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On Doob's Maximal Inequality for Brownian Motion

S. E. GRAVERSEN *and* G. PESKIR

If $B = (B_t)_{t \geq 0}$ is a standard Brownian motion started at x under P_x for $x \geq 0$, and τ is any stopping time for B with $E_x(\tau) < \infty$, then for each $p > 1$ the following inequality is shown to be sharp:

$$E_x\Big(\max_{0\leq t\leq\tau}|B_t|^p\Big) \leq \Big(\frac{p}{p-1}\Big)^p E_x|B_\tau|^p - \Big(\frac{p}{p-1}\Big)x^p .$$

The sharpness is realized through the stopping times of the form:

$$\tau_{\lambda,\varepsilon} = \inf\Big\{\, t > 0 \;\Big|\; \max_{0\leq s\leq t} B_s - \lambda B_t \geq \varepsilon \,\Big\}$$

for which it is computed:

$$E_0(\tau_{\lambda,\varepsilon}) = \frac{\varepsilon^2}{\lambda(2-\lambda)}$$

whenever $\varepsilon > 0$ and $0 < \lambda < 2$. Hence, for the stopping time:

$$\sigma_{\lambda,\varepsilon} = \inf\Big\{\, t > 0 \;\Big|\; \max_{0\leq s\leq t}|B_s| - \lambda|B_t| \geq \varepsilon \,\Big\}$$

which is shown to be a convolution of $\tau_{\lambda,\lambda\varepsilon}$ with the first hitting time of ε by $|B| = (|B_t|)_{t\geq 0}$, we have:

$$E_0(\sigma_{\lambda,\varepsilon}) = \frac{2\varepsilon^2}{(2-\lambda)}$$

for all $\varepsilon > 0$ and all $0 < \lambda < 2$. The method of proof relies upon the principle of smooth fit and the maximality principle for a Stephan problem with moving (free) boundary, and Itô-Tanaka's formula (being applied two-dimensionally). The main emphasis is on the explicit formulas throughout obtained.

1. Introduction

The main purpose of the paper is to derive and investigate a sharp maximal inequality of Doob's type for linear Brownian motion which may start at any point.

To describe this in more detail, let us assume we are given a standard Brownian motion $B = (B_t)_{t\geq 0}$ which is defined on the probability space $(\Omega, \mathcal{F}, P)$ and which starts at 0 under P. Then the well-known Doob's maximal inequality states:

$$E\Big(\max_{0\leq t\leq\tau}|B_t|^2\Big) \leq 4\, E|B_\tau|^2 \tag{1.1}$$

where τ may be any stopping time for B with finite expectation (see [4] p.353 or [8] p.52). The constant 4 is known to be the best possible in (1.1). For this one can consider the stopping times:

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$$\sigma_{\lambda,\varepsilon} = \inf \left\{ t > 0 \;\middle|\; \max_{0 \le s \le t} |B_s| - \lambda |B_t| \ge \varepsilon \right\} \tag{1.2}$$

where λ , $\varepsilon > 0$. It is well-known that $E(\sigma_{\lambda,\varepsilon})^{p/2} < \infty$ if and only if $\lambda < p/(p-1)$ whenever $\varepsilon > 0$ (see [9]). Applying Doob's maximal inequality with a general constant $K > 0$ to the stopping time in (1.2) with some $\varepsilon > 0$ when $0 < \lambda < 2$, we get:

$$E\left(\max_{0 \le t \le \sigma_{\lambda,\varepsilon}} |B_t|^2 \right) = \lambda^2 E|B_{\sigma_{\lambda,\varepsilon}}|^2 + 2\lambda\varepsilon E|B_{\sigma_{\lambda,\varepsilon}}| + \varepsilon^2 \le KE|B_{\sigma_{\lambda,\varepsilon}}|^2 \tag{1.3}$$

Dividing through in (3.1) by $E|B_{\sigma_{\lambda,\varepsilon}}|^2$ and using that $E|B_{\sigma_{\lambda,\varepsilon}}|^2 = E(\sigma_{\lambda,\varepsilon}) \to \infty$ together with $E|B_{\sigma_{\lambda,\varepsilon}}|/E|B_{\sigma_{\lambda,\varepsilon}}|^2 \le 1/\sqrt{E(\sigma_{\lambda,\varepsilon})} \to 0$ as $\lambda \uparrow 4$, we see that $K \ge 4$.

Motivated by these facts our main aim in this paper is to find an analogue of the inequality (1.1) when the Brownian motion B does not necessarily start from 0 , but may start at any given point $x \ge 0$ under P_x for $x \ge 0$. Thus $P_x(B_0 = x) = 1$ for all $x \ge 0$, and we identify P_0 with P . Our main result (Theorem 2.1) is the inequality:

$$E_x\left(\max_{0 \le t \le \tau} |B_t|^2 \right) \le 4E_x|B_\tau|^2 - 2x^2 \tag{1.4}$$

which is valid for any stopping time τ for B with $E_x(\tau) < \infty$, and which is shown to be sharp as such. This is obtained as a consequence of the following inequality:

$$E_x\left(\max_{0 \le t \le \tau} |B_t|^2 \right) \le c\, E_x(\tau) + \frac{c}{2}\left(1 - \sqrt{1 - \frac{4}{c}} \right) x^2 \tag{1.5}$$

which is valid for all $c \ge 4$. When $c > 4$, the stopping time:

$$\tau_c = \inf \left\{ t > 0 \;\middle|\; \max_{0 \le s \le t} |B_s| - \frac{2}{1 + \sqrt{1 - 4/c}}\, |B_t| \ge 0 \right\} \tag{1.6}$$

is the one at which the equality in (1.5) is attained, and moreover we have:

$$E_x(\tau_c) = \frac{\left(1 - \sqrt{1 - 4/c} \right)^2}{4\sqrt{1 - 4/c}}\, x^2 \tag{1.7}$$

for all $x \ge 0$ and all $c > 4$.

In particular, if we consider the stopping time:

$$\tau_{\lambda,\varepsilon} = \inf \left\{ t > 0 \;\middle|\; \max_{0 \le s \le t} B_s - \lambda B_t \ge \varepsilon \right\} \tag{1.8}$$

then (1.7) can be rewritten to read as follows:

$$E_0\big(\tau_{\lambda,\varepsilon}\big) = \frac{\varepsilon^2}{\lambda(2 - \lambda)} \tag{1.9}$$

for all $\varepsilon > 0$ and all $0 < \lambda < 2$. Quite independently from this formula and its proof, below we present a simple argument for $E(\tau_{2,\varepsilon}) = \infty$ which is based upon Tanaka's formula.

Finally, since $\sigma_{\lambda,\varepsilon}$ defined by (1.2) is shown to be a convolution of $\tau_{\lambda,\lambda\varepsilon}$ and H_ε , where $H_\varepsilon = \inf\{\, t > 0 \;:\; |B_t| = \varepsilon \,\}$, from (1.9) we obtain the formula:

$$E_0\big(\sigma_{\lambda,\varepsilon}\big) \;=\; \frac{2\varepsilon^2}{(2-\lambda)} \tag{1.10}$$

for all $\varepsilon > 0$ and all $0 < \lambda < 2$.

The method of proof relies upon the principle of smooth fit (see [5]) and the maximality principle (see [6]) for a Stephan problem with moving (free) boundary, and Itô-Tanaka's formula (being applied two-dimensionally). The result and method of Theorem 2.1 easily extend to the case $p > 1$ (Corollary 2.2), while this further extend to all non-negative submartingales (Corollary 2.3) by using the maximal embedding theorem of Jacka [7]. The main emphasis is on the explicit formulas throughout obtained.

2. The inequality and proof

In this and next section we present the main results of the paper. After this work was completed we learned from D. Burkholder that the inequalities (2.36) below follow as a by-product from his new proof of Doob's inequality for discrete non-negative submartingales (see [2] p.14). While the proof given there in essence relies on the submartingale property, the proof given here is based on the (strong) Markov property. The advantage of the approach taken here lies in its applicability to all diffusions (see [6]). Yet another advantage is that during the proof we explicitly write down the optimal stopping times (those through which the equality is attained). Recently we also learned that Cox [3] derived the analogue of these inequalities for discrete martingales by a method which is based on results from the theory of moments. In his paper Cox also notes that "the method does have the drawback of computational complexity, which sometimes makes it difficult or impossible to push the calculations through".

Theorem 2.1

Let $B = (B_t)_{t\geq 0}$ be a standard Brownian motion started at x under P_x for $x \geq 0$, and let τ be any stopping time for B such that $E_x(\tau) < \infty$. Then the following inequality is sharp:

$$E_x\bigg(\max_{0\leq t\leq\tau} |B_t|^2\bigg) \;\leq\; 4E_x|B_\tau|^2 - 2x^2 \; . \tag{2.1}$$

The constants 4 and 2 are the best possible.

Proof. We shall begin by considering the following optimal stopping problem:

$$V(x,s) \;=\; \sup_\tau E_{x,s}\big(S_\tau - c\tau\big) \tag{2.2}$$

where the supremum is taken over all stopping times τ for B satisfying $E_{x,s}(\tau) < \infty$, while the maximum process $S = (S_t)_{t\geq 0}$ is defined by:

$$S_t \;=\; \bigg(\max_{0\leq r\leq t} |B_r|^2\bigg) \vee s \tag{2.3}$$

where $s \geq x \geq 0$ are given and fixed. The expectation in (2.2) is taken with respect to the probability measure $P_{x,s}$ under which S starts at s, and the process $X = (X_t)_{t \geq 0}$ defined by:

(2.4) $$X_t = |B_t|^2$$

starts at x. The Brownian motion B from (2.3) and (2.4) may be realized as:

(2.5) $$B_t = \tilde{B}_t + \sqrt{x}$$

where $\tilde{B} = (\tilde{B}_t)_{t \geq 0}$ is a standard Brownian motion started at 0 under P. Thus the (strong) Markov process (X, S) starts at (x, s) under P, and $P_{x,s}$ may be identified with P.

By Itô formula we find:

(2.6) $$dX_t = dt + 2\sqrt{X_t}\, dB_t\ .$$

Hence we see that the infinitesimal operator of the (strong) Markov process X in $]0, \infty[$ acts like:

(2.7) $$\mathbf{L_X} = \frac{\partial}{\partial x} + 2x \frac{\partial^2}{\partial x^2}$$

while the boundary point 0 is a point of the instantaneous reflection.

If we assume that the supremum in (2.2) is attained at the exit time from an open set by the (strong) Markov process (X, S) which is degenerated in the second component, then by the general Markov processes theory (see [8] p.287-299) it is plausible to assume that the payoff $x \mapsto V(x,s)$ satisfies the following equation:

(2.8) $$\mathbf{L_X} V(x,s) = c$$

for $x \in]g_*(s), s[$ with $s > 0$ given and fixed, where $s \mapsto g_*(s)$ is an optimal stopping boundary to be found. The boundary conditions which may be fulfilled are the following:

(2.9) $$V(x,s)\Big|_{x=g_*(s)+} = s \quad \text{(instantaneous stopping)}$$

(2.10) $$\frac{\partial V}{\partial x}(x,s)\Big|_{x=g_*(s)+} = 0 \quad \text{(smooth fit)}$$

(2.11) $$\frac{\partial V}{\partial s}(x,s)\Big|_{x=s-} = 0 \quad \text{(normal reflection).}$$

The general solution to the equation (2.8) is given by:

(2.12) $$V(x,s) = A(s)\sqrt{x} + B(s) + cx$$

where $A(s)$ and $B(s)$ are unspecified constants. From (2.9)+(2.10) we find that:

(2.13) $$A(s) = -2c\sqrt{g_*(s)}$$

(2.14) $$B(s) = s + c g_*(s)\ .$$

Inserting this into (2.12) gives:

$$V(x,s) = -2c\sqrt{g_*(s)}\sqrt{x} + s + cg_*(s) + cx \ . \tag{2.15}$$

By (2.11) we find that $s \mapsto g_*(s)$ is to satisfy the (nonlinear) differential equation:

$$cg'(s)\left(1-\sqrt{\frac{s}{g(s)}}\right) + 1 = 0 \ . \tag{2.16}$$

The general solution of the equation (2.16) may be expressed in a closed form. Instead of going into this direction we shall rather note that this equation admits a linear solution of the form:

$$g_*(s) = \alpha\, s \tag{2.17}$$

where the given $\alpha > 0$ is to satisfy:

$$\alpha - \sqrt{\alpha} + 1/c = 0 \ . \tag{2.18}$$

Motivated by the maximality principle (see [6]) we shall chose the greater α satisfying (2.18) as our candidate:

$$\alpha = \left(\frac{1+\sqrt{1-4/c}}{2}\right)^2 . \tag{2.19}$$

Inserting this into (2.15) gives:

$$\begin{aligned} V_*(x,s) &= -2c\sqrt{\alpha x s} + (1+c\alpha)s + cx \quad , \ \text{if} \ \ \alpha s \le x \le s \\ &= s \quad , \ \text{if} \ \ 0 \le x \le \alpha s \end{aligned} \tag{2.20}$$

as a candidate for the payoff $V(x,s)$ defined in (2.2). The optimal stopping time is then to be:

$$\tau_* = \inf\{\, t > 0 \mid X_t \le g_*(S_t)\,\} \tag{2.21}$$

where $s \mapsto g_*(s)$ is defined by (2.17)+(2.19).

To verify that the formulas (2.20) and (2.21) are indeed correct, we shall use Itô-Tanaka's formula being applied two-dimensionally (see [6] for a formal justification of its use in this context; note that $(x,s) \mapsto V_*(x,s)$ is C^2 outside $\{\,(g_*(s),s) \mid s > 0\,\}$ while $x \mapsto V_*(x,s)$ is convex and C^2 on $]0,s[$ but at $g_*(s)$ where it is only C^1 whenever $s > 0$ is given and fixed). In this way we obtain:

$$\begin{aligned} V_*(X_t,S_t) = V_*(X_0,S_0) &+ \int_0^t \frac{\partial V_*}{\partial x}(X_r,S_r)\ dX_r + \int_0^t \frac{\partial V_*}{\partial s}(X_r,S_r)\ dS_r \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2 V_*}{\partial x^2}(X_r,S_r)\ d\langle X,X\rangle_r \end{aligned} \tag{2.22}$$

where we set $(\partial^2 V_*/\partial x^2)(g_*(s),s) = 0$. Since the increment dS_r equals zero outside the diagonal $x = s$, and $V_*(x,s)$ at the diagonal satisfies (2.11), we see that the second integral in (2.22) is identically zero. Thus by (2.6)+(2.7) and the fact that $d\langle X,X\rangle_t = 4X_t\,dt$, we see that

(2.22) can be equivalently written as follows:

$$V_*(X_t,S_t)=V_*(x,s)+\int_0^t \mathbf{L_X}V_*(X_r,S_r)\;dr+2\int_0^t\sqrt{X_r}\;\frac{\partial V_*}{\partial x}(X_r,S_r)\;dB_r\;. \tag{2.23}$$

Next note that $\mathbf{L_X}V_*(y,s)=c$ for $g_*(s)<y<s$, and $\mathbf{L_X}V_*(y,s)=0$ for $0\le y\le g_*(s)$. Moreover, due to the normal reflection of X , the set of those $r>0$ for which $X_r=S_r$ is of Lebesgue measure zero. This by (2.23) shows that:

$$V_*(X_\tau,S_\tau)\;\le\;V_*(x,s)+c\tau+M_\tau \tag{2.24}$$

for any stopping time τ for B , where $M=(M_t)_{t\ge0}$ is a continuous local martingale defined by:

$$M_t\;=\;2\int_0^t\sqrt{X_r}\;\frac{\partial V_*}{\partial x}(X_r,S_r)\;dB_r\;. \tag{2.25}$$

Moreover, this also shows that:

$$V_*(X_\tau,S_\tau)\;=\;V_*(x,s)+c\tau+M_\tau \tag{2.26}$$

for any stopping time τ for B satisfying $\tau\le\tau_*$.

Next we show that:

$$E_{x,s}(M_\tau)\;=\;0 \tag{2.27}$$

whenever τ is a stopping time for B with $E_{x,s}(\tau)<\infty$. For (2.27), by Burkholder-Gundy's inequality for continuous local martingales (see [8] p.153), it is sufficient to show that:

$$E_{x,s}\left(\int_0^\tau\left(\sqrt{X_r}\;\frac{\partial V_*}{\partial x}(X_r,S_r)\right)^2 1_{\{X_r\ge g_*(S_r)\}}\;dr\right)^{1/2}:=I<\infty\;. \tag{2.28}$$

From (2.20) we compute:

$$\frac{\partial V_*}{\partial x}(y,s)\;=\;-\frac{c\sqrt{\alpha s}}{\sqrt{y}}+c \tag{2.29}$$

for $\alpha s\le y\le s$. Inserting this into (2.28) we get:

$$\begin{aligned}I&=c\,E_{x,s}\left(\int_0^\tau\left(\sqrt{X_r}-\sqrt{\alpha S_r}\right)^2 1_{\{X_r\ge\alpha S_r\}}\;dr\right)^{1/2}\\&\le c\,(1-\sqrt{\alpha})\,E_{x,s}\left(\int_0^\tau S_r\;dr\right)^{1/2}\le c\,(1-\sqrt{\alpha})\,E_{x,s}\left(\sqrt{S_\tau}\sqrt{\tau}\right)\\&\le c\,(1-\sqrt{\alpha})\;\sqrt{E_{x,s}(S_\tau)}\;\sqrt{E_{x,s}(\tau)}\\&=c\,(1-\sqrt{\alpha})\left(E_{x,s}\left(\left(\max_{0\le t\le\tau}|\tilde{B}_t+\sqrt{x}|^2\right)\vee s\right)\right)^{1/2}\sqrt{E_{x,s}(\tau)}\end{aligned} \tag{2.30}$$

$$\leq c\,(1-\sqrt{\alpha})\Big(2E_{x,s}\Big(\max_{0\leq t\leq\tau}|\tilde{B}_t|^2\Big)+2x+s\Big)\Big)^{1/2}\sqrt{E_{x,s}(\tau)}$$

$$\leq c\,(1-\sqrt{\alpha})\Big(8E_{x,s}(\tau)+2x+s\Big)^{1/2}\sqrt{E_{x,s}(\tau)}<\infty$$

where we used Hölder's inequality, Doob's inequality (1.1), and the fact that $E_{x,s}|\tilde{B}_\tau|^2=E_{x,s}(\tau)$ whenever $E_{x,s}(\tau)<\infty$.

Since $V_*(x,s)\geq s$, from (2.24)+(2.27) we find:

$$V(x,s)=\sup_\tau E_{x,s}\big(S_\tau-c\tau\big)\leq\sup_\tau E_{x,s}\big(S_\tau-V_*(X_\tau,S_\tau)\big) + \sup_\tau E_{x,s}\big(V_*(X_\tau,S_\tau)-c\tau\big)\leq V_*(x,s)\ . \tag{2.31}$$

Moreover, from (2.26)+(2.27) with $\tau=\tau_*$ we see that:

$$E_{x,s}\big(S_{\tau_*}-c\tau_*\big)=E_{x,s}\big(V_*(X_{\tau_*},S_{\tau_*})-c\tau_*\big)=V_*(x,s) \tag{2.32}$$

provided that $E_{x,s}(\tau_*)<\infty$, which is known to be true if and only if $c>4$ (see [9]). (Below we present a different proof of this fact and moreover compute the value $E_{x,s}(\tau_*)$ exactly.) Matching (2.31) and (2.32) we see that the payoff (2.2) is indeed given by the formula (2.20), and an optimal stopping time for (2.2) (stopping time at which the supremum is attained) is given by (2.21) with $s\mapsto g_*(s)$ from (2.17) and $\alpha\in\,]0,1[$ from (2.19).

In particular, note that from (2.20) with α from (2.19) we get:

$$V_*(x,x)=\frac{c}{2}\bigg(1-\sqrt{1-\frac{4}{c}}\bigg)x\ . \tag{2.33}$$

Applying the very definition of $V(x,x)=V_*(x,x)$ and letting $c\downarrow 4$, this yields:

$$E_x\Big(\max_{0\leq t\leq\tau}|B_t|^2\Big)\leq 4E_x(\tau)+2x\ . \tag{2.34}$$

Finally, standard arguments show that:

$$E_x|B_\tau|^2=E_x|\tilde{B}_\tau+\sqrt{x}|^2=E_x|\tilde{B}_\tau|^2+2\sqrt{x}\,E_x(\tilde{B}_\tau)+x=E_x(\tau)+x\ . \tag{2.35}$$

Inserting this into (2.34) we obtain (2.1). The sharpness clearly follows from the definition of the payoff in (2.2). This completes the proof of the theorem. □

The previous result and method easily extend to the case $p>1$. For reader's convenience we state this extension and sketch the proof.

Corollary 2.2

Let $B=(B_t)_{t\geq 0}$ *be a standard Brownian motion started at* x *under* P_x *for* $x\geq 0$ *, let* $p>1$ *be given and fixed, and let* τ *be any stopping time for* B *such that* $E_x(\tau^{p/2})<\infty$ *. Then the following inequality is sharp:*

$$E_x\left(\max_{0\le t\le\tau}|B_t|^p\right) \le \left(\frac{p}{p-1}\right)^p E_x|B_\tau|^p - \left(\frac{p}{p-1}\right)x^p \ . \tag{2.36}$$

The constants $(p/(p-1))^p$ *and* $p/(p-1)$ *are the best possible.*

Proof. In parallel to (2.2) one is to consider the following optimal stopping problem:

$$V(x,s) = \sup_\tau E_{x,s}\big(S_\tau - cI_\tau\big) \tag{2.37}$$

where the supremum is taken over all stopping times τ for B satisfying $E_{x,s}(\tau^{p/2}) < \infty$, and the underlying processes are given as follows:

$$S_t = \left(\max_{0\le r\le t} X_r\right) \vee s \tag{2.38}$$

$$I_t = \int_0^t \big(X_r\big)^{(p-2)/p} dr \tag{2.39}$$

$$X_t = |B_t|^p \tag{2.40}$$

$$B_t = \tilde{B}_t + x^{1/p} \tag{2.41}$$

where $\tilde{B} = (\tilde{B}_t)_{t\ge 0}$ is a standard Brownian motion started at 0 under $P = P_{x,s}$. This problem can be solved in exactly the same way as the problem (2.2) along the following lines.

The infinitesimal operator of X equals:

$$\mathbf{L_X} = \frac{p(p-1)}{2}\, x^{1-2/p}\, \frac{\partial}{\partial x} + \frac{p^2}{2}\, x^{2-2/p}\, \frac{\partial^2}{\partial x^2} \ . \tag{2.42}$$

The analogue of the equation (2.8) is:

$$\mathbf{L_X} V(x,s) = c\, x^{(p-2)/p} \ . \tag{2.43}$$

The conditions (2.9)-(2.11) are to be satisfied again. The analogue of the solution (2.15) is:

$$V(x,s) = -\frac{2c}{p-1}\, g_*^{1-1/p}(s)\, x^{1/p} + s + \frac{2c}{p} g_*(s) + \frac{2c}{p(p-1)}\, x \tag{2.44}$$

where $s \mapsto g_*(s)$ is to satisfy the equation:

$$\frac{2c}{p}\, g'(s)\left(1 - \left(\frac{s}{g(s)}\right)^{1/p}\right) + 1 = 0 \ . \tag{2.45}$$

Again, like in (2.16), this equation admits a linear solution of the form:

$$g_*(s) = \alpha s \tag{2.46}$$

where $0 < \alpha < 1$ is the maximal root (out of two possible ones) of the equation:

$$\alpha - \alpha^{1-1/p} + p/2c = 0 \ . \tag{2.47}$$

By the standard argument one can verify that (2.47) admits such a root if and only if $c \geq p^{p+1}/2(p-1)^{(p-1)}$. The optimal stopping time is then to be:

$$\tau_* = \inf\{\, t>0 \mid X_t \leq g_*(S_t)\,\} \tag{2.48}$$

where $s \mapsto g_*(s)$ is from (2.46).

To verify that the guessed formulas (2.44) and (2.48) are indeed correct we can use exactly the same procedure as in the proof of Theorem 2.1. For this, it should be recalled that $E_{x,s}(\tau_*^{p/2}) < \infty$ if and only if $c > p^{p+1}/2(p-1)^{(p-1)}$ (see [9]). Note also that the analogue of (2.35) is by Itô formula and the optional sampling theorem given by:

$$E_{x,s}\big(X_\tau\big) = x + \frac{p(p-1)}{2} E_{x,s}(I_\tau) \tag{2.49}$$

whenever $E_{x,s}(\tau^{p/2}) < \infty$, which was the motivation for considering the problem (2.37) with (2.39). The remaining details are easily completed and will be left to the reader. □

Due to the extreme properties of Brownian motion, the inequality (2.36) extend to all non-negative submartingales. This can be obtained by using the maximal embedding result of Jacka [7].

Corollary 2.3

Let $X = (X_t)_{t\geq 0}$ be a non-negative cadlag (right continuous with left limits) uniformly integrable submartingale started at $x \geq 0$ under P . Let X_∞ denote the P-a.s. limit of X_t for $t \to \infty$. Then the following inequality is satisfied and sharp:

$$E\Big(\sup_{t>0} X_t^p\Big) \leq \Big(\frac{p}{p-1}\Big)^p E\big(X_\infty^p\big) - \Big(\frac{p}{p-1}\Big)\, x^p \tag{2.50}$$

for all $p > 1$.

Proof. Given such a submartingale $X = (X_t)_{t\geq 0}$ satisfying $E(X_\infty) < \infty$, and a Brownian motion $B = (B_t)_{t\geq 0}$ started at $X_0 = x$ under P_x , by the result of Jacka [7] we know that there exists a stopping time τ for B , such that $|B_\tau| \sim X_\infty$ and $P\{\, \sup_{t\geq 0} X_t \geq \lambda \,\} \leq P_x\{\, \max_{0\leq t\leq \tau} |B_\tau| \geq \lambda \,\}$ for all $\lambda > 0$, with $(B_{t\wedge\tau})_{t\geq 0}$ being uniformly integrable. The result then easily follows from Corollary 2.2 by using the integration by parts formula. Note that by the submartingale property of $(|B_{t\wedge\tau}|)_{t\geq 0}$ we get $\sup_{t\geq 0} E_x|B_{t\wedge\tau}|^p = E_x|B_\tau|^p$ for all $p > 1$, so that $E_x(\tau^{p/2})$ is finite if and only if $E_x|B_\tau|^p$ is so.

□

3. The expected waiting time

It is our main aim in this section to derive an explicit formula for the expectation of the optimal stopping time τ_* constructed in the proof of Theorem 2.1 (or Corollary 2.2). This extend a result of Wang [9] who showed its finiteness. The stopping times of the form of τ_* have been investigated by several people. Instead of going into a historical exposition on this subject, we shall rather refer the reader to the paper of Azéma and Yor [1] where more information in this direction can be found. We'd like to point out however that as far as one is only concerned with the expectation of such a stopping time, the Laplace transform method developed in some of these works may have the drawback of computational complexity in comparison with our method presented below which elegantly applies to all diffusions.

Throughout we shall work within the setting and notation of Theorem 2.1 and its proof. By (2.21) with (2.17) we have:

$$\tau_* = \inf \{ t>0 \mid X_t \le \alpha S_t \} \tag{3.1}$$

where $\alpha = \alpha(c)$ is a constant defined in (2.19) for $c > 4$. Note that $1/4 < \alpha(c) \uparrow 1$ as $c \uparrow \infty$. Our main task in this section is to compute explicitly the function:

$$m(x,s) = E_{x,s}(\tau_*) \tag{3.2}$$

for $0 \le x \le s$, where $E_{x,s}$ denotes the expectation with respect to $P_{x,s}$ under which X starts at x and S starts at s. Since clearly $m(x,s) = 0$ for $0 \le x \le \alpha s$, we shall assume throughout that $\alpha s < x \le s$ are given and fixed.

Because τ_* may be viewed as the exit time from an open set by the (strong) Markov process (X,S) which is degenerated in the second component, by the general Markov processes theory (see [8] p.287-299) it is plausible to assume that $x \mapsto m(x,s)$ satisfies the equation:

$$\mathbf{L_X}\, m(x,s) = -1 \tag{3.3}$$

for $\alpha s < x < s$ with $\mathbf{L_X}$ given by (2.7). The following two boundary conditions seem apparent:

$$m(x,s)\Big|_{x=\alpha s+} = 0 \quad \text{(instantaneous stopping)} \tag{3.4}$$

$$\frac{\partial m}{\partial s}(x,s)\Big|_{x=s-} = 0 \quad \text{(normal reflection).} \tag{3.5}$$

The general solution to (3.3) is given by:

$$m(x,s) = A(s)\sqrt{x} + B(s) - x \tag{3.6}$$

where $A(s)$ and $B(s)$ are unspecified constants. By (3.4) and (3.5) we find:

$$A(s) = C\, s^{\Delta} + \frac{2\alpha}{2\sqrt{\alpha}-1}\sqrt{s} \tag{3.7}$$

$$B(s) = -C\sqrt{\alpha}\, s^{\Delta+1/2} - \frac{\alpha}{2\sqrt{\alpha}-1}\, s \tag{3.8}$$

where $C = C(\alpha)$ is a constant to be determined, and where:

$$\Delta = \frac{\sqrt{\alpha}}{2(1-\sqrt{\alpha})}\ . \tag{3.9}$$

In order to determine the constant C, we shall note by (3.6)-(3.9) that:

$$m(x,x) = C(1-\sqrt{\alpha})\, x^{1/2(1-\sqrt{\alpha})} \;+\; \frac{(\sqrt{\alpha}-1)^2}{2\sqrt{\alpha}-1}\, x \;. \tag{3.10}$$

Observe now that the power $1/2(1-\sqrt{\alpha}) > 1$, due to the fact that $\alpha = \alpha(c) > 1/4$ when $c > 4$. However, the payoff in (2.33) is linear and given by:

$$V_*(x,x) \;:=\; V_*(x;c) \;=\; K(c)\cdot x \tag{3.11}$$

where $K(c) = (c/2)(1-\sqrt{1-4/c})$. This indicates that the constant C must be identically zero. Formally, this is verified as follows.

Since $c > 4$ there is $\lambda \in\,]0,1[$ such that $\lambda c > 4$. By definition of the payoff we have:

$$\begin{aligned} 0 < V_*(x;c) &= E_{x,x}\big(S_{\tau_*(c)} - c\,\tau_*(c)\big) = E_{x,x}\big(S_{\tau_*(c)} - \lambda c\,\tau_*(c)\big) - (1-\lambda)c\,E_{x,x}(\tau_*(c)) \\ &\le V_*(x;\lambda c) - (1-\lambda)c\,E_{x,x}(\tau_*(c)) \le K(\lambda c)\cdot x - (1-\lambda)c\,E_{x,x}(\tau_*(c)) \;. \end{aligned} \tag{3.12}$$

This shows that $x \mapsto m(x,x)$ is at most linear:

$$m(x,x) \;=\; E_{x,x}(\tau_*(c)) \le \frac{K(\lambda c)}{(1-\lambda)c}\, x \;. \tag{3.13}$$

Looking back to (3.10) we conclude $C \equiv 0$.

Thus by (3.6)-(3.8) with $C \equiv 0$ we finish up with the following candidate for $E_{x,s}(\tau_*)$:

$$m(x,s) \;=\; \frac{2\alpha}{2\sqrt{\alpha}-1}\sqrt{xs} \;-\; \frac{\alpha}{2\sqrt{\alpha}-1}\, s \;-\; x \tag{3.14}$$

when $\alpha s < x \le s$. In order to verify that this formula is indeed correct we shall use Itô-Tanaka's formula in the proof below.

Theorem 3.1

Let $B = (B_t)_{t\ge 0}$ be a standard Brownian motion, and let $X = (X_t)_{t\ge 0}$ and $S = (S_t)_{t\ge 0}$ be associated with B by formulas (2.3)+(2.4). Then for the stopping time τ_* defined in (3.1):

$$\begin{aligned} E_{x,s}(\tau_*) \;&=\; \frac{2\alpha}{2\sqrt{\alpha}-1}\sqrt{xs} \;-\; \frac{\alpha}{2\sqrt{\alpha}-1}\, s \;-\; x \;\;,\;\; \textit{if} \;\; \alpha s \le x \le s \\ &=\; 0 \;\;,\;\; \textit{if} \;\; 0 \le x \le \alpha s \;. \end{aligned} \tag{3.15}$$

where $\alpha > 1/4$.

Proof. Denote the function on the right-hand side of (3.15) by $m(x,s)$. Note that $x \mapsto m(x,s)$ is concave and non-negative on $[\alpha s, s]$ for each fixed $s > 0$. By Itô-Tanaka's formula (see [6] for a justification of its use in this context) we get:

$$\begin{aligned} m(X_t,S_t) \;=\; m(X_0,S_0) &+ \int_0^t \mathbf{L_X}\, m(X_r,S_r)\; dr + 2\int_0^t \sqrt{X_r}\, \frac{\partial m}{\partial x}(X_r,S_r)\; dB_r \\ &+ \int_0^t \frac{\partial m}{\partial s}(X_r,S_r)\; dS_r \;. \end{aligned} \tag{3.16}$$

Due to (3.5) the last integral in (3.16) is identically zero. In addition, let us consider the region $G = \{ (x,s) \mid \alpha s < x < s+1 \}$. Given $(x,s) \in G$ chose bounded open sets $G_1 \subset G_2 \subset \ldots$ such that $\cup_{n=1}^{\infty} G_n = G$ and $(x,s) \in G_1$. Denote the exit time of (X,S) from G_n by τ_n , then clearly $\tau_n \uparrow \tau_*$ as $n \to \infty$. Denote further the second integral in (3.16) by M_t . Then $M = (M_t)_{t \geq 0}$ is a continuous local martingale, and we have:

(3.17) $$E_{x,s}(M_{\tau_n}) \, = \, 0$$

for all $n \geq 1$. For this note that:

(3.18) $$E_{x,s}\left(\int_0^{\tau_n} \left(\sqrt{X_r} \, \frac{\partial m}{\partial x}(X_r, S_r) \right)^2 dr \right) \leq K E_{x,s}(\tau_n) < \infty$$

with some $K > 0$, since $(x,s) \mapsto \sqrt{x} \, (\partial m / \partial x)(x,s)$ is bounded on the closure of G_n .

By (3.3) from (3.16)+(3.17) we find:

(3.19) $$E_{x,s} m(X_{\tau_n}, S_{\tau_n}) \, = \, m(x,s) - E_{x,s}(\tau_n) \; .$$

Since $(x,s) \mapsto m(x,s)$ is non-negative, hence first of all we may deduce:

(3.20) $$E_{x,s}(\tau_*) \, = \, \lim_{n \to \infty} E_{x,s}(\tau_n) \, \leq \, m(x,s) \, < \, \infty \; .$$

This proves the finiteness of the expectation of τ_* (see [9] for another proof based upon random walk). Moreover, motivated by a uniform integrability argument we may note that:

(3.21) $$m(X_{\tau_n}, S_{\tau_n}) \, \leq \, \frac{2\alpha}{2\sqrt{\alpha} - 1} \, \sqrt{X_{\tau_n} S_{\tau_n}} \, \leq \, \frac{2\alpha}{2\sqrt{\alpha} - 1} S_{\tau_*}$$

uniformly over all $n \geq 1$. Moreover, by Doob's inequality (1.1) and (3.20) we find:

(3.22) $$E_{x,s}(S_{\tau_*}) \, \leq \, 2\big(4E_{x,s}(\tau_*) + x\big) + s \, < \, \infty \; .$$

Thus the sequence $(m(X_{\tau_n}, S_{\tau_n}))_{n \geq 1}$ is uniformly integrable, while it clearly converges to zero pointwise. For this reason we may conclude:

(3.23) $$\lim_{n \to \infty} E_{x,s} m(X_{\tau_n}, S_{\tau_n}) \, = \, 0 \; .$$

This shows that we have an equality in (3.20), and the proof is complete. □

Corollary 3.2

Let $B = (B_t)_{t \geq 0}$ *be a standard Brownian motion started at* 0 *under* P *. Consider the stopping times:*

(3.24) $$\tau_{\lambda,\varepsilon} = \inf \left\{ \, t > 0 \; \Big| \; \max_{0 \leq s \leq t} B_s - \lambda B_t \, \geq \, \varepsilon \, \right\}$$

(3.25) $$\sigma_{\lambda,\varepsilon} = \inf \left\{ \, t > 0 \; \Big| \; \max_{0 \leq s \leq t} |B_s| - \lambda |B_t| \geq \varepsilon \, \right\}$$

for $\varepsilon > 0$ *and* $0 < \lambda < 2$ *. Then* $\sigma_{\lambda,\varepsilon}$ *is a convolution of* $\tau_{\lambda,\lambda\varepsilon}$ *and* H_ε *, where* $H_\varepsilon = \inf \{ t > 0 : |B_t| = \varepsilon \}$ *, and the formulas are valid:*

$$E\big(\tau_{\lambda,\varepsilon}\big) = \frac{\varepsilon^2}{\lambda(2-\lambda)} \tag{3.26}$$

$$E\big(\sigma_{\lambda,\varepsilon}\big) = \frac{2\varepsilon^2}{(2-\lambda)} \tag{3.27}$$

for all $\varepsilon > 0$ *and all* $0 < \lambda < 2$ *.*

Proof. Consider the definition rule for $\sigma_{\lambda,\varepsilon}$ in (3.25). Clearly $\sigma_{\lambda,\varepsilon} > H_\varepsilon$ and after hitting ε , the reflected Brownian motion $|B| = (|B_t|)_{t \geq 0}$ does not hit zero before $\sigma_{\lambda,\varepsilon}$. Thus its absolute value sign may be dropped out during the time interval between H_ε and $\sigma_{\lambda,\varepsilon}$, and the claim about the convolution identity follows by the reflection property and the strong Markov property of Brownian motion.

(3.26): Consider the stopping time τ_* defined in (3.1) for $s = x$. By the very definition it can be rewritten to read as follows:

$$\begin{aligned} \tau_* &= \inf \left\{ t > 0 \;\Big|\; |B_t|^2 \leq \alpha \max_{0 \leq s \leq t} |B_s|^2 \right\} \\ &= \inf \left\{ t > 0 \;\Big|\; \max_{0 \leq s \leq t} |B_s| - \frac{1}{\sqrt{\alpha}} |B_t| \geq 0 \right\} \\ &= \inf \left\{ t > 0 \;\Big|\; \max_{0 \leq s \leq t} |\tilde{B}_s + \sqrt{x}| - \frac{1}{\sqrt{\alpha}} |\tilde{B}_t + \sqrt{x}| \geq 0 \right\} \\ &= \inf \left\{ t > 0 \;\Big|\; \max_{0 \leq s \leq t} \big(\tilde{B}_s + \sqrt{x}\big) - \frac{1}{\sqrt{\alpha}} \big(\tilde{B}_t + \sqrt{x}\big) \geq 0 \right\} \\ &= \inf \left\{ t > 0 \;\Big|\; \max_{0 \leq s \leq t} \tilde{B}_s - \frac{1}{\sqrt{\alpha}} \tilde{B}_t \geq \Big(\frac{1}{\sqrt{\alpha}} - 1 \Big) \sqrt{x} \right\} . \end{aligned} \tag{3.28}$$

Setting $\lambda = 1/\sqrt{\alpha}$ and $\varepsilon = (1/\sqrt{\alpha} - 1)\sqrt{x}$, by (3.15) hence we find:

$$E\big(\tau_{\lambda,\varepsilon}\big) = E_{x,x}(\tau_*) = \frac{(\sqrt{\alpha} - 1)^2}{2\sqrt{\alpha} - 1}\, x = \frac{\varepsilon^2}{\lambda(2-\lambda)} . \tag{3.29}$$

(3.27): Since $E(H_\varepsilon) = \varepsilon^2$, by (3.26) we get:

$$E\big(\sigma_{\lambda,\varepsilon}\big) = E(\tau_{\lambda,\lambda\varepsilon}) + E(H_\varepsilon) = \frac{2\varepsilon^2}{(2-\lambda)} . \tag{3.30}$$

The proof is complete. □

Remark 3.3

Let $B = (B_t)_{t \geq 0}$ be a standard Brownian motion started at 0 under P . Consider the stopping time:

$$\tau_{2,\varepsilon} = \inf \left\{ t > 0 \;\Big|\; \max_{0 \leq s \leq t} B_s - 2B_t \geq \varepsilon \right\} \tag{3.31}$$

for $\varepsilon \geq 0$. It follows from (3.26) in Corollary 3.2 that:

$$E(\tau_{2,\varepsilon}) = +\infty \tag{3.32}$$

if $\varepsilon > 0$. Here we present yet another argument based upon Tanaka's formula which implies (3.32).

For this consider the process:

$$\beta_t = \int_0^t sign(B_s)\ dB_s \tag{3.33}$$

where $sign(x) = -1$ for $x \leq 0$ and $sign(x) = 1$ for $x > 0$. Then $\beta = (\beta_t)_{t \geq 0}$ is a standard Brownian motion, and Tanaka's formula states (see [8] p.214-215):

$$|B_t| = \beta_t + L_t \tag{3.34}$$

where $L = (L_t)_{t \geq 0}$ is the local time process of B at 0 given by:

$$L_t = \max_{0 \leq s \leq t} (-\beta_s) \ . \tag{3.35}$$

Thus $\tau_{2,\varepsilon}$ is equally distributed as:

$$\begin{aligned} \sigma &= \inf \Big\{ t > 0 \ \Big| \ \max_{0 \leq s \leq t} (-\beta_s) - 2(-\beta_t) \geq \varepsilon \Big\} \\ &= \inf \Big\{ t > 0 \ \Big| \ |B_t| \geq \varepsilon - \beta_t \Big\} \ . \end{aligned} \tag{3.36}$$

Note that σ is an $(\mathcal{F}_t^{\beta})$-stopping time, and since $\mathcal{F}_t^{\beta} = \mathcal{F}_t^{|B|} \subset \mathcal{F}_t^{B}$, we see that σ is an $(\mathcal{F}_t^{B})$-stopping time too. Assuming now that $E(\tau_{2,\varepsilon})$ which equals $E(\sigma)$ is finite, by the standard Wald's type identity for Brownian motion we obtain:

$$E(\sigma) = E|B_\sigma|^2 = E(\varepsilon - \beta_\sigma)^2 = \varepsilon^2 - 2\varepsilon E(\beta_\sigma) + E|\beta_\sigma|^2 = \varepsilon^2 + E(\sigma) \ . \tag{3.37}$$

Hence we see that ε must be zero. This completes the proof of (3.32).

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Svend Erik Graversen
Department of Mathematical Sciences
University of Aarhus, Denmark
Ny Munkegade, DK-8000 Aarhus
matseg@imf.au.dk

Goran Peskir
Department of Mathematical Sciences
University of Aarhus, Denmark
Ny Munkegade, DK-8000 Aarhus
home.imf.au.dk/goran
goran@imf.au.dk