Proc. Math. Control Theory Finance (Lisbon 2007), Springer, 2008, (95-112)
Research Report No. 4, 2007, *Probab. Statist. Group Manchester* (16 pp)

Predicting the Time of the Ultimate Maximum for Brownian Motion with Drift

J. du Toit & G. Peskir

Given a standard Brownian motion $B^\mu = (B_t^\mu)_{0 \le t \le 1}$ with drift $\mu \in I\!R$, letting $S_t^\mu = \max_{0 \le s \le t} B_s^\mu$ for $t \in [0,1]$, and denoting by θ the time at which S_1^μ is attained, we consider the optimal prediction problem

$$V_* = \inf_{0 \le \tau \le 1} \mathsf{E}|\theta - \tau|$$

where the infimum is taken over all stopping times τ of B^μ. Reducing the optimal prediction problem to a parabolic free-boundary problem and making use of local time-space calculus techniques, we show that the following stopping time is optimal:

$$\tau_* = \inf\{\, 0 \le t \le 1 \mid S_t^\mu - B_t^\mu \ge b(t)\,\}$$

where $b : [0,1] \to I\!R$ is a continuous decreasing function with $b(1) = 0$ that is characterised as the unique solution to a nonlinear Volterra integral equation. This also yields an explicit formula for V_* in terms of b. If $\mu = 0$ then there is a closed form expression for b. This problem was solved in [14] and [4] using the method of time change. The latter method cannot be extended to the case when $\mu \neq 0$ and the present paper settles the remaining cases using a different approach. It is also shown that the shape of the optimal stopping set remains preserved for all Lévy processes.

1. Introduction

Stopping a stochastic process as close as possible to its ultimate maximum is of great practical and theoretical interest. It has numerous applications in the fields of engineering, physics, finance and medicine, for example determining the best time to sell an asset or the optimal time to administer a drug. In recent years the area has attracted considerable interest (and has yielded some counter-intuitive results), and the problems have collectively become known as *optimal prediction problems* (within optimal stopping).

In particular, a number of different variations on the following prediction problem have been studied: let $B = (B_t)_{0 \le t \le 1}$ be a standard Brownian motion started at zero, set $S_t = \max_{0 \le s \le t} B_s$ for $t \in [0,1]$, and consider the optimal prediction problem

$$\inf_{0 \le \tau \le 1} \mathsf{E}(S_1 - B_\tau)^2 \tag{1.1}$$

where the infimum is taken over all stopping times τ of B.

Mathematics Subject Classification 2000. Primary 60G40, 62M20, 35R35. Secondary 60J65, 60G51, 45G10.
Key words and phrases: Brownian motion, optimal prediction, optimal stopping, ultimate-maximum time, parabolic free-boundary problem, smooth fit, normal reflection, local time-space calculus, curved boundary, nonlinear Volterra integral equation, Markov process, diffusion process, Lévy process.

This problem was solved in [4] where the optimal stopping time was found to be

$$\tau_* = \inf\big\{\, 0 \le t \le 1 \,\big|\, S_t - B_t \ge z_* \sqrt{1-t} \,\big\} \tag{1.2}$$

with $z_* > 0$ being a specified constant. The result was extended in [8] where two different formulations were considered: firstly the problem (1.1) above for $p > 1$ in place of 2, and secondly a probability formulation maximising $\mathsf{P}(S_1 - B_\tau \le \varepsilon)$ for $\varepsilon > 0$. Both these were solved explicitly: in the first case, the optimal stopping time was shown to be identical to (1.2) except that the value of z_* was now dependent on p, and in the second case the optimal stopping time was found to be $\tau_* = \inf\{\, t_* \le t \le 1 \mid S_t - B_t = \varepsilon \,\}$ where $t_* \in [0,1]$ is a specified constant.

Setting $B_t^\mu = B_t + \mu t$ and $S_t^\mu = \max_{0 \le s \le t} B_s^\mu$ for $t \in [0,1]$ and $\mu \in I\!R$, one can formulate the analogue of the problem (1.1) for Brownian motion with drift, namely

$$\inf_{0 \le \tau \le 1} \mathsf{E}(S_1^\mu - B_\tau^\mu)^2 \tag{1.3}$$

where the infimum is taken over all stopping times τ of B^μ. This problem was solved in [2] where it was revealed that (1.3) is fundamentally more complicated than (1.1) due to its highly nonlinear time dependence. The optimal stopping time was found to be $\tau_* = \inf\{\, 0 \le t \le 1 \mid b_1(t) \le S_t^\mu - B_t^\mu \le b_2(t) \,\}$ where b_1 and b_2 are two specified functions of time giving a more complex shape to the optimal stopping set (which moreover appears to be counter-intuitive when $\mu > 0$).

The variations on the problem (1.1) summarised above may all be termed *space domain* problems, since the measures of error are all based on a distance from B_τ to S_1. In each case they lead us to stop as 'close' as possible to the maximal value of the Brownian motion. However, the question can also be asked in the *time domain*: letting θ denote the time at which B attains its maximum S_1, one can consider

$$\inf_{0 \le \tau \le 1} \mathsf{E}|\,\theta - \tau| \tag{1.4}$$

where the infimum is taken over all stopping times τ of B. This problem was firstly considered in [12] and then further examined in [14] where the following identity was derived

$$\mathsf{E}(B_\theta - B_\tau)^2 = \mathsf{E}|\,\theta - \tau| + \tfrac{1}{2} \tag{1.5}$$

for any stopping time τ of B satisfying $0 \le \tau \le 1$. Recalling that $B_\theta = S_1$ it follows that the time domain problem (1.4) is in fact equivalent to the space domain problem (1.1). Hence stopping optimally in time is the same as stopping optimally in space (when distance is measured in mean square). This fact, although intuitively appealing, is mathematically quite remarkable.

It is interesting to note that (with the exception of the probability formulation) all the space domain problems above have trivial solutions when distance is measured in mean. Indeed, in (1.1) (with 1 in place of 2) any stopping time is optimal, while in (1.3) one either waits until time 1 or stops immediately (depending on whether $\mu > 0$ or $\mu < 0$ respectively). The error has to be distorted to be seen by the expectation operator, and this introduces a parameter dependence into these problems. While the mean square distance may seem a natural setting (due to its close links with the conditional expectation), there is no reason a priori to prefer one penalty measure over any other. The problems are therefore all based on *parameterised measures of error*, and the solutions are similarly parameter dependent.

The situation becomes even more acute when one extends these space domain problems to other stochastic processes, since there are many processes for which the higher order moments simply do not exist. Examples of these include stable Lévy processes of index $\alpha \in (0,2)$, for which (1.1) would only make sense for powers p strictly smaller than α. This leads to further loss of transparency in the problem formulation. By contrast, the time domain formulation is free from these difficulties as it deals with bounded random variables. One may therefore use any measure of error, including mean itself, and it is interesting to note that even in this case the problem (1.4) above yields a non-trivial solution.

Motivated by these facts, our aim in this paper will be to study the analogue of the problem (1.4) for Brownian motion with drift, namely

$$\inf_{0 \le \tau \le 1} \mathsf{E} |\theta^\mu - \tau| \tag{1.6}$$

where θ^μ is the time at which $B^\mu = (B^\mu_t)_{0 \le t \le 1}$ attains its maximal value S^μ_1, and the infimum is taken over all stopping times τ of B^μ. This problem is interesting not only because it is a *parameter free* measure of optimality, but also because it is unclear what form the solution will take: whether it will be similar to that of (1.1) or (1.3), or whether it will be something else entirely. There are also several applications where stopping close to the maximal *value* is less important than detecting the *time* at which this maximum occurred as accurately as possible.

Our main result (Theorem 1) states that the optimal stopping time in (1.6) is given by

$$\tau_* = \inf \{ \, 0 \le t \le 1 \mid S^\mu_t - B^\mu_t \ge b(t) \, \} \tag{1.7}$$

where $b : [0,1] \to I\!R$ is a continuous decreasing function with $b(1) = 0$ that is characterised as the unique solution to a nonlinear Volterra integral equation. The shape of the optimal stopping set is therefore quite different from that of the problem (1.3), and more closely resembles the shape of the optimal stopping set in the problem (1.1) above. This result is somewhat surprising and it is not clear how to explain it through simple intuition.

However, by far the most interesting and unexpected fact to emerge from the proof is that this problem considered for any *Lévy process* will yield a similar solution. That is, for *any* Lévy process X, the problem (1.6) of finding the closest stopping time τ to the time θ at which X attains its supremum (approximately), will have a solution

$$\tau_* = \inf \{ \, 0 \le t \le 1 \mid S_t - X_t \ge c(t) \, \} \tag{1.8}$$

where $S_t = \sup_{0 \le s \le t} X_s$ and $c : [0,1] \to \bar{I\!R}$ is a decreasing function with $c(1) = 0$. This result is remarkable indeed considering the breadth and depth of different types of Lévy processes, some of which have extremely irregular behaviour and are analytically quite unpleasant. In fact, an analogous result holds for a certain broader class of Markov processes as well, although the state space in this case is three-dimensional (time-space-supremum). Our aim in this paper will not be to derive (1.8) in all generality, but rather to focus on Brownian motion with drift where the exposition is simple and clear. The facts indicating the general results will be highlighted as we progress (cf. Lemma 1 and Lemma 2).

2. Reduction to standard form

As it stands, the optimal prediction problem (1.6) falls outside the scope of standard optimal stopping theory (see e.g. [11]). This is because the gain process $(|\theta - t|)_{0\le t\le 1}$ is not adapted to the filtration generated by B^μ . The first step in solving (1.6) aims therefore to reduce it to a standard optimal stopping problem. It turns out that this reduction can be done not only for the process B^μ but for any Markov process.

1. To see this, let $X = (X_t)_{t\ge 0}$ be a right-continuous Markov process (with left limits) defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathsf{P}_x)$ and taking values in $I\!R$. Here P_x denotes the measure under which X starts at $x \in I\!R$. Define the stochastic process $S = (S_t)_{t\ge 0}$ by $S_t = \sup_{0\le s\le t} X_s$ and the random variable θ by

$$\theta = \inf\{\, 0 \le t \le 1 \mid S_t = S_1 \,\} \tag{2.1}$$

so that θ denotes the first time that X “attains” its ultimate supremum over the interval $[0,1]$. We use “attains” rather loosely here, since if X is discontinuous the supremum might not actually be attained, but will be approached arbitrarily closely. Indeed, since X is right-continuous it follows that S is right-continuous, and hence $S_\theta = S_1$ so that S attains its ultimate supremum over the interval $[0,1]$ at θ . This implies that either $X_\theta = S_1$ or $X_{\theta-} = S_1$ depending on whether $X_\theta \ge X_{\theta-}$ or $X_\theta < X_{\theta-}$ respectively.

The reduction to standard form may now be described as follows (stopping times below refer to stopping times with respect to the filtration $(\mathcal{F}_t)_{t\ge 0}$).

Lemma 1. *Let X , S and θ be as above. Define the function F by*

$$F(t,x,s) = \mathsf{P}_x(S_t \le s) \tag{2.2}$$

for $t \in [0,1]$ and $x \le s$ in $I\!R$. Then the following identity holds:

$$\mathsf{E}_x|\,\theta - \tau\,| = \mathsf{E}_x\left(\int_0^\tau \Big(2\,F(1{-}t, X_t, S_t) - 1\Big)\,dt\right) + \mathsf{E}_x(\theta) \tag{2.3}$$

for any stopping time τ satisfying $0 \le \tau \le 1$.

Proof. Recalling the argument from the proof of Lemma 1 in [14] (cf. [12] & [13]) one has

$$\begin{aligned} |\,\theta - \tau\,| &= (\tau-\theta)^+ + (\tau-\theta)^- = (\tau-\theta)^+ + \theta - \theta\wedge\tau \\ &= \int_0^\tau I(\theta \le t)\,dt + \theta - \int_0^\tau I(\theta > t)\,dt = \theta + \int_0^\tau \Big(2I(\theta\le t) - 1\Big)\,dt\,. \end{aligned} \tag{2.4}$$

Examining the integral on the right hand side, one sees by Fubini's theorem that

$$\begin{aligned} \mathsf{E}_x\left(\int_0^\tau I(\theta\le t)\,dt\right) &= \mathsf{E}_x\left(\int_0^\infty I(\theta\le t)\,I(t<\tau)\,dt\right) \\ &= \int_0^\infty \mathsf{E}_x\Big(I(t<\tau)\,\mathsf{E}_x\big(I(\theta\le t)\,\big|\,\mathcal{F}_t\big)\Big)\,dt \\ &= \mathsf{E}_x\left(\int_0^\tau \mathsf{P}_x(\theta \le t\,|\,\mathcal{F}_t)\,dt\right). \end{aligned} \tag{2.5}$$

We can now use the Markov structure of X to evaluate the conditional probability. For this note that if $S_t = \sup_{0 \le s \le t} X_s$, then $\sup_{t \le s \le 1} X_s = S_{1-t} \circ \theta_t$ where θ_t is the shift operator at time t . We therefore see that

$$
\text{(2.6)} \qquad \mathsf{P}_x(\theta \le t \,|\, \mathcal{F}_t) = \mathsf{P}_x\Big(\sup_{0 \le s \le t} X_s \ge \sup_{t \le s \le 1} X_s \,\Big|\, \mathcal{F}_t \Big) = \mathsf{P}_x(S_{1-t} \circ \theta_t \le s \,|\, \mathcal{F}_t)\Big|_{s=S_t}
$$

$$
= \mathsf{P}_{X_t}(S_{1-t} \le s)\Big|_{s=S_t} \;=\; F(1-t, X_t, S_t)
$$

where F is the function defined in (2.2) above. Inserting these identities back into (2.4) after taking E_x on both sides, we conclude the proof. □

Lemma 1 reveals the rich structure of the optimal prediction problem (1.6). Two key facts are to be noted. Firstly, for any Markov process X the problem is inherently three dimensional and has to be considered in the time-space-supremum domain occupied by the stochastic process $(t, X_t, S_t)_{t \ge 0}$. Secondly, for any two values $x \le s$ fixed, the map $t \mapsto 2F(1-t, x, s) - 1$ is increasing. This fact is important since we are considering a minimisation problem so that the passage of time incurs a hefty penalty and always forces us to stop sooner rather than later. This property will be further explored in Section 4 below.

2. If X is a Lévy process then the problem reduces even further.

Lemma 2. *Let* X , S *and* θ *be as above, and let us assume that* X *is a Lévy process. Define the function* G *by*

$$
\text{(2.7)} \qquad G(t, z) = \mathsf{P}_0(S_t \le z)
$$

for $t \in [0, 1]$ *and* $z \in I\!R_+$. *Then the following identity holds:*

$$
\text{(2.8)} \qquad \mathsf{E}_x |\, \theta - \tau | = \mathsf{E}_x \left(\int_0^\tau \Big(2G(1-t, S_t - X_t) - 1 \Big)\, dt \right) + \mathsf{E}_x(\theta)
$$

for any stopping time τ *satisfying* $0 \le \tau \le 1$.

Proof. This result follows directly from Lemma 1 above upon noting that

$$
\text{(2.9)} \qquad F(1-t, X_t, S_t) = P_0\Big(\sup_{0 \le s \le 1-t} (x + X_s) \le s \Big)\Big|_{x=X_t,\, s=S_t} \;=\; \mathsf{P}_0(S_{1-t} \le z)\Big|_{z=S_t - X_t}
$$

since X under P_x is realised as $x + X$ under P_0 . □

If X is a Lévy process then the reflected process $(S_t - X_t)_{0 \le t \le 1}$ is Markov. This is not true in general and means that for Lévy processes the optimal prediction problem is inherently two-dimensional (rather than three-dimensional as in the general case). It is also important to note that for a Lévy process we have the additional property that the map $z \mapsto 2G(1-t, z) - 1$ is increasing for any $t \in [0, 1]$ fixed. Further implications of this will also be explored in Section 4 below.

3. The free-boundary problem

Let us now formally introduce the setting for the optimal prediction problem (1.6). Let $B = (B_t)_{t \geq 0}$ be a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ with $B_0 = 0$ under P. Given $\mu \in I\!R$ set $B_t^\mu = B_t + \mu t$ and $S_t^\mu = \max_{0 \leq s \leq t} B_s^\mu$ for $t \in [0,1]$. Let θ denote the first time at which the process $B^\mu = (B_t^\mu)_{0 \leq t \leq 1}$ attains its maximum S_1^μ.

Consider the optimal prediction problem

$$V_* = \inf_{0 \leq \tau \leq 1} \mathsf{E} |\theta - \tau| \tag{3.1}$$

where the infimum is taken over all stopping times τ of B^μ. By Lemma 2 above this problem is equivalent to the standard optimal stopping problem

$$V = \inf_{0 \leq \tau \leq 1} \mathsf{E} \left(\int_0^\tau H(t, X_t)\, dt \right) \tag{3.2}$$

where the process $X = (X_t)_{0 \leq t \leq 1}$ is given by $X_t = S_t^\mu - B_t^\mu$, the infimum is taken over all stopping times τ of X, and the function $H : [0,1] \times I\!R_+ \rightarrow [-1,1]$ is computed as

$$H(t,x) = 2\,\mathsf{P}(S_{1-t}^\mu \leq x) - 1 = 2 \left[\Phi\left(\frac{x - \mu(1-t)}{\sqrt{1-t}} \right) - e^{2\mu x}\, \Phi\left(\frac{-x - \mu(1-t)}{\sqrt{1-t}} \right) \right] - 1 \tag{3.3}$$

using the well-known identity for the law of S_{1-t}^μ (cf. [1, p. 397] and [7, p. 526]). Note that $V_* = V + \mathsf{E}(\theta)$ where

$$\begin{aligned} \mathsf{E}(\theta) = \int_0^1 \mathsf{P}(\theta > t)\, dt = \sqrt{\frac{2}{\pi}} \int_0^1 \int_0^\infty \int_{-\infty}^s \frac{2s-b}{t^{3/2}} \left[1 - \Phi\left(\frac{s-b-\mu(1-t)}{\sqrt{1-t}} \right) \right. \\ \left. + e^{2\mu(s-b)}\, \Phi\left(\frac{b-s-\mu(1-t)}{\sqrt{1-t}} \right) \right] e^{-\frac{(2s-b)^2}{2t} + \mu(b - \frac{\mu t}{2})}\, db\, ds\, dt \end{aligned} \tag{3.4}$$

which is readily derived using (2.6), (2.9), (3.3) and (4.1) below.

It is known (cf. [3]) that the strong Markov process X is equal in law to $|Y| = (|Y_t|)_{0 \leq t \leq 1}$ where $Y = (Y_t)_{0 \leq t \leq 1}$ is the unique strong solution to $dY_t = -\mu\, \text{sign}(Y_t)\, dt + dB_t$ with $Y_0 = 0$. It is also known (cf. [3]) that under $Y_0 = x$ the process $|Y|$ has the same law as a Brownian motion with drift $-\mu$ started at $|x|$ and reflected at 0. Hence the infinitesimal generator $I\!L_X$ of X acts on functions $f \in C_b^2\big([0,\infty)\big)$ satisfying $f'(0) = 0$ as $I\!L_X f(x) = -\mu f'(x) + \frac{1}{2} f''(x)$. Since the infimum in (3.2) is attained at the first entry time of X to a closed set (this follows from general theory of optimal stopping and will be made more precise below) there is no restriction to replace the process X in (3.2) by the process $|Y|$.

It is especially important for the analysis of optimal stopping to see how X depends on its starting value x. Although the equation for Y is difficult to solve explicitly, it is known (cf. [2, Lemma 2.2] & [10, Theorem 2.1]) that the Markov process $X^x = (X_t^x)_{0 \leq t \leq 1}$ defined under P as $X_t^x = x \vee S_t^\mu - B_t^\mu$ also realises a Brownian motion with drift $-\mu$ started at $x \geq 0$ and reflected at 0. Following the usual approach to optimal stopping for Markov processes (see e.g. [11]) we may therefore extend the problem (3.2) to

$$V(t,x) = \inf_{0 \leq \tau \leq 1-t} \mathsf{E}_{t,x} \left(\int_0^\tau H(t+s, X_{t+s})\, ds \right) \tag{3.5}$$

where $X_t = x$ under $\mathsf{P}_{t,x}$ for $(t,x) \in [0,1] \times I\!R_+$ given and fixed, the infimum is taken over all stopping times τ of X, and the process X under $\mathsf{P}_{t,x}$ can be identified with either $|Y|$ (under the same measure) or $X^x_{t+s} = x \vee S^\mu_s - B^\mu_s$ under the measure P for $s \in [0, 1-t]$. We will freely use either of these representations in the sequel without further mention.

We will show in the proof below that the value function V is continuous on $[0,1] \times I\!R_+$. Defining the (open) continuation set $C = \{\, (t,x) \in [0,1] \times I\!R_+ \mid V(t,x) < 0 \,\}$ and the (closed) stopping set $D = \{\, (t,x) \in [0,1] \times I\!R_+ \mid V(t,x) = 0 \,\}$, standard results from optimal stopping theory (cf. [11, Corollary 2.9]) indicate that the stopping time

$$\tau_D = \inf \{\, 0 \le t \le 1 \mid (t, X_t) \in D \,\} \tag{3.6}$$

is optimal for the problem (3.2) above. We will also show in the proof below that the value function $V : [0,1] \times I\!R_+ \to I\!R$ solves the following free-boundary problem:

$$(V_t - \mu V_x + \tfrac{1}{2} V_{xx} + H)(t,x) = 0 \quad \text{for } (t,x) \in C \tag{3.7}$$

$$V(t,x) = 0 \quad \text{for } (t,x) \in D \text{ (instantaneous stopping)} \tag{3.8}$$

$$x \mapsto V_x(t,x) \quad \text{is continuous over } \partial C \text{ for } t \in [0,1) \text{ (smooth fit)} \tag{3.9}$$

$$V_x(t, 0+) = 0 \quad \text{for } t \in [0,1) \text{ (normal reflection).} \tag{3.10}$$

Our aim will not be to tackle this free-boundary problem directly, but rather to express V in terms of the boundary ∂C, and to derive an analytic expression for the boundary itself. This approach dates back to [6] in a general setting (for more details see [11]).

4. The result and proof

The function H from (3.3) and the set $\{H \ge 0\} := \{\, (t,x) \in [0,1] \times I\!R_+ \mid H(t,x) \ge 0 \,\}$ will play prominent roles in our discussion. A direct examination of H reveals the existence of a continuous decreasing function $h : [0,1] \to I\!R_+$ satisfying $h(1) = 0$ such that $\{H \ge 0\} = \{\, (t,x) \in [0,1] \times I\!R_+ \mid x \ge h(t) \,\}$. Recall (see e.g. [5, p. 368]) that the joint density function of (B^μ_t, S^μ_t) under P is given by

$$f(t,b,s) = \sqrt{\frac{2}{\pi}}\, \frac{(2s-b)}{t^{3/2}}\, e^{-\frac{(2s-b)^2}{2t} + \mu(b - \frac{\mu t}{2})} \tag{4.1}$$

for $t > 0$, $s \ge 0$ and $b \le s$. Define the function

$$\begin{aligned} K(t,x,r,z) &= \mathsf{E}\Big(H(t+r, X^x_r)\, I(X^x_r < z) \Big) \\ &= \int_0^\infty \int_{-\infty}^s H\big(t+r, x \vee s - b\big)\, I\big(x \vee s - b < z\big)\, f(r,b,s)\, db\, ds \end{aligned} \tag{4.2}$$

for $t \in [0,1]$, $r \in [0, 1-t]$ and $x, z \in I\!R_+$. We may now state the main result of this paper.

Theorem 1. *Consider the optimal stopping problem* (3.5). *Then there exists a continuous decreasing function* $b : [0,1] \to I\!R_+$ *satisfying* $b(1) = 0$ *such that the optimal stopping set is given by* $D = \{\, (t,x) \in [0,1] \times I\!R_+ \mid x \ge b(t) \,\}$. *Furthermore, the value function* V *defined in* (3.5) *is given by*

$$V(t,x) = \int_0^{1-t} K\big(t, x, r, b(t+r)\big)\, dr \tag{4.3}$$

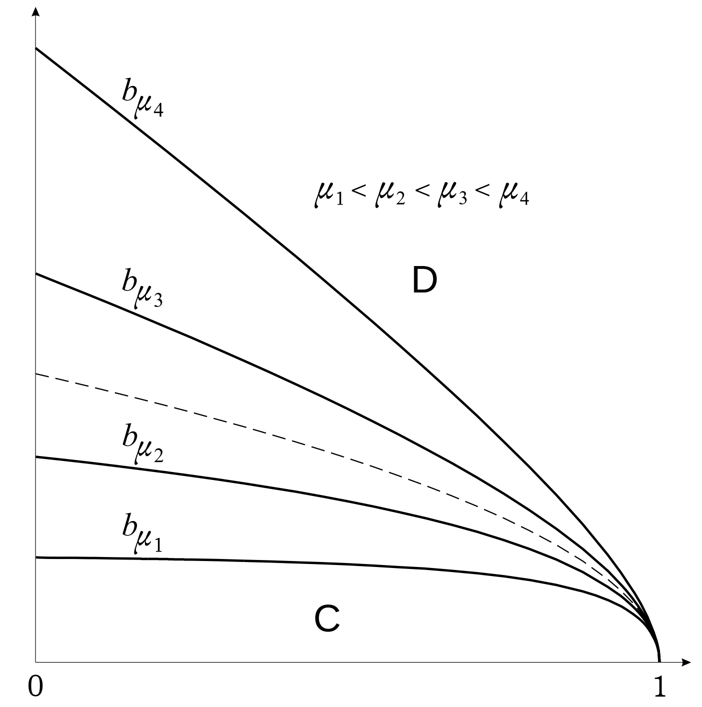

Figure 1. A computer drawing of the optimal stopping boundaries for Brownian motion with drift $\mu_1 = -1.5$, $\mu_2 = -0.5$, $\mu_3 = 0.5$ and $\mu_4 = 1.5$. The dotted line is the optimal stopping boundary for Brownian motion with zero drift.

for all $(t, x) \in [0,1] \times I\!R_+$ *, and the optimal boundary* b *itself is uniquely determined by the nonlinear Volterra integral equation*

$$\int_0^{1-t} K\big(t, b(t), r, b(t+r)\big)\, dr = 0 \tag{4.4}$$

for $t \in [0,1]$ *, in the sense that it is the unique solution to* (4.4) *in the class of continuous functions* $t \mapsto b(t)$ *on* $[0,1]$ *satisfying* $b(t) \ge h(t)$ *for all* $t \in [0,1]$ *. It follows therefore that the optimal stopping time in* (3.5) *is given by*

$$\tau_D := \tau_D(t,x) = \inf\{\, 0 \le r \le 1-t \mid x \vee S_r^\mu - B_r^\mu \ge b(t+r) \,\} \tag{4.5}$$

for $(t,x) \in [0,1] \times I\!R_+$ *. Finally, the value* V_* *defined in* (3.1) *equals* $V(0,0) + \mathsf{E}(\theta)$ *where* $\mathsf{E}(\theta)$ *is given in* (3.4)*, and the optimal stopping time in* (3.1) *is given by* $\tau_D(0,0)$ *(see Figure 1).*

Proof. *Step 1.* We first show that an optimal stopping time for the problem (3.5) exists, and then determine the shape of the optimal stopping set D . Since H is continuous and bounded, and the flow $x \mapsto x \vee S_t^\mu - B_t^\mu$ of the process X^x is continuous, it follows that for any stopping time τ the map $(t,x) \mapsto \mathsf{E}\big(\int_0^\tau H(t+s, x \vee S_s^\mu - B_s^\mu)\, ds\big)$ is continuous and thus upper semicontinuous (usc) as well. Hence we see that V is usc (recall that the infimum of usc functions is usc) and so by general results of optimal stopping (cf. [11, Corollary 2.9]) it follows that τ_D from (3.6) is optimal in (3.5) with C open and D closed.

Next we consider the shape of D . Our optimal prediction problem has a particular internal structure, as was noted in the comments following Lemmas 1 and 2 above, and we now expose this structure more fully. Take any point $(t,x) \in \{H<0\}$ and let $U \subset \{H<0\}$ be an open set containing (t,x) . Define σ_U to be the first exit time of X from U under the measure $\mathsf{P}_{t,x}$ where $\mathsf{P}_{t,x}(X_t = x) = 1$. Then clearly

$$V(t,x) \le \mathsf{E}_{t,x}\left(\int_0^{\sigma_U} H(t+s, X_{t+s})\, ds\right) < 0 \tag{4.6}$$

showing that it is not optimal to stop at (t,x) . Hence the entire region below the curve h is contained in C .

As was observed following Lemma 1, the map $t \mapsto H(t,x)$ is increasing, so that taking any $s<t$ in $[0,1]$ and setting $\tau_s = \tau_D(s,x)$ and $\tau_t = \tau_D(t,x)$, it follows that

$$\begin{aligned} V(t,x) - V(s,x) &= \mathsf{E}\left(\int_0^{\tau_t} H(t+r, X_r^x)\, dr\right) - \mathsf{E}\left(\int_0^{\tau_s} H(s+r, X_r^x)\, dr\right) \\ &\ge \mathsf{E}\left(\int_0^{\tau_t} H(t+r, X_r^x) - H(s+r, X_r^x)\, dr\right) \ge 0 \end{aligned} \tag{4.7}$$

for any $x \ge 0$. Hence $t \mapsto V(t,x)$ is increasing so that if $(t,x) \in D$ then $(t+s,x) \in D$ for all $s \in [0, 1-t]$ when $x \ge 0$ is fixed. Similarly, since $x \mapsto H(t,x)$ is increasing we see for any $x<y$ in $I\!R_+$ that

$$\begin{aligned} V(t,y) - V(t,x) &= \mathsf{E}\left(\int_0^{\tau_y} H(t+s, X_s^y)\, ds\right) - \mathsf{E}\left(\int_0^{\tau_x} H(t+s, X_s^x)\, ds\right) \\ &\ge \mathsf{E}\left(\int_0^{\tau_y} H(t+s, X_s^y) - H(t+s, X_s^x)\, ds\right) \ge 0 \end{aligned} \tag{4.8}$$

for all $t \in [0,1]$ where $\tau_x = \tau_D(t,x)$ and $\tau_y = \tau_D(t,y)$. Hence $x \mapsto V(t,x)$ is increasing so that if $(t,x) \in D$ then $(t,y) \in D$ for all $y \ge x$ when $t \in [0,1]$ is fixed. We therefore conclude that in our problem (and indeed for any Lévy process) there is a single boundary function $b : [0,1] \to \bar{I\!R}$ separating the sets C and D where b is formally defined as

$$b(t) = \inf\{\, x \ge 0 \mid (t,x) \in D \,\} \tag{4.9}$$

for all $t \in [0,1]$, so that $D = \{\, (t,x) \in [0,1] \times I\!R_+ \mid x \ge b(t) \,\}$. It is also clear that b is decreasing with $b \ge h$ and $b(1) = 0$.

We now show that b is finite valued. For this, define $t_* = \sup\{\, t \in [0,1] \mid b(t) = \infty \,\}$ and suppose that $t_* \in (0,1)$ with $b(t_*) < \infty$ (the cases $t_* = 1$ and $b(t_*) = \infty$ follow by a small modification of the argument below). Let $\tau_x = \tau_D(0,x)$ and set $\sigma_x = \inf\{\, t \in [0,1] \mid X_t^x \le h(0)+1 \,\}$. Then from the properties of X^x we have $\tau_x \to t_*$ and $\sigma_x \to 1$ as $x \to \infty$. On the other hand, from the properties of H and h we see that there exists $\varepsilon > 0$ such that $H(t,x) \ge \varepsilon$ for all $x \ge h(0)+1$ and all $t \in [0, t_*]$. Hence we find that

$$0 \ge \lim_{x \to \infty} V(0,x) = \lim_{x \to \infty} \mathsf{E}\left[\int_0^{\tau_x} H(t, X_t^x)\, dt\; I(\tau_x \le \sigma_x)\right] \tag{4.10}$$

$$+\lim_{x\to\infty} \mathsf{E}\left[\int_0^{\tau_x} H(t,X_t^x)\,dt\; I(\tau_x\!>\!\sigma_x)\right] \ge \varepsilon\, t_* > 0$$

which is a contradiction. The case $t_*=0$ can be disproved similarly by enlarging the horizon from 1 to a strictly greater number. Hence b must be finite valued as claimed.

Step 2. We show that the value function $(t,x)\mapsto V(t,x)$ is continuous on $[0,1]\times I\!\!R_+$. For this, take any $x\le y$ in $I\!\!R_+$ and note by the mean value theorem that for every $t\in[0,1]$ there is $z\in(x,y)$ such that

$$\begin{aligned}(4.11)\qquad 0 &\le H(t,y)-H(t,x) = (y\!-\!x)\,H_x(t,z)\\ &= 2(y\!-\!x)\left[\frac{2}{\sqrt{1\!-\!t}}\,\varphi\Big(\frac{z\!-\!\mu(1\!-\!t)}{\sqrt{1\!-\!t}}\Big) - 2\mu\, e^{2\mu z}\,\Phi\Big(\frac{-z-\mu(1-t)}{\sqrt{1-t}}\Big)\right]\\ &\le 4(y\!-\!x)\left[\frac{1}{\sqrt{1\!-\!t}}+|\mu|\right].\end{aligned}$$

Since $0\le y\vee S_t^\mu - x\vee S_t^\mu \le y-x$ it follows that

$$\begin{aligned}(4.12)\qquad 0 &\le V(t,y)-V(t,x) \le \mathsf{E}\left(\int_0^{\tau_x} H(t\!+\!s,X_x^y)-H(t\!+\!s,X_s^x)\,ds\right)\\ &\le \mathsf{E}\left(4\int_0^{\tau_x}(X_s^y\!-\!X_s^x)\left[\frac{1}{\sqrt{1\!-\!t\!-\!s}}+|\mu|\right]ds\right)\\ &\le \mathsf{E}\left(8(y\!-\!x)\Big[\sqrt{1\!-\!t}-\sqrt{1\!-\!t\!-\!\tau_x}+|\mu|\,\tau_x\Big]\right)\\ &\le 8(y\!-\!x)(1\!+\!|\mu|)\end{aligned}$$

where we recall that $\tau_x=\tau_D(t,x)$. Letting $y-x\to 0$ we see that $x\mapsto V(t,x)$ is continuous on $I\!\!R_+$ uniformly over all $t\in[0,1]$.

To conclude the continuity argument it is enough to show that $t\mapsto V(t,x)$ is continuous on $[0,1]$ for every $x\in I\!\!R_+$ given and fixed. To do this, take any $s\le t$ in $[0,1]$ and let $\tau_s=\tau_D(s,x)$. Define the stopping time $\sigma=\tau_s\wedge(1-t)$ so that $0\le\sigma\le 1-t$. Note that $0\le\tau_s-\sigma\le t-s$ so that $\tau_s-\sigma\to 0$ as $t-s\to 0$. We then have

$$\begin{aligned}(4.13)\qquad 0 &\le V(t,x)-V(s,x) \le \mathsf{E}\left(\int_0^{\sigma} H(t\!+\!r,X_r^x)\,dr\right)-\mathsf{E}\left(\int_0^{\tau_s} H(s\!+\!r,X_r^x)\,dr\right)\\ &= \mathsf{E}\left(\int_0^{\sigma} H(t\!+\!r,X_r^x)-H(s\!+\!r,X_r^x)\,dr\right)-\mathsf{E}\left(\int_\sigma^{\tau_s} H(s\!+\!r,X_r^x)\,dr\right).\end{aligned}$$

Letting $t-s\to 0$ and using the fact that $|H|\le 1$ it follows by the dominated convergence theorem that both expectations on the right-hand side of (4.13) tend to zero. This shows that the map $t\mapsto V(t,x)$ is continuous on $[0,1]$ for every $x\in I\!\!R_+$, and hence the value function $(t,x)\mapsto V(t,x)$ is continuous on $[0,1]\times I\!\!R_+$ as claimed.

Step 3. We show that V satisfies the smooth fit condition (3.9). Take any t in $[0,1)$, set $c=b(t)$ and define the stopping time $\tau_\varepsilon=\tau_D(t,c\!-\!\varepsilon)$ for $\varepsilon>0$. Then from the second last inequality in (4.12) we see that

$$(4.14)\qquad 0\le \frac{V(t,c)-V(t,c\!-\!\varepsilon)}{\varepsilon} \le 8\,\mathsf{E}\left(\sqrt{1\!-\!t}-\sqrt{1\!-\!t\!-\!\tau_\varepsilon}+|\mu|\,\tau_\varepsilon\right)$$

for $\varepsilon > 0$. We now show that $\tau_\varepsilon \to 0$ as $\varepsilon \downarrow 0$. To see this, consider the stopping time $\sigma = \inf\{\, 0 \le s \le 1{-}t \mid X_{t+s} \ge c \,\}$ under the measure $\mathsf{P}_{t,c-\varepsilon}$. The process X started at $c{-}\varepsilon$ at time t will always hit the boundary b before hitting the level c since b is decreasing. Hence $0 \le \tau_\varepsilon \le \sigma$ and thus it is enough to show that $\sigma \to 0$ under $\mathsf{P}_{t,c-\varepsilon}$ as $\varepsilon \downarrow 0$. For this, note by the Itô-Tanaka formula that

$$dX_t = -\mu\, dt + \text{sign}(Y_t)\, dB_t + d\ell^0_t(Y) \tag{4.15}$$

where $\ell^0(Y) = (\ell^0_t(Y))_{0 \le t \le 1}$ is the local time of Y at zero. It follows that

$$\begin{aligned} \sigma &= \inf\{\, 0 \le s \le 1{-}t \mid c - \varepsilon - \mu s + \beta_s + \ell^0_s(Y) \ge c \,\} \\ &\le \inf\{\, s \ge 0 \mid -\varepsilon + \beta_s \ge \mu s \,\} \end{aligned} \tag{4.16}$$

where $\beta_s = \int_0^s \text{sign}(Y_r)\, dB_r$ is a standard Brownian motion for $s \ge 0$. Letting $\varepsilon \downarrow 0$ and using the fact that $s \mapsto \mu s$ is a lower function of $\beta = (\beta_s)_{s \ge 0}$ at $0+$, we see that $\sigma \to 0$ under $\mathsf{P}_{t,c-\varepsilon}$ and hence $\tau_\varepsilon \to 0$ as claimed. Passing to the limit in (4.14) for $\varepsilon \downarrow 0$, and using the dominated convergence theorem, we conclude that $x \mapsto V(t,x)$ is differentiable at c and $V_x(t,c) = 0$. Moreover, a small modification of the preceding argument shows that $x \mapsto V(t,x)$ is continuously differentiable at c . Indeed, for $\delta > 0$ define the stopping time $\tau_\delta = \tau_D(t, c{-}\delta)$. Then as in (4.14) we have

$$0 \le \frac{V(t, c{-}\delta{+}\varepsilon) - V(t, c{-}\delta)}{\varepsilon} \le 8\, \mathsf{E}\left(\sqrt{1{-}t} - \sqrt{1{-}t{-}\tau_\delta} + |\mu|\, \tau_\delta\right) \tag{4.17}$$

for $\varepsilon > 0$. Letting first $\varepsilon \downarrow 0$ (upon using that $V_x(t, x{-}\delta)$ exists) and then $\delta \downarrow 0$ (upon using that $\tau_\delta \to 0$) we see as above that $x \mapsto V_x(t,x)$ is continuous at c . This establishes the smooth fit condition (3.9).

Standard results on optimal stopping for Markov processes (see e.g. [11, Section 7]) show that V is $C^{1,2}$ in C and satisfies $V_t + I\!\!L_X V + H = 0$ in C . This, together with the result just proved, shows that V satisfies (3.7)-(3.9) of the free-boundary problem (3.7)-(3.10). We now establish the last of these conditions.

Step 4. We show that V satisfies the normal reflection condition (3.10). For this, note first since $x \mapsto V(t,x)$ is increasing on $I\!\!R_+$ that $V_x(t, 0+) \ge 0$ for all $t \in [0,1]$ where the limit exists since V is $C^{1,2}$ on C . Suppose now that there exists $t \in [0,1)$ such that $V_x(t,0+) > 0$. Recalling again that V is $C^{1,2}$ on C so that $t \mapsto V_x(t,0+)$ is continuous on $[0,1)$, we see that there exist $\delta > 0$ and $\varepsilon > 0$ such that $V_x(t{+}s, 0+) \ge \varepsilon$ for all $s \in [0,\delta]$ where $t + \delta < 1$. Setting $\tau_\delta = \tau_D(t,0) \wedge \delta$ we see by Itô's formula using (4.15) and (3.7) that

$$\begin{aligned} V(t{+}\tau_\delta, X_{t+\tau_\delta}) &= V(t,0) + \int_0^{\tau_\delta} (V_t {-} \mu V_x {+} \tfrac{1}{2} V_{xx})(t{+}r, X_{t+r})\, dr \\ &\quad + \int_0^{\tau_\delta} V_x(t{+}r, X_{t+r})\, \text{sign}(Y_{t+r})\, dB_{t+r} + \int_0^{\tau_\delta} V_x(t{+}r, X_{t+r})\, d\ell^0_{t+r}(Y) \\ &\ge V(t,0) - \int_0^{\tau_\delta} H(t{+}r, X_{t+r})\, dr + M_{\tau_\delta} + \varepsilon\, \ell^0_{t+\tau_\delta}(Y) \end{aligned} \tag{4.18}$$

where $M_s = \int_0^s V_x(t{+}r, X_{t+r})\, \text{sign}(Y_{t+r})\, dB_{t+r}$ is a continuous martingale for $s \in [0, 1{-}t]$ (note from (4.17) that V_x is uniformly bounded). By general theory of optimal stopping for

Markov processes (see e.g. [11]) we know that $V(t+s\wedge\tau_\delta, X_{t+s\wedge\tau_\delta}) + \int_0^{s\wedge\tau_\delta} H(t{+}r, X_{t+r})\, dr$ is a martingale starting at $V(t,0)$ under $\mathsf{P}_{t,0}$ for $s\in[0,1{-}t]$. Hence by taking $\mathsf{E}_{t,0}$ on both sides of (4.18) and using the optional sampling theorem to deduce that $\mathsf{E}_{t,0}(M_{\tau_\delta})=0$, we see that $\mathsf{E}_{t,0}(\ell^0_{t+\tau_\delta}(Y))=0$. Since the properties of the local time clearly exclude this possibility, we must have $V_x(t,0+)=0$ for all $t\in[0,1]$ as claimed.

Step 5. We show that the boundary function $t\mapsto b(t)$ is continuous on $[0,1]$. Let us first show that b is right-continuous. For this, take any $t\in[0,1)$ and let $t_n\downarrow t$ as $n\to\infty$. Since b is decreasing we see that $\lim_{n\to\infty} b(t_n) =: b(t+)$ exists, and since each $(t_n, b(t_n))$ belongs to D which is closed, it follows that $(t, b(t+))$ belongs to D as well. From the definition of b in (4.9) we must therefore have $b(t)\le b(t+)$. On the other hand, since b is decreasing, we see that $b(t)\ge b(t_n)$ for all $n\ge 1$, and hence $b(t)\ge b(t+)$. Thus $b(t)=b(t+)$ and consequently b is right-continuous as claimed.

We now show that b is left-continuous. For this, let us assume that there is $t\in(0,1]$ such that $b(t-)>b(t)$, and fix any $x\in(b(t),b(t-))$. Since $b\ge h$ we see that $x>h(t)$ so that by continuity of h we have $h(s)<x$ for all $s\in[s_1,t]$ with some $s_1\in(0,t)$ sufficiently close to t . Hence $c:=\inf\{\, H(s,y)\mid s\in[s_1,t],\ y\in[x,b(s)]\,\}>0$ since H is continuous. Moreover, since V is continuous and $V(t,y)=0$ for all $y\in[x,b(t-)]$, it follows that

$$|\,\mu V(s,y)|\le \frac{c}{4}\big(b(t-){-}x\big) \tag{4.19}$$

for all $s\in[s_2,t]$ and all $y\in[x,b(s)]$ with some $s_2\in(s_1,t)$ sufficiently close to t . For any $s\in[s_2,t]$ we then find by (3.7) and (3.9) that

$$\begin{aligned} V(s,x) &= \int_x^{b(s)}\!\int_y^{b(s)} V_{xx}(s,z)\,dz\,dy = 2\int_x^{b(s)}\!\int_y^{b(s)} \big(-V_t+\mu V_x - H\big)(s,z)\,dz\,dy \\ &\le 2\int_x^{b(s)} \big[-\mu V(s,y) - c(b(s){-}y)\big]\,dy \\ &\le \frac{c}{2}\big(b(t-){-}x\big)\big(b(s){-}x\big) - c\big(b(s){-}x\big)^2 \end{aligned} \tag{4.20}$$

where in the second last inequality we use that $V_t\ge 0$ and in the last inequality we use (4.19). Letting $s\uparrow t$ we see that $V(t,x)\le(-c/2)(b(t-){-}x)^2<0$ which contradicts the fact that (t,x) belongs to D . Thus b is left-continuous and hence continuous on $[0,1]$. Note that the preceding proof also shows that $b(1)=0$ since $h(1)=0$ and $V(1,x)=0$ for $x\ge 0$.

Step 6. We may now derive the formula (4.3) and the equation (4.4). From (4.12) we see that $0\le V_x(t,x)\le 8(1{+}|\mu|) =: K$ for all $(t,x)\in[0,1]\times I\!\!R_+$. Hence by (3.7) we find that $V_{xx}=2(-V_t+\mu V_x-H)\le 2(|\mu|K{-}H)$ in C . Thus if we let

$$f(t,x) = 2\int_0^x\!\int_0^y \big(1{+}|\mu|K{-}H(t,z)\big)\,dz\,dy \tag{4.21}$$

for $(t,x)\in[0,1]{\times}I\!\!R_+$, then $V_{xx}\le f_{xx}$ on $C\cup D^o$ since $V\equiv 0$ in D and $|H|\le 1$. Defining the function $F:[0,1]{\times}I\!\!R_+\to I\!\!R$ by $F=V{-}f$ we see by (3.9) that $x\mapsto F(t,x)$ is concave on $I\!\!R_+$ for every $t\in[0,1]$. Moreover, it is evident that (i) F is $C^{1,2}$ on $C\cup D^o$ and $F_x(t,0+)=V_x(t,0+)=f_x(t,0+)=0$; (ii) $F_t-\mu F_x+\frac{1}{2}F_{xx}$ is locally bounded on $C\cup D^o$;

and (iii) $t \mapsto F_x(t, b(t)\pm) = -f_x(t, b(t)\pm)$ is continuous on $[0,1]$. Since b is decreasing (and thus of bounded variation) on $[0,1]$ we can therefore apply the local time-space formula [9] to $F(t{+}s, X_{t+s})$, and since f is $C^{1,2}$ on $[0,1]\times I\!\!R_+$ we can apply Itô's formula to $f(t{+}s, X_{t+s})$. Adding the two formulae, making use of (4.15) and the fact that $F_x(t,0+) = f_x(t,0+) = 0$, we find by (3.7)-(3.9) that

$$
\begin{aligned}
(4.22) \qquad V(t{+}s, X_{t+s}) &= V(t,x) + \int_0^s \Big(V_t - \mu V_x + \frac{1}{2}V_{xx}\Big)(t{+}r, X_{t+r})\, I\big(X_{t+r} \neq b(t{+}r)\big)\, dr \\
&\quad + \int_0^s V_x(t{+}r, X_{t+r})\, \mathrm{sign}(Y_{t+r})\, I\big(X_{t+r} \neq b(t{+}r)\big)\, dB_{t+r} \\
&\quad + \frac{1}{2}\int_0^s \big(V_x(t{+}r, X_{t+r}+) - V_x(t{+}r, X_{t+r}-)\big)\, I\big(X_{t+r} = b(t{+}r)\big)\, d\ell^b_{t+r}(X) \\
&= V(t,x) - \int_0^s H\big(t{+}r, X_{t+r}\big) I\big(X_{t+r} < b(t{+}r)\big)\, dr \\
&\quad + \int_0^s V_x(t{+}r, X_{t+r})\, \mathrm{sign}(Y_{t+r})\, dB_{t+r}
\end{aligned}
$$

under $\mathsf{P}_{t,x}$ for $(t,x) \in [0,1]\times I\!\!R_+$ and $s \in [0, 1{-}t]$, where $\ell^b_{t+r}(X)$ is the local time of X on the curve b for $r \in [0, 1{-}t]$. Inserting $s = 1{-}t$ and taking $\mathsf{E}_{t,x}$ on both sides, we see that

$$(4.23) \qquad V(t,x) = \mathsf{E}_{t,x}\bigg(\int_0^{1-t} H\big(t{+}s, X_{t+s}\big) I\big(X_{t+s} < b(t{+}s)\big)\, ds\bigg)$$

which after exchanging the order of integration is exactly (4.3). Setting $x = b(t)$ in the resulting identity we obtain

$$(4.24) \qquad \int_0^{1-t} \mathsf{E}_{t,b(t)}\Big(H\big(t{+}s, X_{t+s}\big) I\big(X_{t+s} < b(t{+}s)\big)\Big)\, ds = 0$$

which is exactly (4.4) as claimed.

Step 7. We show that b is the unique solution to the integral equation (4.4) in the class of continuous functions $t \mapsto b(t)$ on $[0,1]$ satisfying $b(t) \geq h(t)$ for all $t \in [0,1]$. This will be done in the four steps below.

Take any continuous function c satisfying $c \geq h$ which solves (4.24) on $[0,1]$, and define the continuous function $U^c : [0,1]\times I\!\!R_+ \rightarrow I\!\!R$ by

$$(4.25) \qquad U^c(t,x) = \mathsf{E}_{t,x}\bigg(\int_0^{1-t} H\big(t{+}s, X_{t+s}\big) I\big(X_{t+s} < c(t{+}s)\big)\, ds\bigg).$$

Note that c solving (4.24) means exactly that $U^c\big(t, c(t)\big) = 0$ for all $t \in [0,1]$. We now define the closed set $D_c := \{\, (t,x) \in [0,1] \times I\!\!R_+ \mid x \geq c(t) \,\}$ which will play the role of a "stopping set" for c. To avoid confusion we will denote by D_b our original optimal stopping set D defined by the function b in (4.9).

We show that $U^c = 0$ on D_c. The Markovian structure of X implies that the process

$$(4.26) \qquad N_s := U^c(t{+}s, X_{t+s}) + \int_0^s H\big(t{+}s, X_{t+s}\big) I\big(X_{t+s} < c(t{+}s)\big)\, ds$$

is a martingale under $\mathsf{P}_{t,x}$ for $s \in [0,1-t]$ and $(t,x) \in [0,1]\times I\!R_+$. Take any point $(t,x) \in D_c$ and consider the stopping time

$$\sigma_c = \inf\{\, 0 \le s \le 1-t \mid X_{t+s} \notin D_c \,\} = \inf\{\, 0 \le s \le 1-t \mid X_{t+s} \le c(t+s) \,\} \tag{4.27}$$

under the measure $\mathsf{P}_{t,x}$. Since $U^c\big(t,c(t)\big) = 0$ for all $t \in [0,1]$ and $U^c(1,x) = 0$ for all $x \in I\!R_+$, we see that $U^c(t+\sigma_c, X_{t+\sigma_c}) = 0$. Inserting σ_c in (4.26) above and using the optional sampling theorem (upon recalling that H is bounded), we get

$$U^c(t,x) = \mathsf{E}_{t,x}\big(U^c(t+\sigma_c, X_{t+\sigma_c})\big) = 0 \tag{4.28}$$

showing that $U^c = 0$ on D_c as claimed.

Step 8. We show that $U^c(t,x) \ge V(t,x)$ for all $(t,x) \in [0,1]\times I\!R_+$. To do this, take any $(t,x) \in [0,1]\times I\!R_+$ and consider the stopping time

$$\tau_c = \inf\{\, 0 \le s \le 1-t \mid X_{t+s} \in D_c \,\} \tag{4.29}$$

under $\mathsf{P}_{t,x}$. We then claim that $U^c(t+\tau_c, X_{t+\tau_c}) = 0$. Indeed, if $(t,x) \in D_c$ then $\tau_c = 0$ and we have $U^c(t,x) = 0$ by our preceding argument . Conversely, if $(t,x) \notin D_c$ then the claim follows since $U^c\big(t,c(t)\big) = U(1,x) = 0$ for all $t \in [0,1]$ and all $x \in I\!R_+$. Therefore inserting τ_c in (4.26) and using the optional sampling theorem, we see that

$$\begin{aligned} U^c(t,x) &= \mathsf{E}_{t,x} \int_0^{\tau_c} H\big(t+s, X_{t+s}\big) I\big(X_{t+s} \notin D_c\big)\, ds \\ &= \mathsf{E}_{t,x} \int_0^{\tau_c} H\big(t+s, X_{t+s}\big)\, ds \ge V(t,x) \end{aligned} \tag{4.30}$$

where the second identity follows by the definition of τ_c . This shows that $U^c \ge V$ as claimed.

Step 9. We show that $c(t) \le b(t)$ for all $t \in [0,1]$. Suppose that this is not the case and choose a point $(t,x) \in [0,1)\times I\!R_+$ so that $b(t) < c(t) < x$. Defining the stopping time

$$\sigma_b = \inf\{\, 0 \le s \le 1-t \mid X_{t+s} \notin D_b \,\} \tag{4.31}$$

under $\mathsf{P}_{t,x}$ and inserting it into the identities (4.22) and (4.26), we can take $\mathsf{E}_{t,x}$ on both sides and use the optional sampling theorem to see that

$$\mathsf{E}_{t,x}\big(V(t+\sigma_b, X_{t+\sigma_b})\big) = V(t,x) \tag{4.32}$$

$$\mathsf{E}_{t,x}\big(U^c(t+\sigma_b, X_{t+\sigma_b})\big) = U^c(t,x) - \mathsf{E}_{t,x}\left(\int_0^{\sigma_b} H\big(t+s, X_{t+s}\big) I\big(X_{t+s} \notin D_c\big)\, ds\right). \tag{4.33}$$

The fact that (t,x) belongs to both D_c and D_b implies that $V(t,x) = U^c(t,x) = 0$, and since $U^c \ge V$ we must have $U^c(t+\sigma_b, X_{t+\sigma_b}) \ge V(t+\sigma_b, X_{t+\sigma_b})$. Hence we find that

$$\mathsf{E}_{t,x}\left(\int_0^{\sigma_b} H\big(t+s, X_{t+s}\big) I\big(X_{t+s} \notin D_c\big)\, ds\right) \le 0\,. \tag{4.34}$$

The continuity of b and c , however, implies that there is a small enough $[t,u] \subseteq [t,1]$ such that $b(s) < c(s)$ for all $s \in [t,u]$. With strictly positive probability, therefore, the process X

will spend non-zero time in the region between $b(s)$ and $c(s)$ for $s \in [t,u]$, and this combined with the fact that both D_c and D_b are contained in $\{H \ge 0\}$, forces the expectation in (4.34) to be strictly positive and provides a contradiction. Hence we must have $c \le b$ on $[0,1]$ as claimed.

Step 10. We finally show that $c = b$ on $[0,1]$. Suppose that this is not the case. Choose a point $(t,x) \in [0,1) \times I\!R_+$ such that $c(t) < x < b(t)$ and consider the stopping time

$$\tau_D = \inf \{\, 0 \le s \le 1-t \mid X_{t+s} \in D_b \,\} \tag{4.35}$$

under the measure $\mathsf{P}_{t,x}$. Inserting τ_D in (4.22) and (4.26), taking $\mathsf{E}_{t,x}$ on both sides and using the optional sampling theorem, we see that

$$\mathsf{E}_{t,x} \left(\int_0^{\tau_D} H\big(t+s, X_{t+s}\big)\, ds \right) = V(t,x) \tag{4.36}$$

$$\mathsf{E}_{t,x}\big(U^c(t+\tau_D, X_{t+\tau_D})\big) = U^c(t,x) - \mathsf{E}_{t,x} \left(\int_0^{\tau_D} H\big(t+s, X_{t+s}\big) I\big(X_{t+s} \notin D_c\big)\, ds \right). \tag{4.37}$$

Since D_b is contained in D_c and $U^c = 0$ on D_c , we must have $U^c(t+\tau_D, X_{t+\tau_D}) = 0$, and using the fact that $U^c \ge V$ we get

$$\mathsf{E}_{t,x} \left(\int_0^{\tau_D} H\big(t+s, X_{t+s}\big) I\big(X_{t+s} \in D_c\big)\, ds \right) \le 0\,. \tag{4.38}$$

Then, as before, the continuity of b and c implies that there is a small enough $[t,u] \subseteq [t,1]$ such that $c(s) < b(s)$ for all $s \in [t,u]$. Since with strictly positive probability the process X will spend non-zero time in the region between $c(s)$ and $b(s)$ for $s \in [t,u]$, the same argument as before forces the expectation in (4.38) to be strictly positive and provides a contradiction. Hence we conclude that $b(t) = c(t)$ for all $t \in [0,1]$ completing the proof. □

References

[1] Doob, J. L. (1948). Heuristic approach to the Kolmogorov-Smirnov theorems. *Ann. Math. Statist.* 20 (393–403).

[2] Du Toit, J. *and* Peskir, G. (2007). The trap of complacency in predicting the maximum. *Ann. Probab.* 35 (340–365).

[3] Graversen, S. E. *and* Shiryaev, A. N. (2000). An extension of P. Lévy's distributional properties to the case of a Brownian motion with drift. *Bernoulli* 6 (615–620).

[4] Graversen, S. E. Peskir, G. *and* Shiryaev, A. N. (2001). Stopping Brownian motion without anticipation as close as possible to its ulimate maximum. *Theory Probab. Appl.* 45 (125–136).

[5] Karatzas, I. *and* Shreve, S. E. (1998). *Methods of Mathematical Finance.* Springer.

[6] Kolodner, I. I. (1956). Free boundary problem for the heat equation with applications to problems of change of phase I. General method of solution. *Comm. Pure Appl. Math.* 9 (1–31).

[7] Malmquist, S. (1954). On certain confidence contours for distribution functions. *Ann. Math. Statist.* 25 (523–533).

[8] Pedersen, J. L. (2003). Optimal prediction of the ultimate maximum of Brownian motion. *Stoch. Stoch. Rep.* 75 (205–219).

[9] Peskir, G. (2005). A change-of-variable formula with local time on curves. *J. Theoret. Probab.* 18 (499–535).

[10] Peskir, G. (2006). On reflecting Brownian motion with drift. *Proc. Symp. Stoch. Syst.* (Osaka, 2005), ISCIE Kyoto (1–5).

[11] Peskir, G. *and* Shiryaev, A. N. (2006). *Optimal Stopping and Free-Boundary Problems.* Lectures in Mathematics, ETH Zürich, Birkhäuser.

[12] Shiryaev, A. N. (2002). Quickest detection problems in the technical analysis of the financial data. *Proc. Math. Finance Bachelier Congress* (Paris, 2000), Springer (487–521).

[13] Shiryaev, A. N. (2004). A remark on the quickest detection problems. *Statist. Decisions* 22 (79–82).

[14] Urusov, M. A. (2005). On a property of the moment at which Brownian motion attains its maximum and some optimal stopping problems. *Theory Probab. Appl.* 49 (169–176).

Jacques du Toit
School of Mathematics
The University of Manchester
Oxford Road
Manchester M13 9PL
United Kingdom
Jacques.Du-Toit@postgrad.manchester.ac.uk

Goran Peskir
School of Mathematics
The University of Manchester
Oxford Road
Manchester M13 9PL
United Kingdom
goran@maths.man.ac.uk