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On the Exponential Orlicz Norms of Stopped Brownian Motion

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Necessary and sufficient conditions are found for the exponential Orlicz norm (generated by $\psi_p(x) = \exp(|x|^p) - 1$ with $0 < p \le 2$) of $\max_{0 \le t \le \tau} |B_t|$ or $|B_\tau|$ is finite, where $B = (B_t)_{t \ge 0}$ is standard Brownian motion and τ is a stopping time for B. The conditions are in terms of the moments of the stopping time τ. For instance, we find that $\big\| \max_{0 \le t \le \tau} |B_t| \big\|_{\psi_1} < \infty$ as soon as we have:

$$E(\tau^k) = O\big(C^k k^k\big)$$

for some constant $C > 0$ as $k \to \infty$ (or equivalently $\|\tau\|_{\psi_1} < \infty$). In particular, if $\tau \sim Exp(\lambda)$ or $|N(0, \sigma^2)|$ then the last condition is fulfilled, and we obtain:

$$\Big\| \max_{0 \le t \le \tau} |B_t| \Big\|_{\psi_1} \le K \sqrt{E(\tau)}$$

with some universal constant $K > 0$. Moreover, this inequality remains valid for any class of stopping times τ for B satisfying $E(\tau^k) \le C(E\tau)^k k^k$ for all $k \ge 1$ with some fixed constant $C > 0$. The method of proof relies upon Taylor expansion, Burkholder-Gundy's inequality, best constants in Doob's maximal inequality, Davis' best constants in the L^p-inequalities for stopped Brownian motion, and estimates of the smallest and largest positive zero of Hermite polynomials. The results extend to cover the case of any continuous local martingale (by applying the time change method of Dubins and Schwarz).

1. Introduction

The main aim of the paper is to investigate and establish necessary and sufficient conditions for the exponential integrability of the supremum of reflecting Brownian motion taken over a random time interval (as well as of stopped Brownian motion itself).

More precisely, let $B = (B_t)_{t \ge 0}$ be standard Brownian motion, let τ be a stopping time for B, and let $\| \cdot \|_\psi$ denote the Orlicz norm generated by the Young function ψ. Thus, if X is a random variable, then we have:

$$\|X\|_\psi = \inf \big\{ c > 0 \mid E\big(\psi(|X|/c)\big) \le 1 \big\} \tag{1.1}$$

with $\inf(\emptyset) = \infty$. In this paper we are interested in Young functions of exponential growth, and therefore choose to work with $\psi_p(x) = \exp(|x|^p) - 1$ for $0 < p \le 2$. (To handle the small convexity problem around zero when $0 < p < 1$, we can let $\psi_p(x) = \exp(|x|^p) - 1$ for $x \ge (1/p - 1)^{1/p}$ and take $\psi_p(x)$ to be linear on $[\,0, (1/p - 1)^{1/p}]$.) The main problem under

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consideration is to find out when the quantities $\| \max_{0 \le t \le \tau} |B_t| \, \|_{\psi_p}$ (or $\|B_\tau\|_{\psi_p}$) are finite, and to obtain sharp estimates of those for $0 < p \le 2$. In view of the Burkholder-Davis-Gundy (and related) inequalities, where the integrability is usually established for Young functions of a moderated growth, we think that this problem appears worthy of consideration and its solution by itself might be of theoretical and practical interest.

In the very beginning of thinking about this problem, it was not clear to us in what terms the conditions (we should look for) are to be expressed. For this reason we find it convenient here to explain our minding in this direction in more detail. In this context we were firstly motivated with some fundamental (closely related) results in the discrete parameter case. If $\varepsilon = \{\varepsilon_k\}_{k \ge 1}$ is a Rademacher sequence of random variables and $\{a_k\}_{k \ge 1}$ is a sequence of real numbers, then the *Khintchine inequality* states (see [7]):

$$\Big\| \sum_{k=1}^n a_k \varepsilon_k \Big\|_p \le A_p \Big(\sum_{k=1}^n |a_k|^2 \Big)^{1/2} \tag{1.2}$$

for all $0 < p < \infty$ and all $n \ge 1$, where the constant A_p depends only on p . The best value for A_p in (1.2) is known (see [6]):

$$A_p = \begin{cases} \sqrt{2} \Big(\Gamma\big((p+1)/2\big) / \sqrt{\pi} \Big)^{1/p} & \text{if} \quad 2 \le p < \infty \\ 1 & \text{if} \quad 0 < p \le 2 \, . \end{cases} \tag{1.3}$$

In particular, taking $a_1 = \ldots = a_n = 1$ in (1.2), we obtain:

$$\Big\| \sum_{k=1}^n \varepsilon_k \Big\|_p \le A_p \sqrt{n} \tag{1.4}$$

for all $0 < p < \infty$ and all $n \ge 1$.

Having (1.2) with (1.3) for $2 \le p < \infty$, and using Taylor expansion of $x \mapsto \exp(|x|^2)$, the passage from the power Orlicz norms (generated by $\varphi_p(x) = |x|^p$) in (1.2) to the exponential Orlicz norm generated by ψ_2 is rather smooth (see [11], [12], [13]):

$$\Big\| \sum_{k=1}^n a_k \varepsilon_k \Big\|_{\psi_2} \le \sqrt{\tfrac{8}{3}} \Big(\sum_{k=1}^n |a_k|^2 \Big)^{1/2} \tag{1.5}$$

being valid for all $a_1, \ldots, a_n \in \mathbf{R}$ with $n \ge 1$. Moreover, in this way we obtain a very precise information: the constant $\sqrt{8/3}$ is the best possible in (1.5). By using exactly the same procedure one could deduce the inequality (see [7]):

$$\Big\| \sum_{k=1}^n a_k \varepsilon_k \Big\|_{\psi_p} \le B_p \Big(\sum_{k=1}^n |a_k|^2 \Big)^{1/2} \tag{1.6}$$

which is valid for all $0 < p \le 2$ and all $a_1, \ldots, a_n \in \mathbf{R}$ with $n \ge 1$, where the constant B_p depends only on p . In particular, taking $a_1 = \ldots = a_n = 1$ in (1.6), we obtain:

$$\Big\| \sum_{k=1}^n \varepsilon_k \Big\|_{\psi_p} \le B_p \sqrt{n} \tag{1.7}$$

for all $0 < p \le 2$ and all $n \ge 1$.

In view of our main problem stated above, two questions arise naturally in this setting. The first one is: What happens with the right-hand side in (1.4) when n on the left-hand side is replaced by a stopping time τ for ε ? (It would be ideal that $\sqrt{n}$ on the right-hand side may be replaced by $\sqrt{E(\tau)}$ or $E(\sqrt{\tau})$, but this is not true in general as we shall state below). The second one is: Provided that we have an affirmative answer to the first question, is it possible to imitate the passage from (1.2) to (1.5)+(1.6) via Taylor expansion and obtain an analogous inequality in (1.7), with n on the right-hand side being replaced by $\sqrt{E(\tau)}$ or $E(\sqrt{\tau})$? (In view of the negative answer to the first question, this also fails in general.)

The answer to the first question is well-known (see [3] and [5]):

$$\Big\| \sum_{k=1}^{\tau} \varepsilon_k \Big\|_p \leq C_p E\big(\tau^{p/2}\big) \tag{1.8}$$

being valid for any stopping time τ for ε and all $0 < p < \infty$, where the constant C_p depends only on p . It should be also well-known that this is the best we can in the context of the first question. Thus, in view of the second question, this indicates that in order to pass from (1.8) to an analogue of (1.7) via Taylor expansion, one has to take care about all moments of the stopping time τ . This observation significantly clarifies what should be our approach towards solution for our main problem stated above. In other words, it becomes clear that our necessary and sufficient conditions should be expressed in terms of all moments of the stopping time τ , or better to say, in terms of the asymptotic behaviour of the quantity $E(\tau^k)$ when $k \to \infty$. Still, it does not become clear (and remains to be found), what could be an analogous inequality of (1.7) after the extension of (1.8) via Taylor expansion as explained above. In particular, how large is (and how can be described) the class of stopping times for which the analogue of (1.7) remains valid? We find these questions of interest and leave them worthy of consideration. *In this paper we shall focus only to the continuous parameter case, and in this context provide answers to analogous questions.* In this process the preceding discussion will serve mainly as a motivation.

The preceding conclusion about the necessity of taking into account all moments of the stopping time (in order to gain exponential integrability) becomes particularly transparent in the continuous parameter case, after recalling *Burkholder-Gundy's inequality* (see [2]):

$$G_p E\big(\tau^{p/2}\big) \leq E\Big(\max_{0 \leq t \leq \tau} |B_t|^p \Big) \leq H_p E\big(\tau^{p/2}\big) \tag{1.9}$$

which is valid for all stopping times τ for B and all $0 < p < \infty$ with some universal constants G_p and H_p depending only on p . Thus in order to deduce the exponential integrability of $\max_{0 \leq t \leq \tau} |B_t|$ via Taylor expansion one should have a precise information on the constants H_p for $0 < p < \infty$. However, the best values for H_p's in (1.9) do not seem to be known by now. Nonetheless, we will see that despite this fact we can approximate these numbers in an accurate way by using Doob's maximal inequality (with best constants) and Davis's best constants for an analogous inequality of (1.9) where $\max_{0 \leq t \leq \tau} |B_t|$ is replaced by $|B_\tau|$. This leads to the necessity of estimating the (largest and smallest positive) zero of Hermite polynomials and is the approach which is taken in this paper. The details about the procedure just explained will be presented later. Here we find it convenient only to indicate results obtained in this way.

The main result of the paper states that we have (see Theorems 3.4, 3.12 and 3.14):

$$\Big\| \max_{0 \leq t \leq \tau} |B_t| \Big\|_{\psi_p} < \infty \tag{1.10}$$

for any given and fixed stopping time τ for B , as soon as the condition is fulfilled:

$$E(\tau^k) = O\big(C^k k^{k(2-p)/p}\big) \tag{1.11}$$

with some constant $C > 0$ as $k \to \infty$, where $0 < p \leq 2$ is given and fixed. Moreover, the following estimate is valid (see Theorem 3.12):

$$\Big\| \max_{0 \leq t \leq \tau} |B_t| \Big\|_{\psi_p} \leq \sqrt{10}\ \frac{p}{p-1}\, e^{1/p} \sqrt{D} \left(1 + \frac{\big(\Delta_p(D)\big)^{p/2}}{\sqrt{2\pi}} \right) \tag{1.12}$$

for all $1 < p \leq 2$ and any $D > 0$, where we set:

$$\Delta_p(D) = \sup_{k \geq 1} \left(\frac{E\big(\tau^k\big)}{D^k\, k^{k(2-p)/p}} \right) . \tag{1.13}$$

In the case $p = 1$, we have the following estimate (see Theorem 3.4):

$$\Big\| \max_{0 \leq t \leq \tau} |B_t| \Big\|_{\psi_1} \leq 6e \sqrt{D} \left(1 + \frac{\sqrt{\Delta_1(D)}}{\sqrt{2\pi}} \right) \tag{1.14}$$

being valid for any $D > 0$. In particular, if $\tau \sim Exp(\lambda)$ or $|N(0, \sigma^2)|$ with some λ, $\sigma^2 > 0$, then (1.11) is satisfied, and moreover in this case (1.14) yields (see Examples 3.8 and 3.9):

$$\Big\| \max_{0 \leq t \leq \tau} |B_t| \Big\|_{\psi_1} \leq K \sqrt{E(\tau)} \tag{1.15}$$

with some universal constant $K > 0$. We remark that the given stopping time $\tau \sim Exp(\lambda)$ or $|N(0, \sigma^2)|$ is not necessarily independent from Brownian motion B, and in view of (1.7) think that the *inequality (1.15) is by itself of theoretical and practical interest.* Moreover, this inequality remains valid for any class of stopping times τ for B satisfying $E(\tau^k) \leq C(E\tau)^k k^k$ for all $k \geq 1$ with some fixed constant $C > 0$ (see Corollary 3.7). Finally, if (1.10) is satisfied, then the following condition is fulfilled (see Theorem 3.2):

$$E(\tau^k) = O\Big(C^k k^{k(2+p)/p}\Big) \tag{1.16}$$

for some constant $C > 0$ as $k \to \infty$, where $0 < p \leq 2$ is given and fixed. We note that there is a gap in between sufficient condition (1.11) and necessary condition (1.16), but this was the optimum we could obtain by using our method here (see Problem 3.10).

To conclude the introduction let us mention that the results just indicated extend from the Brownian motion case to cover the case of any continuous local martingale (Ito's integral). This is achieved by applying the standard time change method of Dubins and Schwarz. The results in this context are presented in more detail in Section 4.

2. Power integrability of stopped Brownian motion

In this section we shall introduce the notation and collect without proof several facts which will be used in the proofs of our main results in the next section.

In this paper we work with a fixed probability space $(\Omega, \mathcal{F}, P)$ which is large enough to support all random functions under consideration. Moreover, whenever a filtration $(\mathcal{F}_t)_{t \geq 0}$ of $(\Omega, \mathcal{F}, P)$ is considered, it is assumed to satisfy the usual conditions: $\mathcal{F}$ is P-complete, $\mathcal{F}_0$ contains all P-

null sets in $\mathcal{F}$, and $\mathcal{F}_{t+} = \mathcal{F}_t$ for all $t \geq 0$. The main object under investigation in this paper is Brownian motion. We recall that (standard) *Brownian motion* is the process $B = (B_t)_{t\geq 0}$ defined on $(\Omega, \mathcal{F}, P)$ with $B_0 = 0$ P-a.s. for which there exists a filtration $(\mathcal{F}_t)_{t\geq 0}$ such that $B_{t+h} - B_t$ is independent from $\mathcal{F}_t$, and has the Gaussian distribution with expectation 0 and variance h whenever t , $h \geq 0$. A random variable $\tau : \Omega \to [0, \infty]$ is said to be a *stopping time* for B , if $\{ \tau \leq t \} \in \mathcal{F}_t$ for all $t \geq 0$. If this condition holds for a filtration $(\mathcal{F}_t)_{t\geq 0}$ of $(\Omega, \mathcal{F}, P)$ which is not necessarily linked with a Brownian motion, we will say that τ is an $(\mathcal{F}_t)$-stopping time.

Let τ be a stopping time for Brownian motion B such that $\{ B_{t\wedge\tau} \mid t \geq 0 \}$ is uniformly integrable (which holds if $E(\sqrt{\tau}) < \infty$ for instance). Then *Doob's maximal inequality* states:

$$\Big\| \max_{0\leq t\leq\tau} |B_t| \Big\|_p \leq \frac{p}{p-1} \, \|B_\tau\|_p \tag{2.1}$$

for all $1 < p < \infty$ with the constant $p/(p-1)$ being the best possible (see [16]).

Let τ be a stopping time for Brownian motion B . Then the *Burkholder-Gundy's inequality* (1.9) is valid for all $0 < p < \infty$ with some universal constants G_p and H_p depending only on p . The best values for G_p and H_p are not known. However, it is known that we have (see [8]):

$$\frac{2-p}{4-p} \, E\big(\tau^{p/2}\big) \leq E\Big(\max_{0\leq t\leq\tau} |B_t|^p \Big) \leq \frac{4-p}{2-p} \, E\big(\tau^{p/2}\big) \tag{2.2}$$

whenever $0 < p < 2$.

Let τ is a stopping time for Brownian motion B . Then *Burkholder-Davis'* inequality states:

$$E\big(|B_\tau|^p\big) \leq A_p E\big(\tau^{p/2}\big) \; , \text{ if } \; 0 < p < \infty \tag{2.3}$$

$$a_p E\big(\tau^{p/2}\big) \leq E\big(|B_\tau|^p\big) \; , \text{ if } \; 1 < p < \infty \; \text{ and } \; E\big(\tau^{p/2}\big) < \infty \tag{2.4}$$

with the best values for A_p's and a_p's in (2.3)+(2.4) given by (see [3]):

$$A_p = \begin{cases} (z_p^*)^p & \text{if } \; 2 \leq p < \infty \\ (z_p)^p & \text{if } \; 0 < p \leq 2 \end{cases} \tag{2.5}$$

$$a_p = \begin{cases} (z_p)^p & \text{if } \; 2 \leq p < \infty \\ (z_p^*)^p & \text{if } \; 1 < p \leq 2 \end{cases} \tag{2.6}$$

where z_p^* denotes the largest positive zero of the *parabolic cylinder function* $D_p(x)$, while z_p denotes the smallest positive zero of the *confluent hypergeometric function* $x \mapsto M(-p/2, 1/2, x^2/2)$.

We recall that $M(a,b,z)$ denotes the *Kummer's function*; it is a solution of the differential equation (see [1] p.189):

$$z y''(z) + (b-z) y'(z) - a y(z) = 0 \; . \tag{2.7}$$

The Kummer's function is explicitly given by the expression (see [1] p.189):

$$M(a,b,z) = 1 + \sum_{k=0}^{\infty} \frac{a(a+1)\cdots(a+k)}{b(b+1)\cdots(b+k)} \, \frac{z^k}{k!} \; . \tag{2.8}$$

Moreover, we have (see [1] p.194):

$$M(-n, 1/2, x^2/2) = (-1)^n \, 2^n \frac{n!}{(2n)!} \, He_{2n}(x) \tag{2.9}$$

where $He_n(x)$ denotes the *Hermite polynomial* of degree n :

(2.10) $$He_n(x) \;=\; n! \sum_{k=0}^{[n/2]} (-1)^k \frac{1}{k!\, 2^k\, (n-2k)!}\, x^{n-2k} \;.$$

We recall that the *parabolic cylinder function* $D_p(x)$ is defined by (see [1] p.281-282):

(2.11) $$D_p(x) \;=\; Y_1(x) \cos\big(p\pi/2\big) + Y_2(x) \sin\big(p\pi/2\big)$$

where $Y_1(x)$ and $Y_2(x)$ are given by:

(2.12) $$Y_1(x) \;=\; \big(2^{p/2}/\sqrt{\pi}\big)\, \Gamma\big((p+1)/2\big)\, y_1(x)$$

(2.13) $$Y_2(x) \;=\; \big(2^{(p+1)/2}/\sqrt{\pi}\big)\, \Gamma\big((p+2)/2\big)\, y_2(x)$$

where $y_1(x)$ and $y_2(x)$ are linearly independent solutions of the differential equation:

(2.14) $$y''(x) + (ax^2 + bx + c)\, y(x) \;=\; 0$$

with $a = -1/4$, $b = 0$, $c = p + 1/2$. Actually, we have:

(2.15) $$y_1(x) \;=\; \exp(-x^2/4)\, M(-p/2, 1/2, x^2/2)$$

(2.16) $$y_2(x) \;=\; x \exp(-x^2/4)\, M(-p/2 + 1/2, 3/2, x^2/2) \;.$$

In view of (2.9), (2.11), (2.12) and (2.15) we see that z_{2n}^* is the largest positive zero (z_{2n} is the smallest positive zero) of the Hermite polynomial $He_{2n}(x)$. Our next aim is to state estimates of these numbers. This will be used heavily in the proof of our main result in the next section.

For this, let $L_n^{(\alpha)}(x)$ denote the *generalized Laguerre polynomial*:

(2.17) $$L_n^{(\alpha)}(x) \;=\; \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{1}{k!}\, x^k$$

and let $x_m^{(n,\alpha)}$ denote the m-th (positive) zero of $L_n^{(\alpha)}(x)$. Let $J_\nu(z)$ denote the *Bessel* function:

(2.18) $$J_\nu(z) \;=\; \frac{(z/2)^\nu e^{-iz}}{\Gamma(\nu+1)}\, M(\nu+1/2, 2\nu+1, 2iz)$$

and let $j_m^{(\alpha)}$ denote the m-th positive zero of $J_\alpha(x)$. Then we have (see [1] p.346):

(2.19) $$\frac{\big(j_m^{(\alpha)}\big)^2}{4k_n} \;<\; x_m^{(n,\alpha)} \;<\; \frac{k_m}{k_n} \left(2k_m + \sqrt{4k_m^2 + \frac{1}{4} - \alpha^2}\;\right)$$

where $k_r = r + (\alpha+1)/2$.

In order to apply this result in our context, one should note that we have:

(2.20) $$He_{2n}(x) \;=\; (-1)^n 2^n n!\, L_n^{(-1/2)}\big(x^2/2\big)$$

(2.21) $$J_{-1/2}(x) \;=\; \sqrt{\frac{2}{\pi x}}\; \cos(x) \;.$$

Thus $j_1^{(-1/2)} = \pi/2$, and from (2.19)-(2.21) we obtain:

(2.22) $$\frac{\pi}{\sqrt{2(4n+1)}} < z_{2n}$$

(2.23) $$z_{2n}^* < \sqrt{2(4n+1)}$$

being valid for all $n \geq 1$.

A useful application of (2.22) and (2.23) is based upon the following two facts:

(2.24) $$p \mapsto z_p \text{ is (strictly) decreasing on }]0,\infty[$$

(2.25) $$p \mapsto z_p^* \text{ is (strictly) increasing on }]1,\infty[\; .$$

This can be easily seen by recalling some results on the square root stopping boundaries. Namely, let $B = (B_t)_{t\geq 0}$ be standard Brownian motion, and let us consider the stopping times defined by:

(2.26) $$\tau_a = \inf \left\{ t > 0 : |B_t| = a\sqrt{t+1} \right\}$$

(2.27) $$\sigma_a = \inf \left\{ t > 0 : B_t = a\sqrt{t} - 1 \right\}$$

where $a > 0$. Then we have (see [15] and [10] respectively):

(2.28) $$E\big(\tau_a\big)^p < \infty \text{ if and only if } a < z_{2p} \text{ whenever } 0 < p < \infty$$

(2.29) $$E\big(\sigma_a\big)^p < \infty \text{ if and only if } a > z_{2p}^* \text{ whenever } 1/2 < p < \infty \; .$$

Hence (2.24) and (2.25) are easily verified.

3. Exponential integrability of stopped Brownian motion

In this section we present the main results of the paper. Throughout $B = (B_t)_{t\geq 0}$ denotes standard Brownian motion defined on the probability space $(\Omega, \mathcal{F}, P)$, and τ denotes a stopping time for B with $E\big(\tau^k\big) < \infty$ for all $k \geq 1$. Our main aim is to find necessary and sufficient conditions for the exponential Orlicz norms:

(3.1) $$\Big\| \max_{0\leq t\leq \tau} |B_t| \Big\|_{\psi_p}$$

are finite, and obtain estimates of those for $0 < p \leq 2$ in terms of the moments of the stopping time τ . By Taylor expansion and (2.1) it is easily seen that the quantity in (3.1) is finite, if and only if $\|B_\tau\|_{\psi_p}$ is finite. Here we shall concentrate to the quantity (3.1), but the proofs and estimates obtained are easily adapted to cover the case of the quantity $\|B_\tau\|_{\psi_p}$ as well. We begin by exploring necessary conditions. In this context the following lemma is shown to be useful.

Lemma 3.1

There exist universal constants $\varepsilon_{p,k} > 0$ such that the inequality holds:

(3.2) $$\varepsilon_{p,k} \big\|X\big\|_k \leq \big\|X\big\|_{\psi_p}$$

whenever X is a random variable and $p, k \geq 1$ are given and fixed. In fact, one can take:

(3.3) $$\varepsilon_{p,k} = \left(\frac{pe}{k}\right)^{1/p}.$$

Proof. Let $p, k \geq 1$ be given and fixed. It is clear that the problem is reduced to finding the largest constant $c > 0$ for which the following inequality holds true:

$$E\left(\exp\left(\frac{|X|^p}{c^p\,\|X\|_k^p}\right)\right) > 2 \tag{3.4}$$

whenever X is a random variable. By Taylor expansion (3.4) can be rewritten in the form:

$$\sum_{n=1}^{\infty} \frac{E|X|^{np}}{c^{np}\,\|X\|_k^{np}\, n!} = \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{\|X\|_{np}}{c\,\|X\|_k}\right)^{np} > 1\ . \tag{3.5}$$

Let m denote the smallest $n \geq 1$ satisfying $np \geq k$. Then by Jensen's inequality we will have (3.5) as soon as we have:

$$\frac{1}{m!}\left(\frac{1}{c}\right)^{mp} \geq 1\ .$$

Hence by Stirling's formula we can conclude:

$$c \leq \left(\frac{1}{m!}\right)^{1/mp} = \left(\frac{1}{\sqrt{2\pi m}\ m^m\ e^{-m}\ e^{r_m}}\right)^{1/mp} \leq \left(\frac{e}{m}\right)^{1/p} \leq \left(\frac{pe}{k}\right)^{1/p}\ .$$

where $1/(12m{+}1) < r_m < 1/12m$. This completes the proof of the lemma. □

We may now state a necessary condition for the exponential integrability of the supremum of reflecting Brownian motion taken over a random time interval (as well as of stopped Brownian motion itself).

Theorem 3.2

Let τ be a stopping time for Brownian motion $B = (B_t)_{t\geq 0}$ satisfying $E(\tau^k) < \infty$ for all $k \geq 1$, such that we have:

$$\Big\| \max_{0\leq t\leq \tau} |B_t| \Big\|_{\psi_p} < \infty \tag{3.6}$$

for some $0 < p < \infty$. Then the condition is satisfied:

$$E(\tau^k) = O\Big(C^k k^{k(2+p)/p}\Big) \tag{3.7}$$

for some $C > 0$ as $k \to \infty$.

Proof. Put $A = \|\max_{0\leq t\leq\tau} |B_t|\,\|_{\psi_p}$ where $p \geq 1$ is given (and fixed) such that (3.6) is satisfied. Then by (3.2)+(3.3) we have the inequality:

$$A \geq \left(\frac{pe}{k}\right)^{1/p} \|\max_{0\leq t\leq\tau} |B_t|\,\|_k \tag{3.8}$$

for all $k \geq 1$. From (2.2) with $p = 1$ we get:

$$E\Big(\max_{0\leq t\leq \tau} |B_t|\Big) \geq \frac{1}{3}\, E\big(\sqrt{\tau}\big)\ . \tag{3.9}$$

By (2.4)+(2.6), (2.22) and (2.24) we easily find:

$$E\Big(\max_{0\le t\le\tau}|B_t|^k\Big) \;\ge\; \Big(\frac{\pi}{11\sqrt{k}}\Big)^k E\big(\tau^{k/2}\big) \tag{3.10}$$

for all $k\ge 2$. Since $\pi/11<1/3$, then by (3.9) we see that (3.10) remains valid for $k=1$ as well. By (3.8) and (3.10) we have:

$$\begin{aligned} 1 &\ge \left(\frac{(pe)^{1/p}}{A}\,\frac{\pi}{11}\right)^k \frac{1}{k^{k/p+k/2}}\,E\big(\tau^{k/2}\big) \\ &= \left(\frac{(pe)^{1/p}}{A}\,\frac{\pi}{11}\,\frac{1}{2^{(1/2)(1+2/p)}}\right)^k \frac{E\big(\tau^{k/2}\big)}{\Big(\frac{k}{2}\Big)^{(k/2)(1+2/p)}} \\ &= \left(\frac{(pe/2)^{1/p}}{A}\,\frac{\pi}{11\sqrt{2}}\right)^k \frac{E\big(\tau^{k/2}\big)}{\Big(\frac{k}{2}\Big)^{(k/2)(1+2/p)}} \end{aligned} \tag{3.11}$$

for all $k\ge 1$. Hence we conclude:

$$\sup_{k\ge 1}\left(\frac{(pe/2)^{2/p}\pi^2}{242\,A^2}\right)^k \frac{E\big(\tau^k\big)}{k^{k(2+p)/p}} \le 1\ . \tag{3.12}$$

The case $0<p<1$ is treated similarly. This completes the proof of the theorem. $\square$

Example 3.3

Let $\tau\sim\log N(\mu,\sigma^2)$ be a stopping time for Brownian motion $B=(B_t)_{t\ge 0}$ with the log-normal density function:

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}}\,\exp\left(-\frac{(\mu-\log x)^2}{2\sigma^2}\right) 1_{]0,\infty[}(x)$$

where $\mu\in\mathbf{R}$ and $\sigma^2>0$. Thus, if $X\sim N(\mu,\sigma^2)$ is a Gaussian random variable with mean μ and variance σ^2 , then $\tau\sim\exp(X)$. Then we have:

$$E\big(\tau^k\big) = e^{\mu k+\frac{\sigma^2}{2}k^2}$$

for $k\ge 1$. Hence by (3.7) we easily find that:

$$\Big\|\max_{0\le t\le\tau}|B_t|\Big\|_{\psi_p} = \infty$$

for all $0<p<\infty$. In other words, there is no number $\varepsilon>0$ such that:

$$E\left(\exp\Big(\varepsilon\Big(\max_{0\le t\le\tau}|B_t|^p\Big)\Big)\right) < \infty$$

whenever $0<p<\infty$. $\square$

Next we pass to explore sufficient conditions. First we investigate the case $p=1$. The main result in this context is as follows.

Theorem 3.4

Let $B = (B_t)_{t \ge 0}$ *be standard Brownian motion, and let* τ *be a stopping time for* B *satisfying the following condition:*

$$E\big(\tau^k\big) \;=\; O\big(C^k k^k\big) \tag{3.13}$$

for some $C > 0$ *as* $k \to \infty$ *. Then we have:*

$$\Big\| \max_{0 \le t \le \tau} |B_t| \Big\|_{\psi_1} < \; \infty \; . \tag{3.14}$$

Moreover, let us define the function:

$$\Delta_1(D) \;=\; \sup_{k \ge 1} \left(\frac{E\big(\tau^k\big)}{D^k \, k^k} \right) \tag{3.15}$$

for $D > 0$ *. Then the following estimate is valid:*

$$\Big\| \max_{0 \le t \le \tau} |B_t| \Big\|_{\psi_1} \le \; 6e\sqrt{D} \left(1 + \frac{\sqrt{\Delta_1(D)}}{\sqrt{2\pi}} \right) \tag{3.16}$$

for all $D > 0$ *.*

Proof. Given a stopping time τ for B satisfying (3.13), denote $B_\tau^* = \max_{\, 0 \le t \le \tau} |B_t|$. Then by (2.1), (2.3)+(2.5), (2.23) and (2.25) we have:

$$\begin{aligned} E\big(B_\tau^*\big)^k &\le \left(\frac{k}{k-1} \right)^k E|B_\tau|^k \le \left(\frac{k}{k-1} \right)^k \big(z_k^*\big)^k E\big(\tau^{k/2}\big) \\ &\le \left(\frac{k}{k-1} \right)^k \big(z_{2k}^*\big)^k E\big(\tau^{k/2}\big) \le \left(\frac{k}{k-1} \right)^k \left(\sqrt{2(4k+1)} \right)^k E\big(\tau^{k/2}\big) \end{aligned} \tag{3.17}$$

for all $k \ge 2$. Moreover, by (2.2) with $p = 1$ we get:

$$E\big(B_\tau^*\big) \;\le\; 3\,E\big(\sqrt{\tau}\big) \;\; . \tag{3.18}$$

Since $k/(k-1) \le 2$ and $2(4k+1) \le 9k$ for $k \ge 2$, then from (3.17) and (3.18) by Taylor expansion we easily obtain:

$$\begin{aligned} E\left(\exp\left(\frac{B_\tau^*}{c} \right) \right) - 1 &= \sum_{k=1}^{\infty} \frac{E(B_\tau^*)^k}{c^k \, k!} \\ &\le \frac{3E(\sqrt{\tau})}{c} + \sum_{k=2}^{\infty} \left(\frac{6}{c} \right)^k \frac{k^{k/2}}{k!} \, E\big(\tau^{k/2}\big) \le \sum_{k=1}^{\infty} \left(\frac{6}{c} \right)^k \frac{k^{k/2}}{k!} \, E\big(\tau^{k/2}\big) \end{aligned} \tag{3.19}$$

for all $c > 0$. Further, by Stirling's formula we find:

$$\frac{k^{k/2}}{k!} \;=\; \frac{k^{k/2}}{\sqrt{2\pi k} \;\, k^k \, e^{-k} \, e^{r_k}} \;\le\; \frac{e^k}{\sqrt{2\pi} \, k^{(k+1)/2}} \tag{3.20}$$

where $1/(12k+1) < r_k < 1/(12k)$ for $k \ge 1$. Inserting (3.20) into (3.19), and using Jensen's inequality, we obtain the estimate:

$$E\left(\exp\left(\frac{B_\tau^*}{c}\right)\right) - 1 \le \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{\infty} \left(\frac{6e}{c}\right)^k \frac{E\big(\tau^{k/2}\big)}{k^{(k+1)/2}} \tag{3.21}$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{\infty} \left(\frac{6e\sqrt{D}}{c}\right)^k \frac{E\big(\tau^{k/2}\big)}{D^{k/2}\, k^{(k+1)/2}} \le \frac{\sqrt{\Delta_1(D)}}{\sqrt{2\pi}} \sum_{k=1}^{\infty} \left(\frac{6e\sqrt{D}}{c}\right)^k$$

for all c , $D > 0$. Identifying the last term in (3.21) with 1 , we easily find out that:

$$c = 6\,e\sqrt{D}\left(1 + \frac{\sqrt{\Delta_1(D)}}{\sqrt{2\pi}}\right) .$$

This proves (3.16), and hence (3.14) follows as well. The proof is complete. □

Remark 3.5

It should be noted that condition (3.13) may be equivalently formulated as follows:

$$\|\tau\|_{\psi_1} < \infty \; . \tag{3.13'}$$

This is easily verified by applying Stirling's formula. Thus, in short, the result of Theorem 3.4 may be summarized: If τ is a stopping time for Brownian motion $B = (B_t)_{t\ge 0}$ such that $\|\tau\|_{\psi_1} < \infty$, then $\|\max_{0\le t\le\tau} |B_t|\|_{\psi_1} < \infty$ as well.

Remark 3.6

Under the hypotheses of Theorem 3.4, let us refine the definition of $\Delta_1(D)$ by setting:

$$\Delta_1^*(D) = \sup_{k\ge 1}\left(\frac{E\big(\tau^{k/2}\big)}{D^k\, k^{(k+1)/2}}\right)$$

for $D > 0$. Then the preceding proof (without application of Jensen's inequality in (3.21)) shows that the following refinement of the estimate (3.16) is valid:

$$\Big\| \max_{0\le t\le\tau} |B_t| \Big\|_{\psi_1} \le 6\,e D\left(1 + \frac{\Delta_1^*(D)}{\sqrt{2\pi}}\right) \tag{3.16'}$$

for all $D > 0$. Although not as fine as (3.16'), the estimate (3.16) seems to be more applicable, since $\Delta_1(D)$ is generally computed easier. We shall illustrate this in the next corollary and two examples, which serve important applications of Theorem 3.4.

Corollary 3.7

Let $B = (B_t)_{t\ge 0}$ be standard Brownian motion, and let τ be a stopping time for B satisfying the following condition:

$$E\big(\tau^k\big) \le C\big(E\tau\big)^k k^k \tag{3.22}$$

for all $k \ge 1$ with some constant $C > 0$. Then the inequality is satisfied:

$$\Big\| \max_{0\le t\le\tau} |B_t| \Big\|_{\psi_1} \le 6e\left(1+\sqrt{\tfrac{C}{2\pi}}\right)\sqrt{E(\tau)} \; . \tag{3.23}$$

Proof. In the notation of Theorem 3.4 take $D = E(\tau)$, then by (3.22) we get $\Delta_1(D) \leq C$. Hence (3.23) follows straightforward from (3.16). This completes the proof. □

Example 3.8

Let the stopping time $\tau \sim Exp(\lambda)$ for Brownian motion $B = (B_t)_{t \geq 0}$ be exponentially distributed with parameter $\lambda > 0$. Then $E(\tau^k) = k!/\lambda^k$ for $k \geq 1$. Thus (3.22) is satisfied with $C = 1$, and by (3.23) we see that τ satisfies the inequality:

$$\Big\| \max_{0 \leq t \leq \tau} |B_t| \Big\|_{\psi_1} \leq 6e \left(1 + \frac{1}{\sqrt{2\pi}} \right) \sqrt{E(\tau)} \tag{3.24}$$

which is valid simultaneously for all $\lambda > 0$. It should be noted here that τ is not supposed to be independent from B (as it is occasionally assumed in the literature). □

Further applications of Theorem 3.4 are available. We will not pursue this in more detail here, but instead consider another typical example.

Example 3.9

Let the stopping time $\tau \sim |N(0,\sigma^2)|$ for Brownian motion $B = (B_t)_{t \geq 0}$ be from reflecting Gaussian distribution with mean 0 and variance σ^2 . Thus, if $X \sim N(0,\sigma^2)$, then $\tau \sim |X|$. It is well-known that we have:

$$E(\tau^k) = \frac{2^{k/2}\sigma^k}{\sqrt{\pi}} \, \Gamma((k+1)/2) \leq \frac{\big(\sqrt{2}\sigma\big)^k}{\sqrt{\pi}} \, k^k \tag{3.25}$$

for all $k \geq 1$. In the notation of Theorem 3.4 take $D = \sqrt{2}\sigma = \sqrt{\pi}E(\tau)$, then by (3.25) we get $\Delta_1(D) \leq 1/\sqrt{\pi}$. Thus by (3.16) we see that τ satisfies the inequality:

$$\Big\| \max_{0 \leq t \leq \tau} |B_t| \Big\|_{\psi_1} \leq 6e \left(\sqrt{\sqrt{\pi}} + \frac{1}{\sqrt{2\pi}} \right) \sqrt{E(\tau)} \tag{3.26}$$

which is valid simultaneously for all $\sigma^2 > 0$. Note that the constant in (3.26) is only "slightly" increased in comparison with the constant appearing in (3.24). □

Problem 3.10

It should be noted that our method above, although optimally performed in accordance with our main idea which relies upon Taylor expansion and best constants in the L^p-inequalities, leaves an uncovered gap in between necessary and sufficient conditions which are obtained. To explain this in more detail, note that Theorem 3.2 states that a necessary condition for $\| \max_{0 \leq t \leq \tau} |B_t| \, \|_{\psi_1} < \infty$ is $E(\tau^k) = O(C^k k^{3k})$, while by Theorem 3.4 a sufficient condition is $E(\tau^k) = O(C^k k^k)$. We want to make it clear here that no attempt will be made in the paper to single out a common necessary and sufficient condition, due partially to the fact that our sufficient condition is relatively satisfactory (recall Example 3.8 and Example 3.9) and partially to the fact that such an attempt would require a new method (which is far from being evident). Our main aim in this paper was to extract as much as possible from the method which relies upon Taylor expansion and best constants in the L^p-inequalities, and it is clear from our proof that this goal is achieved. Anyway, from the general point of view, the following problem appears worthy of consideration. *Consider the family of stopping times* τ *for Brownian motion* $B = (B_t)_{t \geq 0}$ *satisfying* $E(\tau^k) < \infty$ *for all*

$k \geq 1$ *, and find out the number* $\alpha > 0$ *such that the condition* $E(\tau^k) = O(C^k k^{\alpha k})$ *is equivalent to the fact* $\| \max_{0 \leq t \leq \tau} |B_t| \, \|_{\psi_1} < \infty$. Note that our results above show that such a number α belongs to the interval $[1, 3]$.

In the context of the preceding problem, the following example is of interest.

Example 3.11

Let $B = (B_t)_{t \geq 0}$ be standard Brownian motion, and consider the stopping time for B :

$$\tau \, = \, \inf \, \{ \, t \geq 1 \, : \, |B_t| \, = \, a t^\beta \, \}$$

where $a > 0$ and $0 < \beta < 1/2$ are given and fixed. Set $\tau_n = \tau \wedge n$ for $n \geq 1$. Note that by Jensen's inequality we have:

$$E|B_{\tau_n}|^{2k} \, = \, a^{2k} E(\tau_n)^{2\beta k} \, \leq \, a^{2k} \big(E(\tau_n)^k \big)^{2\beta} \tag{3.27}$$

for all n , $k \geq 1$. On the other hand, by (2.4)+(2.6) and (2.22) we have:

$$\left(\frac{\pi}{\sqrt{2(4k+1)}} \right)^{2k} E(\tau_n)^k \, \leq \, E|B_{\tau_n}|^{2k} \tag{3.28}$$

for all n , $k \geq 1$. From (3.27) and (3.28) by letting $n \to \infty$, we obtain:

$$\Big(E(\tau^k) \Big)^{1-2\beta} \leq \left(\frac{10}{\pi^2} a^2 \right)^k k^k$$

for all $k \geq 1$. This shows that:

$$E(\tau^k) \, = \, O\big(C^k k^{k/(1-2\beta)} \big) \tag{3.29}$$

for some $C > 0$ as $k \to \infty$. In view of (2.1) it is clear that $\| \max_{0 \leq t \leq \tau} |B_t| \, \|_{\psi_1} < \infty$ if and only if $\|B_\tau\|_{\psi_1} < \infty$, which is by Stirling's formula easily verified to be equivalent to $E(\tau^k) = O(C^k k^{k/\beta})$ for some $C > 0$ as $k \to \infty$. Thus, putting $1/(1-2\beta) \leq 1/\beta$, or equivalently $\beta \leq 1/3$, by (3.29) we see that $E(\tau^k) = O(C^k k^{k/\beta})$, showing that:

$$\Big\| \max_{0 \leq t \leq \tau} |B_t| \Big\|_{\psi_1} < \, \infty \tag{3.30}$$

for all $0 < \beta \leq 1/3$. (We do not know what is going on with (3.30) for $\beta \in \,]1/3, 1/2[$.)

It is not known to us is the estimate (3.29) optimal, which if so would show (take $\beta = 1/3$) that there exists a stopping time τ for B with $E(\tau^k) \geq C^k k^{3k}$ such that $\| \max_{0 \leq t \leq \tau} |B_t| \|_{\psi_1} < \infty$ (recall Theorem 3.2 with $p = 1$). This sort of argument would pass through in the case of any optimal improvement upon (3.29) (by taking another β as above). Note, however, that the estimate (3.29) is asymptotically efficient in the following sense. The case $\beta \downarrow 0$ corresponds to the hitting time of the point $a > 0$ given by $\tau = \inf \{ \, t \geq 1 : |B_t| = a \, \}$. By the i.i.d. increments we have $P\{ \, \tau > n+1 \, \} = P\{ \, \max_{1 \leq t \leq n+1} |B_t| \, < \, a \, \} \leq P\{ \, |B_2 - B_1| < 2a \, , \, |B_3 - B_2| < 2a \, , \, \dots \, , \, |B_{n+1} - B_n| < 2a \, \} = \big(P\{ \, |B_1| < 2a \, \} \big)^n$ for all $n \geq 1$. Thus $P\{ \, \tau > n \, \} \leq \varepsilon^n$ for all $n \geq 1$ with some $\varepsilon > 0$, which easily implies that $E\big(\exp(\tau/c) \big) < \infty$ for some $c > 0$. This is equivalent to $\|\tau\|_{\psi_1} < \infty$, or in other words $E(\tau^k) = O(C^k k^k)$ which suits with (3.29) when $\beta \downarrow 0$. The case $\beta \uparrow 1/2$ corresponds to the case of the hitting time of the square root stopping boundaries

(2.26), for which we know by (2.28) (and (2.24) with $z_p \downarrow 0$ as $p \to \infty$) that only finitely many moments are finite, thus again agreeing with (3.29) when $\beta \uparrow 1/2$. $\square$

In the next theorem we generalize and extend the result of Theorem 3.4 to the case $1 < p \le 2$.

Theorem 3.12

Let $B = (B_t)_{t\ge 0}$ be standard Brownian motion, and let τ be a stopping time for B satisfying the following condition:

$$E\big(\tau^k\big) \;=\; O\big(C^k k^{k(2-p)/p}\big) \tag{3.31}$$

for some $C > 0$ as $k \to \infty$, where $1 < p \le 2$ is given and fixed. Then we have:

$$\Big\| \max_{0\le t\le \tau} |B_t| \Big\|_{\psi_p} < \; \infty \; . \tag{3.32}$$

Moreover, let us define the function:

$$\Delta_p(D) \;=\; \sup_{k\ge 1} \left(\frac{E\big(\tau^k\big)}{D^k \, k^{k(2-p)/p}} \right) \tag{3.33}$$

for $D > 0$. Then the following estimate is valid:

$$\Big\| \max_{0\le t\le \tau} |B_t| \Big\|_{\psi_p} \le \sqrt{10}\; \frac{p}{p-1}\, e^{1/p} \sqrt{D} \left(1 + \frac{\big(\Delta_p(D)\big)^{p/2}}{\sqrt{2\pi}} \right) \tag{3.34}$$

for all $D > 0$.

Proof. Given a stopping time τ for B satisfying (3.31), denote $B_\tau^* = \max_{\,0\le t\le \tau} |B_t|$. Then by (2.1), (2.3)+(2.5), (2.23) and (2.25) we have:

$$\begin{aligned} E\big(B_\tau^*\big)^{kp} &\le \left(\frac{kp}{kp-1}\right)^{kp} E|B_\tau|^{kp} \le \left(\frac{kp}{kp-1}\right)^{kp} \big(z_{kp}^*\big)^{kp} E\big(\tau^{kp/2}\big) \\ &\le \left(\frac{kp}{kp-1}\right)^{kp} \big(z_{2k}^*\big)^{kp} E\big(\tau^{kp/2}\big) \le \left(\frac{kp}{kp-1}\right)^{kp} \Big(\sqrt{2(4k+1)}\,\Big)^{kp} E\big(\tau^{kp/2}\big) \end{aligned} \tag{3.35}$$

for all $k \ge 1$. Since $kp/(kp-1) \le p/p-1$ and $2(4k+1) \le 10k$ for $k \ge 1$, then from (3.35) by Taylor expansion we easily obtain:

$$\begin{aligned} E\left(\exp\left(\frac{B_\tau^*}{c}\right)^{p}\right) - 1 &= \sum_{k=1}^{\infty} \frac{E(B_\tau^*)^{kp}}{c^{kp}\, k!} \\ &\le \sum_{k=1}^{\infty} \left(\frac{\sqrt{10}\, p}{c\,(p-1)}\right)^{kp} \frac{k^{kp/2}}{k!}\, E\big(\tau^{kp/2}\big) \end{aligned} \tag{3.36}$$

for all $c > 0$. Further, by Stirling's formula we find:

$$\frac{k^{kp/2}}{k!} = \frac{k^{kp/2}}{\sqrt{2\pi k}\; k^k\, e^{-k}\, e^{r_k}} \le \frac{e^k}{\sqrt{2\pi}\, k^{k(1-p/2)+1/2}} \tag{3.37}$$

where $1/(12k+1) < r_k < 1/(12k)$ for $k \geq 1$. Inserting (3.37) into (3.36), and using Jensen's inequality, we obtain the estimate:

$$E\left(\exp\left(\frac{B_\tau^*}{c}\right)^p\right) - 1 \leq \frac{1}{\sqrt{2\pi}} \sum_{k=1}^\infty \left(\frac{\sqrt{10}\, p\, e^{1/p}}{c\,(p-1)}\right)^{kp} \frac{E\big(\tau^{kp/2}\big)}{k^{k(2-p)/2+1/2}} \tag{3.38}$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{k=1}^\infty \left(\frac{\sqrt{10}\, p\, e^{1/p}\sqrt{D}}{c\,(p-1)}\right)^{kp} \frac{E\big(\tau^{kp/2}\big)}{D^{kp/2}\, k^{k((2-p)/p)(p/2)+1/2}}$$

$$\leq \frac{\big(\Delta_p(D)\big)^{p/2}}{\sqrt{2\pi}} \sum_{k=1}^\infty \left(\frac{\sqrt{10}\, p\, e^{1/p}\sqrt{D}}{c\,(p-1)}\right)^{kp}$$

for all c , $D > 0$. Identifying the last term in (3.38) with 1 , we easily find out that:

$$c = \sqrt{10}\ \frac{p}{p-1}\ e^{1/p}\sqrt{D}\left(1 + \frac{\big(\Delta_p(D)\big)^{p/2}}{\sqrt{2\pi}}\right) .$$

This proves (3.34), and hence (3.32) follows as well. The proof is complete. $\square$

Remark 3.13

It should be noted that the scope of Theorem 3.12 is rather limited, since $1 > (2-p)/p \downarrow 0$ as $p \uparrow 2$. It indicates that the real power of the method which is used throughout in this paper belongs to the cases when p is small. This is stated more explicitly in the next theorem. Finally, it is clear that the analogues of Remark 3.6 and Problem 3.10 can be formulated in the context of Theorem 3.12 as well. The details in this direction are left to the reader.

Theorem 3.14

Let $B = (B_t)_{t \geq 0}$ be standard Brownian motion, and let τ be a stopping time for B satisfying the following condition:

$$E\big(\tau^k\big) = O\big(C^k k^{k(2-p)/p}\big) \tag{3.39}$$

for some $C > 0$ as $k \to \infty$, where $0 < p \leq 2$ is given and fixed. Then we have:

$$E\left(\exp\Big(\varepsilon\Big(\max_{0 \leq t \leq \tau} |B_t|^p\Big)\Big)\right) < \infty \tag{3.40}$$

for some (small enough) $\varepsilon > 0$.

Proof. The case $p = 1$ is proved in Theorem 3.4, while the case $1 < p \leq 2$ is proved in Theorem 3.12. Consider the case $0 < p < 1$. For this, let J denote the smallest $k \geq 2$ such that $kp > 1$. Then by Taylor expansion, Jensen's inequality, and (2.2) (with $p=1$), we have:

$$E\left(\exp\left(\frac{B_\tau^*}{c}\right)^p\right) - 1 = \sum_{k=1}^{J-1} \frac{E(B_\tau^*)^{kp}}{c^{kp}\, k!} + \sum_{k=J}^\infty \frac{E(B_\tau^*)^{kp}}{c^{kp}\, k!} \tag{3.41}$$

$$\leq \frac{(J-1)}{c^{(J-1)p}} \max_{1 \leq k \leq J-1} \frac{\big(3E(\sqrt{\tau})\big)^{kp}}{k!} + \sum_{k=J}^\infty \frac{E(B_\tau^*)^{kp}}{c^{kp}\, k!}$$

$$\leq \frac{(J-1)}{c^{(J-1)p}} \left(\left(3E(\sqrt{\tau})\right)^p + \left(3E(\sqrt{\tau})\right)^{(J-1)p} \right) + \sum_{k=J}^{\infty} \frac{E(B_\tau^*)^{kp}}{c^{kp}\, k!}$$

for all $c > 1$. The first term on the right-hand side in (3.41) is arbitrarily small when c becomes large, while the second term is controlled (and proved to be finite and arbitrarily small) in exactly the same way as in the proof of Theorem 3.12. This completes the proof of the theorem. □

Remark 3.15

Note that (3.41) with (3.35)+(3.37) (as in (3.36)+(3.38)) leaves a possibility of obtaining an estimate of the quantity on the left-hand side in (3.40). The details are left to the reader.

4. Exponential integrability of continuous local martingales

The results obtained in Section 3 for Brownian motion will be extended in this section to continuous local martingales (and Ito's integral). First we want to display the facts which make this extension possible. Recall that a process $M = (M_t)_{t\geq 0}$ is called a *local martingale* (with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ of $(\Omega, \mathcal{F}, P)$), if M_0 is $\mathcal{F}_0$-measurable and there exists an increasing sequence of $(\mathcal{F}_t)$-stopping times $(T_n)_{n\geq 1}$ with $T_n \uparrow \infty$ as $n \uparrow \infty$, such that each "stopped" process $\big(M_{t\wedge T_n} - M_0\big)_{t\geq 0}$ is a martingale (with respect to $(\mathcal{F}_t)_{t\geq 0}$) for $n \geq 1$. If $M = (M_t)_{t\geq 0}$ is a continuous local martingale with $M_0 = 0$, then there exists a unique continuous increasing process $[M] = ([M]_t)_{t\geq 0}$ such that $M^2 - [M] = (M_t^2 - [M]_t)_{t\geq 0}$ is a continuous local martingale (see [14]). The process $[M]$ is called the *quadratic variation process* of M .

The extension mentioned above relies upon a well-known fact that every continuous local martingale is a time-changed Brownian motion. This result is due to Dubins and Schwarz [4], and more precisely may be stated as follows. Let $M = (M_t)_{t\geq 0}$ be a continuous local martingale (with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ of $(\Omega, \mathcal{F}, P)$) such that $M_0 = 0$ and $[M]_t \uparrow \infty$ as $t \uparrow \infty$. Define the stopping time:

(4.1) $$\tau_t = \inf\left\{ s > 0 \,:\, [M]_s = t \right\}$$

and denote $\mathcal{G}_t = \mathcal{F}_{\tau_t}$ for $t \geq 0$. Then $B_t = M_{\tau_t}$ with $t \geq 0$ defines a standard Brownian motion B with respect to the filtration $(\mathcal{G}_t)_{t\geq 0}$. Moreover, for any given and fixed $t \geq 0$, the random variable $[M]_t$ is a $(\mathcal{G}_t)$-stopping time, and we have:

(4.2) $$M_t = B_{[M]_t}$$

for all $t \geq 0$. The result remains valid without the restriction that $[M]_t \uparrow \infty$ as $t \uparrow \infty$, but at the expense of an enlargement of the underlying probability setting $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ (for more details see [14]).

The preceding result has its analogue for Ito's integral as follows (see [9]). Let $B = (B_t)_{t\geq 0}$ be standard Brownian motion, and let $Z = (Z_t)_{t\geq 0}$ be a non-anticipating random process satisfying $\int_0^\infty Z_t^2\, dt < \infty$ P-a.s. Then there exist a standard Brownian motion $\hat{B} = (\hat{B}_t)_{t\geq 0}$ and a stopping time τ for $\hat{B}$ (both depending on Z) such that:

(4.3) $$Law(\tau) = Law\left(\int_0^\infty Z_t^2\, dt \right)$$

$$\text{(4.4)} \qquad Law(\hat{B}_\tau) = Law\Big(\int_0^\infty Z_t \, dB_t \Big) \; .$$

In order to apply the results from Section 3 in the context of a continuous local martingale $(M_t)_{t \geq 0}$, it is enough to recall that $[M]_t$ is a stopping time for Brownian motion $B = (B_t)_{t \geq 0}$ (relative to the filtration $(\mathcal{G}_t)_{t \geq 0}$ defined above), while by (4.2) we clearly have:

$$\text{(4.5)} \qquad \max_{0 \leq s \leq t} |M_s| = \max_{0 \leq s \leq t} \big| B_{[M]_s} \big| = \max_{0 \leq s \leq [M]_t} |B_s|$$

for all $t \geq 0$. These two facts clarify generalizations and extensions of Theorem 3.2, Theorem 3.4, Theorem 3.12 and Theorem 3.14, which are formulated in the following theorem.

Theorem 4.1

Let $M = (M_t)_{t \geq 0}$ *be a continuous local martingale such that its quadratic variation process* $[M] = ([M]_t)_{t \geq 0}$ *satisfies:*

$$\text{(4.6)} \qquad E\big([M]_t\big)^k < \infty$$

for all $k \geq 1$ *with some* $t > 0$ *which is given and fixed. If we have:*

$$\text{(4.7)} \qquad \Big\| \max_{0 \leq s \leq t} |M_s| \Big\|_{\psi_p} < \infty$$

for some $0 < p < \infty$ *, then the condition is satisfied:*

$$\text{(4.8)} \qquad E\big([M]_t\big)^k = O\Big(C^k k^{k(2+p)/p} \Big)$$

for some $C > 0$ *as* $k \to \infty$ *. Moreover, if for some* $0 < p \leq 2$ *the condition is satisfied:*

$$\text{(4.9)} \qquad E\big([M]_t\big)^k = O\big(C^k k^{k(2-p)/p} \big)$$

for some $C > 0$ *as* $k \to \infty$ *, then we have:*

$$\text{(4.10)} \qquad \Big\| \max_{0 \leq s \leq t} |M_s| \Big\|_{\psi_p} < \infty \; .$$

Finally, if for $1 \leq p \leq 2$ *we set:*

$$\text{(4.11)} \qquad \Delta_p(D,t) = \sup_{k \geq 1} \left(\frac{E\big([M]_t\big)^k}{D^k \, k^{k(2-p)/p}} \right)$$

for $D > 0$ *, then the estimates are valid:*

$$\text{(4.12)} \qquad \Big\| \max_{0 \leq s \leq t} |M_s| \Big\|_{\psi_1} \leq 6e\sqrt{D} \left(1 + \frac{\sqrt{\Delta_1(D,t)}}{\sqrt{2\pi}} \right)$$

$$\text{(4.13)} \qquad \Big\| \max_{0 \leq s \leq t} |M_s| \Big\|_{\psi_p} \leq \sqrt{10} \, \frac{p}{p-1} \, e^{1/p} \sqrt{D} \left(1 + \frac{\big(\Delta_p(D,t)\big)^{p/2}}{\sqrt{2\pi}} \right)$$

for all $D > 0$ *and all* $1 < p \leq 2$. □

Remark 4.2

Note that (4.9)+(4.10) with $p = 1$ states: If $M = (M_t)_{t \geq 0}$ is a continuous local martingale such that $\big\| [M]_t \big\|_{\psi_1} < \infty$ for some $t > 0$, then $\| \max_{0 \leq s \leq t} |M_s| \|_{\psi_1} < \infty$ as well. It is moreover clear that many of the facts from Section 3 (which are stated in the remarks and examples) carry over in an obvious way to cover the continuous local martingale case. We will not pursue this in more detail here, but instead concentrate to the Ito's integral as a particular example.

The generalization and extension of the results from Section 3 to the Ito's integral follow immediately by use of (4.3) and (4.4). This is formulated in the following theorem.

Theorem 4.3

Let $B = (B_t)_{t \geq 0}$ *be standard Brownian motion, and let* $Z = (Z_t)_{t \geq 0}$ *be a non-anticipating random process satisfying:*

$$E\Big(\int_0^\infty Z_t^2 \, dt \Big)^k < \infty \tag{4.14}$$

for all $k \geq 1$ *. If we have:*

$$\Big\| \int_0^\infty Z_t \, dB_t \Big\|_{\psi_p} < \infty \tag{4.15}$$

for some $0 < p < \infty$ *, then the condition is satisfied:*

$$E\Big(\int_0^\infty Z_t^2 \, dt \Big)^k = O\Big(C^k k^{k(2+p)/p} \Big) \tag{4.16}$$

for some $C > 0$ *as* $k \to \infty$ *. Moreover, if for some* $0 < p \leq 2$ *the condition is satisfied:*

$$E\Big(\int_0^\infty Z_t^2 \, dt \Big)^k = O\big(C^k k^{k(2-p)/p} \big) \tag{4.17}$$

for some $C > 0$ *as* $k \to \infty$ *, then we have:*

$$\Big\| \int_0^\infty Z_t \, dB_t \Big\|_{\psi_p} < \infty \, . \tag{4.18}$$

Finally, if for $1 \leq p \leq 2$ *we set:*

$$\Delta_p(D, Z) = \sup_{k \geq 1} \left(\frac{E\big(\int_0^\infty Z_t^2 \, dt \big)^k}{D^k \, k^{k(2-p)/p}} \right) \tag{4.19}$$

for $D > 0$ *, then the estimates are valid:*

$$\Big\| \int_0^\infty Z_t \, dB_t \Big\|_{\psi_1} \leq 6e\sqrt{D} \left(1 + \frac{\sqrt{\Delta_1(D, Z)}}{\sqrt{2\pi}} \right) \tag{4.20}$$

$$\Big\| \int_0^\infty Z_t \, dB_t \Big\|_{\psi_p} \leq \sqrt{10} \, \frac{p}{p-1} \, e^{1/p} \sqrt{D} \left(1 + \frac{\big(\Delta_p(D, Z) \big)^{p/2}}{\sqrt{2\pi}} \right) \tag{4.21}$$

for all $D > 0$ *and all* $1 < p \leq 2$ *.* □

Remark 4.4
Note that (4.17)+(4.18) with $p = 1$ states: If $B = (B_t)_{t \geq 0}$ is Brownian motion and $Z = (Z_t)_{t \geq 0}$ is a non-anticipating process such that $\| \int_0^\infty Z_t^2 \, dt \|_{\psi_1} < \infty$, then $\| \int_0^\infty Z_t \, dB_t \|_{\psi_1} < \infty$ as well.

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