Stochastics, Vol. 79, No. 3-4, 2007, (293-302)
Research Report No. 7, 2006, *Probab. Statist. Group Manchester* (11 pp)

Principle of Smooth Fit and Diffusions with Angles

Goran Peskir

We show that there exists a regular diffusion process X and a differentiable gain function G such that the value function V of the optimal stopping problem

$$V(x) = \sup_{\tau} \mathsf{E}_x G(X_\tau)$$

fails to satisfy the smooth fit condition $V'(b) = G'(b)$ at the optimal stopping point b. On the other hand, if the scale function S of X is differentiable at b, then the smooth fit condition $V'(b) = G'(b)$ holds (whenever X is regular and G is differentiable at b). We give an example showing that the latter can happen even when $d^+G/dS < d^+V/dS < d^-V/dS < d^-G/dS$ at b.

1. Introduction

The *principle of smooth fit* states that the optimal stopping point b which separates the continuation set C from the stopping set D in the optimal stopping problem

$$V(x) = \sup_{\tau} \mathsf{E}_x G(X_\tau) \tag{1.1}$$

is characterized by the fact that $V'(b)$ exists and is equal to $G'(b)$. Typically, no other point $\tilde{b}$ separating the candidate sets $\tilde{C}$ and $\tilde{D}$ will satisfy this identity, and most often $V''(b)$ will either fail to exist or will not be equal to $G''(b)$. These unique features of the smooth fit principle make it a powerful tool in solving specific problems of optimal stopping. The same is true in higher dimensions but in the present note we focus on one dimension only.

Regular diffusion processes form a natural class of Markov processes X in (1.1) for which the smooth-fit principle is known to hold in great generality. A number of authors have contributed to understanding of the smooth-fit principle by various means. With no aim to review the full history of these developments we refer to [2]-[14] and [16]-[18] (see also [1] for Lévy processes). These studies contain further references which are useful to consult.

It is easy to construct examples which show that the smooth fit $V'(b) = G'(b)$ can fail if the diffusion process X is not regular (as well as that V need not be differentiable at b if G

Mathematics Subject Classification 2000. Primary 60G40. Secondary 60J60.

Key words and phrases: Optimal stopping, principle of smooth fit, regular diffusion process, scale function, superharmonic function, concave function, local time, stochastic differential equation with reflection.

is not so). Thus *regularity* of the diffusion process X and *differentiability* of the gain function G are minimal conditions under which the smooth fit can hold in greater generality. In this note we address the question of their *sufficiency*.

Our main findings can be summarized as follows. Firstly, we show that there exists a regular diffusion process X and a differentiable gain function G such that the smooth fit condition $V'(b) = G'(b)$ fails to hold at the optimal stopping point b (Example 3.1). Secondly, we show that the latter cannot happen if the scale function S is differentiable at b. In other words, if X is regular and both G and S are differentiable at b, then V is differentiable at b and $V'(b) = G'(b)$ (Theorem 2.3). Thirdly, we give an example showing that the latter can happen even when $d^+G/dS < d^+V/dS < d^-V/dS < d^-G/dS$ at b (Example 2.2). The relevance of this fact will be reviewed in the next section.

Kolmogorov expressed the view that the principle of smooth fit holds because "diffusions do not like angles" (this is one of the famous Shiryaev's tales). It hinges that there must be something special about the diffusion process X in the first example above since the gain function G is differentiable. We will return to this point in the final section below.

2. Smooth fit

1. Let $X = (X_t)_{t \geq 0}$ be a diffusion process with values in an interval J of $I\!R$. For simplicity we will assume that X can be killed only at the end-points of J which do not belong to J. Thus, if ζ denotes the death time of X, then X is a strong Markov process such that $t \mapsto X_t$ is continuous on $[0, \zeta)$, and the end-points of J at which X can be killed act as absorbing boundaries (once such a point is reached X stays there forever). We will denote by $I = (l, r)$ the interior of J.

Given $c \in J$ we will let

$$\tau_c = \inf \{\, t > 0 \mid X_t = c \,\} \tag{2.1}$$

denote the hitting time of X to c. We will assume that X is *regular* in the sense that $\mathsf{P}_b(\tau_c < \infty) > 0$ for every $b \in I$ and all $c \in J$. It means that I cannot be decomposed into smaller intervals from which X could not exit. It also means that b is regular for both $D_1 = (l, b]$ and $D_2 = [b, r)$ in the sense that $\mathsf{P}_b(\tau_{D_i} = 0) = 1$ where $\tau_{D_i} = \inf \{\, t > 0 \mid X_t \in D_i \,\}$ for $i = 1, 2$. In particular, each $b \in I$ is regular for itself in the sense that $\mathsf{P}_b(\tau_b = 0) = 1$.

Let S denote the scale function of X. Recall that $S : J \to I\!R$ is a strictly increasing continuous function such that

$$\mathsf{P}_x(\tau_a < \tau_b) = \frac{S(b) - S(x)}{S(b) - S(a)} \quad \& \quad \mathsf{P}_x(\tau_b < \tau_a) = \frac{S(x) - S(a)}{S(b) - S(a)} \tag{2.2}$$

for $a < x < b$ in J. Recall also that the scale function can be characterized (up to an affine transformation) as a continuous function $S : J \to I\!R$ such that $(S(X_{t \wedge \tau_l \wedge \tau_r}))_{t \geq 0}$ is a continuous local martingale.

2. Let $G : J \to I\!R$ be a measurable function satisfying

$$\mathsf{E} \sup_{0 \leq t < \zeta} |G(X_t)| < \infty. \tag{2.3}$$

Consider the optimal stopping problem

$$V(x) = \sup_{\tau} \mathsf{E}_x G(X_\tau) \tag{2.4}$$

for $x \in J$ where the supremum is taken over all stopping times τ of X (i.e. with respect to the natural filtration $\mathcal{F}_t^X = \sigma(X_s \mid 0 \le s \le t)$ generated by X for $t \ge 0$).

Following the argument of Dynkin and Yushkevich [5, p. 115], take $c < x < d$ in J and choose stopping times τ_1 and τ_2 such that $\mathsf{E}_c G(X_{\tau_1}) \ge V(c) - \varepsilon$ and $\mathsf{E}_d G(X_{\tau_2}) \ge V(d) - \varepsilon$ where $\varepsilon > 0$ is given and fixed. Consider the stopping time $\tau_\varepsilon = (\tau_c + \tau_1 \circ \theta_{\tau_c})\, I(\tau_c < \tau_d) + (\tau_d + \tau_2 \circ \theta_{\tau_d})\, I(\tau_d < \tau_c)$ obtained by applying τ_1 after hitting c (before d) and τ_2 after hitting d (before c). By the strong Markov property of X it follows that

$$\begin{aligned} V(x) &\ge \mathsf{E}_x G(X_{\tau_\varepsilon}) = \mathsf{E}_x G(X_{\tau_c + \tau_1 \circ \theta_{\tau_c}})\, I(\tau_c < \tau_d) + \mathsf{E}_x G(X_{\tau_d + \tau_2 \circ \theta_{\tau_d}})\, I(\tau_d < \tau_c) \\ &= \mathsf{E}_x G(X_{\tau_1}) \circ \theta_{\tau_c}\, I(\tau_c < \tau_d) + \mathsf{E}_x G(X_{\tau_2}) \circ \theta_{\tau_d}\, I(\tau_d < \tau_c) \\ &= \mathsf{E}_x\, \mathsf{E}_{X_{\tau_c}}(G(X_{\tau_1}))\, I(\tau_c < \tau_d) + \mathsf{E}_x\, \mathsf{E}_{X_{\tau_d}}(G(X_{\tau_2}))\, I(\tau_d < \tau_c) \\ &= \mathsf{E}_c(G(X_{\tau_1}))\, \mathsf{P}_x(\tau_c < \tau_d) + \mathsf{E}_d(G(X_{\tau_2}))\, \mathsf{P}_x(\tau_d < \tau_c) \\ &\ge (V(c) - \varepsilon)\, \frac{S(d) - S(x)}{S(d) - S(c)} + (V(d) - \varepsilon)\, \frac{S(x) - S(c)}{S(d) - S(c)} \\ &= V(c)\, \frac{S(d) - S(x)}{S(d) - S(c)} + V(d)\, \frac{S(x) - S(c)}{S(d) - S(c)} - \varepsilon \end{aligned} \tag{2.5}$$

where the first inequality follows by definition of V and the second inequality follows by the choice of τ_1 and τ_2 . Letting $\varepsilon \downarrow 0$ in (2.5) one concludes that

$$V(x) \ge V(c)\, \frac{S(d) - S(x)}{S(d) - S(c)} + V(d)\, \frac{S(x) - S(c)}{S(d) - S(c)} \tag{2.6}$$

for $c < x < d$ in J . This means that V is S-concave (see e.g. [15, p. 546]).

3. In exactly the same way as for concave functions (corresponding to $S(x) = x$ above) one then sees that (2.6) implies that

$$y \mapsto \frac{V(y) - V(x)}{S(y) - S(x)} \quad \text{is decreasing} \tag{2.7}$$

on $J \setminus \{x\}$ for every $x \in I$. It follows that

$$-\infty < \frac{d^+ V}{dS}(x) \le \frac{d^- V}{dS}(x) < +\infty \tag{2.8}$$

for every $x \in I$. It also follows that V is continuous on I since S is continuous.

4. Let us now assume that $b \in I$ is an optimal stopping point in the problem (2.4). Then $V(b) = G(b)$ and hence by (2.7) we get

$$\frac{G(b+\varepsilon) - G(b)}{S(b+\varepsilon) - S(b)} \le \frac{V(b+\varepsilon) - V(b)}{S(b+\varepsilon) - S(b)} \le \frac{V(b-\delta) - V(b)}{S(b-\delta) - S(b)} \le \frac{G(b-\delta) - G(b)}{S(b-\delta) - S(b)} \tag{2.9}$$

for $\varepsilon > 0$ and $\delta > 0$ where the first inequality follows since $G(b+\varepsilon) \le V(b+\varepsilon)$ and the third inequality follows since $-V(b-\delta) \le -G(b-\delta)$ (recalling also that S is strictly increasing). Passing to the limit for $\varepsilon \downarrow 0$ and $\delta \downarrow 0$ this immediately leads to

$$\frac{d^+G}{dS}(b) \le \frac{d^+V}{dS}(b) \le \frac{d^-V}{dS}(b) \le \frac{d^-G}{dS}(b) \tag{2.10}$$

whenever d^+G/dS and d^-G/dS exist at b. In this way we have reached the essential part of Salminen's result [16, p. 96]:

Theorem 2.1 (Smooth fit through scale)
If dG/dS exists at b then dV/dS exists at b and

$$\frac{dV}{dS}(b) = \frac{dG}{dS}(b) \tag{2.11}$$

whenever $V(b) = G(b)$ for $b \in I$.

In particular, if X is on natural scale (i.e. $S(x) = x$) then the smooth fit condition

$$\frac{dV}{dx}(b) = \frac{dG}{dx}(b) \tag{2.12}$$

holds at the optimal stopping point b as soon as G is differentiable at b.

The following example shows that equalities in (2.10) and (2.11) may fail to hold even though the smooth fit condition (2.12) holds.

Example 2.2
Let $X_t = F(B_t)$ where

$$F(x) = \begin{cases} x^{1/3} & \text{if} \quad x \in [0,1] \\ -|x|^{1/3} & \text{if} \quad x \in [-1,0) \end{cases} \tag{2.13}$$

and B is a standard Brownian motion in $(-1,1)$ absorbed (killed) at either -1 or 1. Since F is a strictly increasing and continuous function from $[-1,1]$ onto $[-1,1]$, it follows that X is a regular diffusion process in $(-1,1)$ absorbed (killed) at either -1 or 1.

Consider the optimal stopping problem (2.4) with

$$G(x) = 1 - x^2 \tag{2.14}$$

for $x \in (-1,1)$. Set $X_t^x = F(x+B_t)$ for $x \in (-1,1)$ and let B be defined on $(\Omega, \mathcal{F}, \mathsf{P})$ so that $B_0 = 0$ under P. Since F is increasing (and continuous) it can be verified that

$$\text{Law}(X^x \,|\, \mathsf{P}) = \text{Law}(C \,|\, \mathsf{P}_{F(x)}) \tag{2.15}$$

where $C_t(\omega) = \omega(t)$ is the coordinate process (on a canonical space) that is Markov under the family of probability measures P_c for $c \in (-1,1)$ with $\mathsf{P}_c(C_0 = c) = 1$ (note that each $c \in (-1,1)$ corresponds to $F(x)$ for some $x \in (-1,1)$ given and fixed).

In view of (2.15) let us consider the auxiliary optimal stopping problem

$$\tilde{V}(x) = \sup_{\tau} \mathsf{E}\, \tilde{G}(x+B_\tau) \tag{2.16}$$

where $\tilde{G} = G \circ F$ and the supremum is taken over all stopping times τ of B (up to the time of absorption at -1 or 1). Note that

$$\tilde{G}(x) = 1 - |x|^{2/3} \tag{2.17}$$

for $x \in (-1,1)$. Since $\tilde{V}$ is the smallest superharmonic (i.e. concave) function that dominates $\tilde{G}$ (see [5, pp. 112-126]), and clearly $\tilde{V}(-1) = \tilde{V}(1) = 0$, it follows that

$$\tilde{V}(x) = \begin{cases} 1-x & \text{if } \; x \in [0,1] \\ 1+x & \text{if } \; x \in [-1,0). \end{cases} \tag{2.18}$$

From (2.15) we see that $V(x) = \tilde{V}(F^{-1}(x))$ and since $F^{-1}(x) = x^3$, it follows that

$$V(x) = \begin{cases} 1-x^3 & \text{if } \; x \in [0,1] \\ 1+x^3 & \text{if } \; x \in [-1,0). \end{cases} \tag{2.19}$$

Comparing (2.19) with (2.14) we see that $b=0$ is an optimal stopping point. Moreover, it is evident that the smooth fit (2.12) holds at $b=0$, both derivatives being zero. However, noting that the scale function of X equals $S(x) = x^3$ for $x \in [-1,1]$ (since $S(X) = F^{-1}(F(B)) = B$ is a martingale), it is straightforwardly verified from (2.14) and (2.19) that

$$\frac{d^+G}{dS} = -\infty < \frac{d^+V}{dS} = -1 < \frac{d^-V}{dS} = 1 < \frac{d^-G}{dS} = +\infty \tag{2.20}$$

at the optimal stopping point $b=0$.

5. Note that the scale function S in the preceding example is differentiable at the optimal stopping point b but that $S'(b) = 0$. This motivates the following extension of Theorem 2.1 above.

Theorem 2.3 (Smooth fit)
If both dG/dx and dS/dx exist at b then dV/dx exists at b and

$$\frac{dV}{dx}(b) = \frac{dG}{dx}(b) \tag{2.21}$$

whenever $V(b) = G(b)$ for $b \in I$.

Proof. Assume first that $S'(b) \neq 0$. Multiplying by $(S(b+\varepsilon)-S(b))/\varepsilon$ in (2.9) we get

$$\frac{G(b+\varepsilon)-G(b)}{\varepsilon} \le \frac{V(b+\varepsilon)-V(b)}{\varepsilon} \le \frac{S(b+\varepsilon)-S(b)}{\varepsilon} \, \frac{(G(b-\delta)-G(b))/(-\delta)}{(S(b-\delta)-S(b))/(-\delta)} \, . \tag{2.22}$$

Passing to the limit for $\varepsilon \downarrow 0$ and $\delta \downarrow 0$, and using that $S'(b) \neq 0$, it follows that $d^+V/dx = dG/dx$ at b . (Note that one could take $\varepsilon = \delta$ in this argument.)

Similarly, multiplying by $(S(b-\delta)-S(b))/(-\delta)$ in (2.9) we get

$$\frac{(G(b+\varepsilon)-G(b))/\varepsilon}{(S(b+\varepsilon)-S(b))/\varepsilon}\,\frac{S(b-\delta)-S(b)}{-\delta} \le \frac{V(b-\delta)-V(b)}{-\delta} \le \frac{G(b-\delta)-G(b)}{-\delta}\;. \tag{2.23}$$

Passing to the limit for $\varepsilon \downarrow 0$ and $\delta \downarrow 0$, and using that $S'(b) \neq 0$, it follows that $d^-V/dx = dG/dx$ at b. (Note that one could take $\varepsilon = \delta$ in this argument.) Combining the two conclusions we see that dV/dx exists at b and (2.21) holds as claimed.

To treat the case $S'(b) = 0$ we need the following simple facts of real analysis.

Lemma 2.4
Let $f : I\!R_+ \rightarrow I\!R$ and $g : I\!R_+ \rightarrow I\!R$ be two continuous functions satisfying:

$$f(0) = 0 \;\; and \;\; f(\varepsilon) > 0 \;\; for \;\; \varepsilon > 0\,; \tag{2.24}$$

$$g(0) = 0 \;\; and \;\; g(\delta) > 0 \;\; for \;\; \delta > 0\,. \tag{2.25}$$

Then for every $\varepsilon_n \downarrow 0$ as $n \rightarrow \infty$ there are $\varepsilon_{n_k} \downarrow 0$ and $\delta_k \downarrow 0$ as $k \rightarrow \infty$ such that $f(\varepsilon_{n_k}) = g(\delta_k)$ for all $k \geq 1$. In particular, it follows that

$$\lim_{k\rightarrow\infty} \frac{f(\varepsilon_{n_k})}{g(\delta_k)} = 1\;. \tag{2.26}$$

Proof. Take any $\varepsilon_n \downarrow 0$ as $n \rightarrow \infty$. Since $f(\varepsilon_n) \rightarrow 0$ and $f(\varepsilon_n) > 0$ we can find a subsequence $\varepsilon_{n_k} \downarrow 0$ such that $x_{n_k} := f(\varepsilon_{n_k}) \downarrow 0$ as $k \rightarrow \infty$. Since $g(1) > 0$ there is no restriction to assume that $x_{n_1} < g(1)$. But then by continuity of g and the fact that $x_{n_1} \in (g(0), g(1))$ there must be $\delta_1 \in (0,1)$ such that $g(\delta_1) = x_{n_1}$. Since $x_{n_2} < x_{n_1}$ it follows that $x_{n_2} \in (g(0), g(\delta_1))$ and again by continuity of g there must be $\delta_2 \in (0, \delta_1)$ such that $g(\delta_2) = x_{n_2}$. Continuing likewise by induction we obtain a decreasing sequence $\delta_k \in (0,1)$ such that $g(\delta_k) = x_{n_k}$ for $k \geq 1$. Denoting $\delta = \lim_{k\rightarrow\infty} \delta_k$ we see that $g(\delta) = \lim_{k\rightarrow\infty} g(\delta_k) = \lim_{k\rightarrow\infty} x_{n_k} = 0$. Hence δ must be 0 by (2.25). This completes the proof of Lemma 2.4. □

Let us continue the proof of Theorem 2.3 in the case when $S'(b) = 0$. Take $\varepsilon_n \downarrow 0$ and by Lemma 2.4 choose $\delta_k \downarrow 0$ such that (2.26) holds with $f(\varepsilon) = (S(b+\varepsilon)-S(b))/\varepsilon$ and $g(\delta) = (S(b)-S(b-\delta))/\delta$. Then (2.22) reads

$$\frac{G(b+\varepsilon_{n_k})-G(b)}{\varepsilon_{n_k}} \le \frac{V(b+\varepsilon_{n_k})-V(b)}{\varepsilon_{n_k}} \le \frac{f(\varepsilon_{n_k})}{g(\delta_k)}\,\frac{G(b-\delta_k)-G(b)}{-\delta_k} \tag{2.27}$$

for all $k \geq 1$. Letting $k \rightarrow \infty$ and using (2.26) we see that $(V(b+\varepsilon_{n_k})-V(b))/\varepsilon_{n_k} \rightarrow G'(b)$. Since this is true for any $\varepsilon_n \downarrow 0$ it follows that d^+V/dx exists and is equal to dG/dx at b.

Similarly, take $\varepsilon_n \downarrow 0$ and by Lemma 2.4 choose $\delta_k \downarrow 0$ such that (2.26) holds with $f(\varepsilon) = (S(b)-S(b-\varepsilon))/\varepsilon$ and $g(\delta) = (S(b+\delta)-S(b))/\delta$. Then (2.23) (with ε and δ swaped) reads

$$\frac{G(b+\delta_k)-G(b)}{\delta_k}\,\frac{f(\varepsilon_{n_k})}{g(\delta_k)} \le \frac{V(b-\varepsilon_{n_k})-V(b)}{-\varepsilon_{n_k}} \le \frac{G(b-\varepsilon_{n_k})-G(b)}{-\varepsilon_{n_k}} \tag{2.28}$$

for all $k \geq 1$. Letting $k \to \infty$ and using (2.26) we see that $(V(b-\varepsilon_{n_k})-V(b))/(-\varepsilon_{n_k}) \to G'(b)$. Since this is true for any $\varepsilon_n \downarrow 0$ it follows that d^-V/dx exists and is equal to dG/dx at b . Taken together with the previous conclusion on d^+V/dx this establishes (2.21) and the proof of Theorem 2.3 is complete. $\square$

3. Diffusions with angles

The question arising naturally from the previous considerations is whether differentiability of the gain function G and regularity of the diffusion process X imply the smooth fit $V'(b) = G'(b)$ at the optimal stopping point b .

1. The negative answer to this question is provided by the following example.

Example 3.1

Let $X_t = F(B_t)$ where

$$F(x) = \begin{cases} \sqrt{x} & \text{if} \quad x \in [0,1] \\ -x^2 & \text{if} \quad x \in [-1,0) \end{cases} \tag{3.1}$$

and B is a standard Brownian motion in $(-1,1)$ absorbed (killed) at either -1 or 1 . Since F is a strictly increasing and continuous function from $[-1,1]$ onto $[-1,1]$, it follows that X is a regular diffusion process in $(-1,1)$ absorbed (killed) at either -1 or 1 .

Consider the optimal stopping problem (2.4) with

$$G(x) = 1 - x \tag{3.2}$$

for $x \in (-1,1)$. Set $X_t^x = F(x+B_t)$ for $x \in (-1,1)$ and let B be defined on $(\Omega, \mathcal{F}, \mathsf{P})$ so that $B_0 = 0$ under P . Since F is increasing (and continuous) it follows that

$$\text{Law}(X^x |\, \mathsf{P}) = \text{Law}(C \,|\, \mathsf{P}_{F(x)}) \tag{3.3}$$

where $C_t(\omega) = \omega(t)$ is the coordinate process (on a canonical space) that is Markov under the family of probability measures P_c for $c \in (-1,1)$ with $\mathsf{P}_c(C_0 = c) = 1$ (note that each $c \in (-1,1)$ corresponds to $F(x)$ for some $x \in (-1,1)$ given and fixed).

In view of (3.3) let us consider the auxiliary optimal stopping problem

$$\tilde{V}(x) = \sup_\tau \mathsf{E}\, \tilde{G}(x+B_\tau) \tag{3.4}$$

where $\tilde{G} = G \circ F$ and the supremum is taken over all stopping times τ of B (up to the time of absorption at -1 or 1). Note that

$$\tilde{G}(x) = \begin{cases} 1 - \sqrt{x} & \text{if} \quad x \in [0,1] \\ 1 + x^2 & \text{if} \quad x \in [-1,0)\, . \end{cases} \tag{3.5}$$

Since $\tilde{V}$ is the smallest superharmonic (i.e. concave) function that dominates $\tilde{G}$ (see [5, pp. 112-126]), and clearly $\tilde{V}(-1) = 2$ and $\tilde{V}(1) = 0$, it follows that

$$\tilde{V}(x) = 1 - x \tag{3.6}$$

for $x \in [-1,1]$. From (3.3) we see that $V(x) = \tilde{V}(F^{-1}(x))$, and since $F^{-1}(x) = x^2$ for $x \in [0,1]$ and $F^{-1}(x) = -\sqrt{|x|}$ for $x \in [-1,0)$, it follows that

$$V(x) = \begin{cases} 1 - x^2 & \text{if} \quad x \in [0,1] \\ 1 + \sqrt{|x|} & \text{if} \quad x \in [-1,0)\ . \end{cases} \tag{3.7}$$

Comparing (3.7) with (3.2) we see that $b = 0$ is an optimal stopping point. However, it is evident that the smooth fit $V'(b) = G'(b)$ fails at $b = 0$ (see Figure 1).

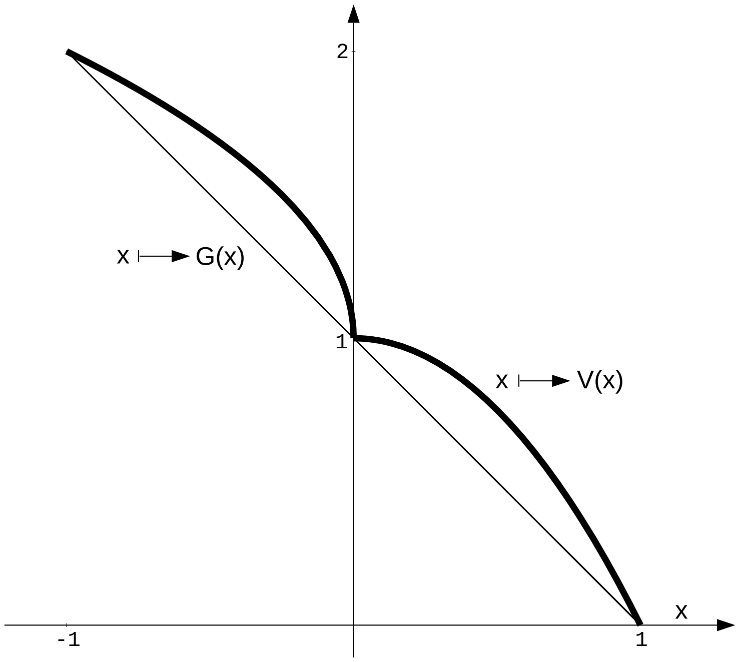

Figure 1. The gain function G and the value function V from Example 3.1. The smooth fit $V'(b) = G'(b)$ fails at the optimal stopping point $b = 0$.

2. Note that the scale function S of X equals F^{-1} given prior to (3.7) above (since $S(X) = F^{-1}(F(B)) = B$ is a martingale) so that $S'_+(0) = 0$ and $S'_-(0) = +\infty$. Note also from (3.1) above that X receives a "strong" push toward $(0,1]$ and a "mild" push toward $[-1,0)$ when at 0 . The two extreme cases of $S'_+(0)$ and $S'_-(0)$ are not the only possible ones to ruin the smooth fit. Indeed, if we slightly modify F in (3.1) above by setting $F(x) = \sqrt{x}$ for $x \in [0,1]$ and $F(x) = x$ for $x \in [-1,0)$, then the same analysis shows that $V(x) = 1 - x^2$ for $x \in [0,1]$ and $V(x) = 1 - x$ for $x \in [-1,0)$, so that the smooth fit $V'(b) = G'(b)$ still fails at the optimal stopping point $b = 0$. In this case the scale function S of X is given by $S(x) = x^2$ for $x \in [0,1]$ and $S(x) = x$ for $x \in [-1,0)$, so that $S'_+(0) = 0$ and $S'_-(0) = 1$.

Moreover, any further speculation that the extreme condition $S'_+(0) = 0$ is needed to ruin the smooth fit is ruled out if we modify F in (3.1) by setting $F(x) = (-1 + \sqrt{1+8x})/2$ for $x \in [0,1]$ and $F(x) = x$ for $x \in [-1,0)$. Then the same analysis shows that $V(x) = 1 - (x^2 + x)/2$

for $x \in [0,1]$ and $V(x) = 1-x$ for $x \in [-1,0)$, so that the smooth fit $V'(b) = G'(b)$ still fails at the optimal stopping point $b = 0$. In this case the scale function S of X is given by $S(x) = (x^2+x)/2$ for $x \in [0,1]$ and $S(x) = x$ for $x \in [-1,0)$, so that $S'_+(0) = 1/2$ and $S'_-(0) = 1$.

3. In order to examine what is "angular" about the diffusion from Example 3.1, let us recall that (2.2) implies that

$$\mathsf{P}_b(\tau_{b-\varepsilon} < \tau_{b+\varepsilon}) = \frac{S(b+\varepsilon)-S(b)}{S(b+\varepsilon)-S(b-\varepsilon)} = \frac{(S(b+\varepsilon)-S(b))/\varepsilon}{(S(b+\varepsilon)-S(b))/\varepsilon + (S(b)-S(b-\varepsilon))/\varepsilon} \longrightarrow \frac{R}{R+L} \tag{3.8}$$

as $\varepsilon \downarrow 0$ whenever $S'_+(b) =: R$ and $S'_-(b) =: L$ exist (and are assumed to be different from zero for simplicity). Likewise, one finds that

$$\mathsf{P}_b(\tau_{b+\varepsilon} < \tau_{b-\varepsilon}) = \frac{S(b)-S(b-\varepsilon)}{S(b+\varepsilon)-S(b-\varepsilon)} = \frac{(S(b)-S(b-\varepsilon))/\varepsilon}{(S(b+\varepsilon)-S(b))/\varepsilon + (S(b)-S(b-\varepsilon))/\varepsilon} \longrightarrow \frac{L}{R+L} \tag{3.9}$$

as $\varepsilon \downarrow 0$ whenever $S'_-(b) =: L$ and $S'_+(b) =: R$ exist (and are assumed to be different from zero for simplicity).

If S is differentiable at b then $R = L$ so that the limit probabilities in (3.8) and (3.9) are equal to $1/2$. Note that these probabilities correspond to X exiting b infinitesimally to either left or right respectively. On the other hand, if S is not differentiable at b , then the two limit probabilities $R/(R+L)$ and $L/(R+L)$ are different and this fact alone may ruin the smooth fit at b as Example 3.1 shows. Thus, regularity of X itself is insufficient for the smooth fit to hold generally, and X requires this sort of "tuned regularity" instead (recall Theorem 2.3 above).

4. Another way of looking at such diffusions is obtained by means of stochastic calculus. The Itô-Tanaka formula implies that the process $X_t = F(B_t)$ solves the integral equation

$$X_t = X_0 + \int_0^t F' \circ F^{-1}(X_s)\, I(X_s \neq 0)\, dB_s + \int_0^t \frac{1}{2} F'' \circ F^{-1}(X_s)\, I(X_s \neq 0)\, ds + \frac{1}{2}\big[F'_+(0) - F'_-(0)\big]\, \ell_t^0(B) \tag{3.10}$$

where $\ell_t^0(B)$ is the local time of B at 0 . Setting

$$A_t = \big[F'_+(0) - F'_-(0)\big]\, \ell_t^0(B) \tag{3.11}$$

we see that (3.10) reads

$$dX_t = \rho(X_t)\, dt + \sigma(X_t)\, dB_t + dA_t \tag{3.12}$$

where $(A_t)_{t\ge 0}$ is continuous, increasing (or decreasing), adapted to $(\mathcal{F}_t^B)_{t\ge 0}$ and satisfies

$$\int_0^t I(X_s \neq 0)\, dA_s = 0 \tag{3.13}$$

with $A_0 = 0$. These conditions usually bear the name of an *SDE with reflection* for (3.12). Note however that X is not necessarily non-negative as additionally required from solutions of SDEs with reflection.

References

[1] Alili, L. *and* Kyprianou, A. E. (2005). Some remarks on first passage of Lévy processes, the American put and pasting principles. *Ann. Appl. Probab.* 15 (2062–2080).

[2] Alvarez, L. H. R. (2001). Reward functionals, salvage values, and optimal stopping. *Math. Methods Oper. Res.* 54 (315–337).

[3] Bather, J. (1970). Optimal stopping problems for Brownian motion. *Adv. in Appl. Probab.* 2 (259–286).

[4] Dayanik, S. and Karatzas, I. (2003). On the optimal stopping problem for one-dimensional diffusions. *Stochastic Process. Appl.* 107 (173–212).

[5] Dynkin, E. B. and Yushkevich, A. A. (1969). *Markov processes: Theorems and Problems.* Plenum Press.

[6] Brekke, K. A. *and* Øksendal, B. (1991). The high contact principle as a sufficiency condition for optimal stopping. *Stochastic Models and Option Values* (Loen, 1989), *Contrib. Econom. Anal.* 200, North-Holland (187–208).

[7] Chernoff, H. (1968). Optimal stochastic control. *Sankhyā Ser. A* 30 (221–252).

[8] Grigelionis, B. I. *and* Shiryaev, A. N. (1966). On Stefan's problem and optimal stopping rules for Markov processes. *Theory Probab. Appl.* 11 (541–558).

[9] Jacka, S. D. (1993). Local times, optimal stopping and semimartingales. *Ann. Probab.* 21 (329–339).

[10] McKean, H. P. Jr. (1965). Appendix: A free boundary problem for the heat equation arising from a problem of mathematical economics. *Ind. Management Rev.* 6 (32–39).

[11] Mikhalevich, V. S. (1958). A Bayes test of two hypotheses concerning the mean of a normal process. (Ukrainian) *Vīsn. Kiïv. Unīv.* No. 1 (254–264).

[12] Øksendal, B. (1990). The high contact principle in optimal stopping and stochastic waves. *Seminar on Stochastic Processes* (San Diego, CA, 1989). *Progr. Probab.* 18, Birkhäuser (177–192).

[13] Øksendal, B. *and* Reikvam, K. (1998). Viscosity solutions of optimal stopping problems. *Stochastics Stochastics Rep.* 62 (285–301).

[14] Peskir, G. *and* Shiryaev, A. N. (2006). *Optimal Stopping and Free-Boundary Problems.* Lectures in Mathematics, ETH Zürich, Birkhäuser.

[15] Revuz, D. *and* Yor, M. (1999). *Continuous Martingales and Brownian Motion.* Springer.

[16] Salminen, P. (1985). Optimal stopping of one-dimensional diffusions. *Math. Nachr.* 124 (85–101).

[17] Shiryaev, A. N. (1978). *Optimal Stopping Rules.* Springer.

[18] van Moerbeke, P. (1976). On optimal stopping and free boundary problems. *Arch. Ration. Mech. Anal.* 60 (101–148).

Goran Peskir
School of Mathematics
The University of Manchester
Sackville Street
Manchester M60 1QD
United Kingdom
www.maths.man.ac.uk/goran
goran@maths.man.ac.uk