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Optimal Stopping in the $L \log L$-Inequality of Hardy and Littlewood

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Let $B = (B_t)_{t \ge 0}$ be standard Brownian motion started at zero. We prove:

$$E\Big(\max_{0 \le t \le \tau} |B_t|\Big) \le 1 + \frac{1}{e^c(c-1)} + cE\Big(|B_\tau| \log^+ |B_\tau|\Big)$$

for all $c > 1$ and all stopping times τ for B satisfying $E(\tau^r) < \infty$ for some $r > 1/2$. This inequality is sharp, and the equality is attained at the stopping time:

$$\tau_* = \inf \big\{ t > 0 \mid S_t \ge u_* , \; X_t = 1 \vee \alpha S_t \big\}$$

where $u_* = 1 + 1/e^c(c-1)$ and $\alpha = (c-1)/c$ for $c > 1$ with $X_t = |B_t|$ and $S_t = \max_{0 \le r \le t} |B_r|$. Likewise, we prove:

$$E\Big(\max_{0 \le t \le \tau} |B_t|\Big) \le \frac{c^2}{e(c-1)} + cE\Big(|B_\tau| \log |B_\tau|\Big)$$

for all $c > 1$ and all stopping times τ for B satisfying $E(\tau^r) < \infty$ for some $r > 1/2$. This inequality is sharp, and the equality is attained at the stopping time:

$$\sigma_* = \inf \big\{ t > 0 \mid S_t \ge v_* , \; X_t = \alpha S_t \big\}$$

where $v_* = c/e(c-1)$ and $\alpha = (c-1)/c$ for $c > 1$. These results contain and refine the results on the $L \log L$-inequality of Gilat [6] which are obtained by analytic methods. The method of proof given here is probabilistic and is based upon solving the optimal stopping problem with the payoff:

$$V = \sup_\tau E\big(S_\tau - cF(X_\tau)\big)$$

where $F(x)$ equals either $x \log^+ x$ or $x \log x$. This optimal stopping problem has some new interesting features, but in essence is solved by applying the principle of smooth fit and the maximality principle. The results extend to the case when B starts at any given point (as well as to all non-negative submartingales).

1. Introduction

The present work was motivated by the paper of Gilat [6] where he settles a question raised by Dubins and Gilat [4], and later by Cox [2], on the $L \log L$-inequality of Hardy and Littlewood. Instead of recalling his results in the analytic framework of the Hardy-Littlewood theory, we shall rather refer the reader to Gilat's paper [6] where a splendid historical exposition on the link between the Hardy-Littlewood theory and probability (martingale theory) can be found too. Despite the fact that Gilat's paper finishes with a comment on the use of his result in the martingale theory, his proof is entirely analytic, and is not satisfactory from the probabilistic point of view. The main aim of this paper is to present a new probabilistic solution to this problem.

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Due to its extreme properties in such a context (see [9]), we choose to work with Brownian motion $B = (B_t)_{t \geq 0}$. For simplicity, we assume for a moment that B starts at zero, but all of the results will extend to the case when B starts at any given point. The $L \log L$-inequality of Hardy and Littlewood [8] formulated in the optimal stopping setting of Brownian motion states:

$$(1.1) \qquad E\Big(\max_{0 \leq t \leq \tau} |B_t|\Big) \leq C_1 \Big(1 + E\Big(|B_\tau| \log^+ |B_\tau|\Big)\Big)$$

for all stopping times τ for B with $E(\tau^r) < \infty$ for some $r > 1/2$, where C_1 is a universal numerical constant (see [3]). The analogue of the problem considered by Gilat [6] may be now stated as follows. Determine the best value for the constant C_1 in (1.1), and find the corresponding optimal stopping time (the one at which the equality in (1.1) is attained). While Gilat's result gives the best value for C_1 , it does not detect the optimal stopping strategy τ_* in (1.1), but rather gives the distribution law of B_{τ_*} and S_{τ_*} (see Remark 2.5 below). In contrast to this deficiency, our proof does both, and together with the extension to the case when B starts at any point, this detection (of the optimal stopping strategy) forms the principal new result of the paper.

As pointed out in Gilat's paper, the inequality (1.1) remains valid if the plus sign is removed from the logarithm sign, so that we have:

$$(1.2) \qquad E\Big(\max_{0 \leq t \leq \tau} |B_t|\Big) \leq C_2 \Big(1 + E\Big(|B_\tau| \log |B_\tau|\Big)\Big)$$

for all stopping times τ for B with $E(\tau^r) < \infty$ for some $r > 1/2$, where C_2 is a universal numerical constant. The problem about (1.1) stated above extends in exactly the same form to (1.2). It turns out that this problem is somewhat easier, but both of them have some new features which make them interesting from the optimal stopping theory point of view.

To describe this in more detail, note that in both cases (1.1) and (1.2) we are given an optimal stopping problem with the payoff:

$$(1.3) \qquad V = \sup_\tau E\big(S_\tau - cF(X_\tau)\big)$$

where $c > 0$, and in the first case $F(x) = x \log^+ x$, while in the second case $F(x) = x \log x$, with $X_t = |B_t|$ and $S_t = \max_{0 \leq r \leq t} |B_r|$. The interesting appearance about the first problem is that the cost $x \mapsto cF(x)$ is somewhat artificially set up zero for $x \leq 1$, while in the second problem the cost is not monotone all over as a function of time. Moreover, in the latter case the Itô formula is not directly applicable to $F(X_t)$, due to the fact that $F''(x) = 1/x$ so that $\int_0^\tau F''(X_t)\, dt = \infty$ for all stopping times τ for which $X_\tau \neq 0$ P-a.s. This makes it difficult to find a "useful" increasing functional $t \mapsto I_t$ with the same expectation as the cost (the fact which enables one to write down a differential equation for the payoff).

Despite these difficulties we solve both optimal stopping problems and in turn get solutions to (1.1) and (1.2) as consequences. The first problem is solved by guessing and then verifying that the guess is correct (Theorem 2.1, Corollary 2.2). The second problem is solved by a truncation method (Theorem 2.3, Corollary 2.4). The results obtained extend to all non-negative submartingales (Corollary 2.6). Instead of describing all of this in more detail, we shall rather refer the reader to the proofs below. Let us just mention that both methods in essence rely upon the principle of smooth fit of Kolmogorov (see [5]) and the maximality principle (see [7]). We believe that both of these methods can be applied in a similar context.

2. The results and proofs

In this section we present the main results and proofs. Since the problem (1.2) is somewhat easier, we begin by stating the main results in this direction (Theorem 2.1). The facts obtained in the proof will be used later (Theorem 2.3) in the solution for the problem (1.1). It is instructive to compare these two proofs and to spot the essential argument needed to conclude in the latter (note that $d\tilde{F}/dx$ from the proof of Theorem 2.1 is continuous at $1/e$, while dF_+/dx from the proof of Theorem 2.3 is not continuous at 1 , thus bringing the local time of X at 1 into the game; this is the crucial difference between these two problems). The Gilat's paper [6] finishes with a concluding remark where a gap between the $L \log L$ and $L \log^+ L$ case is mentioned. The discovery of the exact size of this gap is stressed out to be the main point of his paper. The essential argument mentioned above offers a probabilistic explanation for this gap and gives its exact size in terms of the optimal stopping strategies (compare (2.2) and (2.30), and notice the middle term in (2.45) in comparison with (2.28)).

Theorem 2.1

Let $B = (B_t)_{t \geq 0}$ be standard Brownian motion started at zero under P . Then the following inequality is shown to be satisfied:

$$E\Big(\max_{0 \leq t \leq \tau} |B_t| \Big) \leq \frac{c^2}{e(c-1)} + cE\Big(|B_\tau| \log |B_\tau| \Big) \tag{2.1}$$

for all $c > 1$ and all stopping times τ for B satisfying $E(\tau^r) < \infty$ for some $r > 1/2$. This inequality is sharp, and the equality is attained at the stopping time:

$$\sigma_* = \inf \big\{ t > 0 \mid S_t \geq v_* \, , \, X_t = \alpha S_t \big\} \tag{2.2}$$

where $v_* = c/e(c-1)$ and $\alpha = (c-1)/c$ for $c > 1$ with $X_t = |B_t|$ and $S_t = \max_{0 \leq r \leq t} |B_r|$.

Proof. Given $c > 1$, consider the optimal stopping problem with the payoff:

$$V(x,s) = \sup_\tau E_{x,s}\big(S_\tau - cF(X_\tau) \big) \tag{2.3}$$

where $F(x) = x \log x$, $X_t = |B_t + x|$ and $S_t = s \vee \max_{0 \leq r \leq t} |B_r + x|$ for $0 \leq x \leq s$. Note that the (strong) Markov process (X,S) under $P := P_{x,s}$ starts at (x,s) .

The main difficulty in this problem is that we cannot apply Itô formula to $F(X_t)$. Thus we truncate $F(x)$ by setting $\tilde{F}(x) = F(x)$ for $x \geq 1/e$ and $\tilde{F}(x) = -1/e$ for $0 \leq x \leq 1/e$. Then $\tilde{F} \in C^1$ and $\tilde{F}''$ exists everywhere but $1/e$. Since the time spent by X at $1/e$ is of Lebesgue measure zero, setting $\tilde{F}''(1/e) := e$, by Itô-Tanaka's formula (see [10]) we get:

$$\tilde{F}(X_t) = \tilde{F}(x) + \int_0^t \tilde{F}'(X_r) \, dX_r + \frac{1}{2} \int_0^t \tilde{F}''(X_r) \, d\langle X, X \rangle_r \tag{2.4}$$

$$= \tilde{F}(x) + \int_0^t \tilde{F}'(X_r)\ d(\beta_r + L_r) + \frac{1}{2}\int_0^t \tilde{F}''(X_r)\ dr$$

$$= \tilde{F}(x) + M_t + \frac{1}{2}\int_0^t \tilde{F}''(X_r)\ dr$$

where $\beta = (\beta_t)_{t\geq 0}$ is a standard Brownian motion, $L = (L_t)_{t\geq 0}$ is the local time of X at zero, and $M_t = \int_0^t \tilde{F}'(X_r)\ d\beta_r$ is a continuous local martingale, due to $\tilde{F}'(0) = 0$ and the fact that dL_r is concentrated at $\{ t \mid X_t = 0 \}$ (see [10]). By the optional sampling theorem and Burkholder-Davis-Gundy's inequality for continuous local martingales (see [10]), we easily find:

(2.5) $$E_{x,s}\big(\tilde{F}(X_\tau)\big) = \tilde{F}(x) + \frac{1}{2}E_{x,s}\bigg(\int_0^\tau \tilde{F}''(X_t)\ dt\bigg)$$

for all stopping times τ for B satisfying $E_{x,s}(\tau^r) < \infty$ for some $r > 1/2$. By (2.5) we see that the payoff $V(x,s)$ from (2.3) should be identical to $\tilde{V}(x,s) := W(x,s) - c\tilde{F}(x)$ where:

(2.6) $$W(x,s) = \sup_\tau E_{x,s}\bigg(S_\tau - \frac{c}{2}\int_0^\tau \tilde{F}''(X_t)\ dt\bigg)$$

for $0 \leq x \leq s$. For this, note that clearly $V(x,s) \leq \tilde{V}(x,s)$, so if we prove that the optimal stopping time $\tilde{\sigma}_*$ in (2.6) satisfies $X_{\tilde{\sigma}_*} \geq 1/e$, then due to $E_{x,s}(S_{\tilde{\sigma}_*} - c\tilde{F}(X_{\tilde{\sigma}_*})) = E_{x,s}(S_{\tilde{\sigma}_*} - cF(X_{\tilde{\sigma}_*}))$ this will show $V(x,s) = \tilde{V}(x,s)$ with $\tilde{\sigma}_*$ being optimal in (2.3) too. In the rest of the proof we solve the optimal stopping problem (2.6) and show that the truncation procedure indicated above works as desired.

Supposing that the supremum in (2.6) is attained at the exit time of diffusion (X,S) from an open set, we see that the payoff $W(x,s)$ is to satisfy:

(2.7) $$\mathbf{L_X} W(x,s) = \frac{c}{2}\tilde{F}''(x) \quad (\ g_*(s) < x < s\ )$$

where $\mathbf{L_X} = \partial^2/2\partial x^2$ is the infinitesimal operator of X in $]0,\infty[$ and $s \mapsto g_*(s)$ is the optimal stopping boundary to be found. To solve (2.7) in an explicit form, we shall make use of the following boundary conditions:

(2.8) $$W(x,s)\Big|_{x=g_*(s)+} = s \quad \text{(instantaneous stopping)}$$

(2.9) $$\frac{\partial W}{\partial x}(x,s)\Big|_{x=g_*(s)+} = 0 \quad \text{(smooth fit)}$$

(2.10) $$\frac{\partial W}{\partial s}(x,s)\Big|_{x=s-} = 0 \quad \text{(normal reflection)}\ .$$

Note that (2.7)-(2.10) forms a Stephan problem with moving (free) boundary $s \mapsto g_*(s)$ (for more information on the Stephan problem see [5]). The general solution of (2.7) is given by:

(2.11) $$W(x,s) = C(s)x + D(s) + c\tilde{F}(x)$$

where $s \mapsto C(s)$ and $s \mapsto D(s)$ are unknown functions. By (2.8) and (2.9) we find:

(2.12) $$C(s) = -c\tilde{F}'(g_*(s))$$

(2.13) $$D(s) = s + cg_*(s)\tilde{F}'(g_*(s)) - c\tilde{F}(g_*(s)) \ .$$

Inserting (2.12) and (2.13) into (2.11) we obtain:

(2.14) $$W(x,s) = s - c\big(x - g_*(s)\big)\tilde{F}'(g_*(s)) - c\tilde{F}(g_*(s)) + c\tilde{F}(x)$$

for $g_*(s) \le x \le s$. Clearly $W(x,s) = s$ for $0 \le x \le g_*(s)$. Finally, by (2.10) we find that $s \mapsto g_*(s)$ is to satisfy:

(2.15) $$g_*'(s)\ \tilde{F}''(g_*(s))\ \big(s - g_*(s)\big) = \frac{1}{c}$$

for $s > 0$. Note that this equation has sense only for $F''(g_*(s)) > 0$ or equivalently $g_*(s) \ge 1/e$ when it reads as follows:

(2.16) $$g_*'(s)\left(\frac{s}{g_*(s)} - 1\right) = \frac{1}{c}$$

for $s \ge v_*$ where $g_*(v_*) = 1/e$. Now observe that (2.16) admit a linear solution of the form:

(2.17) $$g_*(s) = \alpha s$$

for $s \ge v_*$ where $\alpha = (c-1)/c$. (Note that this solution is the maximal solution to either (2.15) or (2.16) among all solutions which does not hit the diagonal $x = s$. This is in accordance with the maximality principle (see [7]) and is the main motivation for the candidate (2.17).) This in addition indicates that the formula (2.14) will be valid only if $s \ge v_*$, where v_* is determined from $g_*(v_*) = 1/e$, so that:

(2.18) $$v_* = c/e(c-1) \ .$$

The corresponding candidate for the optimal stopping time is:

(2.19) $$\tilde{\sigma}_* = \inf\ \big\{\ t > 0 \mid X_t \le g_*(S_t)\ \big\}$$

where $s \mapsto g_*(s)$ is given by (2.17) for $s \ge v_*$. The candidate for the payoff (2.6) given by the formula in (2.14) for $g_*(s) \le x \le s$ with $s \ge v_*$ will be denoted in the sequel by $W_*(x,s)$. Clearly $W_*(x,s) = s$ for $0 \le x \le g_*(s)$ with $s \ge v_*$. In the next step we verify that this candidate equals the payoff (2.6), and that $\tilde{\sigma}_*$ from (2.19) is the optimal stopping time.

To verify this, we shall apply Itô-Tanaka's formula to $W_*(X_t,S_t)$. Since $\tilde{F}'' \ge 0$, this gives:

(2.20) $$W_*(X_t,S_t) = W_*(x,s) + \int_0^t \frac{\partial W_*}{\partial x}(X_r,S_r)\ dX_r + \int_0^t \frac{\partial W_*}{\partial s}(X_r,S_r)\ dS_r + \frac{1}{2}\int_0^t \frac{\partial^2 W_*}{\partial x^2}(X_r,S_r)\ d\big\langle X,X\big\rangle_r$$

$$\leq W_*(x,s) + \int_0^t \frac{\partial W_*}{\partial x}(X_r,S_r)\; d(\beta_r + L_r) + \frac{c}{2}\int_0^t \tilde{F}''(X_r)\; dr$$
$$= W_*(x,s) + M_t + \frac{c}{2}\int_0^t \tilde{F}''(X_r)\; dr$$

with $M_t = \int_0^t (\partial W_*/\partial x)(X_r,S_r)\; d\beta_r$ being a continuous local martingale, where we used that dS_r equals zero for $X_r < S_r$, so that by (2.10) the integral over dS_r is equal zero, and since clearly $(\partial W_*/\partial x)(0,s) = 0$, the integral over dL_r is equal zero too. Now since $W_*(x,s) \geq s$ for all $x \geq g_*(s)$ (with the equality if $x = g_*(s)$) hence we obtain:

(2.21) $$S_\tau - \frac{c}{2}\int_0^\tau \tilde{F}''(X_t)\; dt \leq W_*(x,s) + M_\tau$$

for all (bounded) stopping times τ for B with the equality (in (2.20) too) if $\tau = \tilde{\sigma}_*$. Taking the expectation on both sides we get:

(2.22) $$E_{x,s}\Big(S_\tau - \frac{c}{2}\int_0^\tau \tilde{F}''(X_t)\; dt\Big) \leq W_*(x,s)$$

for all (bounded) stopping times τ for B satisfying:

(2.23) $$E_{x,s}(M_\tau) = 0$$

with the equality in (2.22) under the validity of (2.23) if $\tau = \tilde{\sigma}_*$. We first show that (2.23) holds for all stopping times τ for B satisfying $E_{x,s}(\sqrt{\tau}) < \infty$. For this, we compute:

(2.24) $$E_{x,s}\bigg(\int_0^\tau \Big(\frac{\partial W_*}{\partial x}(X_r,S_r)\Big)^2 dr\bigg)^{1/2}$$
$$= cE_{x,s}\bigg(\int_0^\tau \log^2\Big(\frac{X_r}{g_*(S_r)}\Big)\, 1_{\{\, g_*(S_r)\leq X_r\, \}}\; dr\bigg)^{1/2}$$
$$= c\log(1/\alpha)E_{x,s}\big(\sqrt{\tau}\big)$$

so that (2.23) follows by the optional sampling theorem and Burkholder-Davis-Gundy's inequality for continuous local martingales whenever $E_{x,s}\big(\sqrt{\tau}\big) < \infty$. Moreover, it is well-known (see [11]) that $E_{x,s}(\tilde{\sigma}_*^r) < \infty$ for all $r < c/2$. In particular $E_{x,s}\big(\sqrt{\tilde{\sigma}_*}\big) < \infty$, so that (2.23) holds for $\tau = \tilde{\sigma}_*$, and thus we have the equality in (2.22) for $\tau = \tilde{\sigma}_*$. This completes the proof that the payoff (2.6) equals $W_*(x,s)$ for $0 \leq x \leq s$ with $s \geq v_*$, and that σ_* is the optimal stopping time.

Note that $X_{\tilde{\sigma}_*} \geq 1/e$ so that from (2.14) by the remark following (2.6) we get:

(2.25) $$V(x,s) = \tilde{V}(x,s) = W(x,s) - c\tilde{F}(x)$$
$$= s - cx - cx\log g_*(s) + cg_*(s)$$

for all $g_*(s) \leq x \leq s$ with $s \geq v_*$, where $g_*(s) = \alpha s$ with $\alpha = (c-1)/c$ and $v_* = c/e(c-1)$. To complete the proof it remains to compute the payoff $V(x,s)$ for $0 \leq x \leq s$ with $0 \leq s < v_*$.

A simple observation which motivates our formal action in this direction is as follows.

The best point to stop in the region $0 \le x \le s < v_*$ would be $(1/e\,,s)$ with s as close as possible to v_*, since the cost function $x \mapsto cx \log x$ attains its minimal value at $1/e$. The payoff V equals (tends) to $v_* + c/e$ if the process (X,S) is started and stopped at $(1/e\,,s)$ with s being equal (tending) to v_*. However, it is easily seen that the payoff $V(x,s)$ computed above for $s \ge v_*$ satisfies $V(v_*,v_*) = v_* + c/e = c^2/e(c-1)$. This indicates that in the region $0 \le x \le s < v_*$ should be no point of stopping. It is now formally verified as follows. Given a (bounded) stopping time τ for B, define τ' to be τ on $\{\,\tau \ge \tau_{v_*}\,\}$ and $\tilde{\sigma}_*$ on $\{\,\tau < \tau_{v_*}\,\}$. Then τ' is a stopping time for B, and clearly $(X_{\tau'}, S_{\tau'})$ does not belong to the region $0 \le x \le s < v_*$. Moreover, by the strong Markov property:

$$\begin{aligned} (2.26) \qquad E_{x,s}\big(S_{\tau'} - cF(X_{\tau'})\big) &= E_{x,s}\big((S_\tau - cF(X_\tau))\,1_{\{\,\tau \ge \tau_{v_*}\,\}}\big) + E_{x,s}\big((S_{\tilde{\sigma}_*} - cF(X_{\tilde{\sigma}_*}))\,1_{\{\,\tau < \tau_{v_*}\,\}}\big) \\ &= E_{x,s}\big((S_\tau - cF(X_\tau))\,1_{\{\,\tau \ge \tau_{v_*}\,\}}\big) + E_{x,s}\big(E_{v_*,v_*}(S_{\tilde{\sigma}_*} - cF(X_{\tilde{\sigma}_*}))\,1_{\{\,\tau < \tau_{v_*}\,\}}\big) \\ &= E_{x,s}\big((S_\tau - cF(X_\tau))\,1_{\{\,\tau \ge \tau_{v_*}\,\}}\big) + V(v_*,v_*)P_{x,s}\big(\tau < \tau_{v_*}\big) \ge E_{x,s}\big(S_\tau - cF(X_\tau)\big) \end{aligned}$$

for all $0 \le x \le s$ with $0 \le s < v_*$, where $\tau_{v_*} = \inf\{\, t > 0 \mid X_t = v_*\,\}$. Thus $V(x,s) = V(v_*,v_*) = c^2/e(c-1)$ for all $0 \le x \le v_*$, and noting that σ'_* in this case equals σ_* from (2.2), the proof is complete. □

The result of Theorem 2.1 extends to the case when Brownian motion B starts from the points different from zero. This further improves upon the result of Gilat [6].

Corollary 2.2

Let $B = (B_t)_{t \ge 0}$ be standard Brownian motion started at zero under P. Then the following inequality is shown to be satisfied:

$$(2.27) \qquad E\Big(\max_{0 \le t \le \tau} |B_t + x|\Big) \le V(x;c) + cE\Big(|B_\tau + x| \log |B_\tau + x|\Big)$$

for all $c > 1$ and all stopping times τ for B satisfying $E(\tau^r) < \infty$ for some $r > 1/2$, where:

$$\begin{aligned} (2.28) \qquad V(x;c) &= \frac{c^2}{e(c-1)} \quad \text{if } 0 \le x \le v_* \\ &= cx \log\left(\frac{c}{x(c-1)}\right) \quad \text{if } x \ge v_* \end{aligned}$$

with $v_* = c/e(c-1)$. This inequality is sharp: for each $c > 1$ and $x \ge 0$ given and fixed, the equality in (2.27) is attained at the stopping time σ_* defined in (2.2) with $X_t = |B_t + x|$ and $S_t = \max_{0 \le r \le t} |B_r + x|$.

Proof. It follows from the proof of Theorem 2.1. Note that $V(x;c)$ equals $V(x,x)$ in the notation of this proof, so that the explicit expression for $V(x;c)$ is given in (2.25). □

In the next theorem we present the solution in the $L \log^+ L$-case. The first part of the proof is identical to the first part of the proof of Theorem 2.1, and therefore it is omitted.

Theorem 2.3

Let $B = (B_t)_{t \ge 0}$ be standard Brownian motion started at zero under P. Then the following inequality is shown to be satisfied:

$$E\Big(\max_{0\le t\le\tau} |B_t|\Big) \le 1 + \frac{1}{e^c(c-1)} + cE\Big(|B_\tau|\log^+|B_\tau|\Big) \tag{2.29}$$

for all $c > 1$ and all stopping times τ for B satisfying $E(\tau^r) < \infty$ for some $r > 1/2$. This inequality is sharp, and the equality is attained at the stopping time:

$$\tau_* = \inf\ \big\{\, t > 0 \ \big|\ S_t \ge u_* \ ,\ X_t = 1 \vee \alpha S_t \,\big\} \tag{2.30}$$

where $u_* = 1{+}1/e^c(c{-}1)$ and $\alpha = (c-1)/c$ for $c > 1$ with $X_t = |B_t|$ and $S_t = \max_{0\le r\le t}|B_r|$.

Proof. Given $c > 1$, consider the optimal stopping problem with the payoff:

$$V_+(x,s) = \sup_\tau E_{x,s}\big(S_\tau - cF_+(X_\tau)\big) \tag{2.31}$$

where $F_+(x) = x\log^+ x$, $X_t = |B_t{+}x|$ and $S_t = s \vee \max_{0\le r\le t}|B_r{+}x|$ for $0 \le x \le s$. Since $F_+(x) = F(x)$ for all $x \ge 1$, it is clear that $V_+(x,s)$ coincides (with the same optimal stopping strategy given by either (2.2) or (2.30)) with the payoff $V(x,s)$ from (2.3) for $0 \le x \le s$ with $s \ge s_*$, where s_* is determined from $g_*(s_*) = 1$ with $g_*(s) = \alpha s$ and $\alpha = (c{-}1)/c$, so that $s_* = 1/\alpha = c/(c{-}1)$. It is also clear that the process (X,S) cannot be optimally stopped at some τ with $X_\tau < 1$ since $F_+(x) = 0$ for $x < 1$. This shows that $V_+(0,0) = V_+(x,s) = V_+(1,1)$ for all $0 \le x \le s \le 1$. So it remains to compute the payoff $V_+(x,s)$ for $0 \le x \le s$ with $1 \le s < s_*$. This evaluation is the main content of the proof. We begin by giving some intuitive arguments which are followed by a rigorous justification.

The best place to stop in the region $0 \le x \le s$ with $1 \le s \le s_*$ is clearly at $(1,s)$, so that there should exist a point $1 \le u_* \le s_*$ such that the process (X,S) is to be stopped at the vertical line $\{(1,s) \mid u_* \le s \le s_*\}$, as well as on the left from it (if started there). We also expect that $V_+(u_*,u_*) = u_*$ (since we do not stop at $(1, u_*{-}\varepsilon)$ where the payoff V_+ would be equal $u_*{-}\varepsilon$ for $\varepsilon > 0$ as small as desired). Clearly, the payoff V_+ should be constant in the region $0 \le x \le s \le u_*$ (note that there is no running cost), and then (when restricted to the diagonal $x = s$ for $u_* \le s \le s_*$) it should drop down. Note from (2.25) that $V_+(s_*,s_*) = V(s_*,s_*) = 0$. So let us try to determine such a point u_*.

Thus we shall try to compute:

$$\sup_{1\le s\le s_*} E_{s,s}\big(S_{\tau_*'} - cF_+(X_{\tau_*'})\big) \tag{2.32}$$

where $\tau_*' = \tau_*'(s) = \inf\{\, t > 0 \mid X_t = 1 \vee \alpha S_t \,\}$. For this, note by the strong Markov property (and $V_+(s_*,s_*) = 0$) if τ_*' is to be an optimal stopping time (for some $s = u_*$), we are to have:

$$\begin{aligned} V_{+,*}(x,s) &:= E_{x,s}\big(S_{\tau_*'} - cF_+(X_{\tau_*'})\big) \\ &= E_{x,s}\big(S_{\tau_*'}\, 1_{\{\, \tau_*' < \tau_{s_*}\,\}}\big) + E_{x,s}\big((S_{\tau_*'} - cF_+(X_{\tau_*'}))\, 1_{\{\, \tau_*' \ge \tau_{s_*}\,\}}\big) \end{aligned} \tag{2.33}$$

$$= E_{x,s}\big(S_{\tau_*'}\, 1_{\{\, \tau_*' < \tau_{s_*}\,\}}\big) + E_{x,s}\big(E_{s_*,s_*}(S_{\tau_*'} - cF_+(X_{\tau_*'}))\, 1_{\{\, \tau_*' \geq \tau_{s_*}\,\}}\big)$$

$$= E_{x,s}\big(S_{\tau_*'}\, 1_{\{\, \tau_*' < \tau_{s_*}\,\}}\big)$$

for all $1 \leq x \leq s \leq s_*$ where $\tau_{s_*} = \inf\{\, t > 0 \mid X_t = s_*\}$. Note further that τ_*' (on $\{\, \tau_*' < \tau_{s_*}\}$) may be viewed as the exit time of (X, S) from an open set, so that $V_{+,*}(x,s)$ is to satisfy:

(2.34) $$\mathbf{L_X} V_{+,*}(x,s) = 0 \quad (\ 1 < x < s\ )$$

(2.35) $$V_{+,*}(x,s)\Big|_{x=1} = s \quad \text{(instantaneous stopping)}$$

(2.36) $$\frac{\partial V_{+,*}}{\partial s}(x,s)\Big|_{x=s-} = 0 \quad \text{(normal reflection)}$$

(2.37) $$V_{+,*}(x,s)\Big|_{x=s=s_*} = 0 \quad \text{(strong Markov property)}$$

for $1 \leq x \leq s \leq s_*$. The system (2.34)-(2.37) has a unique solution given by:

(2.38) $$V_{+,*}(x,s) = s + (1-x)\log(s-1) + K(x-1)$$

for $1 \leq x \leq s \leq s_*$ where:

(2.39) $$K = -c - \log(c-1)\ .$$

Solving $(\partial V_{+,*}/\partial s)(s,s) = 0$ we find the point at which the supremum in (2.32) is to be attained:

(2.40) $$u_* = 1 + e^K = 1 + 1/e^c(c-1)\ .$$

Thus the candidate $V_{+,*}(x,s)$ for the payoff (2.31) is to be given by the rule (2.38) (only) for all $1 \leq x \leq s$ with $u_* \leq s \leq s_*$. Clearly we have to put $V_{+,*}(x,s) = s$ for $0 \leq x \leq 1$ with $u_* \leq s \leq s_*$. Note moreover that $V_{+,*}(x,s) = V_{+,*}(u_*, u_*) = u_* = 1 + 1/e^c(c-1)$ for all $0 \leq x \leq s \leq u_*$ as suggested above. So if we show in the sequel that this candidate is indeed the payoff with the optimal stopping time $\tau_* = \tau_*'(u_*)$, the proof of the theorem will be complete.

That there should be no point of stopping in the region $0 \leq x \leq s \leq u_*$ is verified in exactly the same way as in (2.26). So let us concentrate to the case when $u_* \leq s \leq s_*$. To apply Itô formula we shall redefine $V_{+,*}(x,s)$ for $x < 1$ by the rule (2.38). This extension will be denoted by $\tilde{V}_{+,*}(x,s)$. Obviously $\tilde{V}_{+,*} \in C^2$ and $\tilde{V}_{+,*}(x,s) = V_{+,*}(x,s)$ for $1 \leq x \leq s$ with $u_* \leq s \leq s_*$. Applying now Itô formula to $\tilde{V}_{+,*}(X_t, S_t)$ and noting that $(\partial \tilde{V}_{+,*}/\partial x)(0,s) \leq 0$ for $u_* \leq s \leq s_*$ (any such C^2-extension would do) we get:

(2.41) $$\tilde{V}_{+,*}(X_t, S_t) = \tilde{V}_{+,*}(x,s) + \int_0^t \frac{\partial \tilde{V}_{+,*}}{\partial x}(X_r, S_r)\ dX_r$$

$$+ \int_0^t \frac{\partial \tilde{V}_{+,*}}{\partial s}(X_r, S_r)\ dS_r + \frac{1}{2}\int_0^t \frac{\partial^2 \tilde{V}_{+,*}}{\partial x^2}(X_r, S_r)\ d\langle X, X\rangle_r$$

$$= \tilde{V}_{+,*}(x,s) + \int_0^t \frac{\partial \tilde{V}_{+,*}}{\partial x}(X_r, S_r)\ d(\beta_r + L_r)$$

$$= \tilde{V}_{+,*}(x,s) + M_t + \int_0^t \frac{\partial \tilde{V}_{+,*}}{\partial x}(0,S_r)\; dL_r \le V_{+,*}(x,s) + M_t$$

for all $0 \le x \le s$ with $u_* \le s \le s_*$ where $M_t = \int_0^t (\partial \tilde{V}_{+,*}/\partial x)(X_r, S_r)\; d\beta_r$ is a continuous local martingale. Moreover, hence we find:

$$\tilde{V}_{+,*}(X_\tau, S_\tau) \le \tilde{V}_{+,*}(x,s) + M_\tau \tag{2.42}$$

for all stopping times τ for B with the equality if $\tau \le \tau_*$. Due (in the end) to $ey \le e^y$ it is elementary verified that $\tilde{V}_{+,*}(x,s) = V_{+,*}(x,s) \ge s - cF_+(x)$ for $1 \le x \le s$ with $u_* \le s \le s_*$ (with the equality if $x = 1$). Now given a stopping time τ for B , define τ' to be τ on $\{ X_\tau \ge 1 \}$ and τ_1 on $\{ X_\tau < 1 \}$, where $\tau_1 = \inf \{ t > 0 \mid X_t = 1 \}$. Then τ' is a new stopping time for B , and by (2.42) and the remark following it, clearly:

$$\begin{aligned} E_{x,s}\big(S_\tau - cF_+(X_\tau)\big) &= E_{x,s}\big(S_{\tau'} - cF_+(X_{\tau'})\big) \\ &\le E_{x,s}\big(V_{+,*}(X_{\tau'}, S_{\tau'})\big) \le V_{+,*}(x,s) + E_{x,s}(M_{\tau'}) = V_{+,*}(x,s) \end{aligned} \tag{2.43}$$

for all $1 \le x \le s$ with $u_* \le s \le s_*$ whenever $E_{x,s}((\tau')^r) < \infty$ for some $r > 1/2$ with the equalities if $\tau = \tau_*$ (recall that $E_{x,s}(\tau_*^r) < \infty$ for all $r < c/2$). The proof is complete. □

The result of Theorem 2.3 also extends to the case when Brownian motion B starts from the points different from zero, thus further improving upon the result of Gilat [6].

Corollary 2.4

Let $B = (B_t)_{t \ge 0}$ be standard Brownian motion started at zero under P . Then the following inequality is shown to be satisfied:

$$E\Big(\max_{0 \le t \le \tau} |B_t + x|\Big) \le V_+(x;c) + cE\Big(|B_\tau + x| \log^+ |B_\tau + x|\Big) \tag{2.44}$$

for all $c > 1$ and all stopping times τ for B satisfying $E(\tau^r) < \infty$ for some $r > 1/2$, where:

$$\begin{aligned} V_+(x;c) &= 1 + \frac{1}{e^c(c-1)} \quad \text{if } 0 \le x \le u_* \\ &= x + (1-x)\log(x-1) - (c + \log(c-1))(x-1) \quad \text{if } u_* \le x \le s_* \\ &= cx \log\left(\frac{c}{x(c-1)}\right) \quad \text{if } x \ge s_* \end{aligned} \tag{2.45}$$

with $u_* = 1 + 1/e^c(c-1)$ and $s_* = c/(c-1)$. This inequality is sharp: for each $c > 1$ and $x \ge 0$ given and fixed, the equality in (2.44) is attained at the stopping time τ_* defined in (2.30) with $X_t = |B_t + x|$ and $S_t = \max_{0 \le r \le t} |B_r + x|$.

Proof. It follows from the proof of Theorem 2.3. Note that $V_+(x;c)$ equals $V_+(x,x)$ in the notation of this proof, so that the explicit expression for $V_+(x;c)$ is given in (2.38). □

Remark 2.5

We would like to point out that the distribution law of X_{τ_*} and S_{τ_*} from Theorem 2.1 (Corollary 2.2) and Theorem 2.3 (Corollary 2.4) can be computed explicitly (see [1]). For this one should use the fact that $H(S_t)-(S_t-X_t)H'(S_t)$ is a (local) martingale before X hits zero for sufficiently many functions H . The details are left to the reader.

Due to the extreme properties of Brownian motion, the inequalities (2.27) and (2.44) extend to all non-negative submartingales. This can be obtained by using the maximal embedding result of Jacka [9]. For convenience of the reader, we state the result and outline the proof.

Corollary 2.6

Let $X=(X_t)_{t\geq 0}$ *be a non-negative cadlag (continuous on the right with left limits) uniformly integrable submartingale started at* $x\geq 0$ *under* P . *Let* X_∞ *denote the* P*-a.s. limit of* X *for* $t\to\infty$. *Then the following inequality is satisfied:*

$$E\Big(\sup_{t>0} X_t\Big) \leq W_G(x;c) + cE\big(G(X_\infty)\big) \tag{2.46}$$

for all $c>1$ *, where* $G(y)$ *is either* $y\log y$ *and in this case* $W_G(x;c)$ *is given by (2.28), or* $G(y)$ *is* $y\log^+ y$ *and in this case* $W_G(x;c)$ *is given by (2.45). This inequality is sharp.*

Proof. Given such a submartingale $X=(X_t)_{t\geq 0}$ satisfying $E(G(X_\infty))<\infty$, and a Brownian motion $B=(B_t)_{t\geq 0}$ started at $X_0=x$ under P_x , by the result of Jacka [9] we know that there exists a stopping time τ for B , such that $|B_\tau|\sim X_\infty$ and $P\{\sup_{t\geq 0} X_t\geq\lambda\}\leq P_x\{\max_{0\leq t\leq\tau}|B_\tau|\geq\lambda\}$ for all $\lambda>0$, with $(B_{t\wedge\tau})_{t\geq 0}$ being uniformly integrable. The inequality (2.46) then easily follows from Corollary 2.2 and Corollary 2.4 by using the integration by parts formula. Note that by the submartingale property of $(|B_{t\wedge\tau}|)_{t\geq 0}$ we have $\sup_{t\geq 0} E_x(G(|B_{t\wedge\tau}|)) = E_x(G(|B_\tau|))$. The proof is complete. □

REFERENCES

[1] AZEMA, J. *and* YOR, M. (1979). Une solution simple au probleme de Skorokhod. *Lecture Notes in Math.* 721, Springer-Verlag Berlin Heidelberg (90-115).

[2] COX, D. C. (1984). Some sharp martingale inequalities related to Doob's inequality. *IMS Lecture Notes Monograph Ser.* 5, USA (78-83).

[3] DOOB, J. L. (1953). *Stochastic Processes.* John Wiley & Sons, Inc. New York.

[4] DUBINS, L. E. *and* GILAT, D. (1978). On the distribution of maxima of martingales. *Proc. Amer. Math. Soc.* 68 (337-338).

[5] DUBINS, L. E. SHEPP, L. A. *and* SHIRYAEV, A. N. (1993). Optimal stopping rules and maximal inequalities for Bessel processes. *Theory Probab. Appl.* 38 (226-261).

[6] GILAT, D. (1986). The best bound in the $L\log L$ inequality of Hardy and Littlewood and its martingale counterpart. *Proc. Amer. Math. Soc.* 97 (429-436).

[7] GRAVERSEN. S. E. *and* PESKIR, G. (1995). Optimal stopping and maximal inequalities for linear diffusions. *Research Report* No. 335, *Dept. Theoret. Statist. Aarhus*, (18 pp). *J. Theoret. Probab.* 11, 1998 (259-277).

[8] HARDY, G. H. *and* LITTLEWOOD, J. E. (1930). A maximal theorem with function-theoretic applications. *Acta Math.* 54 (81-116).

[9] JACKA, S. D. (1988). Doob's inequalities revisited: A maximal H^1-embedding. *Stochastic Process. Appl.* 29 (281-290).

[10] REVUZ, D. *and* YOR, M. (1991). *Continuous Martingales and Brownian Motion*. Springer-Verlag.

[11] WANG, G. (1991). Sharp maximal inequalities for conditionally symmetric martingales and Brownian motion. *Proc. Amer. Math. Soc.* 112 (579-586).

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