Theory Probab. Math. Statist. No. 57, 1997, (140-159)
Preprint Ser. No. 11, 1993, *Math. Inst. Aarhus*

On the Central Limit Theorem for Aperiodic Dynamical Systems and Applications

T. De La Rue, S. Ladouceur, G. Peskir, M. Weber

If $(X, \mathcal{A}, \mu, T)$ is an aperiodic dynamical system, then there exists $f \in L^2(\mu)$ with $\int f \, d\mu = 0$ satisfying the central limit theorem:

$$\mu\Big\{ x \in X \mid \frac{\sum_{j=0}^{n-1} T^j f(x)}{\| \sum_{j=0}^{n-1} T^j f \|_2} \le u \Big\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} \exp\Big(\frac{-v^2}{2}\Big) \, dv$$

as $n \to \infty$, for all $u \in \mathbf{R}$. The method of proof is based on Burton-Denker's construction [1] which in turn relies upon Kakutani-Rochlin's lemma and imitates the analogous result for irrational rotations of the unit circle which is obtained by using Fourier series. A fundamental fact in the background of the entire construction is provided by using Rochlin's result on a factor space of Lebesgue space. The final step in the proof uses a variant of Berry-Esséen's theorem. In addition, the result is extended to a more general case involving orbits of aperiodic dynamical systems. A complete description of weak Gaussian limits in this case is obtained. Applications to the Gaussian *GB* concept in the study of almost everywhere properties of sequences of linear operators on $L^2(\mu)$ are given.

1. Introduction and preliminary results

A main purpose of the paper is to exhibit a real valued function f defined on the phase space X of a given aperiodic dynamical system $(X, \mathcal{A}, \mu, T)$ such that the natural long-term ratio satisfies *the central limit theorem*:

$$\mu\Big\{ x \in X \mid \frac{\sum_{j=0}^{n-1} T^j f(x)}{\| \sum_{j=0}^{n-1} T^j f \|_2} \le u \Big\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} \exp\Big(\frac{-v^2}{2}\Big) \, dv \tag{1.1}$$

as $n \to \infty$, for all $u \in \mathbf{R}$. For the meaning and importance of this problem, as well as for the historical background, we refer the reader to the original Burton-Denker's paper [1]. It appears, however, that the original proof of (1.1) contains a mistaken argument, see the proof of Lemma 4 (p.722) in [1], and the present paper is partly devoted to its amendment, see Proposition 1.1, Corollary 1.2 and Remark 1.3 below. In the rest of the paper we present a modified construction of the desired function and some of its extensions and applications. To conclude the preliminary part of this section we use the opportunity to mention M. Kac as an initiator of investigations on the central limit theorem in dynamical systems, see [7] with discussion and references, while for recent results in this direction we refer to [10] with references. Besides these contributions, we would also like to point out [4], [6] and [12] where some classical facts on the subject appeared. For recent results related to [4] we refer to [13].

AMS 1980 *subject classifications*. Primary 28D05, 28D10, 60F05. Secondary 40C05, 47A35, 47B38, 60G15.
Key words and phrases: Central limit theorem, aperiodic dynamical system, Kakutani-Rochlin lemma, factor space, Berry-Esséen's theorem, continuity theorem, orbit, Gaussian, *GB*-set. (Second edition)

We recall that $(X, \mathcal{A}, \mu, T)$ is said to be *a dynamical system*, if $(X, \mathcal{A}, \mu)$ is *a Lebesgue space*, and T is *a measure-preserving transformation* of X. Let us clarify that a complete finite measure space is called *a nonatomic Lebesgue space*, if it is isomorphic (mod 0) to the ordinary Lebesgue space $([0,\gamma], \mathcal{L}([0,\gamma]), \lambda)$ for some $\gamma > 0$. In other words, there exist sets of measure zero $Z_1 \subset X$ and $Z_2 \subset [0,\gamma]$, and a measurable bijection $\psi : X \setminus Z_1 \to [0,\gamma] \setminus Z_2$, such that ψ^{-1} is measurable and $\mu = \lambda\psi$. A nonatomic Lebesgue space joined with finitely or countably many point masses of finite total mass is called *a Lebesgue space*. Note that, when we speak of a Lebesgue space, we always assume that $\mu(X) = 1$. For basic results concerning Lebesgue spaces we refer to [2]. In addition, we recall that the given T is said to be *aperiodic*, if $\mu\{\, x \in X \mid T^n x = x \,\} = 0$ for all $n \geq 1$. It is instructive to observe that in this case the measure space $(X, \mathcal{A}, \mu)$ is nonatomic. This fact follows straightforwardly by Poincaré recurrence theorem, see [5]. It can be also easily verified by using the following well-known and in the sequel useful result, see [5]:

(1.2) (*Kakutani-Rochlin's lemma*)

If T is aperiodic, then for every $\varepsilon > 0$ and for every $n \geq 1$ there exists $F \in \mathcal{A}$ such that the sets F, $T^{-1}(F)$ ... $T^{-(n-1)}(F)$ are mutually disjoint, and such that we have:

$$\mu\big(\, F \cup T^{-1}(F) \cup \ldots \cup T^{-(n-1)}(F) \,\big) > 1 - \varepsilon .$$

Any set $F \in \mathcal{A}$ satisfying the conclusions of (1.2) with the given and fixed $\varepsilon > 0$ and $n \geq 1$ will be called *an* (ε, n)*-Kakutani-Rochlin set*. The essential fact on such sets needed in the Burton-Denker construction is proved in Corollary 1.2 below, see also Remark 1.3. The next proposition establishes the main step in its proof. As a preliminary fact in this direction we recall a well-known result on Lebesgue spaces due to Rochlin, see [11] (p.31):

(1.3) The factor space of a Lebesgue space with respect to a measurable decomposition is a Lebesgue space.

In particular, consider as in Proposition 1.1 below, a Lebesgue space $(X, \mathcal{A}, \mu)$ and a σ-algebra $\mathcal{B}$ without atom, generated by a countable family $(B_n)_{n\geq 1}$ of elements from $\mathcal{A}$. Let ζ be the decomposition of X generated by $\mathcal{B}$. That is, we introduce the equivalence relation on X by putting $x' \sim x''$, if and only if $1_B(x') = 1_B(x'')$ for all $B \in \mathcal{B}$, and we put $\zeta = \{\, [x] \mid x \in X \,\}$ to denote the set of all equivalence classes. Since for any two points $x', x'' \in X$ we have $1_B(x') = 1_B(x'')$ for all $B \in \mathcal{B}$, if and only if $1_{B_n}(x') = 1_{B_n}(x'')$ for all $n \geq 1$, we see that ζ is measurable in the sense of Rochlin, see [11] (p.4-5,26). That is, the decomposition ζ is generated by countable family of measurable sets $(B_n)_{n\geq 1}$. Notice that $\zeta \subset \mathcal{B} \subset \mathcal{A}$. Moreover, since $\mathcal{B}$ is without atom, we have $\mu(C) = 0$ for all $C \in \zeta$. Hence we easily find that $(X, \mathcal{A}, \mu)$ is nonatomic. Let $(X_\zeta, \mathcal{A}_\zeta, \mu_\zeta)$ be the factor space of $(X, \mathcal{A}, \mu)$ with respect to ζ. That is, we have $X_\zeta = \{\, [x] \mid x \in X \,\}$, $\mathcal{A}_\zeta = \{\, \tilde{A} = \bigcup_{[x]\in X_\zeta} \{[x]\} \mid \tilde{A} \in \mathcal{A} \,\}$ and $\mu_\zeta(\tilde{A}) = \mu(\tilde{A})$ for all $\tilde{A} \in \mathcal{A}_\zeta$. By (1.3) we know that $(X_\zeta, \mathcal{A}_\zeta, \mu_\zeta)$ is a Lebesgue space. Moreover, since $\mathcal{B}$ is without atom, we see that $(X_\zeta, \mathcal{A}_\zeta, \mu_\zeta)$ is nonatomic. In other words, the Lebesgue space $(X_\zeta, \mathcal{A}_\zeta, \mu_\zeta)$ is isomorphic (mod 0) to the ordinary Lebesgue space $([0,1], \mathcal{L}([0,1]), \lambda)$. This fact turns out to be of a vital importance in step 1 of the proof of Proposition 1.1 below. In the sequel the following notation is shown useful. Given a finite measure space $(X, \mathcal{A}, \mu)$ and $A \in \mathcal{A}$, *the trace of* μ *on*

A is a finite measure on $\mathcal{A}$, denoted and defined by $tr(\mu, A)(B) = \mu(A \cap B)$ for all $B \in \mathcal{A}$. If $\mathcal{B}$ is a σ-algebra on X and C is a subset of X, then *the trace of* $\mathcal{B}$ *on* C is a σ-algebra on C, denoted and defined by $tr(\mathcal{B}, C) = \{ B \cap C \mid B \in \mathcal{B} \}$. It is instructive to observe, that if $(X, \mathcal{A}, \mu)$ is a nonatomic Lebesgue space, and A belongs to $\mathcal{A}$ with $\mu(A) > 0$, then $\big(A\, ,\, tr(\mathcal{A}, A)\, ,\, \mu(A)^{-1} tr(\mu, A)\big)$ forms a nonatomic Lebesgue space as well.

Proposition 1.1

Let $(X, \mathcal{A}, \mu)$ *be a Lebesgue space, and let* $\mathcal{B}$ *be a* σ*-algebra without atom, generated by a countable family* $(B_n)_{n \geq 1}$ *of elements from* $\mathcal{A}$*. Then for any finite partition* $\mathcal{P}$ *of* X*, measurable with respect to* $\mathcal{A}$*, there exists a* σ*-algebra* $\mathcal{C}$ *without atom, independent of* $\mathcal{P}$*, and generated by a countable family of elements from* $\mathcal{B}$.

Proof. The construction of C is divided into four steps as follows.

Step 1: Let P be an arbitrary element from $\mathcal{A}$. We show that for any $\theta \in \,]0, 1[$, there exists $A \in \mathcal{B}$ satisfying:

(1.4) $\qquad \mu(A) = \theta$

(1.5) $\qquad \mu(A \cap P) \,=\, \theta \cdot \mu(P)$.

Let ζ be the measurable decomposition generated by $\mathcal{B}$, and let $(X_\zeta, \mathcal{A}_\zeta, \mu_\zeta)$ be the factor space of $(X, \mathcal{A}, \mu)$ with respect to ζ. Denote by f the conditional μ-measure of P with respect to $\mathcal{B}$. Then f can be regarded as a measurable map from X_ξ into $[0, 1]$. According to the remarks stated after (1.3) above, we know that $(X_\zeta, \mathcal{A}_\zeta, \mu_\zeta)$ is isomorphic $(\text{mod } 0)$ to the ordinary Lebesgue space $([0, 1], \mathcal{L}([0, 1]), \lambda)$. Therefore we can restate step 1 as follows.

Step 1': Let f be a measurable function from $[0, 1]$ into itself, and let θ be an element from $]0, 1[$. Then we show that there exists $A \in \mathcal{L} := \mathcal{L}([0, 1])$ satisfying:

(1.4') $\qquad \lambda(A) = \theta$

(1.5') $\qquad \int_A f \, d\lambda \,=\, \theta \cdot \int_{[0,1]} f \, d\lambda$.

Let $\mathcal{L}_\theta$ denote the family of all finite unions of intervals of total length θ, and let $\bar{\mathcal{L}_\theta}$ denote its closure in the topology generated by the metric $d(A, B) = \lambda(A \triangle B)$ for $A, B \in \mathcal{L}$. It is easily seen that $\bar{\mathcal{L}_\theta} = \{ A \in \mathcal{L} \mid \lambda(A) = \theta \}$, and since obviously $\mathcal{L}_\theta$ is connected, then $\bar{\mathcal{L}_\theta}$ is connected as well. Let us define a map ψ from $\bar{\mathcal{L}_\theta}$ into $\mathbf{R}_+$ by:

$$\psi(A) \,=\, \tfrac{1}{\theta} \int_A f \, d\lambda$$

for all $A \in \bar{\mathcal{L}_\theta}$, and let us put $M \,=\, \int_{[0,1]} f \, d\lambda$. Then we claim that there exists $A_+ \in \bar{\mathcal{L}_\theta}$ such that $\psi(A_+) \geq M$. Indeed, put $B = f^{-1}([M, 1])$, and first suppose that $\lambda(B) \geq \theta$. Then obviously we can take for the desired A_+ any measurable $A \subset B$ satisfying $\lambda(A) = \theta$. Next suppose $\lambda(B) < \theta$; then we can fix any measurable $C \subset B^c$ satisfying $\lambda(C) = \theta - \lambda(B)$. Put $A_+ = B \cup C$; then $A_+ \in \bar{\mathcal{L}_\theta}$, and we have:

$$\begin{aligned} \tfrac{1}{\theta} \int_{A_+} f \, d\lambda &= \tfrac{1}{\theta}\big(\int_B f \, d\lambda + M - \int_{C^c} f \, d\lambda \,\big) \\ &\geq \tfrac{1}{\theta}(M \cdot \lambda(B) + M - \lambda(C^c)) \end{aligned}$$

$$= (M-1)\cdot \frac{(1+\lambda(B))}{\theta} + 1 \geq M \ .$$

In a similar way we can find $A_- \in \bar{\mathcal{L}}_\theta$ such that $\psi(A_-) \leq M$. Since ψ is continuous, then $\psi(\bar{\mathcal{L}}_\theta)$ is an interval, and the claim follows.

Step 2: Consider the case where $\mathcal{P} = \{P, P^c\}$, and let $A \in \mathcal{B}$ with $\mu(A) > 0$ be independent of $\mathcal{P}$. Then we claim, that for any $\theta \in]0, \mu(A)[$ there exists $B \in \mathcal{B}$ with $B \subset A$, independent of $\mathcal{P}$, such that $\mu(B) = \theta$. Indeed, this fact follows straightforwardly by applying step 1 to the Lebesgue space $\big(A\, ,\, tr(\mathcal{A}, A)\, ,\, \mu(A)^{-1} tr(\mu, A)\big)$ with the nonatomic σ-algebra $tr(\mathcal{B}, A)$.

Step 3: Consider the case where $\mathcal{P} = \{P, P^c\}$. Then we claim, that there exists a σ-algebra $\mathcal{C}$ without atom, independent of $\mathcal{P}$, and generated by a countable family of elements from $\mathcal{B}$. Indeed, by using step 1 and step 2, we can recursively construct an increasing sequence of finite partitions $\{\mathcal{C}_n\}_{n\geq 1}$ of X whose elements are from $\mathcal{B}$, such that:

(1.6) Each $\mathcal{C}_n$ consists of 2^n atoms of measure 2^{-n}

(1.7) Each atom in $\mathcal{C}_n$ is the union of two atoms from $\mathcal{C}_{n+1}$

(1.8) Each $\mathcal{C}_n$ is independent of $\mathcal{P}$.

Let $\mathcal{C}$ be the smallest σ-algebra on X containing all the atoms of each partition $\mathcal{C}_n$ for $n \geq 1$. Then by (1.6) and (1.7) we see that $\mathcal{C}$ is without atom, and by (1.8) we may easily verify that $\mathcal{C}$ is independent of $\mathcal{P}$. Thus the claim follows.

Step 4: Consider the general case where $\mathcal{P} = \{P_1, \ldots, P_n\}$. Then we claim, that we can find the σ-algebra $\mathcal{C}$ as it is stated in Proposition 1.1. For this, apply step 3 with the partition $\mathcal{P}_1 = \{P_1, P_1^c\}$ and the σ-algebra $\mathcal{B}$. In this way, we can find a σ-algebra $\mathcal{C}_1$ without atom, independent of $\mathcal{P}_1$, and generated by a countable family of elements from $\mathcal{B}$. Then step 3 may be applied with the partition $\mathcal{P}_2 = \{P_2, P_2^c\}$ and the σ-algebra $\mathcal{C}_1$. In this way we can find a σ-algebra $\mathcal{C}_2$ without atom, generated by a countable family of elements from $\mathcal{C}_1$, independent of $\mathcal{P}_2$, and thus of $\mathcal{P}_1 \cup \mathcal{P}_2$ as well. Continuing in this way, we shall at the end obtain a σ-algebra $\mathcal{C}_n$ without atom, generated by a countable family of elements from $\mathcal{B}$, and independent of $\mathcal{P}$. The claim then follows with $\mathcal{C} = \mathcal{C}_n$. These facts complete the proof.

□

Corollary 1.2

Let $(X, \mathcal{A}, \mu)$ *be a nonatomic Lebesgue space, let* T *be a measure-preserving transformation of* X *, let* $F \in \mathcal{A}$ *with* $\mu(F) > 0$ *be a given and fixed set, and let* π_l *be a finite partition of* $T^{-l}(F)$ *with elements from* $\mathcal{A}$ *for all* $0 \leq l < n$ *with some* $n \geq 1$ *. Then for any* $\gamma_1, \ldots, \gamma_p \geq 0$ *with* $\sum_{k=1}^{p} \gamma_k = 1$ *and* $p \geq 1$ *, there exists a partition* $\alpha = \{\, A_1, \ldots, A_p \,\}$ *of* F *with elements from* $\mathcal{A}$ *such that:*

$$\mu\left(T^{-l}(A_k) \cap B\right) = \gamma_k \cdot \mu(B)$$

for all $1 \leq k \leq p$ *, all* $B \in \pi_l$ *, and all* $0 \leq l < n$ *.*

Proof. Applying Proposition 1.1 to the nonatomic Lebesgue space:

$$\big(F,\ tr(\mathcal{A}, F)\ ,\ \mu(F)^{-1} tr(\mu, F)\big)$$

with the partition π_0 and the σ-algebra $tr(\mathcal{A}, F)$, we can find a countably generated σ-algebra $\mathcal{A}_0 \subset tr(\mathcal{A}, F)$ without atom and independent of π_0 with respect to $\mu(F)^{-1} tr(\mu, F)$. We proceed by induction. Suppose that the σ-algebra $\mathcal{A}_l$ is already constructed for some $0 \leq l < n-1$. Then $T^{-(l+1)}(\mathcal{A}_l)$ is a countably generated σ-algebra without atom on $T^{-(l+1)}(F)$, and we can apply Proposition 1.1 to the Lebesgue space:

$$\big(T^{-(l+1)}(F) \, , \, tr(\mathcal{A}, T^{-(l+1)}(F)) \, , \, \mu(F)^{-1} tr(\mu, T^{-(l+1)}(F)) \big)$$

with the partition π_{l+1} and the σ-algebra $T^{-(l+1)}(\mathcal{A}_l)$. In this way we get a countably generated σ-algebra $\mathcal{B}_{l+1} \subset T^{-(l+1)}(\mathcal{A}_l)$ without atom, independent of π_{l+1} with respect to $\mu(F)^{-1} tr(\mu, T^{-(l+1)}(F))$. Then we may define the countably generated σ-algebra $\mathcal{A}_{l+1}$ as follows:

$$\mathcal{A}_{l+1} = \{ \; A \in \mathcal{A}_l \;\; | \;\; T^{-(l+1)}(A) \in \mathcal{B}_{l+1} \; \} \; .$$

The σ-algebra $\mathcal{A}_{n-1} \subset tr(\mathcal{A}, F)$ obtained at the end is clearly such one, that $T^{-l}(\mathcal{A}_{n-1})$ is independent of π_l with respect to $\mu(F)^{-1} tr(\mu, T^{-l}(F))$ for all $0 \leq l < n$. Besides, as it is without atom, we can find a partition $\alpha = \{ A_1, \dots, A_p \} \subset \mathcal{A}_{n-1}$ of F such that $\mu(F)^{-1} \mu(A_k) = \gamma_k$ for all $1 \leq k \leq p$. Hence we get:

$$\mu(F)^{-1} \mu(T^{-l}(A_k) \cap B) = \mu(F)^{-1} \mu(T^{-l}(A_k)) \cdot \mu(F)^{-1} \mu(B) = \gamma_k \cdot \mu(F)^{-1} \mu(B)$$

for all $1 \leq k \leq p$, all $B \in \pi_l$, and all $0 \leq l < n$. This fact completes the proof.

□

Remark 1.3

We shall use Corollary 1.2 with $n = NL$ and F being an (ε, NL)-Kakutani-Rochlin set of an aperiodic dynamical system $(X, \mathcal{A}, \mu, T)$, where $\varepsilon > 0$ and $N, L \geq 1$ with $N \geq 1$ being even. Putting $p = 2^{N/2}$ and $\gamma_k = 2^{-N/2}$ for all $1 \leq k \leq p$ in this way we may conclude the following: *If $F \in \mathcal{A}$ is an (ε, NL)-Kakutani-Rochlin set, and π_l is a finite partition of $T^{-l}(F)$ with elements from $\mathcal{A}$ for all $0 \leq l < NL$, then there exists a finite partition α of F into $2^{N/2}$ sets from $\mathcal{A}$ such that:*

$$\mu \, \big(\, T^{-l}(A) \cap B \, \big) \; = \; 2^{-N/2} \cdot \mu(B)$$

for all $A \in \alpha$, all $B \in \pi_l$, and all $0 \leq l < NL$. It is instructive to observe that in this case we have $\mu(A) = 2^{-N/2} \mu(F)$ for all $A \in \mathcal{A}$.

2. The central limit theorem

Given a dynamical system $(X, \mathcal{A}, \mu, T)$, we shall use $CLT(\mu, T)$ to denote the set of all functions $f \in L^2(\mu)$ with $\int f \; d\mu = 0$ satisfying the central limit theorem as stated in (1.1) above. The main result of this section establishes that $CLT(\mu, T) \neq \emptyset$ whenever T is aperiodic. More precisely, we have:

Theorem 2.1

If $(X, \mathcal{A}, \mu, T)$ is an aperiodic dynamical system, then there exists $f \in L^2(\mu)$ with $\int f \; d\mu = 0$

satisfying the central limit theorem:

$$\mu\Big\{ x \in X \mid \frac{\sum_{j=0}^{n-1} T^j f(x)}{\| \sum_{j=0}^{n-1} T^j f \|_2} \leq u \Big\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} \exp\Big(\frac{-v^2}{2}\Big)\, dv$$

as $n \to \infty$ *, for all* $u \in \mathbf{R}$ *.*

Proof. Let $N, K, L \geq 1$ be given and fixed positive integers, such that N is even and $1 \leq K < N$ is odd. Let $F \in \mathcal{A}$ be an (ε, NL)-Kakutani-Rochlin set, and let π_l be a finite partition of $T^{-l}(F)$ for all $0 \leq l < NL$. Then by Remark 1.3 there exists a partition α of F into $2^{N/2}$ sets from $\mathcal{A}$ satisfying:

$$\mu\,\big(\, T^{-l}(A) \cap B \,\big) \,=\, 2^{-N/2} \cdot \mu(B) \tag{2.1}$$

for all $A \in \alpha$, all $B \in \pi_l$, and all $0 \leq l < NL$. The partition α can be written as follows:

$$\alpha = \{\; A(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{\frac{N}{2}-1}) \;\mid\; \varepsilon_i \in \{-1, 1\} \;\} \;.$$

Define a measurable function $g^{2l} : F \to \{-1, 1\}$ by putting:

$$g^{2l}(x) = \varepsilon_l \quad , \quad \text{if} \quad x \in A(\varepsilon_0, \ldots, \varepsilon_l, \ldots, \varepsilon_{\frac{N}{2}-1})$$

for all $0 \leq l < N/2$. Define a measurable function $g^{(2l+K)\ (mod\ N)} : F \to \{-1, 1\}$ by putting:

$$g^{(2l+K)\ (mod\ N)}(x) \,=\, -g^{2l}(x)$$

for all $x \in F$ and all $0 \leq l < N/2$. In this way a function $g^l : F \to \{-1, 1\}$ is defined for all $0 \leq l < N$. Finally, we define a measurable function $g : X \to \{-1, 0, 1\}$ as follows:

$$g(x) = g^l\big(T^{jN+l}(x)\big) \quad , \quad \text{if} \quad x \in T^{-(jN+l)}(F) \quad \text{for some} \quad 0 \leq j < L \quad \text{and} \quad 0 \leq l < N$$

$$g(x) = 0 \quad , \quad \text{if} \quad x \notin \bigcup_{l=0}^{NL-1} T^{-l}(F) \;.$$

We shall use $S_m(f)(x)$ to denote $\sum_{j=0}^{m-1} f(T^j(x))$ whenever $m \geq 1$ and $x \in X$. In addition, the following two facts on the Kakutani-Rochlin tower $F, T^{-1}(F), \ldots, T^{-NL+1}(F)$ instructively clarify the entire construction and will be freely used below. Namely, we have:

$$T(F) \subset \Big(\bigcup_{l=0}^{NL-1} T^{-l}(F)\Big)^c \cup T^{-NL+1}(F) \tag{2.2}$$

$$T\Big(\big(\bigcup_{l=0}^{NL-1} T^{-l}(F)\,\big)^c\Big) \subset \Big(\bigcup_{l=0}^{NL-1} T^{-l}(F)\Big)^c \cup T^{-NL+1}(F) \;. \tag{2.3}$$

Both statements follow straightforwardly.

Lemma 2.2

Under the hypotheses stated above we have:

(2.4) *Random functions* $g^0, g^2, \ldots, g^{N-2}$ *, as well as random functions* $g^1, g^3, \ldots, g^{N-1}$ *,* *are independent and identically distributed with respect to the probability measure* $\mu(F)^{-1}tr(\mu, F)$ *. Moreover, we have:*

$$\mu(F)^{-1}tr(\mu, F)\,\{\, g^l = \pm 1 \,\} \,=\, \tfrac{1}{2}$$

for all $0 \le l < N$ *.*

(2.5) *Random functions* $g \circ T^i$ *for* $0 \le i < K$ *restricted to* $T^{-l}(F)$ *are independent and identically distributed with respect to the probability measure* $\mu(F)^{-1}tr(\mu, T^{-l}(F))$ *whenever* $K \le l < NL$ *. Moreover, we have:*

$$\mu(F)^{-1}tr(\mu, T^{-l}(F))\,\{\, g \circ T^i = \pm 1 \,\} \,=\, \tfrac{1}{2}$$

for all $0 \le i < K$ *.*

(2.6) *Random function* $S_m(g)$ *restricted to* $T^{-l}(F)$ *is independent of* π_l *with respect to the probability measure* $\mu(F)^{-1}tr(\mu, T^{-l}(F))$ *whenever* $1 \le m \le N$ *and* $N \le l < NL$ *.*

(2.7) *Let* $K \le m < N - K$ *and* $0 \le l < N$ *be given and fixed. Then there exists a set* $J \subset \{\, 0\,, 1\,, \ldots\,, m-1 \,\}$ *of cardinality* $K-1$ *,* K *or* $K+1$ *such that:*

$$\sum_{i=0}^{m-1} g(T^{i+l}(x)) \,=\, \sum_{i \in J} g(T^{i+l}(x))$$

whenever $x \in \bigcup_{j=1}^{L} T^{-jN+1}(F)$ *. Moreover, the random functions* $g \circ T^{i+l}$ *with* $i \in J$ *restricted to* $T^{-l}(F)$ *are independent and identically distributed with respect to the probability measure* $\mu(F)^{-1}tr(\mu, T^{-l}(F))$ *. Moreover, we have:*

$$\mu(F)^{-1}tr(\mu, T^{-l}(F))\,\{\, g \circ T^{i+l} = \pm 1 \,\} \,=\, \tfrac{1}{2}$$

for all $i \in J$ *.*

(2.8) *The following inequality is valid:*

$$|S_m(g)(x)| \le 2\,(K+1)$$

for all $m \ge 1$ *and all* $x \in X$ *outside of a* μ*-null set.*

(2.9) *The following inequalities are valid:*

$$(1-\delta)\, m \,\le\, \|S_m(g)\|_2^2 \,\le\, (1+\delta)\, m$$

for all $1 \le m \le K$ *, with* $\delta \to 0$ *, if* $K(\varepsilon + L^{-1}) \to 0$ *when* $K, L \to \infty$ *and* $\varepsilon \to 0$ *(like maps from* $\mathbf{N}$ *into* $\mathbf{N}$ *, resp.* $]\,0, \infty[$ *, as* $n \to \infty$*) .*

(2.10) *The following inequalities are valid:*

$$(1-\delta)\, K \,\le\, \|S_m(g)\|_2^2 \,\le\, (1+\delta)\, K$$

for all $K \le m < N-K$ *, with* $\delta \to 0$ *, if* $K(\varepsilon + L^{-1}) \to 0$ *when* $K, L \to \infty$ *and* $\varepsilon \to 0$ *(like maps from* $\mathbf{N}$ *into* $\mathbf{N}$ *, resp.* $]\,0, \infty[$ *, as* $n \to \infty$*) .*

Proof. (2.4): The statements follow straightforwardly by definition and the fact that by (2.1) we have $\mu(A) = 2^{-N/2}\mu(F)$ for all $A \in \alpha$.

(2.5): Let us fix $K \leq l < NL$, then by definition we have $g(T^i(x)) = g^{(l-i)(mod\, N)}(T^l(x))$ for all $x \in T^{-l}(F)$ and all $0 \leq i < K$. Hence we can easily verify that $(g\circ T^0, g\circ T^1, \ldots, g\circ T^{K-1})$ is equally distributed as $(\pm g^{j_1}, \pm g^{j_2}, \ldots, \pm g^{j_K})$ with a choice of signs $\pm$ and for some (different) $j_1, j_2, \ldots, j_K$ from $\{0, 2, \ldots, N-2\}$ (both depending on the given l) . Thus the statements follow straightforwardly by (2.4).

(2.6): Let $A = \{ x \in T^{-l}(F) \mid g(T^j(x)) = \varepsilon_j \ , \ \forall\ 0 \leq j < m \}$ with some given and fixed $\varepsilon_j \in \{-1, 1\}$ for $0 \leq j < m$, and let $B \in \pi_l$. If $x \in T^{-l}(F)$, then $T^j(x) \in T^{-(l-j)}(F)$ and thus by definition we have $g(T^j(x)) = g^{(l-j)\,(mod\, N)}\big(T^l(x)\big)$. Hence we easily find that $A = \bigcup_{C\in\beta} T^{-l}(C)$ for some subfamily $\beta \subset \alpha$. Denote $\nu_l = \mu(F)^{-1} tr(\mu, T^{-l}(F))$, then by (2.1) we have:

$$\begin{aligned}
\nu_l(A\cap B) &= \nu_l\big(\textstyle\bigcup_{C\in\beta} T^{-l}(C)\cap B\big) = \textstyle\sum_{C\in\beta} \nu_l(T^{-l}(C)\cap B)\\
&= \mu(F)^{-1}\textstyle\sum_{C\in\beta} \mu(T^{-l}(C)\cap B) = \mu(F)^{-1}\textstyle\sum_{C\in\beta} 2^{-N/2}\mu(B) =\\
&= \mu(F)^{-1}\textstyle\sum_{C\in\beta} 2^{-N/2}\mu(F)\cdot\mu(F)^{-1}\mu(B)\\
&= \textstyle\sum_{C\in\beta} \mu(F)^{-1}\mu(T^{-l}(C))\cdot\mu(F)^{-1}\mu(B)\\
&= \textstyle\sum_{C\in\beta} \nu_l(T^{-l}(C))\cdot\nu_l(B) = \nu_l(A)\cdot\nu_l(B)\ .
\end{aligned}$$

This fact easily completes the proof of (2.6).

(2.7): We shall verify the case where $l = 0$ and $x \in T^{-N+1}(F)$. Other cases follow in exactly the same manner by using periodicity in the definition of g . If $m = K$ or $K+1$, there is nothing to be proved. Thus consider the case where $K+2 \leq m < N-K$, and look at any odd number p satisfying $K \leq p < m$. Then $p = 2j + K$ for some $j \geq 0$, and therefore members of the sum $\sum_{i=0}^{m-1} g(T^i(x))$, which correspond to indices $2j$ and $2j+K$, cancel. Arguing in this way for any odd number between K and m , we may cancel $m-K$, $m-K+1$ or $m-K-1$ indices (depending on m and K) . Doing so, at the end we will have only those $g\circ T^i$ which are independent with respect to $\mu(F)^{-1}tr(\mu, F)$. These facts complete the proof of (2.7).

(2.8): By definition of g we easily find that $\sum_{l=0}^{N-1} g(T^l(x)) = 0$ whenever $x \in \bigcup_{j=1}^{L} T^{-jN+1}(F)$. Therefore by (2.2)+(2.3) and the definition of g we may conclude:

$$S_m(g)(x) = \textstyle\sum_{k=0}^{p-1} g^{i_k}(y) + \sum_{k=0}^{p-1} g^{j_k}(z)$$

for all $x \in X$ outside of a μ-null set, with some $y, z \in F$, some $0 \leq p, q \leq K+1$, and some $0 \leq i_0, \ldots, i_{p-1}, j_0, \ldots, j_{q-1} < N$, where the corresponding term on the right-hand side equals zero if p or q equals zero. Hence the claim follows straightforwardly by the fact that $\mid g^l(x) \mid \leq 1$ for all $x \in X$ and all $0 \leq l < N$.

(2.9): Let $G = \bigcup_{l=N}^{NL-1} T^{-l}(F)$, then $\mu(G) = N(L-1)\mu(F) \geq N(L-1)(1-\varepsilon)N^{-1}L^{-1} \geq 1-\varepsilon-L^{-1}$. Thus we have:

$$\int_{G^c} \mid S_m(g) \mid^2 \, d\mu \leq m^2\mu(G^c) \leq m^2(\varepsilon + L^{-1})\ . \tag{2.11}$$

On the other hand we have:

$$(2.12) \qquad \int_G |\, S_m(g)\,|^2 \; d\mu \;=\; \sum_{l=N}^{NL-1} \int_{T^{-l}(F)} |\, S_m(g)\,|^2 \; d\mu \;\; .$$

Fix $N \leq l < NL$. Since $m \leq K$, then by (2.5) above the functions $g \circ T^i$ for $0 \leq i < m$ restricted to $T^{-l}(F)$ are independent with respect to $\mu(F)^{-1} tr(\mu, T^{-l}(F))$ and we have:

$$(2.13) \qquad \int_{T^{-l}(F)} g(T^i(x)) \; \mu(dx) = \int_{T^{-l}(F)} g^{(l-i)(mod\, N)}(T^l(x)) \; \mu(dx)$$
$$= \int_F g^{(l-i)(mod\, N)}(x) \; \mu(dx) = 0 \;\; .$$

Thus we get:

$$(2.14) \qquad \int_{T^{-l}(F)} |\, S_m(g)\,|^2 \; d\mu = \sum_{i=0}^{m-1} \int_{T^{-l}(F)} [\, g(T^i(x))\,]^2 \mu(dx) = m\mu(F) \;\; .$$

Now by (2.11), (2.12) and (2.14) we easily conclude:

$$\|S_m(g)\|_2^2 = \int_{G^c} |\, S_m(g)\,|^2 \; d\mu + \int_G |\, S_m(g)\,|^2 \; d\mu$$
$$\leq m^2(\varepsilon + L^{-1}) + N(L-1)\, m\, \mu(F) \leq m^2(\varepsilon + L^{-1}) + m$$
$$\leq m\,(1 + K(\varepsilon + L^{-1})) \;\; .$$

Moreover, by (2.12) and (2.14) we get:

$$\|S_m(g)\|_2^2 \geq N(L-1)\, m\, \mu(F) \geq N(L-1)\, m\, (1-\varepsilon)\, N^{-1} L^{-1}$$
$$= m\,(1-L^{-1})\,(1-\varepsilon) \geq m\,(1-\varepsilon-L^{-1}) \;\; .$$

These facts complete the proof of (2.9).

(2.10): Let $G = \bigcup_{l=N}^{NL-1} T^{-l}(F)$, then as above $\mu(G) \geq 1-\varepsilon-L^{-1}$. Thus by (2.8) we have:

$$(2.15) \qquad \int_{G^c} |\, S_m(g)\,|^2 \; d\mu \leq 4(K+1)^2 \mu(G^c) \leq 4(K+1)^2(\varepsilon + L^{-1}) \;\; .$$

On the other hand we have:

$$(2.16) \qquad \int_G |\, S_m(g)\,|^2 \; d\mu \;=\; \sum_{l=N}^{NL-1} \int_{T^{-l}(F)} |\, S_m(g)\,|^2 \; d\mu \;\; .$$

Fix $N \leq l < NL$, and define $l' = jN - 1 - l$ for some $j \geq 2$ in such a way that $0 \leq l' < N$. Denote by J the set of all indices determined by (2.7) being applied to m , l' and $T^{-jN+1}(F)$. Let $x \in T^{-l}(F)$ be a point for which there exists $y \in T^{-jN+1}(F)$ satisfying $T^{l'}(y) = x$. Then by (2.7) we have:

$$(2.17) \qquad S_m(g)(x) = \sum_{i=0}^{m-1} g(T^{i+l'}(y)) = \sum_{i \in J} g(T^{i+l'}(y)) = \sum_{i \in J} g(T^i(x)) \;\; .$$

Since T is measure-preserving, the set of all x's from $T^{-l}(F)$ satisfying the property stated above has the μ-outer measure equal to $\mu(F)$. But the functions on the left and right-hand side of (2.17) are measurable, and thus (2.17) remains valid for μ-a.a. $x \in T^{-l}(F)$. Moreover, by (2.7) we may easily conclude that the functions $g \circ T^i$ for $i \in J$ restricted to $T^{-l}(F)$ are independent with respect to $\mu(F)^{-1} tr(\mu, T^{-l}(F))$. Therefore by (2.13) we get:

$$\int_{T^{-l}(F)} |\, S_m(g)\,|^2 \; d\mu = \sum_{i \in J} \int_{T^{-l}(F)} [\, g(T^i(x))\,]^2 \mu(dx) \tag{2.18}$$
$$= card\,(J) \cdot \mu(F) \leq (K+1)\,\mu(F) \; .$$

Now by (2.15), (2.16) and (2.18) we easily conclude:

$$\|S_m(g)\|_2^2 = \int_{G^c} |\, S_m(g)\,|^2 \; d\mu + \int_G |\, S_m(g)\,|^2 \; d\mu$$
$$\leq 4(K+1)^2(\varepsilon + L^{-1}) + N(L-1)\,(K+1)\,\mu(F)$$
$$\leq 4(K+1)^2(\varepsilon + L^{-1}) + (K+1)$$
$$= K\,(\,1 + 4\,(1+K^{-1})^2 \cdot K(\varepsilon + L^{-1}) + K^{-1}\,) \; .$$

Moreover, by (2.16) and (2.18) we get:

$$\|S_m(g)\|_2^2 \geq N(L-1)\,(K-1)\,\mu(F) \geq N(L-1)\,(K-1)\,(1-\varepsilon)\,N^{-1}L^{-1}$$
$$= (K-1)\,(1-L^{-1})\,(1-\varepsilon) \geq K\,(1-\varepsilon-K^{-1}-L^{-1}-\varepsilon\,K^{-1}L^{-1}) \; .$$

These facts complete the proof of Lemma 2.2

□

To conclude the preliminary part of the proof let us notice that by (2.4) in Lemma 2.2 we have:

$$\int_X g \; d\mu = \sum_{l=0}^{NL-1} \int_{T^{-l}(F)} g(x) \; \mu(dx) = \sum_{l=0}^{NL-1} \int_{T^{-l}(F)} g^{l(mod\,N)}(T^l(x)) \; \mu(dx) \tag{2.19}$$
$$= \sum_{l=0}^{NL-1} \int_F g^{l(mod\,N)}(x) \; \mu(dx) = 0 \; .$$

We proceed by constructing a function f satisfying the statement of the theorem.

Let $\{N_n\}_{n \geq 1}$, $\{K_n\}_{n \geq 1}$ and $\{L_n\}_{n \geq 1}$ be increasing sequences of positive integers with N_n being even and $0 \leq K_n < N_n$ being odd for all $n \geq 1$. Let $\{\varepsilon_n\}_{n \geq 1}$ be a decreasing sequence of positive real numbers converging to zero, and let F_n be an $(\varepsilon_n, N_n L_n)$-Kakutani-Rochlin set for every $n \geq 1$. According to the preceding construction and statement (2.6) in Lemma 2.2 we may conclude, that for each $n \geq 1$ there exists a measurable function g_n from X into $\{-1, 0, 1\}$ such that $S_m(g_n)$ restricted to $T^{-l}(F)$ is independent of the partition generated by $\{\, g_{n-1} \circ T^i \mid 0 \leq i < N_n \,\}$ with respect to the probability measure $\mu(F_n)^{-1} tr(\mu, T^{-l}(F_n))$ for all $1 \leq m \leq N_n$ and all $N_n \leq l < N_n L_n$ with g_0 being zero. In addition we assume that the given numbers satisfy the following four conditions:

$$K_n \, < \, 2^{-1} N_{n-1} \quad (\forall n > 1) \tag{2.20}$$

$$K_n \, (\varepsilon_n + L_n^{-1}) \, \rightarrow \, 0 \quad (n \rightarrow \infty) \tag{2.21}$$

$$\frac{1}{a_n \sqrt{K_n}} \sum_{j<n} a_j K_j \, \rightarrow \, 0 \quad (n \rightarrow \infty) \tag{2.22}$$

$$\frac{\sqrt{K_n}}{A_n} \sum_{j>n} a_j \, \rightarrow \, 0 \quad (n \rightarrow \infty) \; . \tag{2.23}$$

It is instructive to observe that by (2.23) we have $(a_j) \in l_1$, and thus $(a_j) \in l_2$ as well. Hence by (2.19) and the fact that $|g_n| \leq 1$ for all $n \geq 1$ we easily find that $f \in L^2(\mu)$ and $\int f \; d\mu = 0$.

Remark 2.3

Conditions (2.22) and (2.23) may seem to exclude each other. In order to show that (2.20)-(2.23) can be satisfied, we may proceed as follows:

Step 1: First choose $\{a_n\}_{n\geq 1}$ and $\{K_n\}_{n\geq 1}$ to satisfy (2.22) and (2.23). For example, put $a_1 = 2^{-1}$ and $a_n = (a_{n-1})^4$ for all $n \geq 1$, and let $K_n = (a_n)^{-4}$ for all $n \geq 1$. Then (2.22) and (2.23) become:

(2.22') $$a_n \sum_{j<n} (a_j)^{-3} \to 0 \quad (n \to \infty)$$

(2.23') $$(a_n)^{-3} \sum_{j>n} a_j \to 0 \quad (n \to \infty)$$

which is easily verified, since $\{a_j\}_{j\geq 1}$ is decreasing fast enough.

Step 2: In this step K_n's are already given, so choose N_n's to satisfy (2.20), and choose ε_n's (small enough) and L_n's (large enough) to satisfy (2.21). This completes the choice.

In addition we shall introduce an auxiliary sequence that will be of a vital importance in the rest. Given $m \geq 1$, we define $n_m = \sup\{\, n \geq 1 \mid K_n \leq m \,\}$. Further, we denote:

$$f_m = a_{n_m} g_{n_m} + a_{n_m+1} g_{n_m+1} \quad \text{and} \quad A_m = K_{n_m} |a_{n_m}|^2 + m\, |a_{n_m+1}|^2$$

for all $m \geq 1$. Then we have:

Lemma 2.4

Under the hypotheses stated above we have:

(2.24) $$\frac{1}{\sqrt{A_m}} \big\| S_m(f_m) \big\|_2 \to 1$$

as $m \to \infty$. *Moreover, we have:*

(2.25) $$\frac{1}{\sqrt{A_m}} \big\| S_m(f) - S_m(f_m) \big\|_2 \to 0$$

as $m \to \infty$.

Proof. (2.24): Fix $m \geq 1$ and put $G = \bigcup_{l=N_{n_m+1}}^{N_{n_m+1} L_{n_m+1} - 1} T^{-l}(F_{n_m+1})$, then we have:

(2.26) $$\|S_m(f_m)\|_2^2 \leq 2 \int_{G^c} [\, S_m(a_{n_m} g_{n_m}) \,]^2 d\mu + 2 \int_{G^c} [\, S_m(a_{n_m+1} g_{n_m+1}) \,]^2 d\mu + \\ + \int_G [\, S_m(f_m) \,]^2 d\mu \; .$$

By (2.8) in Lemma 2.2 we easily find:

(2.27) $$\int_{G^c} [\, S_m(a_{n_m} g_{n_m}) \,]^2 d\mu \leq 2\, |a_{n_m}|^2 (K_{n_m} + 1)^2 \mu(G^c) \\ \leq 8\, |a_{n_m}|^2 K_{n_m} \cdot K_{n_m+1} (\varepsilon_{n_m+1} + L_{n_m+1}^{-1}) \; .$$

Moreover, since $|g_n| \le 1$ for all $n \ge 1$, then we have:

$$\text{(2.28)} \qquad \int_{G^c}[\, S_m(a_{n_m+1}g_{n_m+1})\,]^2 d\mu \le |a_{n_m+1}|^2\, m^2\, \mu(G^c) \le |a_{n_m+1}|^2\, m \cdot K_{n_m+1}(\varepsilon_{n_m+1} + L_{n_m+1}^{-1})\ .$$

Since by definition $m < K_{n_m+1} < N_{n_m+1}$, then by construction $S_m(a_{n_m+1}g_{n_m+1})$ restricted to $T^{-l}(F_{n_m+1})$ is independent of $S_m(a_{n_m}g_{n_m})$ with respect to $\mu(F_{n_m+1})^{-1}tr(\mu, T^{-l}(F_{n_m+1}))$ for all $N_{n_m+1} \le l < N_{n_m+1}L_{n_m+1}$. Moreover it is easily verified that by definition and (2.20) we have $K_{n_m} \le m < N_{n_m} - K_{n_m}$. Therefore (2.10) in Lemma 2.2 may be applied. In this way by (2.19), and (2.9)+(2.10) in Lemma 2.2 with (2.21), we get:

$$\text{(2.29)} \qquad \int_G [\, S_m(f_m)\,]^2 d\mu = \sum_{l=N_{n_m+1}}^{N_{n_m+1}L_{n_m+1}-1} \int_{T^{-l}(F_{n_m+1})} [\, S_m(a_{n_m}g_{n_m}) + S_m(a_{n_m+1}g_{n_m+1})\,]^2 d\mu$$

$$= \sum_{l=N_{n_m+1}}^{N_{n_m+1}L_{n_m+1}-1} \int_{T^{-l}(F_{n_m+1})} [\, S_m(a_{n_m}g_{n_m})\,]^2 d\mu + \int_{T^{-l}(F_{n_m+1})} [\, S_m(a_{n_m+1}g_{n_m+1})\,]^2 d\mu$$

$$\le \|S_m(a_{n_m}g_{n_m})\|_2^2 + \|S_m(a_{n_m+1}g_{n_m+1})\|_2^2 \le |a_{n_m}|^2 K_{n_m}(1+\delta') + |a_{n_m+1}|^2\, m\,(1+\delta'')$$

where $\delta' \vee \delta'' \to 0$ as $m \to \infty$. Moreover, by (2.25)-(2.29) we get:

$$\text{(2.30)} \qquad \|S_m(f_m)\|_2^2 \le K_{n_m}|a_{n_m}|^2 \cdot \delta/2 + m\,|a_{n_m+1}|^2 \cdot \delta/2 + A_m(1+\delta/2) = A_m(1+\delta)\ .$$

where $\delta = 32\,(\delta' \vee \delta'' \vee K_{n_m+1}(\varepsilon_{n_m+1} + L_{n_m+1}^{-1})) \to 0$ as $m \to \infty$. On the other hand by the independence, (2.27)+(2.28), and (2.9)+(2.10) in Lemma 2.2 with (2.20)+(2.21), we similarly get:

$$\text{(2.31)} \qquad \|S_m(f_m)\|_2^2 \ge \int_G [\, S_m(f_m)\,]^2 d\mu \ge \int_G [\, S_m(a_{n_m}g_{n_m})\,]^2 d\mu + \int_G [\, S_m(a_{n_m+1}g_{n_m+1})\,]^2 d\mu$$

$$= \|S_m(a_{n_m}g_{n_m})\|_2^2 + \|S_m(a_{n_m+1}g_{n_m+1})\|_2^2 - \int_{G^c}[\, S_m(a_{n_m}g_{n_m})\,]^2 d\mu -$$

$$- \int_{G^c}[\, S_m(a_{n_m+1}g_{n_m+1})\,]^2 d\mu \ge |a_{n_m}|^2 K_{n_m}(1-\delta') + |a_{n_m+1}|^2\, m\,(1-\delta'') -$$

$$-8\,|a_{n_m}|^2 K_{n_m} \cdot K_{n_m+1}(\varepsilon_{n_m+1} + L_{n_m+1}^{-1}) - |a_{n_m+1}|^2\, m \cdot K_{n_m+1}(\varepsilon_{n_m+1} + L_{n_m+1}^{-1})$$

$$\ge |a_{n_m}|^2 K_{n_m}(1-\delta/2) + |a_{n_m+1}|^2\, m\,(1-\delta/2) - K_{n_m}|a_{n_m}|^2 \cdot \delta/2 - m\,|a_{n_m+1}|^2 \cdot \delta/2$$

$$= A_m(1-\delta)$$

where $\delta = 16\,(\delta' \vee \delta'' \vee K_{n_m+1}(\varepsilon_{n_m+1} + L_{n_m+1}^{-1})) \to 0$ for $m \to \infty$. Thus the statement follows straightforwardly by (2.30)+(2.31), and the proof of (2.24) is complete.

(2.25): We have already noticed that by (2.20) above we have $K_{n_m} \le m < N_{n_m} - K_{n_m}$ for all $m \ge 1$. Thus (2.10) in Lemma 2.2 may be applied. In this way we get:

$$\frac{1}{\sqrt{A_m}} \big\|S_m(f) - S_m(f_m)\big\|_2 \le \frac{1}{\sqrt{A_m}} \sum_{n=1}^{n_m-1} \|S_m(a_n g_n)\|_2 + \frac{1}{\sqrt{A_m}} \sum_{n=n_m+1}^{\infty} \|S_m(a_n g_n)\|_2$$

$$\le \frac{1}{\sqrt{A_m}} \sum_{n=1}^{n_m-1} 2(K_n+1)a_n + \frac{1}{\sqrt{A_m}} \sum_{n=n_m+2}^{\infty} m \cdot a_n$$

$$\le \frac{4}{\sqrt{A_m}} \sum_{n=1}^{n_m-1} a_n K_n + \frac{\sqrt{m}}{a_{n_m+1}} \sum_{n=n_m+2}^{\infty} a_n$$

$$\leq \frac{4}{a_{n_m}\sqrt{K_{n_m}}} \sum_{n=1}^{n_m-1} a_n K_n + \frac{\sqrt{K_{n_m+1}}}{a_{n_m+1}} \sum_{n=n_m+2}^{\infty} a_n$$

being valid for all $m \geq 1$. By (2.22) and (2.23) above the right-hand side tends to zero as $m \to \infty$. This fact completes the proof of (2.25).

□

Let us put $X_m = S_m(f)/\|S_m(f)\|_2$ and $Y_m = S_m(f_m)/\sqrt{A_m}$ for $m \geq 1$, and let φ_{X_m} and φ_{Y_m} denote the characteristic function of X_m and Y_m respectively, for all $m \geq 1$. Then by (2.24) and (2.25) in Lemma 2.4 we may easily conclude that $X_m - Y_m \to 0$ in $L^2(\mu)$ as $m \to \infty$. Therefore we have $\varphi_{X_m}(t) - \varphi_{Y_m}(t) \to 0$ as $m \to \infty$, for all $t \in \mathbf{R}$. Thus by the continuity theorem the main proof will be completed as soon as we show that $\varphi_{Y_m}(t) \to \exp(-t^2/2)$ as $m \to \infty$, for all $t \in \mathbf{R}$. This fact is established in the next lemma.

Lemma 2.5

Under the hypotheses stated above we have:

$$E \exp\Big(it \, \frac{S_m(f_m)}{\sqrt{A_m}} \Big) \to \exp(-t^2/2)$$

as $m \to \infty$ *, for all* $t \in \mathbf{R}$.

Proof. Let $t \in \mathbf{R}$ be given and fixed. Put $G = \bigcup_{l=N_{n_m+1}}^{N_{n_m+1}L_{n_m+1}-1} T^{-l}(F_{n_m+1})$ for $m \geq 1$. Since $\mu(G) \to 1$ as $m \to \infty$, we may be concerned only with the integration over G and proceed as follows. Denote $\nu_l = \mu(F_{n_m+1})^{-1} tr(\mu, T^{-l}(F_{n_m+1}))$ for all $N_{n_m+1} \leq l < N_{n_m+1}L_{n_m+1}$ with $m \geq 1$. As in the proof of Lemma 2.4 we may conclude that $S_m(a_{n_m+1}g_{n_m+1})$ restricted to $T^{-l}(F_{n_m+1})$ is independent of $S_m(a_{n_m}g_{n_m})$ with respect to ν_l for all $N_{n_m+1} \leq l < N_{n_m+1}L_{n_m+1}$ with $m \geq 1$. Hence we get:

$$(2.32) \quad \int_G \exp\big(it \, \frac{S_m(f_m)}{\sqrt{A_m}} \big) \, d\mu = \sum_{l=N_{n_m+1}}^{N_{n_m+1}L_{n_m+1}-1} \mu(F_{n_m+1}) \cdot \int_{T^{-l}(F_{n_m+1})} \exp\big(it \, \frac{S_m(f_m)}{\sqrt{A_m}} \big) \, d\nu_l$$

$$= \sum_{l=N_{n_m+1}}^{N_{n_m+1}L_{n_m+1}-1} \mu(F_{n_m+1}) \cdot \int_{T^{-l}(F_{n_m+1})} \exp\big(it \, \frac{S_m(a_{n_m}g_{n_m})}{\sqrt{A_m}} \big) \, d\nu_l \, \cdot$$

$$\cdot \int_{T^{-l}(F_{n_m+1})} \exp\big(it \, \frac{S_m(a_{n_m+1}g_{n_m+1})}{\sqrt{A_m}} \big) \, d\nu_l$$

being valid for all $m \geq 1$. Since $m < K_{n_m+1}$, then by (2.5) in Lemma 2.2 we see that the random functions $g_{n_m+1} \circ T^i$ for $0 \leq i \leq m$ restricted to $T^{-l}(F_{n_m+1})$ are independent and identically distributed with respect to ν_l for all $N_{n_m+1} \leq l < N_{n_m+1}L_{n_m+1}$ with $m \geq 1$. Moreover, we have $\nu_l \{ g_{n_m+1} \circ T^i = \pm 1 \} = 1/2$ for all $0 \leq i \leq m$ with $m \geq 1$. Thus by Berry-Esséen's theorem, see [3] (p.542), we may conclude:

$$\sup_{u \in \mathbf{R}} \Big| \, \nu_l \Big\{ x \in T^{-l}(F_{N_m+1}) \mid \frac{S_m(g_{n_m+1})(x)}{\sqrt{m}} \leq u \Big\} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} \exp\big(\frac{-v^2}{2}\big) \, dv \, \Big| \leq \frac{3}{\sqrt{m}}$$

for all $N_{n_m+1} \leq l < N_{n_m+1}L_{n_m+1}$ with $m \geq 1$. Hence we can deduce:

$$(2.33) \qquad \int_{T^{-l}(F_{n_m+1})} \exp\big(it \, \frac{S_m(a_{n_m+1}g_{n_m+1})}{\sqrt{A_m}} \big) \, d\nu_l \; = \; \exp\big(-\frac{1}{2} \, \frac{m \, |a_{n_m+1}|^2}{A_m} \, t^2 \big) + \eta_m'$$

for all $N_{n_m+1} \leq l < N_{n_m+1}L_{n_m+1}$ with $m \geq 1$, where $\eta_m' \to 0$ as $m \to \infty$. Moreover, it may be worthwhile to mention that this statement follows even more directly by the fact that the pointwise convergence of a sequence of characteristic functions to a characteristic function is uniform over every bounded interval. (This fact in turn relies upon Prohorov's theorem and the continuity theorem). Indeed, having this it is enough to notice that $| m |a_{n_m+1}|^2/A_m| \leq 1$ for all $m \geq 1$, and then apply the preceding fact to the interval $[-t, t]$ by using the central limit theorem and the continuity theorem. However, note that, even though it is irrelevant for our purposes, in this way we get $\eta'_m = \eta'_m(t)$. As an alternative proof of (2.33) we can also recall that $\sup_{x\in\mathbf{R}} | F_n(x) - F(x) | \to 0$ as $n \to \infty$, provided that $F_n(x) \to F(x)$ for all $x \in \mathbf{R}$ as $n \to \infty$ and F is continuous, and $\sup_{x\in\mathbf{R}} | F_n(x) - F(x) | \to 0$ as $n \to \infty$, if and only if $\sup_{t\in\mathbf{R}} | \varphi_n(t) - \varphi(t) | \to 0$ as $n \to \infty$. Here F_n and F are distribution functions, and φ_n and φ are associated characteristic functions for $n \geq 1$. Now by (2.32) and (2.33) we get:

$$(2.34) \quad \int_G \exp\Big(it \frac{S_m(f_m)}{\sqrt{A_m}} \Big)\, d\mu = \Big(\exp\Big(-\frac{1}{2} \frac{m\, |a_{n_m+1}|^2}{A_m}\, t^2 \Big) + \eta'_m \Big) \cdot \Big(\int_G \exp\Big(it \frac{S_m(a_{n_m} g_{n_m})}{\sqrt{A_m}} \Big)\, d\mu \Big)$$

being valid for $m \geq 1$, where $\eta'_m \to 0$ as $m \to \infty$. Repeating the similar procedure for the n_m level instead of the level $n_m + 1$, and using the fact that by (2.20) above we have $K_{n_m} \leq m < N_{n_m} - K_{n_m}$, according to which by (2.7) in Lemma 2.2 we can extract from the sequence $\{ g_{n_m} \circ T^i \mid 0 \leq i < m \}$ a subsequence of cardinality $K_{n_m} - 1$, K_{n_m} or $K_{n_m} + 1$ containing functions which are, being restricted to $T^{-l}(F_{n_m})$, mutually independent and identically distributed with respect to $\rho_l := \mu(F_{n_m})^{-1} tr(\mu, T^{-l}(F_{n_m}))$, and taking values ± 1 with probability $1/2$, where $N_{n_m} \leq l < N_{n_m} L_{n_m}$, we obtain:

$$(2.35) \quad \int_{T^{-l}(F_{n_m})} \exp\Big(it \frac{S_m(a_{n_m} g_{n_m})}{\sqrt{A_m}} \Big)\, d\rho_l = \exp\Big(-\frac{1}{2} \frac{K_{n_m}\, |a_{n_m}|^2}{A_m}\, t^2 \Big) + \eta''_m$$

for all $N_{n_m} \leq l < N_{n_m} L_{n_m}$ with $m \geq 1$, where $\eta''_m \to \infty$ as $m \to \infty$. Put $H = \bigcup_{l=N_{n_m}}^{N_{n_m} L_{n_m} - 1} T^{-l}(F_{n_m})$ for $m \geq 1$. Then it is easily verified that $\mu(G \triangle H) \to 0$ as $m \to \infty$. Hence by (2.34) and (2.35) we may conclude:

$$(2.36) \quad \int_G \exp\Big(it \frac{S_m(f_m)}{\sqrt{A_m}} \Big)\, d\mu = \Big(\exp\Big(-\frac{1}{2} \frac{m\, |a_{n_m+1}|^2}{A_m}\, t^2 \Big) + \eta'_m \Big) \cdot$$

$$\cdot \Big(\int_H \exp\Big(it \frac{S_m(a_{n_m} g_{n_m})}{\sqrt{A_m}} \Big)\, d\mu + \delta_m \Big)$$

$$= \Big(\exp\Big(-\frac{1}{2} \frac{m\, |a_{n_m+1}|^2}{A_m}\, t^2 \Big) + \eta'_m \Big) \cdot \Big(\Big(\exp\Big(-\frac{1}{2} \frac{K_{n_m}\, |a_{n_m}|^2}{A_m}\, t^2 \Big) + \eta''_m \Big) \cdot \mu(H) + \delta_m \Big)$$

where $\eta'_m \to 0$, $\delta_m \to 0$, $\eta''_m \to 0$, $\mu(G) \to 1$ and $\mu(H) \to 1$ as $m \to \infty$. Thus letting $m \to \infty$ in (2.36) we get:

$$\int \exp\Big(it \frac{S_m(f_m)}{\sqrt{A_m}} \Big) d\mu \to \exp(-t^2/2)$$

as $m \to \infty$, for all $t \in \mathbf{R}$. This fact completes the proofs of Lemma 2.5 and Theorem 2.1

□□

3. The central limit theorem for orbits

Throughout the whole section we suppose that $(X, \mathcal{A}, \mu, T)$ is a given aperiodic dynamical system. We recall that $CLT(\mu,T)$ denote the set of all functions $f \in L^2(\mu)$ with $\int f \, d\mu = 0$ satisfying the central limit theorem as stated in (1.1) above. According to Theorem 2.1 we have $CLT(\mu,T) \neq \emptyset$. Given $f \in CLT(\mu,T)$ we denote $Orb\,(f) = \{ \ f \circ T^j \ \mid \ j \geq 0 \ \}$. The set $Orb\,(f)$ is called *the orbit* of f. The smallest linear subspace of $L^2(\mu)$ containing $Orb\,(f)$ is denoted by $span\,(Orb\,(f))$. The main aim of this section is to investigate the central limit theorems involving elements of $span\,(\,Orb\,(f))$. It turns out that a complete description of weak Gaussian limits in this case can be obtained. The result is presented in Theorem 3.3 below and then extended in Theorem 3.5 below. The proof relies upon the next two lemmas which are also of interest in themselves. We clarify that $N_d(\mu\,,\Gamma)$ denote the d-dimensional Gaussian distribution with mean vector μ and covariance matrix Γ. Symbol $\mathbf{1}$ denotes the matrix having all entries equal to 1. We begin as follows.

Lemma 3.1

Let $f \in L^2(\mu)$ be an arbitrary function, and let $f_1, \ldots, f_d \in Orb\,(f)$ be arbitrary elements for some $d \geq 1$. Put $A_m = \int |\, S_m(f)\,|^2 d\mu$ and $A_m(\alpha) = \int |\, S_m(\sum_{k=1}^d \alpha_k f_k)\,|^2 d\mu$ for all $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbf{R}^d$ and $m \geq 1$. Suppose that $A_m \to \infty$ as $m \to \infty$. Then we have:

$$\text{(3.1)} \qquad \frac{1}{(\sum_{k=1}^d \alpha_k)\,\sqrt{A_m}} \, \big\| S_m\,(\textstyle\sum_{k=1}^d \alpha_k f_k) - S_m\big(\,(\sum_{k=1}^d \alpha_k) \cdot f\,\big) \big\|_2 \longrightarrow 0$$

$$\text{(3.2)} \qquad \frac{\sqrt{A_m(\alpha)}}{\sqrt{A_m}} \longrightarrow \big|\, \textstyle\sum_{k=1}^d \alpha_k \big|$$

as $m \to \infty$, for all $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbf{R}^d$ with $\sum_{k=1}^d \alpha_k \neq 0$.

Proof. Let $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbf{R}^d$ with $\sum_{k=1}^d \alpha_k \neq 0$ be given and fixed. It is no restriction to assume that $f_1 = f \circ T^{p_1}$, $f_2 = f \circ T^{p_2}, \ldots, f_d = f \circ T^{p_d}$ for some $0 \leq p_1 < p_2 < \ldots < p_d$. Then we have:

$$S_m\,(\textstyle\sum_{k=1}^d \alpha_k f_k) - S_m\big(\,(\sum_{k=1}^d \alpha_k) \cdot f\,\big) =$$
$$\textstyle\sum_{k=1}^d \alpha_k \big(\, \sum_{j=0}^{m-1} f \circ T^{j+p_k} - \sum_{j=0}^{m-1} f \circ T^j \,\big) =$$
$$\textstyle\sum_{k=1}^d \alpha_k \big(\, \sum_{j=m}^{m+p_k-1} f \circ T^j - \sum_{j=0}^{p_k-1} f \circ T^j \,\big)$$

being valid for all $m \geq 1$, where the first term of the first sum in the last line reads zero in the case p_1 when equals zero. Hence we get:

$$\text{(3.3)} \qquad \big\| S_m\,(\textstyle\sum_{k=1}^d \alpha_k f_k) - S_m\big(\,(\sum_{k=1}^d \alpha_k) \cdot f\,\big) \big\|_2 \leq 2 \sum_{k=1}^d |\alpha_k|\, p_k \,\|f\|_2$$

for all $m \geq 1$. Thus (3.1) follows straightforwardly by the fact that $A_m \to 0$ as $m \to \infty$. Moreover (3.2) is an immediate consequence of (3.1). The proof is complete.

□

Lemma 3.2

Let $f \in CLT(\mu,T)$ be an arbitrary function, and let $f_1, \ldots, f_d \in Orb\,(f)$ be arbitrary elements for some $d \geq 1$. Put $A_m = \int |\, S_m(f)\,|^2 d\mu$ and $A_m(\alpha) = \int |\, S_m(\sum_{k=1}^d \alpha_k f_k)\,|^2 d\mu$ for all $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbf{R}^d$ and $m \geq 1$. Suppose that $A_m \to \infty$ as $m \to \infty$. Then we have:

(3.4) $\frac{1}{\sqrt{A_m(\alpha)}} S_m(\sum_{k=1}^d \alpha_k f_k) \overset{\sim}{\longrightarrow} N(0,1)$

(3.5) $\frac{1}{\sqrt{A_m}} S_m(\sum_{k=1}^d \alpha_k f_k) \overset{\sim}{\longrightarrow} N(0, |\sum_{k=1}^d \alpha_k|^2)$

as $m \to \infty$ *, for all* $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbf{R}^d$ *with* $\sum_{k=1}^d \alpha_k \neq 0$ *.*

Proof. Let $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbf{R}^d$ with $\sum_{k=1}^d \alpha_k \neq 0$ be given and fixed. Then by (3.1) in Lemma 3.1 we easily find:

$$\int \exp\Big(it \frac{S_m(\sum_{k=1}^d \alpha_k f_k)}{(\sum_{k=1}^d \alpha_k)\sqrt{A_m}} \Big) \, d\mu \; - \int \exp\Big(it \frac{S_m(f)}{\sqrt{A_m}} \Big) \, d\mu \longrightarrow 0$$

as $m \to \infty$, for all $t \in \mathbf{R}$. Thus (3.4) follows by the continuity theorem and the fact that $f \in CLT(\mu, T)$. Moreover (3.5) follows straightforwardly by (3.4) and (3.2) in Lemma 3.1. The proof is complete.

□

Theorem 3.3

Let $f \in CLT(\mu, T)$ *be an arbitrary function, and let* $f_1, \ldots, f_d \in Orb\,(f)$ *be arbitrary elements for some* $d \geq 1$ *. Denote* $F = (f_1, \ldots, f_d)$ *, and put* $A_m = \int |\, S_m(f)\,|^2 d\mu$ *for all* $m \geq 1$ *. Suppose that* $A_m \to \infty$ *as* $m \to \infty$ *. Then we have:*

(3.6) $\frac{1}{\sqrt{A_m}} S_m(F) \overset{\sim}{\longrightarrow} N_d(0, \mathbf{1})$

as $m \to \infty$ *. More generally, suppose that* $f_1, \ldots, f_d \in span\,(Orb\,(f))$ *are arbitrary elements. Then we have:*

$$f_k = \sum_{i=1}^{N_k} \beta_i^k f_i^k$$

for some $f_i^k \in Orb(f)$ *and* $\beta_i^k \in \mathbf{R}$ *with* $1 \leq i \leq N_k$ *and* $1 \leq k \leq d$ *. Suppose moreover that* $\sum_{i=1}^{N_k} \beta_i^k \neq 0$ *for all* $1 \leq k \leq d$ *, and that* $A_m \to \infty$ *as* $m \to \infty$ *. Then we have:*

(3.7) $\frac{1}{\sqrt{A_m}} S_m(F) \overset{\sim}{\longrightarrow} N_d(0, \Gamma)$

as $m \to \infty$ *, where* $\Gamma = (\Gamma_{kl})_{k,l=1}^d$ *is given by:*

$$\Gamma_{kl} = \Big(\sum_{i=1}^{N_k} \beta_i^k \Big) \cdot \Big(\sum_{i=1}^{N_l} \beta_i^l \Big)$$

for all $1 \leq k, l \leq d$ *.*

Proof. First we prove (3.6). Since the set $D = \{ (\alpha_1, \ldots, \alpha_d) \in \mathbf{R}^d \mid \sum_{k=1}^d \alpha_k \neq 0 \}$ is dense in $\mathbf{R}^d$, then by the continuity theorem it suffices to show:

(3.8) $$\int \exp\big(i \frac{\alpha \cdot S_m(F)}{\sqrt{A_m}} \big) \, d\mu \longrightarrow \int \exp\big(i\, \alpha \cdot \bar{X} \big) \, d\mu$$

for all $\alpha \in D$, as $m \to \infty$, where $\bar{X} = (X, X, \ldots, X)$ with $X \sim N(0,1)$. Again by the continuity theorem, for (3.6) it is enough to show that:

$$\frac{\alpha\cdot S_m(F)}{\sqrt{A_m}} \overset{\sim}{\longrightarrow} \alpha\cdot\bar{X} \tag{3.9}$$

for all $\alpha \in D$, as $m \to \infty$. Notice that we have:

$$\alpha\cdot S_m = S_m(\textstyle\sum_{k=1}^d \alpha_k f_k) \quad , \quad \alpha\cdot\bar{X} = (\textstyle\sum_{k=1}^d \alpha_k)\, X$$

for all $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbf{R}^d$ and all $m \geq 1$. Hence we see that for (3.9) it suffices to show that:

$$\frac{1}{\sqrt{A_m}}\, S_m(\textstyle\sum_{k=1}^d \alpha_k f_k) \overset{\sim}{\longrightarrow} (\textstyle\sum_{k=1}^d \alpha_k)\, X \tag{3.10}$$

for all $\alpha = (\alpha_1, \dots, \alpha_d) \in D$, as $m \to \infty$. However it is precisely the statement (3.5) in Lemma 3.2. Thus (3.8)-(3.10) is valid, and the proof of (3.6) is complete.

Next we prove (3.7). We use the same argument as for (3.6). In this way we obtain that it suffices to show that:

$$\frac{1}{\sqrt{A_m}}\, S_m(\textstyle\sum_{k=1}^d \alpha_k f_k) \overset{\sim}{\longrightarrow} \textstyle\sum_{k=1}^d \alpha_k X_k \tag{3.11}$$

for all $\alpha = (\alpha_1, \dots, \alpha_d)$ that belong to a dense subset D of $\mathbf{R}^d$, as $m \to \infty$, where $X = (X_1, \dots, X_d) \sim N_d(0, \Gamma)$. Notice that we have:

$$S_m(\textstyle\sum_{k=1}^d \alpha_k f_k) = \textstyle\sum_{k=1}^d \sum_{i=1}^{N_k} \alpha_k \beta_i^k f_i^k \tag{3.12}$$

for all $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbf{R}^d$ and all $m \geq 1$. Choose D to be the set of all $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbf{R}^d$ such that $\sum_{k=1}^d \sum_{i=1}^{N_k} \alpha_k \beta_i^k \neq 0$. Then D is dense in $\mathbf{R}^d$. Hence by (3.2) and (3.5) in Lemma 3.2 we get:

$$\frac{1}{\sqrt{A_m}}\, S_m(\textstyle\sum_{k=1}^d \alpha_k f_k) \overset{\sim}{\longrightarrow} \big(\textstyle\sum_{k=1}^d \sum_{i=1}^{N_k} \alpha_k \beta_i^k\big)\, X$$

as $m \to \infty$, with $X \sim N(0,1)$. Thus (3.11) follows with $X_k = (\sum_{i=1}^{N_k} \beta_i^k)\, X$ for all $1 \leq k \leq d$, and the proof of (3.7) is complete. These facts complete the proof of the theorem. $\square$

Example 3.4

Let $(X, \mathcal{A}, \mu, T)$ be equal to $(S^{\mathbf{N}}, \mathcal{A}^{\mathbf{N}}, \nu^{\mathbf{N}}, \theta)$ with $S = \{-1, 1\}$, $\mathcal{A} = 2^S$, $\nu\{\pm 1\} = 1/2$ and $\theta(s_1, s_2, \dots) = (s_2, s_3, \dots)$ for $(s_1, s_2, \dots) \in S^{\mathbf{N}}$. Then T is strongly mixing, and thus aperiodic as well. Let f be the projection onto the first coordinate. Then $f \circ T^j = \varepsilon_j$ form a Rademacher sequence for $j \geq 1$. By the classical central limit theorem we have:

$$\frac{\sum_{j=0}^{m-1} T^j f(x)}{\| \sum_{j=0}^{m-1} T^j f \|_2} = \frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} \varepsilon_j \overset{\sim}{\longrightarrow} N(0,1)$$

as $m \to \infty$. Thus $f \in CLT(\mu, T)$. However, if we put $g = f - f \circ T$, then we have:

$$\frac{\sum_{j=0}^{m-1} T^j g(x)}{\| \sum_{j=0}^{m-1} T^j g \|_2} = \frac{1}{\sqrt{2}}\, (\varepsilon_1 - \varepsilon_m) \sim \frac{1}{\sqrt{2}}\, (\varepsilon_1 + \varepsilon_2) \not\sim N(0, \sigma^2)$$

for all $m \geq 1$ and any $\sigma^2 > 0$. Therefore $g \notin CLT(\mu, T)$. Hence we see that the results of Lemma 3.1, Lemma 3.2 and Theorem 3.3 are as optimal as possible in general.

A close look at the method of the proof of Theorem 3.3, through Lemma 3.1 and Lemma 3.2, shows that these results might be generalized in the number of elements in the orbit being involved. The result can be formulated into two steps as is shown in the next theorem. We clarify that $\sum_{k=1}^\infty \gamma_k f_k$ stands for the $L^2(\mu)$-limit of $\sum_{k=1}^m \gamma_k f_k$ as $m \to \infty$ whenever $(\gamma_k)_{k=1}^\infty \in l_1$ and $f_k \in Orb(f)$ for $k \geq 1$ with $f \in L^2(\mu)$. Notice that under these circumstances the limit exists.

Theorem 3.5

Let $f \in CLT(\mu, T)$ *be an arbitrary function, and let* $f_k = f \circ T^{p_k} \in Orb(f)$ *be arbitrary elements for some* $p_k \geq 0$ *with* $k \geq 1$. *Let* $(\alpha_k)_{k=1}^\infty \in l_1$ *satisfying:*

(3.13) $$\sum_{k=1}^\infty |\alpha_k| \, p_k \, < \, \infty$$

(3.14) $$\sum_{k=1}^\infty \alpha_k \, \neq \, 0 \; .$$

Put $A_m = \int |\, S_m(f)\,|^2 d\mu$ *and* $A_m(\alpha) = \int |\, S_m(\sum_{k=1}^\infty \alpha_k f_k)\,|^2 d\mu$ *for all* $m \geq 1$. *Suppose that* $A_m \to \infty$ *as* $m \to \infty$. *Then we have:*

(3.15) $$\frac{1}{(\sum_{k=1}^\infty \alpha_k)\, \sqrt{A_m}} \left\| S_m\, (\textstyle\sum_{k=1}^\infty \alpha_k f_k) - S_m\big((\sum_{k=1}^\infty \alpha_k) \cdot f\, \big) \right\|_2 \longrightarrow 0$$

(3.16) $$\frac{\sqrt{A_m(\alpha)}}{\sqrt{A_m}} \longrightarrow \big|\, \textstyle\sum_{k=1}^\infty \alpha_k \big|$$

(3.17) $$\frac{1}{\sqrt{A_m(\alpha)}}\, S_m(\textstyle\sum_{k=1}^\infty \alpha_k f_k) \overset{\sim}{\longrightarrow} N(0,1)$$

(3.18) $$\frac{1}{\sqrt{A_m}}\, S_m(\textstyle\sum_{k=1}^\infty \alpha_k f_k) \overset{\sim}{\longrightarrow} N(0, |\, \sum_{k=1}^\infty \alpha_k|^2)$$

as $m \to \infty$. *More generally, suppose that:*

$$f_k \, = \, \sum_{i=1}^\infty \beta_i^k f_i^k$$

for some $f_i^k = f \circ T^{p_i^k} \in Orb(f)$ *with* $p_i^k \geq 0$ *and* $(\beta_i^k)_{i=1}^\infty \in l_1$ *for* $1 \leq k \leq d$ *and* $i \geq 1$. *Suppose moreover that:*

(3.19) $$\sum_{i=1}^\infty |\beta_i^k| \, p_i^k \, < \, \infty$$

(3.20) $$\sum_{i=1}^\infty \beta_i^k \, \neq \, 0$$

for all $1 \leq k \leq d$. *Denote* $F = (f_1, \ldots, f_d)$, *and suppose that* $A_m \to \infty$ *as* $m \to \infty$. *Then we have:*

(3.21) $$\frac{1}{\sqrt{A_m}}\, S_m(F) \overset{\sim}{\longrightarrow} N_d(0, \Gamma)$$

as $m \to \infty$, *where* $\Gamma = (\Gamma_{kl})_{k,l=1}^d$ *is given by:*

$$\Gamma_{kl} \, = \, \big(\textstyle\sum_{i=1}^\infty \beta_i^k \big) \cdot \big(\sum_{i=1}^\infty \beta_i^l \big)$$

for all $1 \leq k, l \leq d$.

Proof. We point out (3.3) in the proof of Lemma 3.1 in order to understand how conditions (3.13) and (3.19) come into the presence. The rest of the proof can be carried out along the lines of the proofs of Lemma 3.1, Lemma 3.2 and Theorem 3.3. We shall omit the details.

□

4. The central limit theorem and regularity of summability methods

The existence of a multidimensional central limit theorem for aperiodic dynamical systems has some interesting consequences to the study of regularity of summability methods in the corresponding function spaces. This section presents basic ideas and facts in this direction with the aim to motivate the investigation of the multidimensional central limit theorem for dynamical systems in general. The regularity of summability methods is studied via *GB* and *GC* sets. Therefore we briefly recall this concept. A non-empty subset L of a Hilbert space H is called *a GB* (resp. *GC*) *set* in H , if the isonormal process of H has a version which is sample bounded (resp. continuous) on L . Recall that the isonormal process of H is a centered Gaussian process indexed by H and having the covariance function given by the scalar product of H . These properties can be characterized in terms of majorizing measures, analyzing the local scattering of L , as well as in terms of metric entropy, see [8]. Throughout the whole section we suppose that $(X, \mathcal{A}, \mu, T)$ is a given aperiodic dynamical system. We begin by explaining the basic idea and result.

Let $A = (a_{n,k})_{n,k \geq 1}$ be an infinite real matrix with row vectors $a_n = (a_{n,k})_{k \geq 1}$ for $n \geq 1$. Suppose that for every $n \geq 1$ a linear operator A_n on $L^p(\mu)$ may be well defined by:

$$A_n(f) = \sum_{k=1}^{\infty} a_{n,k} f \circ T^k \tag{4.1}$$

for all $f \in L^p(\mu)$ with some $1 \leq p < \infty$. Suppose for a moment that T is an ergodic automorphism. Then it is shown in [14] that the regularity of the summability method defined by means of $\{ A_n \mid n \geq 1 \}$ is strongly related to the *GB* property of the set $\alpha = \{ a_n \mid n \geq 1 \}$. For instance, it is shown that under certain regularity assumptions we have that α is a *GB* set in l_2 whenever the condition is satisfied:

$$\sup_{n \geq 1} |A_n(f)| < \infty \quad \mu\text{-a.a.} \tag{4.2}$$

for all $f \in L^p(\mu)$, see Theorem 7.7 and Theorem 7.8 in [14]. A close look into the proofs of these results shows that the ergodicity of T plays a vitally important role. It indicates that a quite different approach should be taken into account in the case where T is not longer supposed to be ergodic, the reason being that we can not longer work with the so-called Stein's elements defined as follows:

$$F_{J,f} = \frac{1}{\sqrt{J}} \sum_{j \leq J} g_j \cdot (f \circ T^j) \tag{4.3}$$

where $f \in L^2(\mu)$ and $J \geq 1$, and where $(g_j)_{j \geq 1}$ is an orthogaussian sequence defined on a possibly different probability space. Thus in order to make things working in the non-ergodic case, we have to find a substitute for Stein's elements. It is precisely at this point that a multidimensional central limit theorem could be of use. The result may be stated as follows. We clarify that $S_m(f)$ stands for $\sum_{j=0}^{m-1} f \circ T^j$ whenever $f \in L^2(\mu)$ and $m \geq 1$.

Theorem 4.1

Let $A = (a_{n,k})_{n,k\geq 1}$ *be an infinite real matrix with row vectors* $a_n = (a_{n,k})_{k\geq 1}$ *being from* l_1 *for* $n \geq 1$ *. Let* $\{ A_n \mid n \geq 1 \}$ *be a sequence of linear operators on* $L^2(\mu)$ *defined by (4.1) above. Suppose that there exists a function* $f \in L^2(\mu)$ *and a positive definite symmetric real matrix* $\Gamma = (\Gamma_{kl})_{k,l\geq 1}$ *such that:*

$$(4.4) \qquad \Big(\tfrac{S_m(A_1(f))}{\|S_m(f)\|_2}, \ldots, \tfrac{S_m(A_p(f))}{\|S_m(f)\|_2} \Big) \xrightarrow{\sim} N_p\big(0, (\Gamma_{kl})_{k,l\geq 1}^p\big)$$

as $m \to \infty$ *for all* $p \geq 1$ *. Let* $X = \{ X_n \mid n \geq 1 \}$ *be a centered Gaussian sequence with covariance matrix* Γ *. If the condition is satisfied:*

$$(4.5) \qquad \sup_{n\geq 1} |A_n(f)| < \infty \quad \mu\text{-a.a.}$$

for all $f \in L^2(\mu)$ *, then* X *is sample bounded.*

Proof. Since a_n belongs to l_1 , then A_n is a well-defined continuous linear operator on $L^2(\mu)$ for all $n \geq 1$. In particular, every A_n is continuous in μ-measure for $n \geq 1$, see [14]. Put $Z_{m,n} = S_m(A_n(f))$ for all $m, n \geq 1$. Then we have:

$$\begin{aligned} Z_{m,n} &= \textstyle\sum_{j=0}^{m-1} A_n(f) \circ T^j = \sum_{j=0}^{m-1} \big(\sum_{k=1}^{\infty} a_{n,k} f \circ T^k \big) \circ T^j \\ &= \textstyle\sum_{j=0}^{m-1} \big(\sum_{k=1}^{\infty} a_{n,k} f \circ T^j \circ T^k \big) = \sum_{j=0}^{m-1} A_n(f \circ T^j) = A_n(S_m(f)) \end{aligned}$$

for all $m, n \geq 1$. Hence by (4.5) and Banach's principle there exists a number $C > 0$ large enough to satisfy:

$$\mu \Big\{ \sup_{1\leq n\leq p} \tfrac{|Z_{m,n}|}{\|S_m(f)\|_2} > C \Big\} \leq \mu \Big\{ \sup_{n\geq 1} \tfrac{|Z_{m,n}|}{\|S_m(f)\|_2} > C \Big\} \leq \tfrac{1}{2}$$

for all $m, p \geq 1$. Letting $m \to \infty$ and using (4.4), and then letting $p \to \infty$ we obtain:

$$\mu \{ \sup_{n\geq 1} |X_n| > C \} \leq \tfrac{1}{2} .$$

Thus the claim follows by the zero-one law for Gaussian sequences. This fact completes the proof. □

Acknowledgment. The authors thank M. Denker, S. E. Graversen, J. Hoffmann-Jørgensen and J. P. Thouvenot for useful discussions on the subject.

REFERENCES

[1] BURTON, R. *and* DENKER, M. (1987). On the central limit theorem for dynamical systems. *Trans. Amer. Math. Soc.* 302 (715-726).

[2] DE LA RUE, T. (1993). Espaces de Lebesgue. *Sém. Probab.* XXVII, *Lecture Notes in Math.* 1557 (15-21).

[3] FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications*, Vol II. Second Edition, John Wiley & Sons, Inc.

[4] GORDIN, M. I. *and* LIFŠIC, B. A. (1978). The central limit theorem for stationary Markov processes. *Soviet Math. Dokl.* 19 (392-394).

[5] HALMOS, P. (1956). *Lectures on Ergodic Theory*. Kenkysha Printing Co., Ltd., Tokyo.

[6] IBRAGIMOV, I. A. (1962). Some limit theorems for stationary processes. *Theory Probab. Appl.* 7 (349-382).

[7] KAC, M. (1946). On the distribution of values of sums of the type $\sum f(2^k t)$. *Ann. Math.* 47 (33-49).

[8] LEDOUX, M. *and* TALAGRAND, M. (1991). *Probability in Banach Spaces* (*Isoperimetry and Processes*). Springer-Verlag Berlin Heidelberg.

[9] PETERSEN, K. (1983). *Ergodic Theory*. Cambridge University Press.

[10] PETIT, B. (1992). Le théorème limite central pour des sommes de Riesz-Raikov. *Probab. Theory Relat. Fields* 93 (407-438).

[11] ROCHLIN, V. A. (1962). On the fundamental ideas of measure theory. *Amer. Math. Soc. Transl.* Ser. 1, Vol. 10 (1-54).

[12] ROSENBLATT, M. (1956). A central limit theorem and a strong mixing condition. *Proc. Nat. Acad. Sci. U.S.A.* 42 (43-47).

[13] VOLNÝ, D. (1993). Approximating martingales and the central limit theorem for strictly stationary processes. *Stochastic Process. Appl.* 44 (41-74).

[14] WEBER, M. (1993). GC sets, Stein's elements and matrix summation methods. *IRMA, Strasbourg, Prepubl. No. 27.*

Thierry de la Rue
Laboratoire d'Analyse et
de Modèles Stochastiques
URA CNRS 1378
Université de Rouen B.P. 118
76134 Mont-Saint-Aignan
France

Stéphane Ladouceur
Departement de Mathematique
et de Statistique
Université de Montreal
C.P. 6138, Succursale A
Montreal, Qc, H3C 3J7
Canada

Goran Peskir
Department of Mathematical Sciences
University of Aarhus, Denmark
Ny Munkegade, DK-8000 Aarhus
home.imf.au.dk/goran
goran@imf.au.dk

Michel Weber
I.R.M.A. Unité de Recherche
associée C.N.R.S., 1
7, rue René Descartes
67084 Strasbourg
France