t)-\varepsilon_n)\big) + F_x(t,b(t)-\varepsilon_n)\,(-b'(t))\;. \end{aligned} \tag{2.11}$$

Inserting $x = b(t)$ hence we see that:

$$\begin{aligned} F_t^n(t,b(t)) &= F_t(t,b(t)-\varepsilon_n) + F_x(t,b(t)-\varepsilon_n)\,b'(t) + O(\varepsilon_n) \\ &\quad -\frac{1}{2}\Big(F_x(t,b(t)+\varepsilon_n)-F_x(t,b(t)-\varepsilon_n)\Big)\,b'(t) + O(\varepsilon_n) - F_x(t,b(t)-\varepsilon_n)\,b'(t) \\ &\to F_t(t,b(t)-) - \frac{1}{2}\Big(F_x(t,b(t)+)-F_x(t,b(t)-)\Big)b'(t) \\ &= F_t(t,b(t)-) - \frac{1}{2}\Big(F_t(t,b(t)-)-F_t(t,b(t)+)\Big) \\ &= \frac{1}{2}\Big(F_t(t,b(t)+)+F_t(t,b(t)-)\Big) \end{aligned} \tag{2.12}$$

as $n \to \infty$ for all $t \in I\!R_+$ by means of (1.13). It thus follows from (2.5) using (2.10) and

(2.12) just established that:

$$F_t^n(t,x) \to F_t(t,x) \quad \text{if} \quad x \neq b(t) \tag{2.13}$$
$$\to \frac{1}{2}\Big(F_t(t,b(t)+)+F_t(t,b(t)-)\Big) \quad \text{if} \quad x = b(t)$$

as $n \to \infty$ for all $(t,x) \in I\!R_+ \times I\!R$.

5. Recalling the localization argument above we know that there is no restriction to assume that $|F_t^n|$ is uniformly bounded by a constant not depending on $n \geq 1$, so that by (2.13) and the dominated convergence theorem it follows that:

$$\int_0^t F_t^n(s,X_s)\,ds \to \int_0^t \frac{1}{2}\Big(F_t(s,X_s+)+F_t(s,X_s-)\Big)\,ds \tag{2.14}$$

as $n \to \infty$.

Similarly, we may assume that $|F_x^n|$ is uniformly bounded, so that by (2.9) and the stochastic dominated convergence theorem (see e.g. [6] p. 142) it follows that:

$$\int_0^t F_x^n(s,X_s)\,dX_s \to \int_0^t \frac{1}{2}\Big(F_x(s,X_s+)+F_x(s,X_s-)\Big)\,dX_s \tag{2.15}$$

in probability as $n \to \infty$.

Finally, using (2.2) and (2.3) we see that:

$$\int_0^t F_{xx}^n(s,X_s)\,d\langle X,X\rangle_s = \int_0^t F_{xx}(s,X_s)\,I(X_s \notin [b(s)-\varepsilon_n, b(s)+\varepsilon_n])\,d\langle X,X\rangle_s \tag{2.16}$$
$$+\frac{1}{2\varepsilon_n}\int_0^t \Big(F_x(s,b(s)+\varepsilon_n)-F_x(s,b(s)-\varepsilon_n)\Big)\,I(b(s)-\varepsilon_n < X_s < b(s)+\varepsilon_n)\,d\langle X,X\rangle_s$$

upon recalling that $\int_0^t I(X_s = b(s))\,d\langle X,X\rangle_s = 0$ for b of bounded variation (which we have since b is C^1 by assumption) so that the value of $F_{xx}(s,\,\cdot\,)$ at $b(s)$ can be set arbitrarily.

Again, by the localization argument we may assume that $|F_{xx}|$ is uniformly bounded, so that by the dominated convergence theorem it follows that:

$$\int_0^t F_{xx}(s,X_s)\,I(X_s \notin [b(s)-\varepsilon_n, b(s)+\varepsilon_n])\,d\langle X,X\rangle_s \tag{2.17}$$
$$\to \int_0^t F_{xx}(s,X_s)\,I(X_s \neq b(s))\,d\langle X,X\rangle_s$$

as $n \to \infty$.

6. Letting $n \to \infty$ in (2.7) above, and using (2.8) with (2.14)-(2.17), we can conclude that:

$$\lim_{n\to\infty} \frac{1}{2\varepsilon_n}\int_0^t \Big(F_x(s,b(s)+\varepsilon_n)-F_x(s,b(s)-\varepsilon_n)\Big)\,I(b(s)-\varepsilon_n < X_s < b(s)+\varepsilon_n)\,d\langle X,X\rangle_s \tag{2.18}$$

exists in $I\!R$ as a limit in probability. In particular, if we choose $X_t^c = X_t - c(t)$ to be the semimartingale and set $F(t,x) = x^+$ with $b(t) \equiv 0$, we see that the local time $\ell_t^c(X)$ of X

at c , given by (1.6) above with c instead of b , exists in $I\!R$ as a limit in probability. The proof in the case b is C^1 and F satisfying (1.3)-(1.5) is of the form $F(t,x) = G(t)\,H(x)$ will therefore be completed as soon as we show that:

$$\text{(2.19)} \quad I\!P\text{-}\lim_{n\to\infty} \frac{1}{2\varepsilon_n} \int_0^t \Big(F_x(s,b(s)+\varepsilon_n) - F_x(s,b(s)-\varepsilon_n) \Big)\, I(b(s)-\varepsilon_n < X_s < b(s)+\varepsilon_n)\, d\langle X,X\rangle_s$$
$$= \frac{1}{2} \int_0^t \Delta_x F_x(s,X_s)\ I(X_s = b(s))\ d\ell_s^b(X)$$

where we set $\Delta_x F_x(s,b(s)) = F_x(s,b(s)+) - F_x(s,b(s)-)$ for $s \in [0,t]$.

7. To verify (2.19) we may add and subtract $-F_x(s,b(s)+)$ as well as $F_x(s,b(s)-)$ under the first integral sign in (2.19). Using then that a continuous function on a compact set is uniformly continuous, for given $\delta > 0$ we can find $n_\delta \ge 1$ such that $|F_x(s,b(s)+\varepsilon_n) - F_x(s,b(s)+)| < \delta$ for all $s \in [0,t]$ and all $n \ge n_\delta$. Hence it follows that:

$$\text{(2.20)} \quad \Big| \frac{1}{2\varepsilon_n} \int_0^t \Big(F_x(s,b(s)+\varepsilon_n) - F_x(s,b(s)+) \Big)\, I(b(s)-\varepsilon_n < X_s < b(s)+\varepsilon_n)\, d\langle X,X\rangle_s \Big|$$
$$\le \delta\ \frac{1}{2\varepsilon_n} \int_0^t I(b(s)-\varepsilon_n < X_s < b(s)+\varepsilon_n)\, d\langle X,X\rangle_s \to \delta\, \ell_t^b(X)$$

in probability as $n \to \infty$. Letting $\delta \downarrow 0$ we can conclude:

$$\text{(2.21)} \quad \frac{1}{2\varepsilon_n} \int_0^t \Big(F_x(s,b(s)+\varepsilon_n) - F_x(s,b(s)+) \Big)\, I(b(s)-\varepsilon_n < X_s < b(s)+\varepsilon_n)\, d\langle X,X\rangle_s \to 0$$

in probability as $n \to \infty$. Similarly, we find that:

$$\text{(2.22)} \quad \frac{1}{2\varepsilon_n} \int_0^t \Big(F_x(s,b(s)-) - F_x(s,b(s)-\varepsilon_n) \Big)\, I(b(s)-\varepsilon_n < X_s < b(s)+\varepsilon_n)\, d\langle X,X\rangle_s \to 0$$

in probability as $n \to \infty$. Using (2.21) and (2.22) in the right-hand side of (2.19) we see that:

$$\text{(2.23)} \quad I\!P\text{-}\lim_{n\to\infty} \frac{1}{2\varepsilon_n} \int_0^t \Big(F_x(s,b(s)+\varepsilon_n) - F_x(s,b(s)-\varepsilon_n) \Big)\, I(b(s)-\varepsilon_n < X_s < b(s)+\varepsilon_n)\, d\langle X,X\rangle_s$$
$$= I\!P\text{-}\lim_{n\to\infty} \frac{1}{2\varepsilon_n} \int_0^t \Big(F_x(s,b(s)+) - F_x(s,b(s)-) \Big)\, I(b(s)-\varepsilon_n < X_s < b(s)+\varepsilon_n)\, d\langle X,X\rangle_s\ .$$

Moreover, we know by (1.9) that:

$$\text{(2.24)} \qquad t \mapsto F_x(t,b(t)+) - F_x(t,b(t)-)\ \text{ is continuous}$$

so in view of (2.23) to establish (2.19) it is enough to verify that:

$$\text{(2.25)} \qquad I\!P\text{-}\lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_0^t g(s)\, I\big(b(s)-\varepsilon < X_s < b(s)+\varepsilon\big)\, d\langle X,X\rangle_s = \int_0^t g(s)\ d\ell_s^b(X)$$

for a continuous function $g : [0,t] \to I\!R$.

For this, denote the left-hand side in (2.25) by L and note that:

$$(2.26) \qquad L = I\!\!P\!-\!\lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} g(s)\, I\big(b(s)-\varepsilon < X_s < b(s)+\varepsilon\big)\, d\langle X,X\rangle_s$$
$$= I\!\!P\!-\!\lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \sum_{i=1}^n g(s_i^\varepsilon) \int_{t_{i-1}}^{t_i} I\big(b(s)-\varepsilon < X_s < b(s)+\varepsilon\big)\, d\langle X,X\rangle_s$$
$$\le \sum_{i=1}^n \Big(\limsup_{\varepsilon\downarrow 0} g(s_i^\varepsilon)\Big)\Big(\ell_{t_i}^b(X) - \ell_{t_{i-1}}^b(X)\Big)$$
$$\le \sum_{i=1}^n g(s_i^*)\Big(\ell_{t_i}^b(X) - \ell_{t_{i-1}}^b(X)\Big) \;\rightarrow\; \int_0^t g(s)\; d\ell_s^b(X)$$

for some s_i^ε and $s_i^* = \limsup_{\varepsilon\downarrow 0} s_i^\varepsilon$ from $[t_{i-1}, t_i]$ where $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = t$ so that $\max_{1\le i\le n}(t_i - t_{i-1}) \rightarrow 0$ as $n\rightarrow\infty$. In exactly the same way using a liminf instead of the limsup one derives the reverse inequality, and this establishes (2.25) for continuous g .

Thus (2.19) holds too, and the proof of the theorem in the special case when b is C^1 and F satisfying (1.3)-(1.5) is of the form $F(t,x) = G(t)\, H(x)$ is complete.

8. One could now wish to argue that a simple density argument (such as that all polynomials $\widetilde{F}(t,x) = \sum_{i=1}^n P_i(t)\, Q_i(x)$ are dense in the class of functions F satisfying (1.3)-(1.5) when restricted to a compact set $K \subset I\!\!R_+ \times I\!\!R$ relative to the norm:

$$(2.27) \qquad \|F\|_K = \sup_{(t,x)\in K} \Big(|F(t,x)| + |F_t(t,x\pm)| + |F_x(t,x\pm)| + |F_{xx}(t,x\pm)|\Big)$$

upon recalling the localization argument above; or that C^1 functions $\tilde{b}$ are dense in the class of C^0 functions b relative to the supremum norm on a compact time interval) should be able to complete the proof. However, since we are dealing with *general* continuous semimartingales X it seems apparent that a conceptual difficulty in completing such a proof would lie in the irregularity of the map $x \mapsto \ell_t^x(X)$ which is generally known to be right-continuous only (see e.g. [6], Chapter VI), or even completely discontinuous, if adopting our symmetric definition of the local time (1.6). We thus proceed with Part II of the proof where these difficulties of Part I will be avoided while the new difficulty appearing in the end of Part II can be resolved using the partial result of Part I just established.

Part II: 1. The second method is based on the well-known convolution approximation. To simplify the analysis of the mapping $s \mapsto F(s, b(t))$ around the point t , which can be complicated if b oscillates heavily, let us first replace the function F by the function G defined by:

$$(2.28) \qquad G(t,x) \;=\; F(t, x+b(t))$$

for $(t,x) \in I\!\!R_+ \times I\!\!R$. Approximate then the new map G as follows:

$$(2.29) \qquad G^n(t,x) = \int_{I\!\!R} \frac{1}{2}\Big(G(t,x+y/n) + G(t,x-y/n)\Big)\Omega(y)\, dy$$

for $(t,x) \in I\!\!R_+ \times I\!\!R$, where $\Omega : I\!\!R \rightarrow I\!\!R_+$ is any given and fixed function satisfying the following three conditions: (i) Ω is C^∞ ; (ii) Ω has a compact support which we choose to be $[0,1]$; (iii) $\int_{I\!R} \Omega(y)\, dy = 1$. Such a function is easily specified explicitly.

Introduce the following functions:

$$P^n(t,x) = \int_{I\!R} G(t, x+y/n)\, \Omega(y)\, dy \tag{2.30}$$

$$N^n(t,x) = \int_{I\!R} G(t, x-y/n)\, \Omega(y)\, dy \tag{2.31}$$

and let H^n denote either P^n or N^n . Then the following facts are valid:

$$t \mapsto H^n(t,x) \ \text{ is of bounded variation} \tag{2.32}$$

$$x \mapsto H^n(t,x) \ \text{ is } \ C^\infty \tag{2.33}$$

$$t \mapsto \frac{\partial^k H^n}{\partial x^k}(t,x) \ \text{ is continuous} \tag{2.34}$$

for every $k \in I\!N \cup \{0\}$ and in particular for k equal to 1 and 2 what we shall use.

To establish (2.32) one can use Itô's formula (see e.g. [6] p. 147) with the $C^{1,2}$ function $(t,z) \mapsto F(s, x \pm y/n + z)$ and the semimartingale $(t, b(t))$, since $F(t, x \pm y/n + z) = F_1(t, x \pm y/n + z)$ if $x \pm y/n \le 0$ and $F(t, x \pm y/n + z) = F_2(s, x \pm y/n + z)$ if $x \pm y/n > 0$ for $z = b(t)$ in both cases, where F_1 and F_2 are $C^{1,2}$ functions from (1.5). Integrating the resulting formula with respect to $\Omega(y)\, dy$, and using the Fubini theorem, one obtains a representation for $H^n(t,x)$ from which the claim follows readily.

To verify (2.33) and (2.34) we can introduce substitutions $z = x \pm y/n$ in the two integrals and then differentiate under the integral signs as many times as we please upon using the standard argument to justify this operation (based on the dominated convergence theorem); the continuity of the map in (2.34) follows then easily from the continuity of the map $t \mapsto G(t,z)$.

2. The property (2.34) embodies a clear advantage of the convolution approximation upon the linear approximation from Part I, and we shall now exploit this fact. We will begin by analysing the function P^n first; this will then be followed by a similar analysis of the function N^n .

Let $Z = (Z_t)_{t \ge 0}$ be a continuous semimartingale. Using (2.32) with $H^n = P^n$ and Itô's formula for C^2 functions let us write:

$$\begin{aligned} P^n(t, Z_t) - P^n(0, Z_0) &= \sum_{i=1}^m P^n(t_i, Z_{t_i}) - P^n(t_{i-1}, Z_{t_i}) \\ &+ \sum_{i=1}^m P^n(t_{i-1}, Z_{t_i}) - P^n(t_{i-1}, Z_{t_{i-1}}) = \sum_{i=1}^m \int_{t_{i-1}}^{t_i} P^n(ds, Z_{t_i}) \\ &+ \sum_{i=1}^m \left(\int_{t_{i-1}}^{t_i} P^n_x(t_{i-1}, Z_s)\, dZ_s + \frac{1}{2} \int_{t_{i-1}}^{t_i} P^n_{xx}(t_{i-1}, Z_s)\; d\langle Z, Z\rangle_s \right) \end{aligned} \tag{2.35}$$

where $0 = t_0 < t_1 < \ldots t_{m-1} < t_m = t$ are given so that $\max_{1 \le i \le m} (t_i - t_{i-1}) \to 0$ as $m \to \infty$. By (2.34) with $k=1$ and the stochastic dominated convergence theorem as well as (2.34) with $k=2$ and the dominated convergence theorem, upon using the localization argument if needed, we see that the final sum in (2.35) converges in probability as $m \to \infty$, and so does the first sum too. Denoting the latter limit by $\int_0^t P^n(ds, Z_s)$ (which clearly does not depend on any particular choice of the points t_i) in this way we end up with the following version of the Itô formula:

$$P^n(t, Z_t) = P^n(0, Z_0) + \int_0^t P^n(ds, Z_s) + \int_0^t P^n_x(s, Z_s)\, dZ_s + \frac{1}{2}\int_0^t P^n_{xx}(s, Z_s)\, d\langle Z, Z\rangle_s \tag{2.36}$$

being valid for all $n \geq 1$.

3. To compute the first integral in (2.36) we shall first note that the Itô formula gives:

$$\begin{aligned} F(t_i, x+b(t_i)) = F(t_{i-1}, x+b(t_{i-1})) &+ \int_{t_{i-1}}^{t_i} F_t(s, (x+b(s))\pm)\, ds \\ &+ \int_{t_{i-1}}^{t_i} F_x(s, (x+b(s))\pm)\, db(s) \end{aligned} \tag{2.37}$$

for all $x \in I\!R$ and $i = 1, 2, \ldots, n$. It follows in the same way as the proof of (2.32) sketched above upon using that $F = F_1$ on C and $F = F_2$ on D where F_1 and F_2 are $C^{1,2}$ functions from (1.5).

Recalling the definition of the first integral in (2.36) we find using (2.30)+(2.28) and (2.37) that the following line of identities is true:

$$\begin{aligned} \int_0^t P^n(ds, Z_s) &= I\!P\!-\lim_{m\to\infty} \sum_{i=1}^m \Big(P^n(t_i, Z_{t_i}) - P^n(t_{i-1}, Z_{t_i}) \Big) \\ &= I\!P\!-\lim_{m\to\infty} \sum_{i=1}^m \int_{I\!R} \Big(G(t_i, Z_{t_i}+y/n) - G(t_{i-1}, Z_{t_i}+y/n) \Big) \Omega(y)\, dy \\ &= I\!P\!-\lim_{m\to\infty} \sum_{i=1}^m \int_{I\!R} \Big(F(t_i, Z_{t_i}+y/n+b(t_i)) - F(t_{i-1}, Z_{t_i}+y/n+b(t_{i-1})) \Big) \Omega(y)\, dy \\ &= I\!P\!-\lim_{m\to\infty} \sum_{i=1}^m \int_{I\!R} \bigg(\int_{t_{i-1}}^{t_i} F_t(s, (Z_{t_i}+y/n+b(s))+)\, ds \\ &\qquad\qquad + \int_{t_{i-1}}^{t_i} F_x(s, (Z_{t_i}+y/n+b(s))+)\, db(s) \bigg) \Omega(y)\, dy \\ &= \int_{I\!R} \bigg(\int_0^t F_t(s, (Z_s+y/n+b(s))+)\, ds + \int_0^t F_x(s, (Z_s+y/n+b(s))+)\, db(s) \bigg) \Omega(y)\, dy \end{aligned} \tag{2.38}$$

where the final identity follows from the fact that the maps $z \mapsto F_t(s, (z+y/n+b(s))+)$ and $z \mapsto F_x(s, (z+y/n+b(s))+)$ are right-continuous. Letting $n \to \infty$ in (2.38) we finally get:

$$\lim_{n\to\infty} \int_0^t P^n(ds, Z_s) = \int_0^t F_t(s, (Z_s+b(s))+)\, ds + \int_0^t F_x(s, (Z_s+b(s))+)\, db(s)\ . \tag{2.39}$$

4. Similar calculations can be performed with the map N^n . Instead of (2.35) we can write:

$$N^n(t, Z_t) - N^n(0, Z_0) = \sum_{i=1}^m \Big(N^n(t_i, Z_{t_i}) - N^n(t_i, Z_{t_{i-1}}) \Big) \tag{2.40}$$

$$+\sum_{i=1}^{m}\Big(N^n(t_i,Z_{t_{i-1}})-N^n(t_{i-1},Z_{t_{i-1}})\Big) = \sum_{i=1}^{m}\Big(\int_{t_{i-1}}^{t_i} N_x^n(t_i,Z_s)\,dZ_s$$
$$+\frac{1}{2}\int_{t_{i-1}}^{t_i} N_{xx}^n(t_i,Z_s)\;d\langle Z,Z\rangle_s\Big) + \sum_{i=1}^{m}\int_{t_{i-1}}^{t_i} N^n(ds,Z_{t_{i-1}})\;.$$

Using the same arguments as above, this then leads to the following analogue of (2.36):

$$N^n(t,Z_t) = N^n(0,Z_0) + \int_0^t N^n(ds,Z_s) + \int_0^t N_x^n(s,Z_s)\,dZ_s + \frac{1}{2}\int_0^t N_{xx}^n(s,Z_s)\,d\langle Z,Z\rangle_s \tag{2.41}$$

as well as the following analogue of (2.38):

$$\begin{aligned}\int_0^t N^n(ds,Z_s) &= I\!P\text{-}\lim_{m\to\infty}\sum_{i=1}^{m} N^n(t_i,Z_{t_{i-1}})-N^n(t_{i-1},Z_{t_{i-1}}) \\ &= \int_{I\!R}\Big(\int_0^t F_t(s,(Z_s-y/n+b(s))-)\,ds + \int_0^t F_x(s,(Z_s-y/n+b(s))-)\,db(s)\Big)\Omega(y)\,dy\end{aligned} \tag{2.42}$$

where this time for the final identity it is used that the maps $z\mapsto F_t(s,(z-y/n+b(s))-)$ and $z\mapsto F_x(s,(z-y/n+b(s))-)$ are left-continuous. Letting $n\to\infty$ in (2.42) we finally get:

$$\lim_{n\to\infty}\int_0^n N^n(ds,Z_s) = \int_0^t F_t(s,(Z_s+b(s))-)\,ds + \int_0^t F_x(s,(Z_s+b(s))-)\,db(s)\;. \tag{2.43}$$

5. With the aim of letting $n\to\infty$ in (2.36) and (2.41), let us first note that:

$$P^n(t,x)\to G(t,x+)\quad\text{and}\quad N^n(t,x)\to G(t,x-) \tag{2.44}$$
$$P_x^n(t,x)\to G_x(t,x+)\quad\text{and}\quad N_x^n(t,x)\to G_x(t,x-) \tag{2.45}$$
$$P_{xx}^n(t,x)\to G_{xx}(t,x+)\quad\text{and}\quad N_{xx}^n(t,x)\to G_{xx}(t,x-) \tag{2.46}$$

as $n\to\infty$. It is important to note that the sequences in (2.44) and (2.45) are bounded by a constant on any bounded subset of $I\!R_+\times I\!R$ so that the (stochastic) dominated convergence theorem can be used in (2.36) and (2.41) upon applying the localization argument; this, however, is not the case with the sequences in (2.46) due to the existence of jumps of G_x at $(t,0)$ when such ones exist for the map $F_x(t,\cdot\,)$ at $b(t)$. More explicitly, substituting $z=x\pm y/n$ in (2.30) and (2.31) respectively, differentiating twice under the integral signs (which is justified by the standard arguments recalled in the proof of (2.33) and (2.34) above), and using integration by parts twice, we find that the following formulas are valid:

$$\begin{aligned}P_{xx}^n(t,x) = &\int_{I\!R} G_{xx}(t,x+y/n)\,I(x+y/n\neq 0)\,\Omega(y)\,dy \\ &+ n\Omega(-nx)\Big(G_x(t,0+)-G_x(t,0-)\Big)\end{aligned} \tag{2.47}$$

$$\begin{aligned}N_{xx}^n(t,x) = &\int_{I\!R} G_{xx}(t,x-y/n)\,I(x-y/n\neq 0)\,\Omega(y)\,dy \\ &+ n\Omega(nx)\Big(G_x(t,0+)-G_x(t,0-)\Big)\end{aligned} \tag{2.48}$$

for all $(t,x) \in I\!R_+ \times I\!R$ and all $n \geq 1$, where clearly the final terms present the difficulty as they are not uniformly bounded. From (2.47) and (2.48) we however see that:

$$P^n_{xx}(t,x)\, I\big(x \notin\,]-1/n, 0[\big) \;\to\; G_{xx}(t,x+) \tag{2.49}$$

$$N^n_{xx}(t,x)\, I\big(x \notin\,]0, 1/n[\big) \;\to\; G_{xx}(t,x-) \tag{2.50}$$

as $n \to \infty$, while clearly these sequences are bounded by a constant on any bounded subset of $I\!R_+ \times I\!R$, so that the same arguments can be used for them as for the sequences in (2.44) and (2.45) above.

6. It follows from the preceding arguments that by letting $n \to \infty$ in (2.36)+(2.41) and using (2.39)+(2.43), (2.44)+(2.45), (2.49)+(2.50) together with the identity:

$$G(t,x) = \lim_{n\to\infty} G^n(t,x) = \lim_{n\to\infty} \frac{1}{2}\Big(P^n(t,x) + N^n(t,x)\Big) \tag{2.51}$$

we obtain the following formula:

$$\begin{aligned} G(t,Z_t) = G(0,Z_0) &+ \int_0^t \frac{1}{2}\Big(F_t(s,(Z_s+b(s))+) + F_t(s,(Z_s+b(s))-)\Big)\, ds \\ &+ \int_0^t \frac{1}{2}\Big(F_x(s,(Z_s+b(s))+) + F_x(s,(Z_s+b(s))-)\Big)\, db(s) \\ &+ \int_0^t \frac{1}{2}\Big(G_x(s,Z_s+) + G_x(s,Z_s-)\Big)\, dZ_s + \frac{1}{2}\int_0^t G_{xx}(s,Z_s)\; I(Z_s \neq 0)\; d\langle Z,Z\rangle_s \\ &+ \frac{1}{2}\; I\!P\!-\!\lim_{n\to\infty} \frac{1}{2}\int_0^t \Big(n\,\Omega(-nZ_s)\, I(-1/n < Z_s < 0) + n\,\Omega(nZ_s)\, I(0 < Z_s < 1/n)\Big) \\ &\qquad\qquad \Big(G_x(s,0+) - G_x(s,0-)\Big)\, d\langle Z,Z\rangle_s \end{aligned} \tag{2.52}$$

where the value of $G_{xx}(s,Z_s)$ for $Z_s = 0$ does not matter since $\int_0^t I(Z_s = 0)\, d\langle Z,Z\rangle_s = 0$ by the occupation times formula (see e.g. [6] p. 224).

Applying (2.52) to the semimartingale $Z_t = X_t - b(t)$ and noting that the $db(s)$ integral cancels out, we obtain the following formula:

$$\begin{aligned} F(t,X_t) = F(0,X_0) &+ \int_0^t \frac{1}{2}\Big(F_t(s,X_s+) + F_t(s,X_s-)\Big)\, ds \\ &+ \int_0^t \frac{1}{2}\Big(F_x(s,X_s+) + F_x(s,X_s-)\Big)\, dX_s + \frac{1}{2}\int_0^t F_{xx}(s,X_s)\; I(X_s \neq 0)\; d\langle X,X\rangle_s \\ &+ \frac{1}{2}\; I\!P\!-\!\lim_{n\to\infty} \frac{1}{2}\int_0^t \Delta_x F_x(s,b(s))\, \Big(n\,\Omega(n(b(s)-X_s))\, I(b(s)-1/n < X_s < 0) \\ &\qquad\qquad + n\,\Omega(n(X_s - b(s)))\, I(0 < X_s < b(s)+1/n)\Big)\, d\langle X,X\rangle_s \end{aligned} \tag{2.53}$$

where we set $\Delta_x F_x(s,b(s)) = F_x(s,b(s)+) - F_x(s,b(s)-)$ and use that $\langle X-b, X-b\rangle = \langle X,X\rangle$ since b is of bounded variation.

7. Thus to complete the proof of (2.1) it is enough to establish the following limit representation:

$$\text{(2.54)} \qquad I\!\!P - \lim_{n\to\infty} \frac{1}{2(1/n)} \int_0^t \Delta_x F_x(s,b(s)) \Big(\Omega(n(b(s)-X_s))\, I(b(s)-1/n < X_s < 0) + \Omega(n(X_s-b(s)))\, I(0 < X_s < b(s)+1/n) \Big) d\langle X,X\rangle_s = \int_0^t \Delta_x F_x(s,b(s))\, d\ell_s^b$$

which appears to be the analogue of (2.25) from Part I in the present case. An equivalent way of looking at (2.54) is the following:

$$\text{(2.55)} \qquad \frac{1}{2(1/n)} \Big(\Omega(n(b(s)-X_s))\, I(b(s)-1/n < X_s < 0) + \Omega(n(X_s-b(s)))\, I(0 < X_s < b(s)+1/n) \Big) d\langle X,X\rangle_s \stackrel{\sim}{\to} d\ell_s^b$$

on $[0,t]$ in probability as $n \to \infty$. In view of (2.25) we see that yet another equivalent way of looking at either (2.54) or (2.55) is the following:

$$\text{(2.56)} \qquad \frac{1}{2(1/n)} \Big(\Omega(n(b(s)-X_s))\, I(b(s)-1/n < X_s < 0) + \Omega(n(X_s-b(s)))\, I(0 < X_s < b(s)+1/n) \Big) d\langle X,X\rangle_s - \frac{1}{2(1/n)}\, I(b(s)-1/n < X_s < b(s)+1/n)\, d\langle X,X\rangle_s \stackrel{\sim}{\to} 0$$

on $[0,t]$ in probability as $n \to \infty$.

8. It is at this point that Part I of the proof shows helpful. For this, firstly note that it is enough to prove (2.54) when $\Delta_x F_x(s,b(s))$ is replaced by $h(s)$ where $h : I\!\!R_+ \to I\!\!R$ is any C^1 function. This follows easily since C^1 functions $s \mapsto h(s)$ are dense (by the Weierstrass theorem) in the class of continuous functions $s \mapsto \Delta_x F_x(s,b(s))$ with respect to the supremum norm on the compact set $[0,t]$. Thus to establish (2.54) it is enough to prove that:

$$\text{(2.57)} \qquad I\!\!P - \lim_{n\to\infty} \frac{1}{2(1/n)} \int_0^t h(s) \Big(\Omega(n(b(s)-X_s))\, I(b(s)-1/n < X_s < 0) + \Omega(n(X_s-b(s)))\, I(0 < X_s < b(s)+1/n) \Big) d\langle X,X\rangle_s = \int_0^t h(s)\, d\ell_s^b$$

for any C^1 function $h : I\!\!R_+ \to I\!\!R$.

If such a function h is given, we can consider the function $\widetilde{F} : I\!\!R_+ \times I\!\!R \to I\!\!R$ given by:

$$\text{(2.58)} \qquad \widetilde{F}(t,x) = h(t)\, |x|$$

for $(t,x) \in I\!\!R_+ \times I\!\!R$. This function is then of the type for which the formula (2.1) with $\widetilde{b}(s) \equiv 0$

was proved in Part I above for any continuous semimartingale. Taking the semimartingale to be $\widetilde{X}_t = X_t - b(t)$ and noting that $\Delta_x \widetilde{F}_x(s, \widetilde{b}(s)) = \Delta_x \widetilde{F}_x(s, 0) = 2h(s)$, we see that (2.57) follows by a direct comparison of (2.53) from Part II and (2.1) from Part I, both with $\widetilde{F}$, $\widetilde{X}$ and $\widetilde{b}$ in place of F, X and b, respectively. Thus (2.57) holds for all C^1 functions and therefore for all continuous functions h as well. Since $s \mapsto \Delta_x F_x(s, b(s))$ is continuous by (1.9), this establishes (2.54). Finally, inserting (2.54) in (2.53) we obtain (2.1), and the proof is complete. $\square$

We will conclude the present section with a few remarks which are aimed to clarify some of the points related to the change-of-variable formula (2.1).

Remark 2.2

The following two simple examples may help to obtain a better feeling for the class of functions $F : I\!R_+ \times I\!R \to I\!R$ to which the change-of-variable formula (2.1) is applicable.

1. Let $F_1(t,x) = (t-1)^2$ and $F_2(t,x) = x^2$ for $(t,x) \in I\!R_+ \times I\!R$, and let $C = \{\,(t,x) \mid x < |t-1|\,\}$ and $D = \{\,(t,x) \mid x > |t-1|\,\}$. Then F defined to be F_1 on $\bar{C}$ and being equal F_2 on $\bar{D}$ satisfies (1.3) and (1.4) so that the change-of-variable formula (2.1) can be applied. Note that $b(t) = |t-1|$ in this case, and that b is not differentiable at $t=1$. This fact does not contradict the implicit function theorem (see e.g. [3] p. 8) since $(\partial/\partial t)(F_2 - F_1)(t,x) = -2(t-1) = 0$ for $(t,x) = (1, b(1)) = (1,0)$. Using the same method one can similarly construct many other functions F to which the change-of-variable formula (2.1) is applicable.

2. Let $F(t,x) = (x - b(t))^+$ for $(t,x) \in I\!R_+ \times I\!R$ where $b : I\!R_+ \to I\!R$ is a continuous function. If b is C^1 then F satisfies (1.3) and (1.4) with C and D from (1.1) and (1.2), and the change-of-variable formula (2.1) can be applied. If b is only of bounded variation, then the change-of-variable formula (2.1) can still be applied, however, with the function $\widetilde{F}(t,x) = x^+$ instead of F and the continuous semimartingale $\widetilde{X}_t = X_t - b(t)$ instead of X, as in this case we see that (1.1)-(1.4) are satisfied with $\widetilde{F}$ in place of F and $\widetilde{b} \equiv 0$ in place of b. Finally, if b is only continuous then the change-of-variable formula (2.1) cannot generally be applied in its present form.

Remark 2.3

The change-of-variable formula (2.1) can obviously be extended to the case when instead of one function b we are given finitely many functions $b_1, b_2, \ldots, b_n$ which do not intersect.

More precisely, let us assume that the following conditions are satisfied:

(2.59) $\quad b_i : I\!R_+ \to I\!R$ is continuous and of bounded variation for $1 \le i \le n$

(2.60) $\quad F_i : I\!R_+ \times I\!R \to I\!R$ is $C^{1,2}$ for $1 \le i \le n+1$

(2.61)
$$\begin{aligned} F(t,x) &= F_1(t,x) \quad \text{if} \quad x < b_1(t) \\ &= F_i(t,x) \quad \text{if} \quad b_{i-1}(t) < x < b_i(t) \quad \text{for} \quad 2 \le i \le n \\ &= F_{n+1}(t,x) \quad \text{if} \quad x > b_n(t) \end{aligned}$$

where $F : I\!R_+ \times I\!R \to I\!R$ is continuous. If $X = (X_t)_{t \ge 0}$ is a continuous semimartingale, then the change-of-variable formula (2.1) extends as follows:

$$
\begin{aligned}
(2.62)\qquad F(t,X_t) = F(0,X_0) &+ \int_0^t \frac{1}{2}\Big(F_t(s,X_s+)+F_t(s,X_s-)\Big)\,ds \\
&+ \int_0^t \frac{1}{2}\Big(F_x(s,X_s+)+F_x(s,X_s-)\Big)\,dX_s \\
&+ \frac{1}{2}\int_0^t F_{xx}(s,X_s)\; I\big(X_s \notin \{b_1(s),\ldots,b_n(s)\}\big)\; d\langle X,X\rangle_s \\
&+ \frac{1}{2}\sum_{i=1}^n \int_0^t \Big(F_x(s,X_s+)-F_x(s,X_s-)\Big)\, I(X_s = b_i(s))\; d\ell_s^{b_i}(X)
\end{aligned}
$$

where $\ell_s^{b_i}(X)$ is the local time of X at the curve b_i given by (1.6) above, and $d\ell_s^{b_i}(X)$ refers to the integration with respect to $s \mapsto \ell_s^{b_i}(X)$ for $i = 1,\ldots,n$.

Remark 2.4

It should be noted that an effort is made in the proof above (recall (2.28) and (2.29) in Part II) to establish the change-of-variable formula (2.1) with the one-dimensional limits $F_t(s,X_s\pm)$ and $F_x(s,X_s\pm)$ instead of the two-dimensional limits $F_t(s\pm,X_s\pm)$ and $F_x(s\pm,X_s\pm)$ in the first and second integral, respectively. The latter two limits may require a more complex analysis which in turn may lead to more restrictive conditions on the function F if the curve b oscillates heavily. This is not the case with the former two limits which also more naturally reflect that fact that b is a function of t so that each line parallel to the x-axis intersects b at most once.

Note, however, if the following conditions are satisfied:

$$
(2.63)\qquad \int_0^t \Big(F_t(s,X_s+)+F_t(s,X_s-)\Big) I(X_s = b(s))\, ds = 0
$$

$$
(2.64)\qquad \int_0^t \Big(F_x(s,X_s+)+F_x(s,X_s-)\Big) I(X_s = b(s))\, dX_s = 0
$$

then the first two integrals in (2.1) can be simplified to read as follows:

$$
\begin{aligned}
(2.65)\qquad F(t,X_t) = F(0,X_0) &+ \int_0^t F_t(s,X_s)\; I(X_s \neq b(s))\; ds + \int_0^t F_x(s,X_s)\; I(X_s \neq b(s))\; dX_s \\
&+ \frac{1}{2}\int_0^t F_{xx}(s,X_s)\; I(X_s \neq b(s))\; d\langle X,X\rangle_s \\
&+ \frac{1}{2}\int_0^t \Big(F_x(s,X_s+)-F_x(s,X_s-)\Big)\; I(X_s = b(s))\; d\ell_s^b(X)
\end{aligned}
$$

whenever the other conditions of Theorem 2.1 are fulfilled. [The same fact extends to the formula (2.62) above and the formula (2.70) below.]

Note that (2.63) is satisfied as soon as we know that:

$$
(2.66)\qquad P\big(X_s = b(s)\big) = 0 \quad \text{for} \quad s \in \langle 0,t] \ .
$$

Moreover, if $X = M + A$ is the decomposition of X into a local martingale M and a bounded-variation process A (both being continuous and adapted to the same filtration) then

$\int_0^t (F_x(s,X_s+)+F_x(s,X_s-))\,I(X_s=b(s))\,dX_s = \int_0^t (F_x(s,X_s+)+F_x(s,X_s-))\,I(X_s=b(s))\,dA_s$ since $\int_0^t (F_x(s,X_s+)+F_x(s,X_s-))\,I(X_s=b(s))\,dM_s = 0$ by the extended occupation times formula (see e.g. [6] p. 232) using also that $\langle M,M\rangle = \langle X,X\rangle = \langle X-b,X-b\rangle$ where $X-b$ is a continuous semimartingale. We thus see that (2.64) is equivalent to:

$$\int_0^t \Big(F_x(s,X_s+)+F_x(s,X_s-)\Big)\,I(X_s=b(s))\,dA_s = 0\ . \tag{2.67}$$

For example, if X solves $dX_t = \mu(t,X_t)\,dt + \sigma(t,X_t)\,dB_t$ where B is a standard Brownian motion, then $A_t = \int_0^t \mu(s,X_s)\,ds$ and (2.67) holds whenever (2.66) holds. Thus, in this case we see that (2.66) is sufficient for both (2.63) and (2.64), and consequently (2.1) takes the simpler form (2.65).

Remark 2.5

The change-of-variable formula (2.1) in Theorem 2.1 is expressed in terms of the *symmetric* local time (1.6). The Part II of the proof clearly shows that we could work equally well with the *one-sided* local times defined by:

$$\ell_s^{b+}(X) = I\!P\!-\!\lim_{\varepsilon\downarrow 0}\ \frac{1}{\varepsilon}\int_0^s I(b(r)\le X_r<b(r)+\varepsilon)\,d\langle X,X\rangle_r \tag{2.68}$$

$$\ell_s^{b-}(X) = I\!P\!-\!\lim_{\varepsilon\downarrow 0}\ \frac{1}{\varepsilon}\int_0^s I(b(r)-\varepsilon<X_r\le b(r))\,d\langle X,X\rangle_r \tag{2.69}$$

where only (2.2)+(2.3) in Part I of the proof should be accordingly modified. Then under the same conditions as in Theorem 2.1 we find that the following analogues of (2.1) are valid:

$$\begin{aligned} F(t,X_t) = F(0,X_0) &+ \int_0^t F_t(s,X_s\mp)\,ds + \int_0^t F_x(s,X_s\mp)\,dX_s \\ &+ \frac{1}{2}\int_0^t F_{xx}(s,X_s)\ I(X_s\ne b(s))\ d\langle X,X\rangle_s \\ &+ \frac{1}{2}\int_0^t \Big(F_x(s,X_s+)-F_x(s,X_s-)\Big)\ I(X_s=b(s))\ d\ell_s^{b\pm}(X)\ . \end{aligned} \tag{2.70}$$

It is well-known that a main advantage of the one-sided local time (2.68) upon the symmetric local time (1.6) is that the former generally admits a *right-continuous* modification in the space variable (see e.g. [6] pp. 225 and 234). In particular, if $X-b$ is a continuous local martingale, then the three definitions (1.6), (2.68) and (2.69) coincide.

Remark 2.6

A key point in the proof above was reached through the definition (2.58) which enabled us to connect Part II with Part I of the proof and in this way establish (2.54). When considering (2.54) on its own, however, it is tempting to make use of the extended occupation times formula (see e.g. [6] p. 232) and then integrate by parts in order to make use of the fact that P_x^n and N_n^x converge weakly as $n\to\infty$. Although leading to the same formula (2.1), it seems that this approach may have a drawback of requiring stronger conditions to be imposed on X than those used in Theorem 2.1 above, the aim of which would be to recover the continuity of the map $x\mapsto \ell_t^x(X)$ in order to exploit the weak convergence mentioned. While this may well be working for a special class of continuous semimartingales (such as diffusions driven by Brownian motion via SDE's) it may be difficult to extend this approach to the case of general continuous semimartingales treated in Theorem 2.1. It is exactly at this point where the real power of the argument above rests.

3. Another proof and extensions

The proof of Theorem 2.1 given above makes use of the Itô formula and derives the Tanaka formula. If we make use of the Tanaka formula as well, then a simpler proof can be given as follows. The idea to use (3.1)-(3.4) below is due to Thomas Kurtz.

1. Set $Z_t^1 = X_t \wedge b(t)$ and $Z_t^2 = X_t \vee b(t)$ and note by (1.5) that:

$$F(t, X_t) = F^1(t, Z_t^1) + F^2(t, Z_t^2) - F(t, b(t)) \tag{3.1}$$

where for the notational convenience we set $F^1 = F_1$ and $F^2 = F_2$. The processes $(Z_t^1)_{t\geq 0}$ and $(Z_t^2)_{t\geq 0}$ are continuous semimartingales admitting the following representations:

$$Z_t^1 = \frac{1}{2}\big(X_t + b(t) - |X_t - b(t)|\big) \tag{3.2}$$

$$Z_t^2 = \frac{1}{2}\big(X_t + b(t) + |X_t - b(t)|\big) \ . \tag{3.3}$$

Recalling the Tanaka formula:

$$|X_t - b(t)| = |X_0 - b(0)| + \int_0^t \operatorname{sign}(X_s - b(s))\, d(X_s - b(s)) + \ell_t^b(X) \tag{3.4}$$

where $\operatorname{sign}(0) = 0$, we find that:

$$\begin{aligned} dZ_t^1 &= \frac{1}{2}\Big(d(X_t + b(t)) - \operatorname{sign}(X_t - b(t))\, d(X_t - b(t)) - d\ell_t^b(X)\Big) \\ &= \frac{1}{2}\Big(\big(1 - \operatorname{sign}(X_t - b(t)\big)\, dX_t + \big(1 + \operatorname{sign}(X_t - b(t)\big)\, db(t) - d\ell_t^b(X)\Big) \end{aligned} \tag{3.5}$$

$$\begin{aligned} dZ_t^2 &= \frac{1}{2}\Big(d(X_t + b(t)) + \operatorname{sign}(X_t - b(t))\, d(X_t - b(t)) + d\ell_t^b(X)\Big) \\ &= \frac{1}{2}\Big(\big(1 + \operatorname{sign}(X_t - b(t)\big)\, dX_t + \big(1 - \operatorname{sign}(X_t - b(t)\big)\, db(t) + d\ell_t^b(X)\Big) \ . \end{aligned} \tag{3.6}$$

Hence we also see that:

$$d\langle Z^1, Z^1\rangle_t = \Big(I(X_t < b(t)) + \frac{1}{4} I(X_t = b(t))\Big)\, d\langle X, X\rangle_t = I(X_t < b(t))\, d\langle X, X\rangle_t \tag{3.7}$$

$$d\langle Z^2, Z^2\rangle_t = \Big(I(X_t > b(t)) + \frac{1}{4} I(X_t = b(t))\Big)\, d\langle X, X\rangle_t = I(X_t > b(t))\, d\langle X, X\rangle_t \tag{3.8}$$

where the second identity in (3.7) and (3.8) follows by the occupation times formula.

2. Applying the Itô formula to $F^1(t, Z_t^1)$ and using (3.5)+(3.7) we get:

$$
\begin{aligned}
F^1(t,Z_t^1) &= F^1(0,Z_0^1) + \int_0^t F_t^1(s,Z_s^1)\,ds \\
&\quad + \int_0^t F_x^1(s,Z_s^1)\,dZ_s^1 + \frac{1}{2}\int_0^t F_{xx}^1(s,Z_s^1)\,d\langle Z^1,Z^1\rangle_s \\
&= F^1(0,Z_0^1) + \int_0^t F_t^1(s,Z_s^1)\,ds + \frac{1}{2}\int_0^t \Big(1-\operatorname{sign}(X_s - b(s))\Big) F_x^1(s,Z_s^1)\,dX_s \\
&\quad + \frac{1}{2}\int_0^t \Big(1+\operatorname{sign}(X_s - b(s))\Big) F_x^1(s,Z_s^1)\,db(s) - \frac{1}{2}\int_0^t F_x^1(s,Z_s^1)\,d\ell_s^b(X) \\
&\quad + \frac{1}{2}\int_0^t I(X_s < b(s))\, F_{xx}^1(s,Z_s^1)\,d\langle X,X\rangle_s \ .
\end{aligned}
\tag{3.9}
$$

Applying the Itô formula to $F^2(t, Z_t^1)$ and using (3.6)+(3.8) we get:

$$
\begin{aligned}
F^2(t,Z_t^2) &= F^2(0,Z_0^2) + \int_0^t F_t^2(s,Z_s^2)\,ds \\
&\quad + \int_0^t F_x^2(s,Z_s^2)\,dZ_s^2 + \frac{1}{2}\int_0^t F_{xx}^2(s,Z_s^2)\,d\langle Z^2,Z^2\rangle_s \\
&= F^2(0,Z_0^2) + \int_0^t F_t^2(s,Z_s^2)\,ds + \frac{1}{2}\int_0^t \Big(1+\operatorname{sign}(X_s - b(s))\Big) F_x^2(s,Z_s^2)\,dX_s \\
&\quad + \frac{1}{2}\int_0^t \Big(1-\operatorname{sign}(X_s - b(s))\Big) F_x^2(s,Z_s^2)\,db(s) + \frac{1}{2}\int_0^t F_x^2(s,Z_s^2)\,d\ell_s^b(X) \\
&\quad + \frac{1}{2}\int_0^t I(X_s > b(s))\, F_{xx}^2(s,Z_s^2)\,d\langle X,X\rangle_s \ .
\end{aligned}
\tag{3.10}
$$

3. With the aim of inserting (3.9) and (3.10) in the right-hand side of (3.1) we will proceed by grouping the corresponding terms.

Firstly, note that:

$$
\begin{aligned}
F^1(0,Z_0^1) + F^2(0,Z_0^2) &= F^1(0, X_0 \wedge b(0)) + F^2(0, X_0 \vee b(0)) \\
&= F(0,X_0) + F(0,b(0))
\end{aligned}
\tag{3.11}
$$

upon using (1.5) with $t = 0$.

Secondly, note that:

$$
\begin{aligned}
\int_0^t F_t^1(s,Z_s^1)\,ds + \int_0^t F_t^2(s,Z_s^2)\,ds &= \int_0^t \Big(F_t^1(s,X_s) + F_t^2(s,b(s))\Big) I(X_s < b(s))\,ds \\
&\quad + \int_0^t \Big(F_t^1(s,b(s)) + F_t^2(s,b(s))\Big) I(X_s = b(s))\,ds \\
&\quad + \int_0^t \Big(F_t^1(s,b(s)) + F_t^2(s,X_s)\Big) I(X_s > b(s))\,ds
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
&= \int_0^t \Big(F_t(s,X_s)\, I(X_s < b(s)) + \frac{1}{2}\Big(F_t(s,b(s)-) + F_t(s,b(s)+)\Big)\, I(X_s = b(s)) \\
&\qquad\qquad + F_t(s,X_s)\, I(X_s > b(s))\Big)\, ds \\
&\quad + \int_0^t \Big(F_t(s,b(s)+)\, I(X_s < b(s)) + \frac{1}{2}\Big(F_t(s,b(s)-) + F_t(s,b(s)+)\Big)\, I(X_s = b(s)) \\
&\qquad\qquad + F_t(s,b(s)-)\, I(X_s > b(s))\Big)\, ds \\
&= \int_0^t \frac{1}{2}\Big(F_t(s,X_s+) + F_t(s,X_s-)\Big)\, ds \\
&\quad + \int_0^t \Big(F_t(s,b(s)+)\, I(X_s < b(s)) + \frac{1}{2}\Big(F_t(s,b(s)-) + F_t(s,b(s)+)\Big)\, I(X_s = b(s)) \\
&\qquad\qquad + F_t(s,b(s)-)\, I(X_s > b(s))\Big)\, ds
\end{aligned}
$$

upon using in the final identity that F is $C^{1,2}$ on C and D .

Thirdly, note that:

$$
\begin{aligned}
(3.13)\qquad & \frac{1}{2}\int_0^t \Big(1 - \operatorname{sign}(X_s - b(s))\Big) F_x^1(s, Z_s^1)\, dX_s + \frac{1}{2}\int_0^t \Big(1 + \operatorname{sign}(X_s - b(s))\Big) F_x^2(s, Z_s^2)\, dX_s \\
&= \int_0^t \Big(\frac{1}{2}\Big(1 - \operatorname{sign}(X_s - b(s))\Big) F_x^1(s, X_s) \\
&\qquad\qquad + \frac{1}{2}\Big(1 + \operatorname{sign}(X_s - b(s))\Big) F_x^2(s, b(s))\Big) I(X_s < b(s))\, dX_s \\
&\quad + \int_0^t \Big(\frac{1}{2}\Big(1 - \operatorname{sign}(X_s - b(s))\Big) F_x^1(s, b(s)) \\
&\qquad\qquad + \frac{1}{2}\Big(1 + \operatorname{sign}(X_s - b(s))\Big) F_x^2(s, b(s))\Big) I(X_s = b(s))\, dX_s \\
&\quad + \int_0^t \Big(\frac{1}{2}\Big(1 - \operatorname{sign}(X_s - b(s))\Big) F_x^1(s, b(s)) \\
&\qquad\qquad + \frac{1}{2}\Big(1 + \operatorname{sign}(X_s - b(s))\Big) F_x^2(s, X_s)\Big) I(X_s > b(s))\, dX_s \\
&= \int_0^t F_x^1(s, X_s)\, I(X_s < b(s))\, dX_s + \int_0^t \frac{1}{2}\Big(F_x^1(s,b(s)) + F_x^2(s,b(s))\Big) I(X_s = b(s))\, dX_s \\
&\quad + \int_0^t F_x^2(s, X_s)\, I(X_s > b(s))\, dX_s = \int_0^t \frac{1}{2}\Big(F_x(s, X_s+) + F_x(s, X_s-)\Big)\, dX_s
\end{aligned}
$$

upon using in the final identity that $F = F^1$ on $\bar{C}$ and $F = F^2$ on $\bar{D}$.

Fourthly, note that:

$$
(3.14)\qquad \frac{1}{2}\int_0^t \Big(1 + \operatorname{sign}(X_s - b(s))\Big) F_x^1(s, Z_s^1)\, db(s) + \frac{1}{2}\int_0^t \Big(1 - \operatorname{sign}(X_s - b(s))\Big) F_x^2(s, Z_s^2)\, db(s)
$$

$$
\begin{aligned}
&= \int_0^t \Big(\frac{1}{2}\Big(1+\operatorname{sign}(X_s - b(s))\Big) F_x^1(s,X_s) \\
&\qquad\quad + \frac{1}{2}\Big(1-\operatorname{sign}(X_s - b(s))\Big) F_x^2(s,b(s))\Big) I(X_s < b(s))\, db(s) \\
&\quad + \int_0^t \Big(\frac{1}{2}\Big(1+\operatorname{sign}(X_s - b(s))\Big) F_x^1(s,b(s)) \\
&\qquad\quad + \frac{1}{2}\Big(1-\operatorname{sign}(X_s - b(s))\Big) F_x^2(s,b(s))\Big) I(X_s = b(s))\, db(s) \\
&\quad + \int_0^t \Big(\frac{1}{2}\Big(1+\operatorname{sign}(X_s - b(s))\Big) F_x^1(s,b(s)) \\
&\qquad\quad + \frac{1}{2}\Big(1-\operatorname{sign}(X_s - b(s))\Big) F_x^2(s,X_s)\Big) I(X_s > b(s))\, db(s) \\
&= \int_0^t \Big(F_x^2(s,b(s))\, I(X_s < b(s)) + \frac{1}{2}\Big(F_x^1(s,b(s)) + F_x^2(s,b(s))\Big) I(X_s = b(s)) \\
&\qquad\quad + F_x^1(s,b(s))\, I(X_s > b(s))\Big)\, db(s) \\
&= \int_0^t \Big(F_x(s,b(s)+)\, I(X_s < b(s)) + \frac{1}{2}\Big(F_x(s,b(s)-) + F_x(s,b(s)+)\Big) I(X_s = b(s)) \\
&\qquad\quad + F_x(s,b(s)-)\, I(X_s > b(s))\Big)\, db(s)
\end{aligned}
$$

upon using in the final identity that $F = F^1$ on $\bar{C}$ and $F = F^2$ on $\bar{D}$.

4. Inserting (3.9) and (3.10) in the right-hand side of (3.1) and using (3.11)-(3.14) we see that the change-of-variable formula (2.1) will be obtained if we can verify the following identity:

$$
\begin{aligned}
F(t,b(t)) &= F(0,b(0)) \\
&\quad + \int_0^t \Big(F_t(s,b(s)+)\, I(X_s < b(s)) + \frac{1}{2}\Big(F_t(s,b(s)+) + F_t(s,b(s)-)\Big) I(X_s = b(s)) \\
&\qquad\quad + F_t(s,b(s)-)\, I(X_s > b(s))\Big)\, ds \\
&\quad + \int_0^t \Big(F_x(s,b(s)+)\, I(X_s < b(s)) + \frac{1}{2}\Big(F_x(s,b(s)+) + F_x(s,b(s)-)\Big) I(X_s = b(s)) \\
&\qquad\quad + F_x(s,b(s)-)\, I(X_s > b(s))\Big)\, db(s)\ .
\end{aligned}
\tag{3.15}
$$

To prove (3.15) formally first note that by (1.5) we have:

$$
\begin{aligned}
F(t,b(t)) - F(0,b(0)) &= \int_0^t F_t(s,b(s)+)\, ds + \int_0^t F_x(s,b(s)+)\, db(s) \\
&= \int_0^t F_t(s,b(s)-)\, ds + \int_0^t F_x(s,b(s)-)\, db(s)
\end{aligned}
\tag{3.16}
$$

for all $t \geq 0$. In particular, taking first $A = \langle t_1, t_2]$ for $0 \leq t_1 < t_2 \leq t$ with $t > 0$ fixed, and then using the standard uniqueness argument for finite measures, we see that the second identity in (3.16) extends as follows:

$$\begin{aligned} (3.17) \qquad & \int_0^t F_t(s, b(s)+)\, 1_A(s)\, ds + \int_0^t F_x(s, b(s)+)\, 1_A(s)\, db(s) \\ & \qquad = \int_0^t F_t(s, b(s)-)\, 1_A(s)\, ds + \int_0^t F_x(s, b(s)-)\, 1_A(s)\, db(s) \end{aligned}$$

for every Borel subset A of $[0,t]$. Letting A in (3.17) first to be $\{\, 0 \leq s \leq t \mid X_s < b(s) \,\}$ and then $\{\, 0 \leq s \leq t \mid X_s = b(s) \,\}$, we see that the identity (3.15) reduces to equality between the first and the third term in (3.16). This shows that (3.15) is valid and thus (2.1) is satisfied as well.

5. The preceding proof can be conveniently extended to derive a version of the change-of-variable formula (2.1) that goes beyond the conditions (1.3) and (1.4). Motivated by applications in free-boundary problems of optimal stopping (cf. [4]-[5]) we will now present such an extension of (2.1) where (1.3) and (1.4) are replaced by the conditions:

$$F \text{ is } C^{1,2} \text{ on } C \tag{3.18}$$

$$F \text{ is } C^{1,2} \text{ on } D \tag{3.19}$$

and $X = (X_t)_{t \geq 0}$ is a diffusion process solving:

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dB_t \tag{3.20}$$

in Itô's sense. The latter more precisely means that X satisfies:

$$X_t = X_0 + \int_0^t \mu(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dB_s \tag{3.21}$$

for all $t \geq 0$ where μ and σ are locally bounded (continuous) functions for which the integrals in (3.21) are well-defined (the second being Itô's) so that X itself is a continuous semimartingale (the process $B = (B_t)_{t \geq 0}$ is a standard Brownian motion). To ensure that X is non-degenerate we will assume that $\sigma > 0$. This fact, in particular, implies that (2.66) holds for all $t > 0$ so that (2.1) takes the simpler form (2.65) whenever (1.3) and (1.4) are satisfied.

It turns out, however, that the conditions (1.3) and (1.4) are not always readily verified. The main example we have in mind (arising from the free-boundary problems mentioned above) is:

$$F(t,x) = E_{t,x}\big(G(t + \tau_D, X_{t+\tau_D})\big) \tag{3.22}$$

where $X_t = x$ under $P_{t,x}$, an admissible function G is given and fixed, and:

$$\tau_D = \inf \{\, s > 0 \mid (t+s, X_{t+s}) \in D \,\} \, . \tag{3.23}$$

Then one directly obtains the 'interior condition' (3.18) by standard means while the 'closure condition' (1.3) is harder to verify at b since (unless we know a priori that b is Lipschitz continuous or even differentiable) both F_t and F_{xx} may in principle diverge when b is approached from the interior of C .

The following theorem is designed precisely to handle such cases (without entering into more involved arguments on Lipschitz continuity or differentiability of b which is not given explicitly) provided that one has some basic control over F_x at b . [In free-boundary problems mentioned above such a control is provided by the principle of smooth fit which often follows knowing only that b is increasing or decreasing for instance.] It turns out that in the latter case even if F_t is formally to diverge when the boundary b is approached from the interior of C , this deficiency is counterbalanced by a similar behaviour of F_{xx} through the infinitesimal generator of X , and consequently the first integral in (3.29) below is still well-defined and finite. For specific applications of the theorem below see [4] and [5] (as well as a number of subsequent papers on the topic).

6. Given a subset A of $I\!\!R_+ \times I\!\!R$ and a function $f : A \rightarrow I\!\!R$ we say that f is *locally bounded* on A (in $I\!\!R_+ \times I\!\!R$) if for each a in $\bar{A}$ there is an open set U in $I\!\!R_+ \times I\!\!R$ containing a such that f restricted to $A \cap U$ is bounded. Note that f is locally bounded on A if and only if for each compact set K in $I\!\!R_+ \times I\!\!R$ the restriction of f to $A \cap K \neq \emptyset$ is bounded. Given a function $g : [0,t] \rightarrow I\!\!R$ of bounded variation we let $V(g)(t)$ denote the total variation of g on $[0,t]$.

To grasp the meaning of the condition (3.26) below in the case of F from (3.22) above, letting $I\!\!L_X = \partial/\partial t + \mu\, \partial/\partial x + (\sigma^2/2)\, \partial^2/\partial x^2$ denote the infinitesimal generator of X , note that:

$$I\!\!L_X F = 0 \quad \text{in} \quad C \tag{3.24}$$

$$I\!\!L_X F = I\!\!L_X G \quad \text{in} \quad D\ . \tag{3.25}$$

This shows that $I\!\!L_X F$ is locally bounded on $C \cup D$ as soon as $I\!\!L_X G$ is so on D . The latter condition (in free-boundary problems) is easily verified since G is given explicitly.

The second result of the present paper may now be stated as follows (see also Remark 3.2 below for further sufficient conditions).

Theorem 3.1

Let $X = (X_t)_{t \geq 0}$ be a diffusion process solving (3.20) in Itô's sense, let $b : I\!\!R_+ \rightarrow I\!\!R$ be a continuous function of bounded variation, and let $F : I\!\!R_+ \times I\!\!R \rightarrow I\!\!R$ be a continuous function satisfying (3.18) and (3.19) above.

If the following conditions are satisfied:

$$F_t + \mu F_x + (\sigma^2/2) F_{xx} \ \text{ is locally bounded on } \ C \cup D \tag{3.26}$$

$$F_x(\,\cdot\,, b(\,\cdot\,) \pm \varepsilon) \rightarrow F_x(\,\cdot\,, b(\,\cdot\,)\pm) \ \text{ uniformly on } \ [0,t] \ \text{ as } \ \varepsilon \downarrow 0 \tag{3.27}$$

$$\sup_{0<\varepsilon<\delta} V(F(\,\cdot\,, b(\,\cdot\,) \pm \varepsilon))(t) < \infty \ \text{ for some } \ \delta > 0 \tag{3.28}$$

then the following change-of-variable formula holds:

$$\begin{aligned} F(t,X_t) = F(0,X_0) &+ \int_0^t \big(F_t + \mu F_x + (\sigma^2/2) F_{xx}\big)(s,X_s)\ I(X_s \neq b(s))\ ds \\ &+ \int_0^t (\sigma F_x)(s,X_s)\ I(X_s \neq b(s))\ dB_s \end{aligned} \tag{3.29}$$

$$+ \frac{1}{2} \int_0^t \Big(F_x(s, X_s+) - F_x(s, X_s-) \Big) \, I(X_s = b(s)) \; d\ell_s^b(X)$$

where $\ell_s^b(X)$ *is the local time of* X *at the curve* b *given by (1.6) above, and* $d\ell_s^b(X)$ *refers to the integration with respect to the continuous increasing function* $s \mapsto \ell_s^b(X)$.

Before we pass to the proof of (3.29) we will state a number of *sufficient conditions* which either imply (3.26)-(3.28) or can be used instead. *It will be assumed in the following remark that* F *satisfies (3.18) and (3.19) above*. Fuller arguments for the statements appearing in the remark will be given in the proof below.

Remark 3.2

1. Note that (1.3) and (1.4) imply (3.26)-(3.28) so that (3.29) is a version of (2.1) obtained under weaker conditions. Note also that the following condition:

(3.30) $\quad F_x$ is continuous on $\bar{C}$ and $\bar{D}$

implies (3.27). Finally, if either of the following two sets of conditions is satisfied:

(3.31) $s \mapsto F(s,x)$ is decreasing on $[0,t]$ for each $x \in I\!\!R$; $x \mapsto F(s,x)$ is decreasing (increasing) on $I\!\!R$ for each $s \in [0,t]$; $s \mapsto b(s)$ is increasing (decreasing) on $[0,t]$

(3.32) $s \mapsto F(s,x)$ is increasing on $[0,t]$ for each $x \in I\!\!R$; $x \mapsto F(s,x)$ is increasing (decreasing) on $I\!\!R$ for each $s \in [0,t]$; $s \mapsto b(s)$ is increasing (decreasing) on $[0,t]$

then (3.28) holds as well.

2. If (3.27) holds, then the following condition:

(3.33) $\quad s \mapsto F_t(s, b(s) \pm \varepsilon)$ does not change its sign on $[0,t]$ for $\varepsilon \downarrow 0$

implies (3.28). Moreover, if both (3.26) and (3.27) hold, then the following condition:

(3.34) $\quad s \mapsto F_{xx}(s, b(s) \pm \varepsilon)$ does not change its sign on $[0,t]$ for $\varepsilon \downarrow 0$

implies (3.28) as well. In particular, *if (3.26) and the following two conditions hold*:

(3.35) $x \mapsto F(s,x)$ *is convex or concave on* $[b(s)-\delta, b(s)]$ *and convex or concave on* $[b(s), b(s)+\delta]$ *for each* $s \in [0,t]$ *with some* $\delta > 0$

(3.36) $s \mapsto F_x(s, b(s)\pm)$ *is continuous on* $[0,t]$ *with values in* $I\!\!R$

then both (3.27) and (3.28) hold. This shows that *(3.35) and (3.36) imply (3.29) when (3.26) holds*. The condition (3.35) can further be relaxed to the form where:

(3.37) $\quad F_{xx} = G_1 + G_2$ on $C \cup D$

where G_1 is non-negative (non-positive) and G_2 is continuous on $\bar{C}$ and $\bar{D}$. Thus, if (3.36) and (3.37) hold, then both (3.27) and (3.28) hold implying also (3.29) when (3.26) holds.

3. The condition (3.27) in the above theorem can be replaced by the following condition:

$$\sup_{0\le\varepsilon<\delta} V(F_x(\,\cdot\,,b(\,\cdot\,)\pm\varepsilon))(t) < \infty \quad \text{for some} \quad \delta > 0\ . \tag{3.38}$$

Thus, if (3.26) holds, then (3.28) and (3.38) imply (3.29). Note that ε in (3.38) is required to be 0 as well. [Although it appears to be of theoretical interest, when stated together and compared with (3.28), the condition (3.38) is somewhat harder to verify generally.]

Proof. The proof is an extension of the arguments (3.1)-(3.17) above obtained by replacing b by $b-\varepsilon$ and $b+\varepsilon$ and passing to the limit when $\varepsilon \downarrow 0$ (possibly over a subsequence).

1. For this, set $Z_t^{1,\varepsilon} = X_t \wedge (b(t)-\varepsilon)$ and $Z_t^{2,\varepsilon} = X_t \vee (b(t)+\varepsilon)$ and write down the formulas (3.9) and (3.10) with $b-\varepsilon$ and $b+\varepsilon$ instead of b respectively. Grouping the corresponding terms like in (3.11)-(3.14) we find that:

$$\begin{aligned} F(t,Z_t^{1,\varepsilon}) + F(t,Z_t^{2,\varepsilon}) &= F\big(0,X_0\wedge(b(0)-\varepsilon)\big) + F\big(0,X_0\vee(b(0)+\varepsilon)\big) \\ &+ \int_0^t \big(F_t+\mu F_x+(\sigma^2/2)F_{xx}\big)(s,X_s)\, I\big(X_s\notin[b(s)-\varepsilon,b(s)+\varepsilon]\big)\, ds \\ &+ \int_0^t (\sigma F_x)(s,X_s)\, I\big(X_s\notin[b(s)-\varepsilon,b(s)+\varepsilon]\big)\, dB_s \\ &+ \frac{1}{2}\int_0^t F_x(s,b(s)+\varepsilon)\ d\ell_s^{b+\varepsilon}(X) - \frac{1}{2}\int_0^t F_x(s,b(s)-\varepsilon)\ d\ell_s^{b-\varepsilon}(X) \\ &+ \int_0^t F_t(s,b(s)+\varepsilon)\ I(X_s<b(s)+\varepsilon)\ ds + \int_0^t F_x(s,b(s)+\varepsilon)\ I(X_s<b(s)+\varepsilon)\ db(s) \\ &+ \int_0^t F_t(s,b(s)-\varepsilon)\ I(X_s>b(s)-\varepsilon)\ ds + \int_0^t F_x(s,b(s)-\varepsilon)\ I(X_s>b(s)-\varepsilon)\ db(s) \end{aligned} \tag{3.39}$$

upon using that $P(X_s=b(s)\pm\varepsilon) = 0$ for $0 < s \le t$.

2. Suppose that we can prove that:

$$\begin{aligned} &\frac{1}{2}\int_0^t F_x(s,b(s)+\varepsilon)\ d\ell_s^{b+\varepsilon}(X) - \frac{1}{2}\int_0^t F_x(s,b(s)-\varepsilon)\ d\ell_s^{b-\varepsilon}(X) \\ &\to \frac{1}{2}\int_0^t F_x(s,b(s)+)\ d\ell_s^{b}(X) - \frac{1}{2}\int_0^t F_x(s,b(s)-)\ d\ell_s^{b}(X) \end{aligned} \tag{3.40}$$

as well as that:

$$\begin{aligned} &\int_0^t F_t(s,b(s)+\varepsilon)\ I(X_s<b(s)+\varepsilon)\ ds + \int_0^t F_x(s,b(s)+\varepsilon)\ I(X_s<b(s)+\varepsilon)\ db(s) \\ &+ \int_0^t F_t(s,b(s)-\varepsilon)\ I(X_s>b(s)-\varepsilon)\ ds + \int_0^t F_x(s,b(s)-\varepsilon)\ I(X_s>b(s)-\varepsilon)\ db(s) \\ &\to F(t,b(t)) - F(0,b(0)) \end{aligned} \tag{3.41}$$

both P-a.s. as $\varepsilon \downarrow 0$ (possibly over a subsequence). Then letting $\varepsilon \downarrow 0$ in (3.39) and using the dominated convergence theorem (deterministic and stochastic) to establish a convergence in P-probability of the first and second integral in (3.39), where (3.26) can be used for the former and for the latter it may be noted that (3.27) implies that F_x and thus σF_x as well are both locally bounded on $C \cup D$, we see that the limiting identity obtained is exactly (3.29) above. The proof therefore reduces to derive (3.40) and (3.41).

3. To derive (3.40) let us show that:

$$\int_0^t F_x(s, b(s) \pm \varepsilon)\, d\ell_s^{b\pm\varepsilon}(X) \rightarrow \int_0^t F_x(s, b(s)\pm)\, d\ell_s^b(X) \tag{3.42}$$

P-a.s. as $\varepsilon \downarrow 0$ over a subsequence. For this, note that $\sup_{0\le s\le t} |\ell_s^{b\pm\varepsilon}(X) - \ell_s^b(X)| \rightarrow 0$ P-a.s. as $\varepsilon \downarrow 0$ over a subsequence as shown in (3.59) below. Thus outside a P-null set $d\ell_s^{b\pm\varepsilon} \overset{\sim}{\rightarrow} d\ell_s^b(X)$ on $[0,t]$ as $\varepsilon \downarrow 0$ over a subsequence. From the weak convergence just established and the uniform convergence assumed in (3.27) one easily finds that (3.42) holds as claimed.

4. To derive (3.41) first note that by adding and subtracting the same terms and using (3.45) below we find that (3.41) is equivalent to:

$$\begin{aligned}
&\int_0^t F_t(s, b(s){+}\varepsilon)\ I(X_s \ge b(s){+}\varepsilon)\ ds + \int_0^t F_x(s, b(s){+}\varepsilon)\ I(X_s \ge b(s){+}\varepsilon)\ db(s) \\
&+ \int_0^t F_t(s, b(s){-}\varepsilon)\ I(X_s \le b(s){-}\varepsilon)\ ds + \int_0^t F_x(s, b(s){-}\varepsilon)\ I(X_s \le b(s){-}\varepsilon)\ db(s) \\
&\rightarrow F(t, b(t)) - F(0, b(0))
\end{aligned} \tag{3.43}$$

P-a.s. as $\varepsilon \downarrow 0$ over a subsequence. Note that the relation (3.43) is obvious if X_s stays strictly above or below $b(s)$ for all $0 \le s \le t$ or if $s \mapsto X_s$ crosses $s \mapsto b(s)$ finitely many times on $[0,t]$. To treat the case of a sample path of X we may invoke some results on weak convergence of signed measures and proceed as follows.

Let μ_ε denote the Lebesgue-Stieltjes signed measure associated with $s \mapsto F(s, b(s){+}\varepsilon)$ on $[0,t]$, and let μ denote the Lebesgue-Stieltjes signed measure associated with $s \mapsto F(s, b(s))$ on $[0,t]$. Let μ_ε^+ and μ_ε^- denote the positive and negative part of μ_ε respectively. Since (3.28) holds we can use Helly's selection theorem to conclude that $\mu_{\varepsilon_n}^+ \overset{\sim}{\rightarrow} \mu_1$ and $\mu_{\varepsilon_n}^- \overset{\sim}{\rightarrow} \mu_2$ over a subsequence $\varepsilon_n \downarrow 0$ as $n \rightarrow \infty$, where μ_1 and μ_2 are positive finite measures on $[0,t]$. Moreover, since $F(s, b(s){+}\varepsilon_n) \rightarrow F(s, b(s))$ as $n \rightarrow \infty$ for each $s \in [0,t]$, and $s \mapsto F(s, b(s))$ is (right-)continuous on $[0,t]$, it follows that $\mu = \mu_1 - \mu_2$.

Set $A_n = \{\, 0 \le s \le t \mid X_s \ge b(s){+}\varepsilon_n \,\}$ for $n \ge 1$ and note that $A_n \uparrow A$ as $n \rightarrow \infty$ where $A = \{\, 0 \le s \le t \mid X_s > b(s) \,\}$. Since $\partial A_n \subseteq \{\, 0 \le s \le t \mid X_s = b(s){+}\varepsilon_n \,\}$ we see that:

$$\begin{aligned}
E\Big(\int_0^t 1_{\partial A_n}(s)\ \mu_{1,2}(ds) \Big) &\le E\Big(\int_0^t I(X_s {=} b(s){+}\varepsilon_n)\ \mu_{1,2}(ds) \Big) \\
&= \int_0^t P(X_s {=} b(s){+}\varepsilon_n)\ \mu_{1,2}(ds) = 0
\end{aligned} \tag{3.44}$$

for all $n \ge 1$. This shows that $\mu_{1,2}(\partial A_n) = 0$ for all $n \ge 1$ outside a P-null set. In exactly the same way one finds that $\mu_{1,2}(\partial A) = 0$ outside a P-null set.

Denoting $\mu_n^{\pm} = \mu_{\varepsilon_n}^{\pm}$ for all $n \geq 1$ we claim that $\lim_{n\to\infty} \mu_n^{\pm}(A_n) = \mu_{1,2}(A)$ outside a P-null set. To see this set $a_{nm} = \mu_n^{\pm}(A_m)$ and note that the two limits $a_{n\infty} = \lim_{m\to\infty} a_{nm} = \mu_n^{\pm}(A)$ and $a_{\infty m} = \lim_{n\to\infty} a_{nm} = \mu_{1,2}(A_m)$ exist (the latter outside a P-null set by the weak convergence established) and moreover satisfy $\lim_{n\to\infty} a_{n\infty} = \lim_{m\to\infty} a_{\infty m} = \mu_{1,2}(A) =: a_{\infty\infty}$ (the former outside a P-null set by the weak convergence established). Clearly $a_{nn} \leq a_{n\infty}$ since $A_n \uparrow A$ so that $\limsup_{n\to\infty} a_{nn} \leq a_{\infty\infty}$. On the other hand, since $a_{nm} \leq a_{nn}$ for all $m \leq n$, it follows first by letting $n \to \infty$ that $a_{\infty m} \leq \liminf_{n\to\infty} a_{nn}$ and then again by letting $m \to \infty$ that $a_{\infty\infty} \leq \liminf_{n\to\infty} a_{nn}$. This proves that $\lim_{n\to\infty} a_{nn} = a_{\infty\infty}$ outside a P-null set as claimed. In particular, this implies that $\mu_{\varepsilon_n}(A_{\varepsilon_n}) \to \mu(A)$ as $n \to \infty$ outside a P-null set.

Returning to (3.43) and recalling the formula (with the plus sign):

$$F(t,b(t)\pm\varepsilon) = F(0,b(0)\pm\varepsilon) + \int_0^t F_t(s,b(s)\pm\varepsilon)\;ds + \int_0^t F_x(s,b(s)\pm\varepsilon)\;db(s) \tag{3.45}$$

being true for every $t > 0$, we see by the previous conclusion that:

$$\begin{aligned} &\int_0^t F_t(s,b(s)+\varepsilon_n)\;I(X_s \geq b(s)+\varepsilon_n)\;ds + \int_0^t F_x(s,b(s)+\varepsilon_n)\;I(X_s \geq b(s)+\varepsilon_n)\;db(s) \\ &= \mu_{\varepsilon_n}(A_{\varepsilon_n}) \;\to\; \mu(A) \end{aligned} \tag{3.46}$$

P-a.s. as $n \to \infty$. In exactly the same way one proves that:

$$\begin{aligned} &\int_0^t F_t(s,b(s)-\varepsilon_n)\;I(X_s \leq b(s)-\varepsilon_n)\;ds + \int_0^t F_x(s,b(s)-\varepsilon_n)\;I(X_s \leq b(s)-\varepsilon_n)\;db(s) \\ &= \nu_{\varepsilon_n}(B_{\varepsilon_n}) \;\to\; \mu(B) \end{aligned} \tag{3.47}$$

P-a.s. over some $\varepsilon_n \downarrow 0$ as $n \to \infty$, where ν_{ε_n} is the Lebesgue-Stieltjes signed measure associated with $s \mapsto F(s,b(s)-\varepsilon_n)$ on $[0,t]$ while $B_{\varepsilon_n} = \{\, 0 \leq s \leq t \mid X_s \leq b(s)-\varepsilon_n \,\}$ and $B = \{\, 0 \leq s \leq t \mid X_s < b(s) \,\}$, upon using the formula (3.45) (with the minus sign).

From (3.46) and (3.47) we see that the four integrals on the left-hand side of (3.43) converge to $\mu(A) + \mu(B)$ P-a.s. as $\varepsilon \downarrow 0$ over a subsequence. Moreover, from the fact that:

$$\begin{aligned} E\Big(\int_0^t 1_{A^c \cap B^c}(s)\;\mu_{1,2}(ds) \Big) &= E\Big(\int_0^t I(X_s = b(s))\;\mu_{1,2}(ds) \Big) \\ &= \int_0^t P(X_s = b(s))\;\mu_{1,2}(ds) = 0 \end{aligned} \tag{3.48}$$

it follows that $\mu_{1,2}(A^c \cap B^c) = 0$ outside a P-null set. Hence $\mu(A^c \cap B^c) = 0$ outside a P-null set so that $\mu(A) + \mu(B) = \mu(A \cup B) = \mu([0,t]) = F(t,b(t)) - F(0,b(0))$ P-a.s. as claimed in (3.43). This completes the proof of (3.29) under (3.26)-(3.28).

In the reminder of the proof we present fuller arguments for the statements given in Remark 3.2 above upon recalling that F is assumed to satisfy (3.18) and (3.19).

5. To see that (1.3) and (1.4) imply (3.26)-(3.28) note first that (1.3) and (1.4) imply (3.26). Moreover, noting that (1.3) and (1.4) imply (3.30), and applying the general fact that each continuous function on a compact set is uniformly continuous, we see that (3.30) implies (3.27) and thus so

do (1.3) and (1.4) as claimed. Finally, from (3.45) we find that:

$$V(F(\,\cdot\,,b(\,\cdot\,)\pm\varepsilon))(t) \le \int_0^t |F_t(s,b(s)\pm\varepsilon)|\,ds + \int_0^t |F_x(s,b(s)\pm\varepsilon)|\,dV(b)(s) \tag{3.49}$$

for $\varepsilon > 0$. Since under (1.3) and (1.4) both F_t and F_x are continuous on $\bar{C}$ and $\bar{D}$, and thus bounded on each compact set contained in $\bar{C}$ and $\bar{D}$, it follows by taking supremum over all $0 < \varepsilon < \delta$ on both sides of (3.49) that (3.28) is satisfied for all $\delta > 0$. This shows that (1.3) and (1.4) imply (3.28) as claimed.

If (3.31) holds then $s \mapsto F(s,b(s)\pm\varepsilon)$ is decreasing on $[0,t]$ and therefore of bounded variation. Moreover, since F is continuous and thus locally bounded we see that (3.28) follows as well. This shows that (3.31) implies (3.28) as claimed. Similarly, if (3.32) holds then $s \mapsto F(s,b(s)\pm\varepsilon)$ is increasing on $[0,t]$ and (3.28) follows in the same way. This shows that (3.32) implies (3.28) as claimed.

6. For (3.33) recall first that (3.45) implies (3.49) above. Moreover, using (3.33) and taking supremum over all $0 < \varepsilon < \delta$ in (3.45) we find:

$$\begin{aligned} &\sup_{0<\varepsilon<\delta} \int_0^t |F_t(s,b(s)\pm\varepsilon)|\,ds = \sup_{0<\varepsilon<\delta} \int_0^t F_t(s,b(s)\pm\varepsilon)\,ds \\ &= \sup_{0<\varepsilon<\delta} \Big(F(t,b(t)\pm\varepsilon) - F(0,b(0)\pm\varepsilon) - \int_0^t F_x(s,b(s)\pm\varepsilon)\,db(s) \Big) \\ &\le \sup_{0<\varepsilon<\delta} |F(t,b(t)\pm\varepsilon)| + \sup_{0<\varepsilon<\delta} |F(0,b(0)\pm\varepsilon)| + \sup_{0<\varepsilon<\delta} \int_0^t |F_x(s,b(s)\pm\varepsilon)|\,dV(b)(s) < \infty \end{aligned} \tag{3.50}$$

where the final (strict) inequality follows from the fact that F and F_x are locally bounded on C and D . Taking supremum over all $0 < \varepsilon < \delta$ on both sides of (3.49) and using (3.50) we find that (3.28) is satisfied for all $\delta > 0$. This shows that (3.33) implies (3.28) when (3.27) holds.

For (3.34) note that $F_t = I\!\!L_X F - \mu F - (\sigma^2/2) F_{xx}$ where $L_X F$ and μF_x are locally bounded on C and D by (3.26) and (3.27) respectively. Inserting the former expression for F_t into the right-hand-side of (3.49) we get:

$$\begin{aligned} V(F(\,\cdot\,,b(\,\cdot\,)\pm\varepsilon))(t) &\le \int_0^t |I\!\!L_X F(s,b(s)\pm\varepsilon)|\,ds + \int_0^t |(\mu F_x)(s,b(s)\pm\varepsilon)|\,ds \\ &+ \int_0^t ((\sigma^2/2)|F_{xx}|)(s,b(s)\pm\varepsilon)\,ds + \int_0^t |F_x(s,b(s)\pm\varepsilon)|\,dV(b)(s) \end{aligned} \tag{3.51}$$

for $\varepsilon > 0$. Moreover, using (3.34) and taking supremum over all $0 < \varepsilon < \delta$ in (3.45) we find:

$$\begin{aligned} &\sup_{0<\varepsilon<\delta} \int_0^t ((\sigma^2/2)|F_{xx}|)(s,b(s)\pm\varepsilon)\,ds = \sup_{0<\varepsilon<\delta} \int_0^t ((\sigma^2/2)(\pm F_{xx}))(s,b(s)\pm\varepsilon)\,ds \\ &= \sup_{0<\varepsilon<\delta} \Big(\pm F(0,b(0)\pm\varepsilon) \mp F(t,b(t)\pm\varepsilon) \pm \int_0^t (I\!\!L_X F - \mu F_x)(s,b(s)\pm\varepsilon)\,ds \\ &\qquad \pm \int_0^t F_x(s,b(s)\pm\varepsilon)\,db(s) \Big) \\ &\le \sup_{0<\varepsilon<\delta} |F(0,b(0)\pm\varepsilon)| + \sup_{0<\varepsilon<\delta} |F(t,b(t)\pm\varepsilon)| + \sup_{0<\varepsilon<\delta} \int_0^t |I\!\!L_X F(s,b(s)\pm\varepsilon)|\,ds \end{aligned} \tag{3.52}$$

$$+ \sup_{0<\varepsilon<\delta} \int_0^t |(\mu F_x)(s, b(s) \pm \varepsilon)| \, ds + \sup_{0<\varepsilon<\delta} \int_0^t |F_x(s, b(s) \pm \varepsilon)| \, dV(b)(s) < \infty$$

where the final (strict) inequality follows from the fact that F , $I\!\!L_X F$, μF_x and F_x are locally bounded on C and D . Taking supremum over all $0 < \varepsilon < \delta$ on both sides of (3.51) and using (3.52) we find that (3.28) is satisfied for all $\delta > 0$. This shows that (3.34) implies (3.28) when (3.26) and (3.27) hold.

For (3.35) note first that if $x \mapsto F(s,x)$ is convex (concave) then $x \mapsto F_x(s,x)$ is increasing (decreasing) so that $\varepsilon \mapsto F_x(s, b(s) \pm \varepsilon)$ is increasing (decreasing). Each of the preceding two conclusions together with (3.36) then implies (3.27) by Dini's theorem (note that each $s \mapsto F_x(s, b(s) \pm \varepsilon)$ is continuous on the compact set $[0,t]$). Moreover, since $F_{xx} \geq 0$ or $F_{xx} \leq 0$ depending on if $x \mapsto F(s,x)$ is convex or concave, we see that (3.35) implies (3.34). Taken together with the previous conclusion this shows that (3.35) and (3.36) imply (3.27) and (3.28) when (3.26) holds.

That (3.35) in the previous implication can be relaxed to the form (3.37) follows firstly by a simple modification of the proof given in (3.51)-(3.52) above which yields (3.28). For (3.27) note that by setting $c = \min_{0 \leq s \leq t} b(s) - 1$ and integrating both sides in (3.37) from c to x with respect to the space variable we find that $F_x(s,x) - F_x(s,c) = \int_c^x F_{xx}(s,y) \, dy = \int_c^x G_1(s,y) \, dy + \int_c^x G_2(s,y) \, dy =: H_1(s,x) + H_2(s,x)$ where $x \mapsto H_1(s,x)$ is convex and $(s,x) \mapsto H_2(s,x)$ is continuous on $\bar{C}$ and $\bar{D}$. It thus follows as above that $H_1(\,\cdot\,, b(\,\cdot\,) \pm \varepsilon) \rightarrow H_1(\,\cdot\,, b(\,\cdot\,)\pm)$ and $H_2(\,\cdot\,, b(\,\cdot\,) \pm \varepsilon) \rightarrow H_2(\,\cdot\,, b(\,\cdot\,)\pm)$ both uniformly on $[0,t]$ as $\varepsilon \downarrow 0$, where it should be noted that by (3.36) and the fact that H_2 is continuous on $\bar{C}$ and $\bar{D}$ it follows that $s \mapsto H_1(s, b(s)\pm)$ is continuous on $[0,t]$. This shows that (3.36) and (3.37) imply (3.27) and (3.28) when (3.26) holds.

7. For (3.38) the idea is to transfer the requirement of uniform convergence (in the weak-limit relations yielding (3.40) above) from F_x to $\ell^b(X)$ using integration by parts. The latter gives:

$$\int_0^t F_x(s, b(s) \pm \varepsilon) \; d\ell_s^{b\pm\varepsilon}(X) = F_x(s, b(s) \pm \varepsilon) \ell_s^{b\pm\varepsilon}(X) \Big|_0^t - \int_0^t \ell_s^{b\pm\varepsilon}(X) \; d_s F_x(s, b(s) \pm \varepsilon) \tag{3.53}$$

for $\varepsilon > 0$. Suppose that we can prove that:

$$\int_0^t \ell_s^{b\pm\varepsilon}(X) \; d_s F_x(s, b(s) \pm \varepsilon) \rightarrow \int_0^t \ell_s^b(X) \; d_s F_x(s, b(s)\pm) \tag{3.54}$$

P-a.s. as $\varepsilon \downarrow 0$ over a subsequence. Then letting $\varepsilon \downarrow 0$ in (3.53), using (3.54), and integrating back by parts we see that (3.42) holds and so does (3.40). Thus, the proof of (3.40) in this case reduces to establish (3.54).

For this, since $F_x(s, b(s) \pm \varepsilon) \rightarrow F_x(s, b(s)\pm)$ as $\varepsilon \downarrow 0$ for every $s \in [0,t]$, and the condition (3.38) is assumed to be satisfied, by Helly's theorem it follows that $d_s F_x(s, b(s) \pm \varepsilon) \overset{\sim}{\rightarrow} d_s F_x(s, b(s)\pm)$ on $[0,t]$ in the sense that $\int_0^t g(s) \, d_s F_x(s, b(s) \pm \varepsilon) \rightarrow \int_0^t g(s) \, d_s F_x(s, b(s)\pm)$ as $\varepsilon \downarrow 0$ for every continuous function $g : [0,t] \rightarrow I\!\!R$. In view of (3.54) it is therefore sufficient to show that outside a P-null set $\ell_s^{b\pm\varepsilon}(X) \rightarrow \ell_s^b(X)$ uniformly over s in $[0,t]$ as $\varepsilon \downarrow 0$ (possibly over a subsequence).

For this, apply the Tanaka formula to the semimartingale $Z = X - b$ and the function

$f(z) = |z \pm \varepsilon|$ for $z \in I\!R$ and $\varepsilon \geq 0$. This gives:

$$|Z_t \pm \varepsilon| = |Z_0 \pm \varepsilon| + \int_0^t sign(Z_s \pm \varepsilon)\; dZ_s + \ell_t^{\pm\varepsilon}(Z) \tag{3.55}$$

where $sign(0) = 0$. Noting that $\ell_t^{\pm\varepsilon}(Z) = \ell_t^{b\pm\varepsilon}(X)$ we find from (3.55) that:

$$\begin{aligned} \sup_{0\le s\le t} |\ell_s^{b\pm\varepsilon}(X) - \ell_s^b(X)| &= \sup_{0\le s\le t} |\ell_s^{\pm\varepsilon}(Z) - \ell_s^0(Z)| \\ &\le \sup_{0\le s\le t} \Big(\big||Z_s\pm\varepsilon|-|Z_s|\big| + \big||Z_0|-|Z_0\pm\varepsilon|\big| + \Big|\int_0^s \big(sign(Z_r)-sign(Z_r\pm\varepsilon)\big)\, dZ_r\Big| \Big). \end{aligned} \tag{3.56}$$

Setting $H_s^\varepsilon = sign(Z_s) - sign(Z_s \pm \varepsilon)$ hence we find:

$$\begin{aligned} \sup_{0\le s\le t} |\ell_s^{b\pm\varepsilon}(X) - \ell_s^b(X)| \le 2\varepsilon &+ \int_0^t |H_s^\varepsilon|\, |\mu(s,X_s)|\, ds \\ &+ \int_0^t |H_s^\varepsilon|\, dV(b)(s) + \sup_{0\le s\le t} \Big| \int_0^t H_s^\varepsilon\, \sigma(s,X_s)\, dB_s \Big| \,. \end{aligned} \tag{3.57}$$

The argument of (3.48) with Lebesgue measure λ instead of $\mu_{1,2}$ shows that outside a P-null set $Z_s = X_s - b(s) = 0$ for λ-a.a. s in $[0,t]$. It follows that outside the same P-null set $H_s^\varepsilon \to 0$ for λ-a.a. s in $[0,t]$. Since $|H_s^\varepsilon| \le 2$ for $0 \le s \le t$ and μ is locally bounded it thus follows by the dominated convergence theorem that the first integral in (3.57) tends to zero outside a P-null set as $\varepsilon \downarrow 0$. In exactly the same way we find that the second integral in (3.57) tends to zero outside a P-null set as $\varepsilon \downarrow 0$. To bound the third integral in (3.57) we can make use of the Burkholder-Davis-Gundy inequality which yields:

$$E\Big(\sup_{0\le s\le t} \Big| \int_0^s H_r^\varepsilon\, \sigma(r,X_r)\, dB_r \Big| \Big) \le E\Big(\int_0^t (H_s^\varepsilon)^2 \sigma^2(s,X_s)\, ds \Big)^{1/2} . \tag{3.58}$$

Since σ is locally bounded and $H_s^\varepsilon = 0$ for $X_s \notin [b(s)-\varepsilon, b(s)+\varepsilon]$ when $s \in [0,t]$, it follows as above following (3.57) that $\int_0^t (H_s^\varepsilon)^2 \sigma^2(s,X_s)\, ds \to 0$ outside a P-null set as $\varepsilon \downarrow 0$ so that the right-hand side of (3.58) tends to zero as $\varepsilon \downarrow 0$ by the dominated convergence theorem. This implies that the third integral in (3.57) tends to zero outside P-null set as $\varepsilon \downarrow 0$ over a subsequence.

Summarizing the preceding conclusions in (3.57) we obtain:

$$\sup_{0\le s\le t} |\ell_s^{b\pm\varepsilon}(X) - \ell_s^b(X)| \to 0 \tag{3.59}$$

P-a.s. as $\varepsilon \downarrow 0$ over a subsequence. Thus outside a P-null set $\ell_s^{b\pm\varepsilon}(X) \to \ell_s^b(X)$ uniformly over s in $[0,t]$ as $\varepsilon \downarrow 0$ over a subsequence as claimed. This shows that (3.38) imply that (3.28) is sufficient for (3.29) when (3.26) holds. The proof is complete. □

It appears evident from the proofs above that the change-of-variable formula (2.1) can be extended to the case of a general (not necessarily continuous) semimartingale in the multi-dimensional setting of functions which are smooth above and below surfaces (instead of curves). Some of these extensions will be studied in a subsequent publication.

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