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Extremal Problems in the Maximal Inequalities of Khintchine

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The problem is raised of finding the best possible constant in the maximal Khintchine inequality for Rademacher sequence $\varepsilon = (\varepsilon_k)_{k\geq 1}$:

$$\left(E\left(\max_{1\leq k\leq n} \left| \sum_{i=1}^{k} a_i\varepsilon_i \right|^p \right) \right)^{1/p} \leq \mathbf{B}_p^* \left(\sum_{k=1}^{n} |a_k|^2 \right)^{1/2}$$

being valid for all $a_1, \ldots, a_n \in \mathbf{R}$ with $n \geq 1$, where $0 < p < \infty$ is given and fixed. We conjecture that the best possible constant is:

$$\mathbf{B}_p^* = \left(E\left(\max_{0\leq t\leq 1} |B_t|^p \right) \right)^{1/p}$$

where $B = (B_t)_{t\geq 0}$ is standard Brownian motion. For simplicity, we consider only the case $p = 1$ and prove that this conjecture is as close to the truth as desired in the following asymptotic sense:

$$E\left(\max_{1\leq k\leq n} |S_k| \right) \leq \left(\sqrt{\frac{\pi}{2}} + \left(\sqrt{\pi}+2\right) \frac{\sqrt{\log \|\vec{a}_n\|_2}}{\|\vec{a}_n\|_2} \right) \|\vec{a}_n\|_2$$

being valid for all $|a_1| \leq 1, \ldots, |a_n| \leq 1$ and all $n \geq 1$, where $S_k = \sum_{i=1}^{k} a_i\varepsilon_i$ and $\|\vec{a}_n\|_2 = (\sum_{k=1}^{n} |a_k|^2)^{1/2} \geq 2$. It should be noted here that:

$$E\left(\max_{0\leq t\leq 1} |B_t| \right) = \sqrt{\frac{\pi}{2}} \ .$$

The method of proof relies upon Skorohod's imbedding. Motivated by consequences of this result we deduce in a purely computational way that:

$$E\left(\max_{1\leq k\leq n} \left| \sum_{i=1}^{k} a_i\varepsilon_i \right| \right) \leq \frac{2}{\sqrt{3}} \left(\sum_{k=1}^{n} |a_k|^2 \right)^{1/2}$$

whenever $a_1 = 1$, $a_2 = \lambda$, $a_3 = \lambda^2$, ... , $a_n = \lambda^{n-1}$ and λ belongs to $]0, 1/2]$ with $n \geq 1$. The constant $2/\sqrt{3}$ is shown to be the best possible in this inequality.

1. Description of the problem

Let $\varepsilon = (\varepsilon_k)_{k\geq 1}$ be a Rademacher sequence (ε_k's are independent random variables taking values ± 1 with probability $1/2$) defined on the probability space $(\Omega, \mathcal{F}, P)$, and let $a = (a_k)_{k\geq 1}$ be a sequence of real numbers. Then the *Khintchine inequalities* [4] are formulated as follows:

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$$ (1.1) \qquad \mathbf{A}_p \left(\sum_{k=1}^{n} |a_k|^2 \right)^{1/2} \le \left(E \left| \sum_{k=1}^{n} a_k \varepsilon_k \right|^p \right)^{1/p} \le \mathbf{B}_p \left(\sum_{k=1}^{n} |a_k|^2 \right)^{1/2} $$

where $\mathbf{A}_p$ and $\mathbf{B}_p$ are universal constants, while $0 < p < \infty$ and $n \ge 1$ are arbitrary.

In many respects the inequality (1.1) is fundamental. It has played an important role in building up the chain of best known inequalities in modern probability theory, starting with the discrete (time) case and finishing up with the continuous one (Paley, Marcinkiewicz-Zygmund, Burkholder-Davis-Gundy, Rosenthal, etc). For more details see [7].

An intriguing question regarding (1.1) is how to determine the best possible values for the constants $\mathbf{A}_p$ and $\mathbf{B}_p$ in the case when they don't depend on the given $n \ge 1$ (see [3] and [5]), as well as in the case when they do (see [11]). This question has a long history and for an up-to-date information in this direction we shall refer to [7].

In order to state the main problem in this paper recall that Lévy's inequality states:

$$ (1.2) \qquad P\left\{ \max_{1 \le k \le n} |S_k| > t \right\} \le 2\, P\{\, |S_n| > t \,\} $$

for all $t > 0$, where $S_k = \sum_{i=1}^{k} \xi_i$ for $1 \le k \le n$ with $\xi_1 , \dots , \xi_n$ being independent and symmetrically distributed for $n \ge 1$. By (1.1) and (1.2) integration by parts clearly yields the *maximal inequalities of Khintchine*:

$$ (1.3) \qquad \mathbf{A}_p^* \left(\sum_{k=1}^{n} |a_k|^2 \right)^{1/2} \le \left(E \left(\max_{1 \le k \le n} \left| \sum_{i=1}^{k} a_i \varepsilon_i \right|^p \right) \right)^{1/p} \le \mathbf{B}_p^* \left(\sum_{k=1}^{n} |a_k|^2 \right)^{1/2} $$

for all $a_1, \dots, a_n \in \mathbf{R}$ and all $n \ge 1$, where $\mathbf{A}_p^*$ and $\mathbf{B}_p^*$ are universal constants. In fact, we immediately obtain (1.3) with $\mathbf{A}_p^* = \mathbf{A}_p$ and $\mathbf{B}_p^* = 2^{1/p}\, \mathbf{B}_p$, where $\mathbf{A}_p$ and $\mathbf{B}_p$ are the constants appearing in (1.1).

The main problem we want to formulate in this paper (see also [6]) is to find the best values for the constants $\mathbf{A}_p^*$ and $\mathbf{B}_p^*$ in (1.3), both in the case when they don't depend on the given $n \ge 1$, as well as in the case when they do. Our analysis below shows that *the elegant structure of the single partial sum inequalities (1.1) is seriously ruined when passing to the maximal inequalities (1.3) involving a set of partial sums.*

In view of Donsker's invariance principle it seems natural to guess that the best constant in the right-hand inequality of (1.3), which does not depend on the given $n \ge 1$, should be:

$$ (1.4) \qquad \mathbf{B}_p^* = \left(E \left(\max_{0 \le t \le 1} |B_t|^p \right) \right)^{1/p} $$

where $B = (B_t)_{t \ge 0}$ is standard Brownian motion and $0 < p < \infty$. The main aim of this paper is to show the extent to which this conjecture can be reached by our method. We use *Skorohod's imbedding* (see [9]) as the main tool. For simplicity we only consider the case $p = 1$ but other values of $0 < p < \infty$ could be treated similarly.

It should be noted that the problem of finding the best value for $\mathbf{A}_p^*$ in (1.3) will not be considered here, since it is of a different character and requires another method. However, one should observe that the best value for $\mathbf{A}_p^*$ when $p \geq 2$ is 1 , since the inequality is clearly satisfied by Jensen's inequality, while the choice $a_1 = 1$, $a_2 = \ldots = a_n = 0$ proves the optimality, even in the case where the constant depends on the given $n \geq 1$. Thus, the best values for $\mathbf{A}_p^*$ have only to be found for $0 < p < 2$. This problem is rather intriguing and is certainly worthy of consideration.

From the well-known properties of Brownian motion it follows:

$$E\Big(\max_{0 \leq t \leq 1} |B_t| \Big) = \sqrt{\frac{\pi}{2}} \ . \tag{1.5}$$

Thus the conjecture (1.4) for $p = 1$ is that:

$$\mathbf{B}_1^* = \sqrt{\frac{\pi}{2}} \ . \tag{1.6}$$

The main result of this paper (Theorem 2.1) states that this conjecture is as close to the truth as desired, provided that the l_2-norm of $(a_k)_{k \geq 1}$ is large enough in comparison with the l_∞-norm of $(a_k)_{k \geq 1}$. We do not know how to improve upon this result.

2. The main results

We consider the problem of finding the best possible value for the constant $\mathbf{B}_1^*$ appearing in the maximal Khintchine inequality:

$$E\bigg(\max_{1 \leq k \leq n} \Big| \sum_{i=1}^k a_i \varepsilon_i \Big| \bigg) \leq \mathbf{B}_1^* \bigg(\sum_{k=1}^n |a_k|^2 \bigg)^{1/2} \tag{2.1}$$

being valid for all $a_1, \ldots, a_n \in \mathbf{R}$ and all $n \geq 1$, where $\varepsilon = (\varepsilon_k)_{k \geq 1}$ is a given and fixed Rademacher sequence defined on the probability space $(\Omega, \mathcal{F}, P)$.

Given $a_1, \ldots, a_n \in \mathbf{R}$ we denote $S_n = \sum_{k=1}^n a_k \varepsilon_k$ and $A_n = \sum_{k=1}^n |a_k|^2$. The main tool we use in this paper is Skorohod's imbedding. Let $B = (B_t)_{t \geq 0}$ be standard Brownian motion. Consider independent mean zero random variables $\xi_i = a_i \varepsilon_i$ for $i = 1, \ldots, n$ and define:

$$\begin{aligned} \tau_1 &= \inf\{\, t > 0 : |B_t| = |a_1| \,\} \\ \tau_2 &= \inf\{\, t > 0 : |B_{\tau_1 + t} - B_{\tau_1}| = |a_2| \,\} \\ &\ldots \\ \tau_n &= \inf\{\, t > 0 : |B_{\tau_1 + \ldots + \tau_{n-1} + t} - B_{\tau_1 + \ldots + \tau_{n-1}}| = |a_n| \,\} \ . \end{aligned} \tag{2.2}$$

Then $\tau_1, \ldots, \tau_n$ are independent random variables with $E(\tau_k) = D(\xi_k) = |a_k|^2$, and $T_k := \sum_{i=1}^k \tau_i$ is a stopping time for B for all $1 \leq k \leq n$. Moreover, by the strong Markov property we see that $B^{(k)} = (B_{\tau_1 + \ldots + \tau_k + t} - B_{\tau_1 + \ldots + \tau_k})_{t \geq 0}$ is a Brownian motion itself, and clearly τ_{k+1} is a stopping time for $B^{(k)}$ whenever $1 \leq k < n$. By the scaling property of Brownian motion we find that $\tau_k \sim |a_k|^2 \tau_*$, where we set:

(2.3) $$\tau_* = \inf \{ \, t > 0 \, : \, |B_t| = 1 \, \} \ .$$

In particular, we see that:

(2.4) $$T_n = \sum_{k=1}^n \tau_k \sim \Big(\sum_{k=1}^n |a_k|^2 \Big) \, \tau_*$$

for all $n \geq 1$. Finally, from the construction above we have:

(2.5) $$Law \, (\xi_1, \, \ldots \, , \xi_n) = Law \, (B_{\tau_1}, B_{\tau_1+\tau_2} - B_{\tau_1}, \, \ldots \, , B_{\tau_1+\ldots+\tau_n} - B_{\tau_1+\ldots+\tau_{n-1}})$$

where we recall that $\xi_i = a_i \varepsilon_i$ for $1 \leq i \leq n$. Hence:

(2.6) $$\max_{1 \leq k \leq n} |S_k| \sim \max_{1 \leq k \leq n} |B_{T_k}|$$

for all $n \geq 1$. This implies:

(2.7) $$E\bigg(\max_{1 \leq k \leq n} |S_k| \bigg) = E\bigg(\max_{1 \leq k \leq n} |B_{T_k}| \bigg) \leq E\bigg(\max_{0 \leq t \leq T_n} |B_t| \bigg) \ .$$

Here (at the inequality) we lose a certain amount of information. This certainly matters for small $n's$ but no attempt will be made in the sequel to detect the error more precisely. The inequality (2.7) is the starting point for what follows.

To obtain a bound for the best constant $\mathbf{B}_1^*$ in (2.1) recall that the $\sqrt{2}$-inequality for Brownian motion states (see [1] and [2]):

(2.8) $$E\bigg(\max_{0 \leq t \leq T} |B_t| \bigg) \leq \sqrt{2} \, \sqrt{E(T)} \ ,$$

valid for all stopping times T for B , and the constant $\sqrt{2}$ is the best possible. Applying this to (2.7) above and using (2.4) we get:

(2.9) $$E\bigg(\max_{1 \leq k \leq n} |S_k| \bigg) \leq \sqrt{2} \left(\sum_{k=1}^n |a_k|^2 \right)^{1/2} .$$

This gives the bound $\mathbf{B}_1^* \leq \sqrt{2}$. Note, however, that quite surprisingly, this inequality is already obtainable from Lévy's inequality (1.2) by using Jensen's inequality:

(2.10) $$E\bigg(\max_{1 \leq k \leq n} |S_k| \bigg) \leq \left(E\bigg(\max_{1 \leq k \leq n} |S_k|^2 \bigg) \right)^{1/2} \leq \Big(2 \, E|S_n|^2 \Big)^{1/2} = \sqrt{2} \, \sqrt{A_n} \ .$$

Note, moreover, that from (2.6) by Doob's maximal inequality (see [10]) we obtain:

(2.11) $$\left(E\bigg(\max_{1 \leq k \leq n} |S_k|^2 \bigg) \right)^{1/2} \leq 2 \left(\sum_{k=1}^n |a_k|^2 \right)^{1/2}$$

for all $n \geq 1$. The constant 2 appearing here should be compared with the constant $\mathbf{B}_2^*$ conjectured by (1.4). In fact, it is easily found that $\mathbf{B}_2^* = \int_0^\infty 1/\cosh(\sqrt{2\lambda}) \, d\lambda = 1.83193\ldots$

which is quite close to 2 just obtained.

Finally, recall that by Donsker's invariance principle we have:

$$\frac{1}{\sqrt{n}} \max_{1 \le k \le n} \Big| \sum_{i=1}^{k} \varepsilon_i \Big| \xrightarrow{\sim} \max_{0 \le t \le 1} |B_t| \tag{2.12}$$

as $n \to \infty$. Hence, by uniform integrability we get:

$$\frac{1}{\sqrt{n}} E\bigg(\max_{1 \le k \le n} \Big| \sum_{i=1}^{k} \varepsilon_i \Big| \bigg) \longrightarrow E\Big(\max_{0 \le t \le 1} |B_t| \Big) \tag{2.13}$$

as $n \to \infty$. Linking the facts just presented together (with the choice $a_1 = \ldots = a_n = 1$ in (2.1)) and using (2.13), we may conclude that the best possible constant $\mathbf{B}_1^*$ in (2.1) satisfies:

$$\sqrt{\frac{\pi}{2}} \le \mathbf{B}_1^* \le \sqrt{2} \ . \tag{2.14}$$

Motivated by the conjecture (1.4) we show that $\mathbf{B}_1^*$ is as close to $\sqrt{\pi/2}$ as desired, provided that $\|\overrightarrow{a_n}\|_2 / \|\overrightarrow{a_n}\|_\infty$ is large enough. This is described more precisely in Theorem 2.1 below. The following few remarks are aimed to increase its readability.

Given $a_1, \ldots, a_n \in \mathbf{R}$, we find it convenient to set $\overrightarrow{a_n} = (a_1, \ldots, a_n)$ and denote:

$$\|\overrightarrow{a_n}\|_2 = \bigg(\sum_{k=1}^{n} |a_k|^2 \bigg)^{1/2} \quad ; \quad \|\overrightarrow{a_n}\|_\infty = \max_{1 \le k \le n} |a_k| \ . \tag{2.15}$$

The sequences $a = (a_k)_{k \ge 1}$ of main interest in applications of the theorem below are those which satisfy the following condition:

$$\|\overrightarrow{a_n}\|_2 \Big/ \|\overrightarrow{a_n}\|_\infty \to \infty \tag{2.16}$$

as $n \to \infty$. Having a sequence $b = (b_k)_{k \ge 1}$ satisfying this condition, we may define:

$$a_k = b_k \Big/ \Big(\max_{1 \le j \le n} |b_j| \Big) \tag{2.17}$$

for $1 \le k \le n$, and in this way we obtain a sequence $a = (a_k)_{k \ge 1}$ satisfying both (2.16) and:

$$\max_{1 \le k \le n} |a_k| = 1 \ . \tag{2.18}$$

(It should be noted here that the coefficient change (2.17) is allowed in the maximal Khintchine inequalities (2.1).) This explains why it is no restriction to assume that condition (2.18) is satisfied as well. For this reason, and to simplify notation, we shall assume its validity in the theorem below, even in a weaker form.

Theorem 2.1

Let $\varepsilon = (\varepsilon_k)_{k \ge 1}$ be a Rademacher sequence, and let there be given $a_1, \ldots, a_n \in \mathbf{R}$ satisfying $|a_k| \le 1$ for all $1 \le k \le n$. Then the following inequality is satisfied:

(2.19) $$E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_i\varepsilon_i\right|\right) \le \left(\sqrt{\frac{\pi}{2}} + \mathbf{R}_n\right)\left(\sum_{k=1}^{n}|a_k|^2\right)^{1/2}$$

where the error term is given by:

(2.20) $$\mathbf{R}_n = \left(\sqrt{\pi}+2\right)\frac{\sqrt{\log\|\overrightarrow{a_n}\|_2}}{\|\overrightarrow{a_n}\|_2}$$

whenever $\|\overrightarrow{a_n}\|_2 \ge 2$ *. The constant in (2.19)+(2.20) is asymptotically the best possible.*

Proof. Having $a_1,\dots,a_n \in \mathbf{R}$ given and fixed, we shall denote $S_n = \sum_{k=1}^{n} a_k\varepsilon_k$. Then by (2.7) with T_n from (2.4) we have:

(2.21) $$\frac{E\left(\max_{1\le k\le n}|S_k|\right)}{\left(\sum_{k=1}^{n}|a_k|^2\right)^{1/2}} \le \mathbf{K}_n := \frac{E\left(\max_{0\le t\le T_n}|B_t|\right)}{\sqrt{E(T_n)}} .$$

By Brownian scaling and (1.5) we find:

(2.22) $$\mathbf{K}_n = E\left(\max_{0\le t\le T_n/E(T_n)}|B_t|\right) = E\left(\max_{0\le t\le T_n/E(T_n)}|B_t|\cdot 1_{\{\,T_n/E(T_n)\le\Delta\,\}}\right)$$
$$+E\left(\max_{0\le t\le T_n/E(T_n)}|B_t|\cdot 1_{\{\,T_n/E(T_n)>\Delta\,\}}\right) \le E\left(\max_{0\le t\le\Delta}|B_t|\right) + \mathbf{M}_n(\Delta)$$
$$= \sqrt{\Delta}\,E\left(\max_{0\le t\le 1}|B_t|\right) + \mathbf{M}_n(\Delta) = \sqrt{\Delta}\,\sqrt{\frac{\pi}{2}} + \mathbf{M}_n(\Delta)$$

where we denote:

(2.23) $$\mathbf{M}_n(\Delta) = E\left(\max_{0\le t\le T_n/E(T_n)}|B_t|\cdot 1_{\{\,T_n/E(T_n)>\Delta\,\}}\right)$$

with any $\Delta > 1$. (This relation, and most of those to come, hold for all $\Delta > 0$, but we are motivated by the fact that $T_n/E(T_n)\to 1$ as shown below, and thus consider only such Δ's.)

To estimate $\mathbf{M}_n(\Delta)$ from (2.23), we shall use Hölder's inequality and Doob's maximal inequality respectively:

(2.24) $$\mathbf{M}_n(\Delta) \le \sqrt{E\left(\max_{0\le t\le T_n/E(T_n)}|B_t|^2\right)}\,\sqrt{P\Big\{\,T_n/E(T_n)>\Delta\,\Big\}}$$
$$\le 2\,\frac{\sqrt{E|B_{T_n}|^2}}{\sqrt{E(T_n)}}\,\sqrt{P\Big\{\,T_n/E(T_n)>\Delta\,\Big\}} = 2\sqrt{P\Big\{\,T_n/E(T_n)>\Delta\,\Big\}}\,.$$

In this way we have obtained the estimate:

(2.25) $$\mathbf{K}_n \le \sqrt{\Delta}\,\sqrt{\frac{\pi}{2}}\ +\ 2\sqrt{P\Big\{\,T_n/E(T_n) > \Delta\,\Big\}}$$

which is valid for all $\Delta > 1$. It turns out that the last term in (2.25) tends to zero when $n \to \infty$. The next lemma gives a precise estimate on how fast this happens.

Lemma 2.2
The following inequality is satisfied:

(2.26) $$P\Big\{\,\frac{T_n}{E(T_n)} > \Delta\,\Big\} \le \exp\bigg(-\,\frac{(\Delta-1)^2}{4}\,E(T_n)\bigg)$$

for all $1 < \Delta \le 3/2$.

Proof. According to (2.4) we know that $T_n = \sum_{k=1}^n \tau_k$, where $\tau_k \sim |a_k|^2 \tau_*$ are independent for $1 \le k \le n$, and τ_* is given by (2.3). It is well-known that:

(2.27) $$E\Big(e^{\lambda\tau_*}\Big) = \frac{1}{\cos(\sqrt{2\lambda})}$$

for all $0 < \lambda < \pi^2/8$ (see [8] p.69). Let $\Delta > 1$ be given and fixed. Then from (2.27) by Markov's inequality we get:

(2.28) $$P\Big\{\,\frac{T_n}{E(T_n)} > \Delta\,\Big\} = P\Big\{\ \exp\big(\lambda T_n\big) > \exp\big(\lambda\Delta E(T_n)\big)\Big\}$$
$$\le \frac{E\big(e^{\lambda T_n}\big)}{e^{\lambda\Delta E(T_n)}} = \prod_{k=1}^n \frac{E\big(e^{\lambda|a_k|^2\tau_*}\big)}{e^{\lambda\Delta|a_k|^2}} = \prod_{k=1}^n \frac{e^{-\Delta\left(\sqrt{\lambda}\,|a_k|\right)^2}}{\cos\big(\sqrt{2}\,\sqrt{\lambda}\,|a_k|\big)}$$

for any $\lambda > 0$ small enough which is to be determined.

In order to proceed further in (2.28), we set $d = (\Delta-1)/2$ and verify that:

(2.29) $$\frac{e^{-\Delta x^2}}{\cos\big(\sqrt{2}\,x\big)} \le e^{-dx^2}$$

for all $0 < x \le \sqrt{d}$ whenever $0 < d \le 1/4$. For this, note that (2.29) is equivalent to:

(2.30) $$e^{(\Delta-d-1)\,x^2} e^{x^2} \cos\big(\sqrt{2}\,x\big) - 1 \ge 0\ .$$

Now, by using the Taylor expansion:

(2.31) $$\log\big(\cos x\big) = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \frac{17\,x^8}{2520} - \ \ldots$$

which is valid for $|x| \le \pi/2$, one can easily verify that:

(2.32) $$e^{x^2}\cos\big(\sqrt{2}\,x\big) \ge e^{x^4/2}$$

for all $0 < x \le 1/2$. Thus, to obtain (2.29)-(2.30), it is enough to check that:

$$e^{(\Delta - d - 1)\, x^2} e^{x^4/2} - 1 \geq 0 \ . \tag{2.33}$$

This is equivalent to the inequality:

$$(\Delta - d - 1)\, x^2 - \frac{x^4}{2} \geq 0 \tag{2.34}$$

which is evident for $0 < x \leq \sqrt{2d}$ since $(\Delta - d - 1) = d$. This proves (2.29) for $0 < x \leq \sqrt{d}$.

We may now proceed in (2.28) by using (2.29) and taking $\lambda = d$. In this way we get:

$$P\Big\{ \frac{T_n}{E(T_n)} > \Delta \Big\} \leq \prod_{k=1}^{n} \frac{e^{-\Delta \left(\sqrt{d}\, |a_k|\right)^2}}{\cos\left(\sqrt{2}\,\sqrt{d}\, |a_k|\right)} \leq \prod_{k=1}^{n} e^{-d^2 |a_k|^2} = e^{-d^2 \sum_{k=1}^{n} |a_k|^2} \tag{2.35}$$

for all $1 < \Delta \leq 3/2$ or equivalently $0 < d \leq 1/4$. This completes the proof of the lemma.

□

We continue with the main proof by observing that from (2.25) with (2.26) we obtain:

$$\mathbf{K}_n \leq \sqrt{\Delta}\, \sqrt{\frac{\pi}{2}} \ + \ 2 \ \exp\left(- \frac{(\Delta - 1)^2}{8}\, E(T_n)\right) = \sqrt{\frac{\pi}{2}} \ + \ \mathbf{L}_n \tag{2.36}$$

where we set:

$$\mathbf{L}_n = \left(\sqrt{\Delta} - 1\right) \sqrt{\frac{\pi}{2}} \ + \ 2 \ \exp\left(- \frac{(\Delta - 1)^2}{8}\, E(T_n)\right) \tag{2.37}$$

with $1 < \Delta \leq 3/2$. We put $\Delta := \Delta_n = 1 + \delta_n$ with $\delta_n > 0$ to be chosen, and use that $\sqrt{1+\delta} - 1 \leq \delta/2$ for $\delta > 0$. Then we get:

$$\mathbf{L}_n = \left(\sqrt{1+\delta_n} - 1\right) \sqrt{\frac{\pi}{2}} + 2 \exp\left(- \frac{\delta_n^2}{8} E(T_n)\right) \leq \frac{1}{2} \sqrt{\frac{\pi}{2}}\, \delta_n + 2 \exp\left(- \frac{\delta_n^2}{8} E(T_n)\right) . \tag{2.38}$$

In the next we shall take:

$$\delta_n = \sqrt{8} \left(\frac{\log \sqrt{E(T_n)}}{E(T_n)} \right)^{1/2} \tag{2.39}$$

which is to be less or equal $1/2$ in accordance with (2.37) above. It is easily verified that this is fulfilled as soon as $\sqrt{E(T_n)} \geq 9$. Inserting (2.39) into (2.38) we get:

$$\mathbf{L}_n \leq \sqrt{\pi} \left(\frac{\log \sqrt{E(T_n)}}{E(T_n)} \right)^{1/2} + \frac{2}{\sqrt{E(T_n)}} \tag{2.40}$$

whenever $\sqrt{E(T_n)} \geq 9$. It remains to note that $1/\sqrt{x} \leq ((\log \sqrt{x})/x)^{1/2}$ for $x \geq 9$, which together with (2.40) establishes the estimate:

$$\mathbf{L}_n \leq \left(\sqrt{\pi} + 2\right) \left(\frac{\log \sqrt{E(T_n)}}{E(T_n)} \right)^{1/2} \tag{2.41}$$

whenever $\sqrt{E(T_n)} \geq 9$. Finally, by (2.4) and the fact $E(\tau_*) = 1$ we note that $\sqrt{E(T_n)} = \|\overrightarrow{a_n}\|_2$. This proves (2.19) with (2.20) for $\|\overrightarrow{a_n}\|_2 \geq 9$. The cases $2 \leq \|\overrightarrow{a_n}\|_2 < 9$ are verified straightforwardly using (2.9). This completes the proof of the theorem. □

Corollary 2.3

Let $\varepsilon = (\varepsilon_k)_{k \geq 1}$ be a Rademacher sequence. Then the following estimate is valid:

$$(2.42) \qquad E\left(\max_{1 \leq k \leq n} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{k} \varepsilon_i \right| \right) \leq \left(\sqrt{\frac{\pi}{2}} + \left(\sqrt{\pi} + 2 \right) \frac{\sqrt{\log \sqrt{n}}}{\sqrt{n}} \right)$$

for all $n \geq 1$. The constant on the right-hand side is asymptotically the best possible.

Proof. The inequality (2.42) is the inequality (2.19) with (2.20) specialized to the case where $a_1 = \ldots = a_n = 1$ for $n \geq 4$. The cases $1 \leq n \leq 3$ are trivial. The last statement follows from (2.13) with (1.5). This completes the proof. □

Problem 2.4 (Corollary)

Let $\varepsilon = (\varepsilon_k)_{k \geq 1}$ be a Rademacher sequence, and let $\overrightarrow{\mathbf{a}_{n,n}} = (\mathbf{a}_{1n}, \ldots, \mathbf{a}_{nn})$ denote a vector in $\mathbf{R}^n$ at which the maximum is attained:

$$(2.43) \qquad Z_n := \max \left\{ E\left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} a_i \varepsilon_i \right| \right) \Big/ \left(\sum_{k=1}^{n} |a_k|^2 \right)^{1/2} : \ (a_1, \ldots, a_n) \in \mathbf{R}^n \right\}$$

where $n \geq 1$. Suppose one could prove that:

$$(2.44) \qquad \|\overrightarrow{\mathbf{a}_{n,n}}\|_2 \Big/ \|\overrightarrow{\mathbf{a}_{n,n}}\|_\infty \longrightarrow \infty$$

as $n \to \infty$. Then (2.19) would hold with $\mathbf{R}_n \equiv 0$, and this would prove the conjecture (1.4) for $p = 1$. We were unable to derive (2.44) but feel that it is worthy of further attempts (see Example 2.5 below).

To prove the statement just indicated, it is enough to note that the sequence $(Z_n)_{n \geq 1}$ is increasing, and that by (2.19) we have:

$$(2.45) \qquad Z_n \leq \sqrt{\frac{\pi}{2}} + \mathbf{R}_n$$

with $\mathbf{R}_n \to 0$ as $n \to \infty$ (note that the rate of convergence doesn't matter at all) whenever (2.44) is fulfilled. Now letting $n \to \infty$ in (2.45) we obtain the statement. We note that various modifications of the argument just presented may be applicable as well. □

Motivated by the preceding problem, we considered the case where $a_n = \lambda^{n-1}$ with $0 < \lambda < 1$ for $n \geq 1$. In this case we have shown, using elementary but lengthy calculations that:

$$(2.46) \qquad E\left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} a_i \varepsilon_i \right| \right) \leq \frac{2}{\sqrt{3}} \left(\sum_{k=1}^{n} |a_k|^2 \right)^{1/2}$$

and that $2/\sqrt{3}$ is the best possible constant in this case.

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