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A Note on the Call-Put Parity and a Call-Put Duality

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Along with the well-known "call-put parity" relation, that makes it possible to express the rational price of a put option in terms of the rational price of a call option, we introduce a "call-put duality" relation. This new concept offers a simple explanation of the relationship between the rational price of a put option and a call option, not only for options of the European type, but also for options of the American type.

1. The call-put parity

Consider the (B, S)-model of a financial market where $B = (B_t)_{t \geq 0}$ is the value of a bank account and $S = (S_t)_{t \geq 0}$ is the price of a stock. Assume for simplicity that $B_t \equiv 1$ and let the stock price evolve as a geometric Wiener process:

$$S_t = S_0 \exp\left(\sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right)t\right) \tag{1.1}$$

where $S_0 > 0$, $\sigma > 0$, $\mu \in I\!R$ and $W = (W_t)_{t \geq 0}$ is a standard Wiener process. We know by Itô formula that $S = (S_t)_{t \geq 0}$ solves:

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t \ . \tag{1.2}$$

Let $f_T = (S_T - K)^+$ be the payoff function of a standard European call option with a strike price $K > 0$ and a maturity time $T > 0$, and let:

$$\mathbf{C}_T = \mathbf{C}_T(S_0, K; \sigma) \tag{1.3}$$

denote the rational (arbitrage) price of the option. Then by the general option pricing theory (see e.g. [1, page 710]) we know that:

$$\mathbf{C}_T = \widehat{E}(f_T) \tag{1.4}$$

where $\widehat{E}$ is the expectation with respect to the martingale measure $\widehat{P}_T$ defined by:

$$d\widehat{P}_T = \exp\left(-\frac{\mu}{\sigma} W_T - \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 T\right) dP_T \tag{1.5}$$

and P_T is the 'physical' measure. A direct calculation of $\mathbf{C}_T$ based on (1.4) and (1.5) yields

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the Black-Scholes formula:

$$\mathbf{C}_T = S_0\,\Phi\left(\frac{\log(\frac{S_0}{K}) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right) - K\Phi\left(\frac{\log(\frac{S_0}{K}) - \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right) \tag{1.6}$$

where $\Phi(x) = (1/\sqrt{2\pi})\int_{-\infty}^{x} e^{-t^2/2}\,dt$ for $x \in I\!R$.

Similarly, if $g_T = (K-S_T)^+$ is the payoff function of a standard European put option with a strike price $K > 0$ and a maturity time $T > 0$, then the same arguments show that the rational (arbitrage) price of the option satisfies the identity:

$$\mathbf{P}_T = \mathbf{P}_T(S_0, K;\sigma) = \widehat{E}(g_T) \tag{1.7}$$

and the following explicit formula is valid:

$$\mathbf{P}_T = S_0\,\Phi\left(\frac{\log(\frac{S_0}{K}) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right) - K\Phi\left(\frac{\log(\frac{S_0}{K}) - \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right) + (K-S_0)\ . \tag{1.8}$$

For more computational details on (1.6) and (1.8) see e.g. [1; Chapter VIII, § 1b].

From (1.6) and (1.8) we see that the following relation holds:

$$\mathbf{P}_T = \mathbf{C}_T + (K-S_0) \tag{1.9}$$

which is termed the "*call-put parity*" in financial theory. In fact, a quick way to derive (1.8) when (1.6) is known is to note that (1.9) follows immediately from the obvious identity:

$$(K-S_T)^+ = (S_T-K)^+ + (K-S_T) \tag{1.10}$$

upon applying $\widehat{E}$ and using that $\widehat{E}(S_T) = S_0$. All these facts are well-known (see e.g. [1]).

2. The call-put duality

1. We now state the key observation of this note. Using that $\Phi(-x) = 1 - \Phi(x)$ we see from (1.6) and (1.8) that the rational prices $\mathbf{P}_T$ and $\mathbf{C}_T$ satisfy the following relation:

$$\mathbf{P}_T(S_0, K;\sigma) = \mathbf{C}_T(-S_0, -K; -\sigma) \tag{2.1}$$

which we term the "*call-put duality*". The right-hand side of (2.1) is formally defined by (1.6) for negative values of the three parameters.

Similarly to (1.9) the identity (2.1) shows that in order to find the rational put price $\mathbf{P}_T$ it is sufficient to know the rational call price $\mathbf{C}_T$, and the former is obtained from the latter by a formal replacement of the triple $(S_0, K;\sigma)$ with its negative counterpart $(-S_0, -K; -\sigma)$.

In order to examine this fact closer, let us consider the following identity:

$$(K-S_T)^+ = (-S_T-(-K))^+ \tag{2.2}$$

in parallel to (1.10). Denoting $\widetilde{S}_t = -S_t$, $\widetilde{K} = -K$, $\widetilde{\sigma} = -\sigma$ and introducing a new Wiener process $\widetilde{W}_t = -W_t$, we see from (1.2) that:

$$d\widetilde{S}_t = \mu \widetilde{S}_t \, dt + \widetilde{\sigma} \widetilde{S}_t \, d\widetilde{W}_t \ . \tag{2.3}$$

Solving this equation we find that:

$$\widetilde{S}_t = \widetilde{S}_0 \exp\left(\widetilde{\sigma}\widetilde{W}_t + \left(\mu - \frac{\widetilde{\sigma}^2}{2}\right)t\right) = -S_t \ . \tag{2.4}$$

Rewriting (2.2) in the form:

$$(K - S_T)^+ = (\widetilde{S}_T - \widetilde{K})^+ \tag{2.5}$$

we may conclude the following: *The problem of finding the rational put price* $\mathbf{P}_T(S_0, K; \sigma)$ *in the model (1.1)-(1.2) is equivalent to the problem of finding the rational call price* $\mathbf{C}_T(\widetilde{S}_0, \widetilde{K}; \widetilde{\sigma})$ *in the model (2.3)-(2.4).*

This statement may be viewed as an explanation of the "call-put duality" relation (2.1) as observed above by a direct comparison of the closed form expressions (1.6) and (1.8). A simple economic interpretation of this equivalence appears to be transparent.

2. It is interesting to note that the "call-put duality" relation, unlike the "call-put parity" relation, remains also valid for call and put options of the American type.

For this consider the (B, S)-model of a financial market as above. Let $f_\tau = e^{-\lambda\tau}(S_\tau - K)^+$ and $g_\tau = e^{-\lambda\tau}(K - S_\tau)^+$ be the payoff functions of a standard American call and put option, respectively, with a strike price $K > 0$, a maturity time $T > 0$, a discount factor $\lambda \geq 0$, evaluated at a stopping time τ . We shall for simplicity assume that $T = \infty$, i.e. the options are perpetual, but the conclusions are generally valid.

By the general option pricing theory (see e.g. [1, Chapter VIII, § 2a]) we know that the rational (arbitrage) prices of these options are respectively given as follows:

$$\mathbf{C}^*(S_0, K; \sigma) = \sup_\tau \widehat{E}\Big(e^{-\lambda\tau}(S_\tau - K)^+\Big) \tag{2.6}$$

$$\mathbf{P}^*(S_0, K; \sigma) = \sup_\tau \widehat{E}\Big(e^{-\lambda\tau}(K - S_\tau)^+\Big) \tag{2.7}$$

where the supremums are taken over all stopping times τ of S .

Note that from (1.10) we obtain:

$$e^{-\lambda\tau}(K - S_\tau)^+ = e^{-\lambda\tau}(S_\tau - K)^+ + e^{-\lambda\tau}(K - S_\tau) \tag{2.8}$$

and hence it follows:

$$\sup_\tau \widehat{E}\Big(e^{-\lambda\tau}(K - S_\tau)^+\Big) = \sup_\tau \widehat{E}\Big(e^{-\lambda\tau}(S_\tau - K)^+ + e^{-\lambda\tau}(K - S_\tau)\Big) \ . \tag{2.9}$$

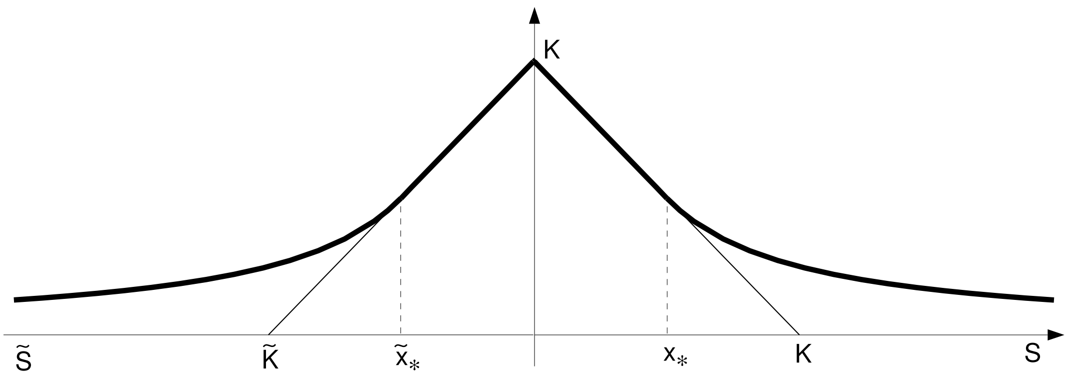
From this identity we see that it is difficult to formulate an analogue of the "call-put parity" relation for the rational prices $\mathbf{C}^*$ and $\mathbf{P}^*$. It turns out, however, that we may still establish some analogue of the "call-put duality" relation for these prices, and this is seen as follows.

Consider the optimal stopping problem:

$$\mathbf{C}^*(\widetilde{S}_0, \widetilde{K}; \widetilde{\sigma}) = \sup_\tau \widehat{E}\Big(e^{-\lambda\tau}(\widetilde{S}_\tau - \widetilde{K})^+\Big) \tag{2.10}$$

with negative $\widetilde{S}_0 = -S_0$, $\widetilde{K} = -K$, $\widetilde{\sigma} = -\sigma$ and negative process $\widetilde{S} = -S$ solving (2.3) with $\widetilde{W} = -W$. The solution of the problem (2.10) can be found by the same method which is used to find solutions of the optimal stopping problems (2.6) and (2.7) upon reducing them to a free-boundary problem (for more details see e.g. [1; Chapter VIII, §§ 2a, 2b]).

The following drawing:



makes it clear that the solution of the problem (2.10) is a mirror image of the solution for the problem (2.7). For the solution of the problem (2.10) we only need to consider the left-hand part i.e. negative values $\widetilde{S}$, and for the solution of the problem (2.7) we only need to consider the right-hand part i.e. positive values S .

By this symmetry argument it is clear without any calculation that $\widetilde{x}_* = -x_*$, i.e. the optimal stopping times for the problems (2.7) and (2.10) are respectively given as follows:

$$\tau_* = \inf \{ t > 0 \mid S_t \le x_* \} \tag{2.11}$$

$$\widetilde{\tau}_* = \inf \{ t > 0 \mid \widetilde{S}_t \ge \widetilde{x}_* \} . \tag{2.12}$$

Since $\widetilde{S}_t = -S_t$ and $\widetilde{x}_* = -x_*$ we moreover see that $\tau_* = \widetilde{\tau}_*$. It is also evident from the same symmetry argument that:

$$\mathbf{P}^*(S_0, K; \sigma) = \mathbf{C}^*(-S_0, -K; -\sigma) \tag{2.13}$$

showing that the "call-put duality" relation holds for the call and put options of the American type.

REFERENCES

[1] SHIRYAEV, A. N. (1999). *Essentials of Stochastic Finance*. World Scientific.

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